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# $U(1)$ symmetry of the nontribimaximal pattern in the degenerate mass spectrum case of the neutrino mass matrix

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On account of the new neutrino oscillation data signaling a nonzero value for the smallest mixing angle ($\theta_z$), we present an explicit realization of the underlying $U(1)$ symmetry characterizing the maximal atmospheric mixing angle ($\theta_y = \frac{\pi}{4}$) pattern with two degenerate masses but now with generic values of $\theta_z$. We study the effects of the form invariance with respect to $U(1)$, and/or $Z_3$, $Z_2$ subgroups, on the Yukawa couplings and the mass terms. Later on, we specify $\theta_z$ to its experimental best fit value ($\sim 8^o$), and impose the symmetry in an entire model which includes charged leptons, and many Higgs doublets or standard model singlet heavy scalars, to show that it can make room for the charged lepton mass hierarchies. In addition, we show for the non-tribimaximal value of $\theta_z \neq 0$ within a type-I seesaw mechanism enhanced with flavor symmetry that neutrino mass hierarchies can be generated. Furthermore, lepton/baryogenesis can be interpreted via a type-II seesaw mechanism within a setup meeting the flavor $U(1)$ symmetry.

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## I. INTRODUCTION

The lepton sector has quite a different pattern from that of the quarks mainly in two respects. First, the mixing among gauge eigenstates in leptons to form mass eigenstates is more pronounced than in the quark sector, although the mass spectrum of the charged leptons exhibits a similar hierarchy to the quarks' spectrum [1]. Second, the neutrino masses are quite small when compared with all other mass scales, and this invokes some ways to understand this smallness, most popular of which is the seesaw mechanism within grand unification [2]. As to the neutrino mass hierarchies, they are not yet determined experimentally, and many models based on flavor symmetry considerations were constructed in order to account for the experimental data on neutrino masses and mixing (Ref. [3] and references therein).

The "symmetric" neutrino mass matrix $M_\nu$ is diagonalized by a single unitary mixing matrix $U_L^\nu$ as follows:

$$M_\nu = U_L^\nu M_\nu^{\text{diag}} (U_L^\nu)^T, \qquad M_\nu^{\text{diag}} = \text{diag}(m_1, m_2, m_3). \quad (1)$$

There are many possible parametrizations of the neutrino mixing matrix $U_L^\nu$, and we opt for the one in which the Dirac phase $\delta$ does not appear in the effective mass term of the neutrinoless double decay [4]. In this adopted parametrization, the mixing matrix $U_L^\nu$ is parametrized by three rotation angles $(\theta_x, \theta_y, \theta_z)$ and, in addition to $\delta$, two Majorana phases $(\rho, \sigma)$ as follows:

$$\begin{aligned} U_L^\nu &= R_{23}(\theta_y) \cdot R_z(\delta) \cdot R_{12}(\theta_x) \cdot P, \\ P &= \text{diag}(e^{i\rho}, e^{i\sigma}, 1), \\ R_z(\delta) &= \begin{pmatrix} c_z & 0 & s_z \\ 0 & e^{-i\delta} & 0 \\ -s_z & 0 & c_z \end{pmatrix}, \end{aligned} \quad (2)$$

with $s_x \equiv \sin\theta_x$, $c_y \equiv \cos\theta_y$, $t_z \equiv \tan\theta_z$ (for later use), and so on. As to $R_{12}$ and $R_{23}$, they are rotations around the $z$ and $x$ axes respectively, while $m_i$'s are the masses of the neutrino mass states, leading to a mixing matrix of the form

$$U_L^\nu = \begin{pmatrix} c_x c_z & s_x c_z & s_z \\ -c_x s_y s_z - s_x c_y e^{-i\delta} & -s_x s_y s_z + c_x c_y e^{-i\delta} & s_y c_z \\ -c_x c_y s_z + s_x s_y e^{-i\delta} & -s_x c_y s_z - c_x s_y e^{-i\delta} & c_y c_z \end{pmatrix} \cdot \text{diag}(e^{i\rho}, e^{i\sigma}, 1). \quad (3)$$

There is a simple relation, discussed in Refs. [5,6], between this adopted parametrization and the standard one used, say, in the recent data analysis of Fogli *et al.* [7].

In a similar way to the uncharged neutrinos case, the generally nonsymmetric charged lepton mass matrix linking the left handed (LH) leptons to their right handed (RH) counterparts can be diagonalized by a bi-unitary transformation:

$$M_l = U_L^l \cdot \text{diag}(m_e, m_\mu, m_\tau) \cdot (U_R^l)^\dagger. \quad (4)$$

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We work in the flavor basis where the charged leptons mixing matrix is equal to the identity matrix $U_L^l = \mathbf{1}$, whence the flavor mixing matrix $V \equiv (U_L^l)^\dagger U_L^\nu = U_L^\nu$, which can be constrained by observational data, comes wholly from the neutrino sector in the flavor basis.

The authors of Ref. [8] noticed that the experimental data excluding the phases lead approximately to a specific pattern dubbed tribimaximal (TB):

$$U_L^\nu \simeq V^{\mathrm{TB}} \equiv \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & +1/\sqrt{2} \end{pmatrix}, \tag{5}$$

coming down to $\theta_x = \arcsin(\frac{1}{\sqrt{3}})$, $\theta_y = \frac{\pi}{4}$, and $\theta_z = 0$. It has been shown in Ref. [9] that the TB pattern is equivalent to a certain form for the $M_\nu$ in the flavor basis called "tripartite":

$$M_\nu^{\mathrm{diag}} = (V^{\mathrm{TB}})^T M_\nu^{\mathrm{TB}} V^{\mathrm{TB}} \Leftrightarrow M_\nu^{\mathrm{TB}} = M_A + M_B + M_C, \tag{6}$$

where

$$M_A = A\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_B = B\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$M_C = C\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \tag{7}$$

with neutrino eigenmasses related to the tripartite coefficients via

$$m_1 = A - B, \quad m_2 = A - B + 3C, \quad m_3 = A + B,$$
$$A = (m_1 + m_3)/2, \quad B = (m_3 - m_1)/2, \quad C = (m_2 - m_1)/3. \tag{8}$$

Furthermore, a symmetry $(Z_3 \times Z_2)$ for the "bipartite" form $(M_A + M_B)$ corresponding to a degenerate mass spectrum was given:

$$S_3^{\mathrm{TB}} = \begin{pmatrix} -1/2 & -\sqrt{3/8} & \sqrt{3/8} \\ \sqrt{3/8} & 1/4 & 3/4 \\ -\sqrt{3/8} & 3/4 & 1/4 \end{pmatrix} : (S_3^{\mathrm{TB}})^3 = \mathbf{1},$$
$$S_2^{\mathrm{TB}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} : (S_2^{\mathrm{TB}})^2 = 1. \tag{9}$$

The degenerate mass case corresponds to an $(O_2 \times Z_2^z)$ symmetry, where the $O(2)$ group corresponds to the eigenspace of the degenerate mass eigenvalue, whereas the $Z_2^z$ concerns the third mass value representing a reflection [$I_z = \mathrm{diag}(1, 1, -1)$] across its axis ($z$), hence the superscript $z$. In Ref. [10], we presented a specific realization of the "continuous" $U(1)$ symmetry characterizing the degenerate mass bipartite form $(M_A + M_B)$ and studied its phenomenological consequences, be it in the corresponding current and conserved charges or in the possibility to implement it in setups allowing for lepton mass hierarchies and baryogenesis.

Our method to find symmetry realizations characterizing the neutrino mass matrix in the flavor space consists in searching all the unitary matrices $S$ satisfying what was called in Ref. [11] the form invariance formula:

$$S^T M_\nu S = M_\nu. \tag{10}$$

The term $M_C$ in the tripartite form breaks the $U(1)$ part of $O(2) = U(1) \times Z_2^y$,[1] as we shall see, into $Z_2' = \{I = \mathrm{diag}(1, 1, 1), -I_z = \mathrm{diag}(-1, -1, 1)\}$, and we are left with a $Z_2' \times Z_2^y \times Z_2^z$ discrete symmetry characterizing the most general nondegenerate symmetric neutrino mass matrix. Moreover, since the expression $(S^T M_\nu S)$ is the same for a matrix $(S)$ and its opposite $(-S)$, we deduce that $Z_2'$ is implied by $Z_2^z$ and the group $Z_2^y \times Z_2^z$ uniquely characterizes the tripartite form. In addition, as $I_z \circ I_y = -I_x$ (where $I_i$ is the reflection across the $i$ axis) we find that the characterizing group $Z_2^y \times Z_2^z$ is equivalent, as far as Eq. (10) is concerned, to the discrete group $(Z_2^3)$ corresponding to the three reflections $(I_x, I_y, I_z)$ across the axes in the diagonal basis. In Ref. [12], we found specific realizations of this $Z_2^3$ symmetry in any basis defined by the mixing matrix $V$.

Although the TB pattern proved successful phenomenologically [13], it was important to restudy these symmetries in light of the recent neutrino data departing from the TB pattern, in particular, for the $\theta_z$ angle whose vanishing TB value is no longer acceptable [7,14–17]. We would designate the non-tribimaximal pattern with $[\theta_y = \frac{\pi}{4},\ \theta_x = \arcsin(\frac{1}{\sqrt{3}}),\ \theta_z \neq 0]$ the NTB pattern, assigning a subscript zero mark ($\mathrm{NTB}_0$) for the case $\theta_z = \arcsin(\frac{1}{\sqrt{50}}) \sim 8^o$ which approximates well the best fit for $\theta_z$. This rather large value $8^o$ renders it implausible that the TB mixing pattern can be considered as a leading order approximation to be improved by perturbation towards the $\mathrm{NTB}_0$ pattern. Thus the need arises to construct models that can accommodate large $\theta_z$ from the outset.

In Ref. [6], we found the $Z_2^3$-symmetry realizations corresponding to the NTB pattern for generic $\theta_z$ and studied the phenomenological consequences for the $\mathrm{NTB}_0$ one. Since the degenerate mass spectrum case can be considered as a first step approximation perturbed by the $M_C$ part proportional to the neutrino mass splitting $(m_2 - m_1)$, it is of value to restudy this degenerate case,

[1]The $Z_2$ factor in the decomposition $O(2) = U(1) \times Z_2$ corresponds to the group consisting of the identity matrix and the reflection across one of the lines in the plane, say the y axis, hence the notation $Z_2^y$.

but corresponding to the NTB pattern, which is just the objective of this paper. We note that the $\theta_x$ angle is irrelevant in the degenerate case ($m_1 = m_2$), since the corresponding rotation matrix $R_{12}(\theta_x)$ commutes with the degenerate mass matrix, so we call any quantity corresponding to the NTB pattern in the degenerate mass case by the distinctive acronym DNTB specified by ($\theta_y = \frac{\pi}{4}$, $\theta_z \neq 0$, $m_1 = m_2$), whereas the special degenerate nontribimaximal pattern $\mathrm{DNTB}_0$ is specified by $[\theta_y = \frac{\pi}{4}, \theta_z = \arcsin(\frac{1}{\sqrt{50}}), m_1 = m_2]$.

The plan of the paper is as follows. In Sec. II A, which is devoted to the analysis of the $U(1)$ symmetry, we find an explicit realization of the $U(1)$ symmetry, henceforth called the $S$ symmetry, leading to the DNTB pattern, and we deduce the modified bipartite form characterizing it. Also in this subsection, we find the form of the Yukawa couplings dictated by the $S$ symmetry. In Sec. II B, we infer the corresponding constraints on the forms of the mass matrix and the Yukawa couplings imposed by the $Z_3$ and $Z_2$ subgroups of $U(1)$, and justify the phenomenological equivalence between the continuous $U(1)$ $S$ symmetry and the $Z_3$ discrete symmetry. In Sec. II C, we state all the preceding subsections' results but corresponding now to the "observed" $\mathrm{DNTB}_0$ pattern, whose $S$ symmetry would be named $S_0$ symmetry. We use the latter results in Sec. III within models containing charged leptons to show the possibility of generating the observed charged lepton mass hierarchies. Section III A involves many Higgs doublets, while many standard model (SM) singlet heavy scalars are involved in Sec. III B keeping, only one SM-Higgs doublet. We also study the current associated with the continuous $S$ symmetry in Sec. III C. In Sec. IV, we study the lepton $S$ symmetry in setups involving seesaw mechanisms. Section IV A gives a type-I seesaw mechanism, showing how to accommodate all kinds of neutrino mass hierarchies but with no lepton/baryogenesis. We need a type-II seesaw mechanism enriched with flavor symmetry in order to account for baryogenesis, as we show in Sec. IV B. We end with a summary and conclusions in Sec. V.

## II. ANALYSIS OF THE UNDERLYING SYMMETRY OF THE DNTB PATTERN

If we just restrict the atmospheric angle to its maximal value ($\theta_y = \frac{\pi}{4}$), leaving $\theta_x$ and $\theta_z$ general, we get the mixing matrix (ignoring phases)

$$V^{xz} = \begin{pmatrix} c_z c_x & c_z s_x & s_z \\ -\frac{1}{\sqrt{2}}(s_z c_x + s_x) & -\frac{1}{\sqrt{2}}(s_z s_x - c_x) & \frac{1}{\sqrt{2}} c_z \\ -\frac{1}{\sqrt{2}}(s_z c_x - s_x) & -\frac{1}{\sqrt{2}}(s_z s_x + c_x) & \frac{1}{\sqrt{2}} c_z \end{pmatrix}. \tag{11}$$

Then the neutrino mass matrix in Eq. (1) can be cast in a general tripartite form:

$$M_\nu = \begin{pmatrix} A - B + C & \frac{\sqrt{2} c_z s_z}{1-3s_z^2} B + \frac{(s_z c_{2x} - \frac{1}{2} s_{2x} c_z^2) c_z}{\sqrt{2} s_x c_x s_z (1-3s_z^2)} C & \frac{\sqrt{2} c_z s_z}{1-3s_z^2} B + \frac{s_z c_{2x} + \frac{1}{2} s_{2x}(1-5s_z^2)}{\sqrt{2} s_x c_x s_z (1-3s_z^2)} C \\ - & A + C & \frac{c_z^2}{1-3s_z^2} B + \frac{c_{2z} c_{2x} - s_z s_x c_x c_z^2}{s_x c_x s_z (1-3s_z^2)} C \\ - & - & A - C \end{pmatrix}, \tag{12}$$

where the missed entries from this point on are determined from the matrix being symmetric. The coefficients of this general tripartite form are given in terms of $\theta_x$, $\theta_z$ and the neutrino masses as follows:

$$\begin{aligned} A &= \frac{1}{2} c_z^2 \left(\frac{1}{2} - c_x^2\right)(m_1 - m_2) \\ &\quad + \frac{1}{2}\left(1 - \frac{1}{2} c_z^2\right)(m_1 + m_2) + \frac{1}{2} c_z^2 m_3, \\ B &= \frac{1}{2}\left[-3c_z^2\left(-\frac{1}{2} + c_x^2\right) + 2 s_z s_x c_x\right](m_1 - m_2) \\ &\quad + \frac{1}{2}\left(1 - \frac{3}{2} c_z^2\right)(m_1 + m_2) + \left(\frac{3}{2} c_z^2 - 1\right) m_3, \\ C &= s_x c_x s_z (m_1 - m_2). \end{aligned} \tag{13}$$

We restrict our study henceforth to the degenerate nontribimaximal pattern DNTB specified by ($\theta_y = \frac{\pi}{4}$, $m_1 = m_2$). We see directly from Eq. (13) that the $\theta_x$ dependence is dropped, as well as the perturbation part involving $C$, and we are left with a modified symmetric generic bipartite form:

$$M_\nu^{\mathrm{DNTB}} = \begin{pmatrix} A - B & \frac{\sqrt{2} s_z c_z}{1-3s_z^2} B & \frac{\sqrt{2} s_z c_z}{1-3s_z^2} B \\ - & A & \frac{c_z^2}{1-3s_z^2} B \\ - & - & A \end{pmatrix}, \tag{14}$$

where the neutrino eigenmasses and the coefficients of this generic bipartite form are related by

$$\begin{aligned} m_1 &= m_2 = A - \frac{c_z^2}{1 - 3s_z^2} B, \\ A &= \left(1 - \frac{1}{2} c_z^2\right) m_1 + \frac{1}{2} c_z^2 m_3, \\ m_3 &= A + \frac{1 + s_z^2}{1 - 3s_z^2} B, \\ B &= \left(1 - \frac{3}{2} c_z^2\right) m_1 + \left(\frac{3}{2} c_z^2 - 1\right) m_3. \end{aligned} \tag{15}$$

The mass spectrum of this generic bipartite form of the neutrino mass matrix can account for all types of neutrino mass hierarchies as follows (assuming small $\theta_z$):

(i) Normal hierarchy:

$$A \simeq B \Rightarrow m_1 = m_2 \ll m_3, \tag{16}$$

(ii) Inverted hierarchy:

$$A \simeq -B \Rightarrow m_3 \ll m_1 = m_2, \tag{17}$$

(iii) Degenerate case:

$$A \gg B \Rightarrow m_1 = m_2 \simeq m_3. \tag{18}$$

Some "mass relations," characterizing the nontriplicity in the degenerate case, can be deduced here as

$$\begin{aligned} &M_\nu^{\rm DNTB}(1,2) = M_\nu^{\rm DNTB}(1,3),\\ &M_\nu^{\rm DNTB}(1,2) = \sqrt{2}t_z M_\nu^{\rm DNTB}(2,3),\\ &M_\nu^{\rm DNTB}(1,1) = M_\nu^{\rm DNTB}(2,2) - \frac{1-3s_z^2}{\sqrt{2}s_z c_z} M_\nu^{\rm DNTB}(1,2),\\ &M_\nu^{\rm DNTB}(1,1) = M_\nu^{\rm DNTB}(2,2) - \frac{1-3s_z^2}{c_z^2} M_\nu^{\rm DNTB}(2,3), \end{aligned} \tag{19}$$

which for vanishing $\theta_z$ reduce, as expected, to the simpler ones characterizing the triplicity in the degenerate case, namely,

$$\begin{aligned} M_\nu^{\rm DTB}(1,2) = M_\nu^{\rm DTB}(1,3) = 0,\\ M_\nu^{\rm DTB}(2,2) - M_\nu^{\rm DTB}(2,3) = M_\nu^{\rm DTB}(1,1). \end{aligned} \tag{20}$$

Before carrying out the analysis, it is important at this stage to quantify the $C$ term [Eq. (13)] breaking the $O(2)$ symmetry. This $C$ breaking term is proportional to the mass splitting $(m_2 - m_1 \geq 0)$, whereas the other terms conserving the $O(2)$ symmetry are proportional to $(m_1 + m_2)$. This helps in estimating the relative size of the breaking term. In this way, a small value of the ratio

$$r = \frac{m_2 - m_1}{m_2 + m_1} \tag{21}$$

would indicate that the $O(2)$ symmetry is satisfied to a good approximation.

It is worth noting here that, in many theoretically well-justified and experimentally acceptable patterns for the neutrino mass matrix, numerical outcomes lead to $\frac{m_2}{m_1} < 1.05$, implying $r < 1 - \frac{m_1}{m_2} < 5\%$ [5]. In a model-independent way, we write down in Table I the latest global fit carried out in Ref. [7] for the mixing angles and the solar $(\delta m^2)$ and atmospheric $(|\Delta m^2|)$ mass-squared differences defined by

TABLE I. The latest global-fit results of the three neutrino mixing angles $(\theta_x, \theta_y, \theta_z)$ and the two neutrino mass-squared differences $\delta m^2$ and $\Delta m^2$ as defined in Eq. (22). Here, it is assumed that $\cos\delta = \pm 1$ and that new reactor fluxes have been used [7].

| Parameter | Bestfit | $2\sigma$ range |
|---|---|---|
| $\delta m^2 (10^{-5}\ \mathrm{eV}^2)$ | 7.58 | [7.16, 7.99] |
| $\lvert\Delta m^2\rvert (10^{-3}\ \mathrm{eV}^2)$ | 2.35 | [2.17, 2.57] |
| $\theta_x$ | 33.58° | [31.95°, 36.09°] |
| $\theta_y$ | 40.40° | [36.87°, 50.77°] |
| $\theta_z$ | 8.33° | [6.29°, 11.68°] |
| $R_\nu$ | 0.0323 | [0.0279, 0.0368] |

$$\delta m^2 \equiv m_2^2 - m_1^2, \quad |\Delta m^2| \equiv \left| m_3^2 - \frac{1}{2}(m_1^2 + m_2^2) \right|, \tag{22}$$

and also for the parameter $(R_\nu)$ characterizing the hierarchy between these two quantities:

$$R_\nu \equiv \frac{\delta m^2}{|\Delta m^2|}. \tag{23}$$

We can estimate the ratio $(r = \frac{\delta m^2}{(m_1+m_2)^2})$ in all three types of neutrino mass hierarchies as follows. In the normal hierarchy $(m_1 \leq m_2 \ll m_3)$, we note that the experimental data excluding two vanishing neutrino masses forbid, in the case of a degenerate mass spectrum, a zero value for $m_1$ corresponding to $(r = 1)$. However, even when approaching this "extreme" case of $m_1 \sim 0$, the part proportional to $m_3$ in the neutrino mass matrix [Eqs. (12) and (13)], which does not affect the $O(2)$ symmetry, would be preponderant compared to the $O(2)$-conserving term proportional to $(m_2 + m_1)$ and the $O(2)$-breaking term proportional to $(m_2 - m_1)$, as we have here

$$\begin{aligned} (m_2 + m_1)/m_3 \sim (m_2 - m_1)/m_3 \sim m_2/m_3\\ \sim \mathcal{O}(\sqrt{\delta m^2/\Delta m^2}) \sim \mathcal{O}(\sqrt{R_\nu}) \sim 18\%. \end{aligned} \tag{24}$$

For a "large" nonvanishing value of $m_1$, the ratio $r$ can take quite small values, and as an estimate we evaluate $r$ when $m_1^2 \sim \delta m^2$, leading to $m_2^2 \sim 2\delta m^2$, hence $r \sim 1/(1+\sqrt{2})^2 < 18\%$. The larger $m_1$, the smaller $r$, so for example, when $(m_1 + m_2)^2$ is of order $\mathcal{O}(|\Delta m^2|)$, then $r = \mathcal{O}(R_\nu) = \mathcal{O}(3\%)$. In the inverted hierarchy case $(m_3 \ll m_1 \leq m_2)$, one can estimate $(m_3)$ by a very tiny value, so we get $(m_1 \sim m_2 \sim \sqrt{|\Delta m^2|})$, and $r \sim R_\nu/4 \sim 0.8\%$. Finally, in the degeneracy spectrum case $(m_1 \sim m_2 \sim m_3 \sim m_0)$, we should have $(m_0 \geq \sqrt{|\Delta m^2|})$, and so $r < 0.8\%$. Thus, we can say that both experimental and numerical results corroborate the degenerate mass case as a good starting approximation for the nondegenerate spectrum case.

### A. $U(1)$ symmetry in the DNTB pattern

In order to find the symmetries imposing the form of Eq. (14), we see that any unitary matrix $U$ satisfying the form invariance in the "diagonalized" basis,

$$U^{\mathrm{T}} \cdot M_\nu^{\mathrm{D,diag}} \cdot U = M_\nu^{\mathrm{D,diag}} \equiv \mathrm{diag}(m_1, m_1, m_3), \tag{25}$$

corresponds to a unitary matrix

$$S^{\mathrm{DNTB}} = (V^{\mathrm{xz}})^* \cdot U \cdot (V^{\mathrm{xz}})^{\mathrm{T}}, \tag{26}$$

satisfying the form invariance [Eq. (10)] in the degenerate mass spectrum case:

$$(S^{\mathrm{DNTB}})^{\mathrm{T}} \cdot M_\nu^{\mathrm{DNTB}} \cdot S^{\mathrm{DNTB}} = M_\nu^{\mathrm{DNTB}}, \tag{27}$$

where

$$M_\nu^{\mathrm{D,diag}} = (V^{\mathrm{xz}})^{\mathrm{T}} \cdot M_\nu^{\mathrm{DNTB}} \cdot V^{\mathrm{xz}}. \tag{28}$$

It is clear now that the unitary matrices $U$ satisfying Eq. (25) represent a group $O(2) \times Z_2$, where $Z_2 = \{\mathrm{I}, \mathrm{I}_z\}$, while the orthogonal group $O(2)$ is a direct product of rotations ($SO(2) \cong U(1)$) in the degenerate eigenspace and another $Z_2$ representing a reflection in this space. If, for continuity purposes, we restrict ourselves to the connected component of the unity, then we have

$$U(\theta) = R_{12}(\theta) \equiv \begin{pmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{29}$$

We thus deduce the "continuous" $S$ symmetry in the flavor basis by applying Eq. (26):

$$S_\theta^{\mathrm{DNTB}} = \begin{pmatrix} c_\theta c_z^2 + s_z^2 & -\frac{c_z}{\sqrt{2}}[-s_\theta + s_z(c_\theta - 1)] & -\frac{c_z}{\sqrt{2}}[s_\theta + s_z(c_\theta - 1)] \\ -\frac{c_z}{\sqrt{2}}[s_\theta + s_z(c_\theta - 1)] & c_\theta - \frac{1}{2}c_z^2(c_\theta - 1) & s_\theta s_z - \frac{1}{2}c_z^2(c_\theta - 1) \\ -\frac{c_z}{\sqrt{2}}[-s_\theta + s_z(c_\theta - 1)] & -s_\theta s_z - \frac{1}{2}c_z^2(c_\theta - 1) & c_\theta - \frac{1}{2}c_z^2(c_\theta - 1) \end{pmatrix}. \tag{30}$$

We note that the $\theta_x$ dependence in Eq. (26) disappears since the two rotations around the third axis $R_{12}(\theta_x)$ and $R_{12}(\theta)$ commute. One can check now that this $S$ symmetry is equivalent to the DNTB modified generic bipartite form in that for all angles $\theta_z$ we have the following:

$$\{(M = M^{\mathrm{T}}) \wedge [\forall\ \theta, (S_\theta^{\mathrm{DNTB}})^{\mathrm{T}} \cdot M \cdot S_\theta^{\mathrm{DNTB}} = M]\} \Leftrightarrow \left[\exists\ A, B, C\colon M = \begin{pmatrix} A - B & \frac{\sqrt{2}s_z c_z}{1-3s_z^2}B & \frac{\sqrt{2}s_z c_z}{1-3s_z^2}B \\ - & A & \frac{c_z^2}{1-3s_z^2}B \\ - & - & A \end{pmatrix}\right], \tag{31}$$

If we drop the symmetric matrix condition in Eq. (31), then we get for all angles $\theta_z$ the following equivalence (which will prove useful when studying the general form of the Yukawa couplings in the adopted Lagrangian with specific field transformations under $S$ symmetry):

$$[\forall\ \theta, (S_\theta^{\mathrm{DNTB}})^{\mathrm{T}} \cdot f \cdot S_\theta^{\mathrm{DNTB}} = f] \Leftrightarrow \left[\exists\ A, B, C\colon f = \begin{pmatrix} A - B + \sqrt{2}t_z C & C & \frac{\sqrt{2}s_{2z}}{1-3s_z^2}B - \frac{1+s_z^2}{1-3s_z^2}C \\ \frac{\sqrt{2}s_{2z}}{1-3s_z^2}B - \frac{1+s_z^2}{1-3s_z^2}C & A & B \\ C & \frac{1+s_z^2}{1-3s_z^2}B - \frac{\sqrt{2}s_{2z}}{1-3s_z^2}C & A \end{pmatrix}\right]. \tag{32}$$

Note that the equivalence Eq. (31) can be deduced from that of Eq. (32) by the following substitution:

$$B \longrightarrow \frac{c_z^2}{1 - 3s_z^2}B, \qquad C \longrightarrow \frac{\sqrt{2}s_z c_z}{1 - 3s_z^2}B. \tag{33}$$

Also, it is useful to have the following equivalence corresponding to a "left-congruous" form invariance:

$$[\forall\ \theta, (S_\theta^{\mathrm{DNTB}})^{\mathrm{T}} \cdot f = f] \Leftrightarrow \left[\exists\ A, B, C\colon f = \begin{pmatrix} \sqrt{2}t_z A & \sqrt{2}t_z B & \sqrt{2}t_z C \\ A & B & C \\ A & B & C \end{pmatrix}\right]. \tag{34}$$

One last note in this subsection is that we have neglected the Majorana phases in our discussion so far. However, diagonalizing the modified generic bipartite form for the neutrino mass matrix in the DNTB pattern, we have

$$\begin{pmatrix} A-B & \frac{\sqrt{2}s_zc_z}{1-3s_z^2}B & \frac{\sqrt{2}s_zc_z}{1-3s_z^2}B \\ - & A & \frac{c_z^2}{1-3s_z^2}B \\ - & - & A \end{pmatrix} = V_0^x \cdot P \cdot \mathrm{diag}\left( \left| A - \frac{c_z^2}{1-3s_z^2}B \right|, \left| A - \frac{c_z^2}{1-3s_z^2}B \right|, \left| A + \frac{1+s_z^2}{1-3s_z^2}B \right| \right) \cdot P^T \cdot (V_0^x)^T, \tag{35}$$

where $P$, a diagonal phase matrix, is given by

$$P = \mathrm{diag}(e^{i\alpha}, e^{i\alpha}, e^{i\beta}), \qquad 2\alpha = \arg\left(A - \frac{c_z^2}{1-3s_z^2}B\right), \qquad 2\beta = \arg\left(A + \frac{1+s_z^2}{1-3s_z^2}B\right). \tag{36}$$

We can absorb the $\beta$ phase by an "unphysical" global phase shift of the neutrino fields ($\nu_i \to e^{-i\beta}\nu_i$) in the neutrino mass term ($M_{ij}^{\mathrm{DNTB}_0}\nu_i\nu_j$), so when we compare with Eqs. (1)–(3) we find

$$\rho = \sigma, \qquad \delta = 0, \qquad \theta_y = \pi/4. \tag{37}$$

Conversely, starting from the following general expressions of the elements of the degenerate mass matrix [resulting from Eqs. (1)–(3) with $m_1 = m_2$],

$$\begin{aligned}
M_{\nu 11} &= m_1 c_z^2(c_x^2 e^{2i\rho} + s_x^2 e^{2i\sigma}) + m_3 s_z^2,\\
M_{\nu 12} &= m_1[-c_z s_z s_y(c_x^2 e^{2i\rho} + s_x^2 e^{2i\sigma}) + c_z c_x s_x c_y(e^{i(2\sigma-\delta)} - e^{i(2\rho-\delta)})] + m_3 c_z s_z s_y,\\
M_{\nu 13} &= m_1[-c_z s_z(c_x^2 s_y e^{2i\rho} + s_x^2 c_y e^{2i\sigma}) + c_z c_x s_x s_y(e^{i(2\rho-\delta)} - e^{i(2\sigma-\delta)})] + m_3 c_z s_z c_y,\\
M_{\nu 22} &= m_1[s_z^2 s_y^2(c_x^2 e^{2i\rho} + s_x^2 e^{2i\sigma}) + c_y^2(s_x^2 e^{2i(\rho-\delta)} + c_x^2 e^{2i(\sigma-\delta)}) + 2c_y c_x s_x s_z s_y(e^{i(2\rho-\delta)} - e^{i(2\sigma-\delta)})] + m_3 c_z^2 s_y^2,\\
M_{\nu 33} &= m_1[s_z^2 c_y^2(c_x^2 e^{2i\rho} + s_x^2 e^{2i\sigma}) + s_y^2(s_x^2 e^{2i(\rho-\delta)} + c_x^2 e^{2i(\sigma-\delta)}) + 2c_y c_x s_x s_z s_y(-e^{i(2\rho-\delta)} + e^{i(2\sigma-\delta)})] + m_3 c_z^2 c_y^2,\\
M_{\nu 23} &= m_1[c_y s_y s_z^2(c_x^2 e^{2i\rho} + s_x^2 e^{2i\sigma}) + s_z c_x s_x c_{2y}(e^{i(2\rho-\delta)} - e^{i(2\sigma-\delta)}) - c_y s_y(s_x^2 e^{i(2\rho-\delta)} + c_x^2 e^{i(2\sigma-\delta)})] + m_3 s_y c_y c_z^2,
\end{aligned} \tag{38}$$

and requiring them to correspond to the modified generic bipartite form [Eq. (14)], so that the $\theta_x$ dependence in the mass matrix elements drops out, we get $\rho = \sigma$ and $\theta_y = \pi/4$. Moreover, the last mass relation in Eq. (19) for the DNTB bipartite form leads to $\delta = 0$ since $M_{\nu 11}$ in Eq. (38)is $\delta$ independent, whereas $M_{\nu_{22}}$ and $M_{\nu_{23}}$ depend on $\delta$. We conclude then that for the $S$ symmetry to be satisfied by the degenerate neutrino mass matrix, we need to have $\rho = \sigma$ and $\delta = 0$.

### B. The $Z_2$ and $Z_3$ subgroups in the DNTB pattern

As we explained earlier in the Introduction, the symmetry $Z_2 \times Z_3$ mentioned in Ref. [9] to characterize the degenerate mass spectrum case is a special case of the $U(1)$ $S$ symmetry we stated in the previous subsection. The $Z_3$ symmetry corresponding to the DNTB pattern can be found by putting $\theta = -\frac{2\pi}{3}$ in Eq. (30):

$$S_3^{\mathrm{DNTB}} \equiv S_{\theta=-2\pi/3}^{\mathrm{DNTB}} = \begin{pmatrix} -\frac{3}{2}c_z^2 + 1 & -\frac{\sqrt{2}}{4}c_z(\sqrt{3} - 3s_z) & \frac{\sqrt{2}}{4}c_z(\sqrt{3} + 3s_z) \\ \frac{\sqrt{2}}{4}c_z(\sqrt{3} + 3s_z) & -\frac{1}{2} + \frac{3}{4}c_z^2 & -\frac{\sqrt{3}}{2}s_z + \frac{3}{4}c_z^2 \\ -\frac{\sqrt{2}}{4}c_z(\sqrt{3} - 3s_z) & \frac{\sqrt{3}}{2}s_z + \frac{3}{4}c_z^2 & -\frac{1}{2} + \frac{3}{4}c_z^2 \end{pmatrix}. \tag{39}$$

As for the $Z_2$ symmetry corresponding to the DNTB pattern,[2] it can be obtained by substituting $\theta$ by $\pi$:

$$S_2^{\mathrm{DNTB}} \equiv S_{\theta=\pi}^{\mathrm{DNTB}} = \begin{pmatrix} -c_{2z} & \frac{1}{\sqrt{2}}s_{2z} & \frac{1}{\sqrt{2}}s_{2z} \\ \frac{1}{\sqrt{2}}s_{2z} & -s_z^2 & c_z^2 \\ \frac{1}{\sqrt{2}}s_{2z} & c_z^2 & -s_z^2 \end{pmatrix}. \tag{40}$$

[2] We denoted, in the Introduction, this $Z_2$ subgroup of $U(1)$ by $Z_2'$. However, we shall drop the$'$ mark, as it is clear from the context which $Z_2$ is meant.

One can find similar equivalences to Eqs. (31), (32), and (34) corresponding to these subgroups. For the $Z_2$ subgroup we find for all angles $\theta_z$ the following:

$$\begin{aligned}
&[(S_2^{\mathrm{DNTB}})^{\mathrm{T}} \cdot f \cdot S_2^{\mathrm{DNTB}} = f] \Leftrightarrow \Big[\exists\ a_1, a_2, c_1, c_2, c_3:\\
&\quad f(1,1) = a_1, f(1,2) = a_2, f(1,3) = -a_2 + 2t_z^2 c_1 + \sqrt{2}t_z(c_3 + c_2 - a_1),\\
&\quad f(2,1) = -\frac{1-3s_z^2}{c_z^2}c_1 + \sqrt{2}t_z(c_3 + c_2 - a_1), f(2,2) = c_3 + \sqrt{2}t_z(c_1 - a_2),\\
&\quad f(2,3) = \frac{1-3s_z^2}{c_z^2}(c_2 + \sqrt{2}t_z c_1) + \sqrt{2}t_z[a_2 + \sqrt{2}t_z(a_1 - c_3)],\\
&\quad f(3,1) = c_1, f(3,2) = c_2, f(3,3) = c_3\Big]. \qquad (41)
\end{aligned}$$

In order to impose a symmetric matrix condition, which would be useful for a neutrino mass matrix, it suffices to make the following substitution,

$$c_2 \rightarrow a_1 - c_3 + \frac{1}{\sqrt{2}t_z}a_2 + \frac{1-3s_z^2}{\sqrt{2}c_z s_z}c_1, \qquad (42)$$

leaving us with four free parameters that can be cast in the following form for all angles $\theta_z$:

$$\begin{aligned}
&[(M = M^{\mathrm{T}}) \wedge (S_2^{\mathrm{DNTB}})^{\mathrm{T}} \cdot M \cdot S_2^{\mathrm{DNTB}} = M] \Leftrightarrow \Big[\exists\ A, B, C, D:\\
&M(1,1) = A - B + C, M(2,2) = A + C, M(2,3) = M(3,2) = D,\\
&M(1,2) = M(2,1) = \frac{-4\sqrt{2}s_{2z}s_z^3 B - \sqrt{2}c_z^3(10c_z^2 + \sqrt{2}s_z - 8)C + 4\sqrt{2}c_z(1-3s_z^2)D}{4s_z(1-3s_z^2)^2},\\
&M(1,3) = M(3,1) = \frac{2\sqrt{2}s_z^2 c_z B + c_z(5\sqrt{2}c_z^2 + s_z - 4\sqrt{2})C}{2s_z(1-3s_z^2)},\\
&M(3,3) = A - \frac{s_{2z}^2}{2(1-3s_z^2)^2}B - \frac{2s_z c_{2z} - 14\sqrt{2}c_z^2 + 11\sqrt{2}c_z^4 + 4\sqrt{2}}{\sqrt{2}(1-3s_z^2)^2}C + \frac{2s_z^2}{1-3s_z^2}D\Big]. \qquad (43)
\end{aligned}$$

We note that if

$$D = \frac{c_z^2}{1-3s_z^2}B + \frac{-s_{2z}c_z + 2\sqrt{2}c_z^2 - \sqrt{2}}{s_z(1-3s_z^2)}C, \qquad (44)$$

then we get exactly the modified generic tripartite form with three free parameters characterizing the NTB $Z_2^3$ symmetry obtained in Ref. [6]. This means that if a symmetric matrix satisfies the form invariance with respect to $Z_2^3$ symmetry, then it satisfies it for the $Z_2$ symmetry realized by $S_2^{\mathrm{DNTB}}$. This is clear due to the fact that $S_2^{\mathrm{DNTB}}$ is, up to a sign, just a factor of $Z_2^3$ symmetry, as can be seen trivially in the diagonalized basis. Moreover, if we put, in addition, $C = 0$ we get exactly the modified generic bipartite form with two free parameters [Eq. (14)] characterized by the $S$ symmetry. This is also evident since the latter bipartite form corresponds to a degenerate mass spectrum where $m_1 = m_2$, which is a special case of the general mass spectrum for the modified tripartite form. As for the equivalence in Eq. (34), corresponding to the symmetry acting only from the left, we get exactly the same form in both the $Z_2$ symmetry and the $U(1)$ $S$ symmetry.

We may think that we need to impose $Z_3 \times Z_2$ in order to characterize the modified generic bipartite form [Eq. (14)] in line with what was stated in Ref. [9]. However, in accordance with our findings in Ref. [10], we checked that imposing the $Z_3$ symmetry in the degenerate mass spectrum case, which is represented by $S_3^{\mathrm{DNTB}}$, leads precisely to the same equivalences in Eqs. (31), (32), and (34), so we can write, for all angles $\theta_z$, the following:

$$\begin{aligned}
&[(M = M^T) \wedge (\forall\ \theta, (S_\theta^{\mathrm{DNTB}})^{\mathrm{T}} \cdot M \cdot S_\theta^{\mathrm{DNTB}} = M)] \Leftrightarrow [(M = M^T) \wedge (\forall\ \theta, (S_3^{\mathrm{DNTB}})^T \cdot M \cdot S_3^{\mathrm{DNTB}} = M)]\\
&[\forall\ \theta, (S_\theta^{\mathrm{DNTB}})^{\mathrm{T}} \cdot f \cdot S_\theta^{\mathrm{DNTB}} = f] \Leftrightarrow [(S_3^{\mathrm{DNTB}})^T \cdot f \cdot S_3^{\mathrm{DNTB}} = f] \qquad (45)\\
&[\forall\ \theta, (S_\theta^{\mathrm{DNTB}})^T \cdot f = f] \Leftrightarrow [(S_3^{\mathrm{DNTB}})^T \cdot f = f] \Leftrightarrow [(S_2^{\mathrm{DNTB}})^T \cdot f = f].
\end{aligned}$$

Thus, the $Z_3$ symmetry and the $U(1)$ $S$ symmetry are phenomenologically equivalent regarding the form invariance, and the question arises as to what lies behind this fact. For this, we examine again what symmetries would characterize the form invariance formula [Eq. (10)], in the diagonal basis, restricting it to the 2-dim subspace corresponding to the mass eigenvalues $m_1$ and $m_2$. Any "special" unitary matrix in two dimensions is represented by a rotation, so we have

$$R^{\mathrm{T}}(\theta)\cdot \mathrm{diag}(m_1, m_2)R(\theta) = \mathrm{diag}(m_1, m_2)\colon\ R(\theta) = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} = c_\theta \mathrm{I} + i s_\theta \sigma_2, \tag{46}$$

where the identity matrix $I$ and the Pauli matrix $\sigma_2$ are given by

$$\mathrm{I} = \mathrm{diag}(1, 1), \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{47}$$

This leads to

$$c_\theta^2 \mathrm{diag}(m_1, m_2) + \frac{i}{2} s_{2\theta}[\mathrm{diag}(m_1, m_2), \sigma_2] + s_\theta^2 \sigma_2 \mathrm{diag}(m_1, m_2)\sigma_2 = \mathrm{diag}(m_1, m_2). \tag{48}$$

We see directly here that for any fixed given angle $\theta \neq \pi$, the relation in Eq. (48) cannot be met unless we have a degenerate mass spectrum ($m_1 = m_2$), so $Z_n$-symmetry, corresponding to $\theta = \frac{2\pi}{n}$, leads, as long as $n \neq 2$, to a degenerate spectrum and thus to the $U(1)$-symmetry, and vice versa, whence the mentioned equivalence. It is clear also now, that the residual symmetry after the breaking of $U(1)$ due to mass splitting $m_1 \neq m_2$ is the subgroup $Z_2$ corresponding to $\theta = \pi$, the only value satisfying Eq. (48) for a non-degenerate mass spectrum. This equivalence between $Z_n$ and $U(1)$ regarding the form invariance should be contrasted with the case of regular n-polygons which are not $U(1)$-invariant under the whole set of rotations around their centers by arbitrary angles, although they are $Z_n$-symmetric, in that they stay unchanged when the rotation angle is a multiple of $\frac{2\pi}{n}$.

### C. The $\mathrm{DNTB}_0$ pattern

In order to make definite conclusions and precise predictions for the phenomenological analyses in the following sections, we specify the results in this subsection to the experimentally best-fit degenerate mass spectrum case of the NTB pattern, the $\mathrm{DNTB}_0$ pattern characterized by

$$\theta_z = \arcsin\left(\frac{1}{\sqrt{50}}\right), \quad \theta_x \,\text{undetermined}, \quad m_1 = m_2. \tag{49}$$

The mixing matrix becomes

$$V_0^x = \frac{1}{10}\begin{pmatrix} 7\sqrt{2}c_x & 7\sqrt{2}s_x & \sqrt{2} \\ -(c_x + 5\sqrt{2}s_x) & -(s_x - 5\sqrt{2}c_x) & 7 \\ -(c_x - 5\sqrt{2}s_x) & -(s_x + 5\sqrt{2}c_x) & 7 \end{pmatrix}. \tag{50}$$

The modified special bipartite form is

$$M_\nu^{\mathrm{DNTB}_0} = \begin{pmatrix} A - B & \frac{7\sqrt{2}}{47}B & \frac{7\sqrt{2}}{47}B \\ \frac{7\sqrt{2}}{47}B & A & \frac{49}{47}B \\ \frac{7\sqrt{2}}{47}B & \frac{49}{47}B & A \end{pmatrix}, \tag{51}$$

and the eigenmasses can be determined in terms of the bipartite form coefficients as

$$\begin{aligned} m_1 = m_2 = A - \frac{49}{47}B, \qquad A = \frac{51}{100}m_1 + \frac{49}{100}m_3, \\ m_3 = A + \frac{51}{47}B, \qquad B = -\frac{47}{100}m_1 + \frac{47}{100}m_3. \end{aligned} \tag{52}$$

The $U(1)$ $S_0$ symmetry which characterizes this modified special bipartite form is given by

$$S_\theta^{\mathrm{DNTB}_0} = \begin{pmatrix} \frac{1}{50}(1 + 49c_\theta) & \frac{7\sqrt{2}}{100}(1 - c_\theta) + \frac{7}{10}s_\theta & \frac{7\sqrt{2}}{100}(1 - c_\theta) - \frac{7}{10}s_\theta \\ \frac{7\sqrt{2}}{100}(1 - c_\theta) - \frac{7}{10}s_\theta & \frac{1}{100}(49 + 51c_\theta) & \frac{49}{100}(1 - c_\theta) + \frac{\sqrt{2}}{10}s_\theta \\ \frac{7\sqrt{2}}{100}(1 - c_\theta) + \frac{7}{10}s_\theta & \frac{49}{100}(1 - c_\theta) - \frac{\sqrt{2}}{10}s_\theta & \frac{1}{100}(49 + 51c_\theta) \end{pmatrix}. \tag{53}$$

The corresponding $Z_3$ and $Z_2$ symmetries are given by

$$S_3^{\mathrm{DNTB}_0} \equiv S_{\theta=-2\pi/3}^{\mathrm{DNTB}_0} = \begin{pmatrix} -\frac{47}{100} & \frac{21\sqrt{2}}{200} - \frac{7\sqrt{3}}{20} & \frac{21\sqrt{2}}{200} + \frac{7\sqrt{3}}{20} \\ \frac{21\sqrt{2}}{200} + \frac{7\sqrt{3}}{20} & \frac{47}{200} & \frac{147}{200} - \frac{\sqrt{6}}{20} \\ \frac{21\sqrt{2}}{200} - \frac{7\sqrt{3}}{20} & \frac{147}{200} + \frac{\sqrt{6}}{20} & \frac{47}{200} \end{pmatrix}, \tag{54}$$

$$S_2^{\mathrm{DNTB}_0} \equiv S_{\theta=\pi}^{\mathrm{DNTB}_0} = \frac{1}{50}\begin{pmatrix} -48 & 7\sqrt{2} & 7\sqrt{2} \\ 7\sqrt{2} & -1 & 49 \\ 7\sqrt{2} & 49 & -1 \end{pmatrix}. \tag{55}$$

Note that, as expected, the $\theta_x$ dependence should not appear either in the bipartite form or in the characterizing symmetry.

We have the corresponding equivalences

$$\{(M = M^{\mathrm{T}}) \wedge [\forall\ \theta, (S_\theta^{\mathrm{DNTB}_0})^{\mathrm{T}} \cdot M \cdot S_\theta^{\mathrm{DNTB}_0} = M]\} \Leftrightarrow \{(M = M^{\mathrm{T}}) \wedge [(S_3^{\mathrm{DNTB}_0})^{\mathrm{T}} \cdot M \cdot S_3^{\mathrm{DNTB}_0} = M]\}$$
$$\Leftrightarrow \left[\exists\ A, B\colon M = \begin{pmatrix} A - B & \frac{7\sqrt{2}}{47}B & \frac{7\sqrt{2}}{47}B \\ \frac{7\sqrt{2}}{47}B & A & \frac{49}{47}B \\ \frac{7\sqrt{2}}{47}B & \frac{49}{47}B & A \end{pmatrix}\right] \tag{56}$$

and

$$[\forall\ \theta, (S_\theta^{\mathrm{DNTB}_0})^{\mathrm{T}} \cdot f \cdot S_\theta^{\mathrm{DNTB}_0} = f] \Leftrightarrow [(S_3^{\mathrm{DNTB}_0})^{\mathrm{T}} \cdot f \cdot S_3^{\mathrm{DNTB}_0} = f]$$
$$\Leftrightarrow \left[\exists\ A, B, C\colon f = \begin{pmatrix} A - B + \frac{\sqrt{2}}{7}C & C & \frac{14\sqrt{2}}{47}B - \frac{51}{47}C \\ \frac{14\sqrt{2}}{47}B - \frac{51}{47}C & A & B \\ C & \frac{51}{47}B - \frac{14\sqrt{2}}{47}C & A \end{pmatrix}\right] \tag{57}$$

and

$$[\forall\ \theta, (S_\theta^{\mathrm{DNTB}_0})^{\mathrm{T}} \cdot f = f] \Leftrightarrow [(S_3^{\mathrm{DNTB}_0})^{\mathrm{T}} \cdot f = f] \Leftrightarrow \left[\exists\ A, B, C\colon f = \begin{pmatrix} \frac{\sqrt{2}A}{7} & \frac{\sqrt{2}B}{7} & \frac{\sqrt{2}C}{7} \\ A & B & C \\ A & B & C \end{pmatrix}\right]. \tag{58}$$

One can also find the corresponding equivalences for the $Z_2$ symmetry for this special DNTB$_0$ pattern, and deduce that the resulting symmetric form contains the $Z_2^3$ modified special tripartite form attained in Ref. [6] [Eq. (12) with $\theta_x = \arcsin(1/\sqrt{3})$, $\theta_z = \arcsin(1/\sqrt{50})$], which in turn includes the modified special bipartite form.

## III. $S_0$ SYMMETRY IN THE WHOLE LEPTON SECTOR

We impose now the $S_0$ symmetry in a setup involving also the charged leptons, since their LH components couple to the neutrinos, and any symmetry imposed on the latter should be met by the LH charged leptons as well.

### A. Model with many Higgs doublets

We follow here the model presented in Ref. [9] and assume one heavy Higgs triplet $(\xi^{++}, \xi^+, \xi^0)$ and three scalar doublets $(\phi_i^0, \phi_i^-)$ playing the role of the SM Higgs doublet.

$$\mathcal{L}_Y = h_{ij}[\xi^0\nu_i\nu_j - \xi^+(\nu_i l_j + l_i\nu_j)/\sqrt{2} + \xi^{++}l_i l_j] + f_{ij}^k(l_i\phi_j^0 - \nu_i\phi_j^-)l_k^c + \text{H.c.}, \tag{59}$$

where under the $S_0$ symmetry the fields are transformed as

$$(\nu, l)_i \rightarrow (S_\theta)_{ij}(\nu, l)_j, \qquad l_k^c \rightarrow l_k^c, \tag{60}$$

$$(\phi^0, \phi^-)_i \rightarrow (S_\theta)_{ij}(\phi^0, \phi^-)_j,$$
$$(\xi^{++}, \xi^+, \xi^0) \rightarrow (\xi^{++}, \xi^+, \xi^0). \tag{61}$$

Invariance of the Lagrangian means we have

$$S_\theta^T h S_\theta = h, \tag{62}$$

$$S_\theta^T f^k S_\theta = f^k. \tag{63}$$

This Lagrangian has a global symmetry $U(1)_L \otimes S_0$, where $U(1)_L$ is the total lepton number symmetry, where we assign a zero lepton number to the doublets $\phi_i$ and a two-lepton number for the heavy triplet $\xi$. We now add a soft symmetry breaking "$\mu$"-term,

$$\delta\mathcal{L}_Y = \frac{\mu_{ij}}{2}\phi_i^T\xi^\dagger i\tau_2\phi_j + \text{H.c.}, \tag{64}$$

where the symmetric matrix $\mu_{ij}$ is not proportional to either the identity or the form dictated by $S_\theta$, so both the $U(1)_L$ symmetry and the $S_0$ symmetry are broken explicitly. This term is introduced to avoid a Goldstone boson (GB) associated with the spontaneous breaking of total lepton number (called a Majoron) [18]. This will have the effect of inducing a mass to the would-be GB of the order of the mass of the Higgs triplet, hence avoiding an invisible decay of the $Z$ gauge boson. In addition, this tadpole term will generate, upon minimization of the potential with respect to the neutral component of the triplet scalar $\xi^0$, a vacuum expectation value (vev) given by

$$\langle \xi^0 \rangle = \frac{-\mu_{ij} v_i v_j}{M_\xi^2}, \qquad (65)$$

which can be small in the electron volt range, in line with a naturally tiny neutrino mass, for $\mu_{ij} \sim M_\xi$ (the triplet mass) $\sim 10^{12}$ GeV [19,20].[3] Furthermore, the tree level correction to the electroweak (EW) $\rho$ parameter is $\sim (\langle \xi^0 \rangle / v_{\rm EW})^2 \sim (m_\nu / v_{\rm EW})^2$ which is negligible.

Moreover, this $\mu$-term in the scalar sector will not destabilize the structure of the neutrino mass matrix, since the latter is dictated by how the leptonic fields transform under $S_\theta$. However, there will be Yukawa-like interactions between the neutrinos and the pseudo-GB, and between the charged leptons and the electrically charged components of the triplet field. But since all these scalar fields (including the pseudo-GB) are much heavier than the TeV scale, they do not have observable effects on the fermion sector of the model. Said differently, the components of the triplet field will practically decouple from the low energy spectrum, and one is left only with the SM degrees of freedom plus effective higher dimensional operators suppressed by the mass square of the triplet (and a correction to the Higgs self-coupling) of the form

$$\mu \frac{LL\Phi\Phi}{M_\xi^2},$$

which after the EW symmetry breaking gives $m_\nu \sim \mu v^2 / M_\xi^2 \ll v$.

The equivalences [Eqs. (56) and (57)] for the symmetric matrix $h_{ij}$ and the not-necessarily symmetric Yukawa couplings $f_{ij}^k$ lead to the forms

$$h = \begin{pmatrix} A - B & \frac{7\sqrt{2}}{47} B & \frac{7\sqrt{2}}{47} B \\ \frac{7\sqrt{2}}{47} B & A & \frac{49}{47} B \\ \frac{7\sqrt{2}}{47} B & \frac{49}{47} B & A \end{pmatrix},$$
$$f^k = \begin{pmatrix} a_k - b_k + \frac{\sqrt{2}}{7} c_k & c_k & \frac{14\sqrt{2}}{47} b_k - \frac{51}{47} c_k \\ \frac{14\sqrt{2}}{47} b_k - \frac{51}{47} c_k & a_k & b_k \\ c_k & \frac{51}{47} b_k - \frac{14\sqrt{2}}{47} c_k & a_k \end{pmatrix}. \qquad (66)$$

The neutrino mass matrix, when $\xi^0$ gets a vev, is

$$(M_\nu)_{ij} = \langle \xi^0 \rangle h_{ij}, \qquad (67)$$

which expresses the translation of the $S_0$ symmetry from the symmetric Yukawa couplings $h_{ij}$ to the neutrino mass matrix $M_{\nu_{ij}}$. As to the charged leptons, the Yukawa term $(f_{ij}^k l_i l_k^c \phi_j^0)$ leads, when the SM-like Higgs fields take vevs $(v_j = \langle \phi_j^0 \rangle)$, to the mass matrix

$$(M_l)_{ik} = f_{ij}^k v_j. \qquad (68)$$

The Yukawa couplings can be arranged so as to bring, after suitably rotating the charged RH singlet lepton $l^c$, the charged lepton mass matrix into its form in the flavor basis. For example, if $v_{1,2} \ll v_3$ we have

$$M_l = v_3 \begin{pmatrix} A_1' & A_2' & A_3' \\ B_1' & B_2' & B_3' \\ C_1' & C_2' & C_3' \end{pmatrix}, \qquad (69)$$

where

$$A_i' = \frac{14\sqrt{2}}{47} b_i - \frac{51}{47} c_i, \qquad B_i' = b_i, \qquad C_i' = a_i. \qquad (70)$$

In Ref. [6], a charged lepton matrix of precisely the same form was shown to represent the lepton mass matrix in the flavor basis with the right charged lepton mass hierarchies, assuming just the ratios of the magnitudes of the vectors comparable to the lepton mass ratios.

### B. Model with many heavy SM singlets

The model with many Higgs doublets induces dangerous flavor changing neutral currents [21]. For this, we might think of keeping just one SM-Higgs doublet $\Phi$ but at the expense of adding three heavy SM-singlet scalars $\Delta_i$ transforming nontrivially under the $S_0$ symmetry. Again, we assume the SM Higgs and the charged RH leptons $l_j^c$ to be singlets under the $S_0$ symmetry, whereas the lepton LH doublets transform component-wise faithfully:

$$L_i \to S_{ij}^{\rm DNTB_0} L_j, \qquad (71)$$

with $i, j = 1, 2, 3$. Then, the invariance of the SM term,

$$\mathcal{L}_1 = Y_{ij} \bar{L}_i \Phi l_j^c, \qquad (72)$$

under $S_0$ symmetry leads via Eq. (58) to the form

$$Y_{ij} \sim \begin{pmatrix} \frac{\sqrt{2}}{7} A & \frac{\sqrt{2}}{7} B & \frac{\sqrt{2}}{7} C \\ A & B & C \\ A & B & C \end{pmatrix}. \qquad (73)$$

We see here that this term leads, when $\Phi^0$ gets a vev, to a charged lepton squared mass matrix proportional to

$$Y \cdot Y^\dagger \sim \frac{(|A|^2 + |B|^2 + |C|^2)}{7} \begin{pmatrix} \frac{2}{7} & 1 & 1 \\ 1 & 7 & 7 \\ 1 & 7 & 7 \end{pmatrix},$$

with two zero eigenvalues, so we cannot produce the charged lepton mass spectrum with this term. Moreover, $YY^\dagger$ has eigenvectors proportional to $(1, -\frac{\sqrt{2}}{7}, 0)$, $(0, -1, 1)$, $(\frac{\sqrt{2}}{7}, 1, 1)$, which means that $YY^\dagger$ is not

[3] We could, in principle, choose $M_\xi \sim$ TeV, which makes the triplet field accessible to colliders, but that would require choosing the coupling $h_{ij}$ unnaturally small ($\sim 10^{-10}$) unless one chooses $\mu \sim$ eV.

diagonalized trivially, and so we are not in the flavor basis, which would destroy the predictions of the $\mathrm{DNTB}_0$ pattern. We note here that, had we really taken the original full symmetry of the model $[F = U(1) \times Z_2 \times Z_2 = \langle U(1), Z_2^3 \rangle]$, then the equation $F \cdot Y = Y$ could not be met for all elements in $F$ unless $Y = 0$.

The additional heavy SM-singlet scalar fields $\Delta_i$ help in resolving these inconveniences. We assume that they transform under $S_0$ symmetry as

$$\Delta_i \to S_{ij}^{\mathrm{DNTB}_0} \Delta_j \tag{74}$$

and that they are coupled to the lepton LH doublets through a nonrenormalizable dimension-5 operator

$$\mathcal{L}_2 = \frac{f_{ikr}}{\Lambda} \bar{L}_i \Phi \Delta_k l_r^c, \tag{75}$$

where $\Lambda$ is a heavy mass scale. Invariance of $\mathcal{L}_2$ under $S_0$ symmetry leads to

$$(S^{\mathrm{DNTB}_0})^{\mathrm{T}} f_r S^{\mathrm{DNTB}_0} = f_r, \tag{76}$$

where $f_r$, for fixed $r$, is the matrix whose $(i, j)$ entry is $f_{ijr}$. The equivalence [Eq. (57)] leads to

$$f_r = \begin{pmatrix} A_r - B_r + \frac{\sqrt{2}}{7} C_r & C_r & \frac{14\sqrt{2}}{47} B_r - \frac{51}{47} C_r \\ \frac{14\sqrt{2}}{47} B_r - \frac{51}{47} C_r & A_r & B_r \\ C_r & \frac{51}{47} B_r - \frac{14\sqrt{2}}{47} C_r & A_r \end{pmatrix}. \tag{77}$$

When $\Delta_k$ and $\phi^\circ$ take the vevs $\delta_k$ and $v$, respectively, then we get the charged lepton mass matrix:

$$(\mathcal{M}_l)_{ir} = \frac{v f_{ikr}}{\Lambda} \delta_k. \tag{78}$$

One can again arrange the vevs and the Yukawa couplings such that $\mathcal{M}_l$, after suitably rotating the flavor- and SM-singlets $l_j^c$, is the charged lepton mass matrix in the flavor basis. For example, if $\delta_1, \delta_2 \ll \delta_3$ we get

$$M_l = \frac{v\delta_3}{\Lambda} \begin{pmatrix} A_1' & A_2' & A_3' \\ B_1' & B_2' & B_3' \\ C_1' & C_2' & C_3' \end{pmatrix}, \tag{79}$$

where

$$A_i' = \left( \frac{14\sqrt{2}}{47} B^i - \frac{51}{47} C^i \right), \quad B_i' = B^i, \quad C_i' = A^i. \tag{80}$$

The same diagonalization procedure mentioned in the last subsection can be applied here to show that $\mathcal{M}_l$ can be seen, to a good approximation, as the charged lepton mass matrix with the correct mass hierarchies in the flavor basis.

### C. The conserved current associated with $S$ symmetry

One can determine the conserved current and charge corresponding to the continuous $U(1)$ symmetry. In order to stress the generality of the treatment, we shall discuss the $S$ symmetry with generic values of $\theta_z$. Let us, for illustration purposes, consider the neutrino part where the relevant term for computing the current is the kinetic energy (the sum is understood over the flavor index $k$):

$$K_\nu = i \bar{\nu}_k \gamma^\mu \partial_\mu \nu_k. \tag{81}$$

The current associated with the $S$ symmetry [Eq. (30)] is given by

$$J_\nu^\mu \equiv -i \frac{\partial K_\nu}{\partial(\partial_\mu \nu_j)} T_{jk} \nu_k = T_{jk} \bar{\nu}_j \gamma^\mu \nu_k, \tag{82}$$

where $T_{ij}$ is the generator of the $S$ symmetry:

$$T = i \begin{pmatrix} 0 & \frac{c_z}{\sqrt{2}} & \frac{-c_z}{\sqrt{2}} \\ \frac{-c_z}{\sqrt{2}} & 0 & s_z \\ \frac{c_z}{\sqrt{2}} & -s_z & 0 \end{pmatrix}, \tag{83}$$

satisfying

$$S_{\delta\theta}^{\mathrm{DNTB}} \approx \mathrm{I} - i\delta\theta T. \tag{84}$$

Moreover, since $S_\theta^{\mathrm{DNTB}}$ is a three-dimensional representation of the commutative $U(1)$ group whose irreducible representations (irreps) are one dimensioanl, one must be able to reduce $S_\theta^{\mathrm{DNTB}}$ to three one-dimensional irreps obtained by diagonalizing $S_\theta^{\mathrm{DNTB}}$ as follows:

$$S_\theta^{\mathrm{DNTB}} = L \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} L^\dagger, \tag{85}$$

$$L = \begin{pmatrix} s_z & \frac{-c_z(i+s_z)}{\sqrt{2(1+s_z^2)}} & \frac{-c_z(-i+s_z)}{\sqrt{2(1+s_z^2)}} \\ \frac{c_z}{\sqrt{2}} & \frac{-c_z^2+2is_z}{2\sqrt{(1+s_z^2)}} & \frac{-c_z^2-2is_z}{2\sqrt{(1+s_z^2)}} \\ \frac{c_z}{\sqrt{2}} & \frac{\sqrt{1+s_z^2}}{2} & \frac{\sqrt{1+s_z^2}}{2} \end{pmatrix} = \begin{pmatrix} V_0 & V_- & V_+ \end{pmatrix}. \tag{86}$$

The "neutrino" eigenvectors $(V_0, V_-, V_+)$, with expressions in terms of the flavor or "gauge" states given by the columns of $L$, are the neutrino fields with definite $S_\theta^{\mathrm{DNTB}}$ charges equaling respectively to $(0, -1, +1)$. Inverting now, to express the neutrino gauge states $\nu_i$ in terms of $(V_0, V_-, V_+)$, and substituting into Eq. (82) we get

$$J_\nu^\mu = (0\bar{V}_0 \gamma^\mu V_0 - 1\bar{V}_- \gamma^\mu V_- + 1\bar{V}_+ \gamma^\mu V_+), \tag{87}$$

which expresses explicitly the conserved current in terms of the $S$-charge eigenstates. This current corresponds to a global nongauged continuous symmetry, similar to the $U(1)$ baryon number conservation in the SM.

Using now $V^{xz}$ [Eq. (11)] to go from the neutrino gauge states $\nu^{\mathbf{g}} = (\nu_e \nu_\mu \nu_\tau)^{\mathrm{T}}$ to neutrino "mass" states $\nu^{\mathbf{m}} = (\nu_1^m \nu_2^m \nu_3^m)^{\mathrm{T}}$, we express the definite $S_\theta^{\mathrm{DNTB}}$-charge neutrino fields in terms of the mass eigenstates:

$$\mathbf{V} \equiv (V_0 V_- V_+)^{\mathrm{T}} = L^T \cdot \nu^{\mathbf{g}} = L^T \cdot (V^{xz})^* \cdot \nu^{\mathbf{m}}, \tag{88}$$

which gives

$$V_0 = \nu_3^m, \quad V_- = a(\nu_1^m - i\nu_2^m),$$
$$V_- = a^*(\nu_1^m + i\nu_2^m), \text{ where } a = -\frac{i + s_z}{\sqrt{2(1+s_z^2)}} e^{ix}. \tag{89}$$

Although the expressions involve $\theta_x$, as expected, this phase has no "physical" content in the degenerate mass spectrum case. This is because the particular combination of mass eigenstates in Eq. (89) never mixes under free time evolution provided $\nu_1^m$ and $\nu_2^m$ have degenerate mass, which is the case when the $S$ symmetry is exact. The same conclusion still holds if one thinks of the underlying symmetry, in the degenerate two-mass case, as $Z_3 \times Z_2$, due to the compatibility of both $S_\theta^{\mathrm{DNTB}}$ and $Z_3 \times Z_2$ in that they all commute and have common eigenstates. In fact, as we have seen earlier, even in the nondegenerate spectrum case, the mass eigenstates $(\nu_1^m, \nu_2^m, \nu_3^m)$ are the eigenvectors of the residual $Z_2$ symmetry, to which we can attribute "conserved" charges equal, respectively, to $(-1, -1, 1)$.

## IV. DNTB$_0$ NEUTRINO MASS MATRIX IN SEESAW SCENARIOS

We saw in Eqs. (16)–(18) that the modified bipartite form can explain all sorts of neutrino mass hierarchies. In the next subsection, we shall be more specific on the origin of the coefficients of the bipartite form through invoking type-I seesaw scenarios.

### A. Type-I seesaw scenario in the DNTB pattern

The effective light LH neutrino mass matrix is generated through the seesaw formula

$$M_\nu = -M_\nu^D \cdot M_R^{-1} \cdot (M_\nu^D)^{\mathrm{T}}, \tag{90}$$

where $M_R$ is the heavy Majorana RH neutrino mass matrix, whereas the Dirac neutrino mass matrix comes from the Yukawa term

$$g_{ij}\bar{L}_i\tilde{\Phi}\nu_{Rj}, \tag{91}$$

with $\tilde{\Phi} = i\tau_2\Phi^*$. Again, for generality, we will treat, in this subsection, the $S$ symmetry corresponding to the DNTB pattern with generic values of $\theta_z$ since the results are not specific to any particular value of it. We assume the RH neutrinos transforming under $S$ symmetry to be

$$\nu_{Rj} \to S_{j\gamma}\nu_{R\gamma}. \tag{92}$$

Then, the invariance of the Lagrangian under $S$ symmetry leads to

$$S^{\mathrm{T}} \cdot g \cdot S = g. \tag{93}$$

The equivalence in Eq. (31) leads, when $\tilde{\Phi}$ takes a vev $v$, to the following Dirac mass matrix:

$$M_\nu^D = v\begin{pmatrix} A_D - B_D + \sqrt{2}t_z C_D & C_D & \frac{\sqrt{2}s_{2z}}{1-3s_z^2}B_D - \frac{1+s_z^2}{1-3s_z^2}C_D \\ \frac{\sqrt{2}s_{2z}}{1-3s_z^2}B_D - \frac{1+s_z^2}{1-3s_z^2}C_D & A_D & B_D \\ C_D & \frac{1+s_z^2}{1-3s_z^2}B_D - \frac{\sqrt{2}s_{2z}}{1-3s_z^2}C_D & A_D \end{pmatrix}. \tag{94}$$

The invariance under $S$ symmetry of the term $\frac{1}{2}\nu_{iR}^{\mathrm{T}}C(M_R)_{ij}\nu_{jR}$ ($C$ is the charge conjugation matrix) would impose the modified generic bipartite form for the symmetric Majorana RH neutrino mass matrix [Eq. (31)]:

$$M_R = \Lambda_R\begin{pmatrix} A_R - B_R & \frac{\sqrt{2}s_zc_z}{1-3s_z^2}B_R & \frac{\sqrt{2}s_zc_z}{1-3s_z^2}B_R \\ \frac{\sqrt{2}s_zc_z}{1-3s_z^2}B_R & A_R & \frac{c_z^2}{1-3s_z^2}B_R \\ \frac{\sqrt{2}s_zc_z}{1-3s_z^2}B_R & \frac{c_z^2}{1-3s_z^2}B_R & A_R \end{pmatrix}, \tag{95}$$

where $\Lambda_R$ is a high scale characterizing the seesaw mechanism.

Applying the seesaw formula [Eq. (90)] we get the same form characterizing the DNTB pattern:

$$M_\nu = -\frac{v^2}{\Lambda}\begin{pmatrix} A_\nu - B_\nu & \frac{\sqrt{2}s_zc_z}{1-3s_z^2}B_\nu & \frac{\sqrt{2}s_zc_z}{1-3s_z^2}B_\nu \\ \frac{\sqrt{2}s_zc_z}{1-3s_z^2}B_\nu & A_\nu & \frac{c_z^2}{1-3s_z^2}B_\nu \\ \frac{\sqrt{2}s_zc_z}{1-3s_z^2}B_\nu & \frac{c_z^2}{1-3s_z^2}B_\nu & A_\nu \end{pmatrix}, \tag{96}$$

where the bipartite form coefficients $A_\nu$, $B_\nu$ are given in terms of those characterizing the Dirac and Majorana mass matrices as follows.

$$
\begin{aligned}
A_\nu &= \frac{A_{\nu_1}A_R + A_{\nu_2}B_R}{(1-3s_z^2)[(1-3s_z^2)A_R - c_z^2B_R][(1-3s_z^2)A_R + (1+s_z^2)B_R]},\\
A_{\nu_1} &= (1-3s_z^2)^3A_D^2 + (1-3s_z^2)(1+s_z^2)^2C_D^2 - 2\sqrt{2}s_{2z}(1+s_z^2)(1-3s_z^2)B_DC_D + (1+s_z^2)^2(1-3s_z^2)B_D^2,\\
A_{\nu_2} &= 2s_z^2(1-3s_z^2)^2A_D^2 + \sqrt{2}s_{2z}(1+s_z^2)(1-3s_z^2)A_DC_D - 2c_z^2(1+s_z^2)(1-3s_z^2)A_DB_D\\
&\quad + c_z^2(1+s_z^2)^2C_D^2 - \sqrt{2}s_{2z}(1+s_z^2)^2B_DC_D + 2s_z^2(1+s_z^2)^2B_D^2,
\end{aligned}
\tag{97}
$$

$$
\begin{aligned}
B_\nu &= \frac{B_{\nu_1}A_R + B_{\nu_2}B_R}{-c_z^2[(1-3s_z^2)A_R - c_z^2B_R][(1-3s_z^2)A_R + (1+s_z^2)B_R]},\\
B_{\nu_1} &= \sqrt{2}s_{2z}(1-3s_z^2)^2A_DC_D - 2c_z^2(1-3s_z^2)^2A_DB_D + (1+s_z^2)(1-3s_z^2)^2C_D^2 - \sqrt{2}s_{2z}(1-3s_z^2)B_DC_D,\\
B_{\nu_2} &= c_z^2(1-3s_z^2)^2A_D^2 + c_z^2(1+s_z^2)^2C_D^2 - 4\sqrt{2}s_zc_z^3(1+s_z^2)B_DC_D - c_z^2(1+s_z^2)B_D^2.
\end{aligned}
\tag{98}
$$

All types of neutrino mass hierarchies can be accommodated according to relations in Eqs. (16)–(18), which in turn impose constraints on Dirac and RH Majorana neutrino mass matrices as follows.

(i) Normal hierarchy: with

$$A_D \simeq B_D, \qquad C_D \ll B_D(A_D), \qquad A_R \simeq B_R. \tag{99}$$

We get, for most values of $\theta_z$ in the experimentally acceptable range ([6.29°, 11.68°]), the following:

$$A_\nu \simeq \frac{A_D^2}{A_R} \simeq B_\nu \Rightarrow A_\nu \simeq B_\nu. \tag{100}$$

(ii) Inverted hierarchy: with

$$A_D \simeq -B_D, \qquad C_D \ll B_D(A_D), \qquad A_R \simeq -B_R. \tag{101}$$

We get, for most acceptable values of $\theta_z$, the following:

$$A_\nu \simeq \frac{A_D^2}{A_R} \simeq -B_\nu \Rightarrow A_\nu \simeq -B_\nu. \tag{102}$$

(iii) Degenerate case: with

$$A_D \gg B_D \gg C_D, \qquad A_R \gg B_R. \tag{103}$$

We get, for most acceptable values of $\theta_z$, the following:

$$
\begin{aligned}
A_\nu &\simeq \frac{A_D^2}{A_R},\\
B_\nu &\simeq 2\frac{B_DA_D}{A_R} - A_D^2\frac{B_R}{A_R^2} \Rightarrow A_\nu \gg B_\nu.
\end{aligned}
\tag{104}
$$

The RH neutrino mass term violates lepton number by two units and could lead to lepton asymmetry. The produced asymmetry due to the out-of-equilibrium decay of the lightest RH neutrino to SM particles is given by [22]

$$\epsilon \simeq \frac{3}{16\pi v^2}\frac{1}{(\tilde{M}_D^\dagger\tilde{M}_D)_{11}}\sum_{j=2,3}\mathrm{Im}\{[(\tilde{M}_D^\dagger\tilde{M}_D)_{1j}]^2\}\frac{M_{R1}}{M_{Rj}}, \tag{105}$$

where $M_{Ri}$, $i = 1\ldots3$ are the masses for RH neutrinos, and $\tilde{M}_D$ is the Dirac neutrino mass matrix in the basis where the Majorana RH neutrino mass matrix $M_R$ is diagonal. Since the RH neutrino mass matrix [Eq. (96)] has a DNTB modified generic bipartite form, then it is diagonalized by $V^{xz}$ [Eq. (11)]. Thus, under $\nu_R \to (V^{xz})^*\nu_R$ we have $M^D \to M^D(V^{xz})^*$. We still have freedom in multiplying the diagonalizing unitary matrix $V^{xz}$ by diagonal phases $F = \mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$ adjusted normally such that the phases of the spectrum of $M_R$ disappear. Namely, these phases vanish if we choose

$$
\begin{aligned}
(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{2}\arg\bigg(&A_R - \frac{c_z^2}{1-3s_z^2}B_R, A_R\\
&- \frac{c_z^2}{1-3s_z^2}B_R, A_R + \frac{1+s_z^2}{1-3s_z^2}B_R\bigg).
\end{aligned}
\tag{106}
$$

Thus, we have $\tilde{M}_D = M^D \cdot (V^{xz})^* \cdot F^*$, so we can write

$$\tilde{M}_D^\dagger \cdot \tilde{M}_D = F^{\mathrm{T}} \cdot (V^{xz})^{\mathrm{T}} \cdot M^{D\dagger} \cdot M^D \cdot (V^{xz})^* \cdot F^*. \tag{107}$$

The required entries of $\tilde{M}_D^\dagger \cdot \tilde{M}_D$ for calculating the asymmetry are

$$(\tilde{M}_\nu^{D\dagger}\tilde{M}_\nu^D)_{11} = |A_D|^2 + \frac{1}{4-12c_z^2+9c_z^4}[(4s_z^2+c_z^4)|B_D|^2 + (2c_z^2(1+s_z^2))|C_D|^2 - \sqrt{2}s_zc_z(c_z^2+2)(C_DB_D^* + C_D^*B_D) + (2c_z^2-3c_z^4)(A_DB_D^*+A_D^*B_D) + \sqrt{2}s_zc_z(3c_z^2-2)(A_DC_D^*+A_D^*C_D)],$$
$$(\tilde{M}_\nu^{D\dagger}\tilde{M}_\nu^D)_{12} = \frac{1}{1-3s_z^2}[2s_z(B_D^*A_D - B_DA_D^*) + \sqrt{2}c_z(C_D^*B_D - C_DB_D^*) + \sqrt{2}c_z(A_D^*C_D - A_DC_D^*)],$$
$$(\tilde{M}_\nu^{D\dagger}\tilde{M}_\nu^D)_{13} = 0, \tag{108}$$

which leads to a vanishing lepton asymmetry since the entries (1, 1) and (1, 2) are, respectively, real and pure imaginary, whereas the entry (1, 3) assumes the value zero. Thus, we are tempted to look for other phenomenologically motivated venues producing enough lepton asymmetry in the context of the type-II seesaw mechanism.

### B. Type-II seesaw scenario in the DNTB$_0$ pattern

The type-II seesaw scenario can solely accommodate enough lepton/baryogenesis for the observed baryon/photon density in the Universe. For this, we need to do some numerical estimations; that is why we use the special value $\theta_z \sim 8^o$ of the DNTB$_0$ pattern in this section. As in Ref. [6], we introduce two SM triplet fields $\Sigma_A$, $A = 1, 2$ which are singlet under the $S$ symmetry. The Lagrangian part relevant for the neutrino mass matrix is

$$\mathcal{L} = \lambda^A_{\alpha\beta}L^T_\alpha C\Sigma_A i\tau_2 L_\beta + \mathcal{L}(H, \Sigma_A) + \text{H.c.}, \tag{109}$$

where $A = 1, 2$ and

$$\mathcal{L}(H,\Sigma_A) = \mu_H^2 H^\dagger H + \frac{\lambda_H}{2}(H^\dagger H)^2 + M_A\,\mathrm{Tr}(\Sigma_A^\dagger\Sigma_A) + \frac{\lambda_{\Sigma_A}}{2}[\mathrm{Tr}(\Sigma_A^\dagger\Sigma_A)]^2 + \lambda_{H\Sigma_A}(H^\dagger H)\mathrm{Tr}(\Sigma_A^\dagger\Sigma_A) + \mu_A H^T\Sigma_A^\dagger i\tau_2 H + \text{H.c.}, \tag{110}$$

where $H$ and $\Sigma_A$ are written as

$$H = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}, \qquad \Sigma_A = \begin{pmatrix}\frac{\Sigma^+}{\sqrt{2}} & \Sigma^0\\ \Sigma^{++} & -\frac{\Sigma^+}{\sqrt{2}}\end{pmatrix}_A. \tag{111}$$

The neutrino mass matrix due to the exchange of the two triplets, $\Sigma_1$ and $\Sigma_2$, is

$$(M_\nu)_{\alpha\beta} \simeq v^2\left[\lambda^1_{\alpha\beta}\frac{\mu_1}{M^2_{\Sigma_1}} + \lambda^2_{\alpha\beta}\frac{\mu_2}{M^2_{\Sigma_2}}\right], \tag{112}$$

where $M_{\Sigma_i}$ is the mass of the neutral component $\Sigma_i^0$ of the triplet $\Sigma_i$, $i = 1, 2$.

Some remarks are in order here. First, the two matrices $\lambda^A$, and thus also the neutrino mass matrix in Eq. (112), have the modified special bipartite form because of the Lagrangian invariance under the $S_0$ symmetry:

$$\lambda^a = \begin{pmatrix} A_a - B_a & \frac{7\sqrt{2}}{47}B_a & \frac{7\sqrt{2}}{47}B_a\\ \frac{7\sqrt{2}}{47}B^a & A^a & \frac{49}{47}B\\ \frac{7\sqrt{2}}{47}B^a & \frac{49}{47}B^a & A^a\end{pmatrix}, \quad a = 1, 2. \tag{113}$$

Hence, all kinds of neutrino mass hierarchies can be generated. Second, the same remarks mentioned in Sec. III A about the $\mu_A$-term apply here concerning the generation of small neutrino mass and the integrating out of the heavy triplet and the pseudo-GB. Third, the flavor changing neutral current due to the triplet is highly suppressed because of the large value of its mass scale.

We follow now the same steps carried out in Refs. [6,10,12], discussing how to generate baryon asymmetry from leptogenesis using the sphaleron interaction [23]. We note in passing that the would-be GB, which is the imaginary part of the neutral component of the triplet, would contribute to the leptogenesis as any component of this field. The reason behind this is that at temperatures much larger than the EW scale, the vev of the triplet is equal to zero since it is triggered by the spontaneous breaking of the SM gauge symmetry (which at these temperatures is unbroken). That is, the $SU(2)\times U(1)$ gauge symmetry (unbroken at $T \sim M_\Sigma \gg v_{\mathrm{EW}}$) dictates that all three triplet components (charged, doubly charged, and neutral) contribute on equal footing, and are thus taken into account when computing the decay of $\Sigma$ into LL and the $CP$ conjugate process. The choice of having more than one Higgs triplet is essential to generate the lepton asymmetry [24]. In this case, the $CP$ asymmetry in the decay of the lightest Higgs triplet (which we choose to be $\Sigma_1$) is generated at one loop level due to the interference between the tree and the one loop self-energy diagrams, and it is given by

$$\epsilon_{CP} \approx -\frac{1}{8\pi^2}\frac{\mathrm{Im}[\mu_1\mu_2^*\,\mathrm{Tr}(\lambda^1\lambda^{2\dagger})]}{M_2^2}\frac{M_1}{\Gamma_1}. \tag{114}$$

$\Gamma_1$ is the decay rate of the lightest Higgs triplet, and it is given by

$$\Gamma_1 = \frac{M_1}{8\pi}\left[\mathrm{Tr}(\lambda^1\lambda^{1\dagger}) + \frac{\mu_1^2}{M_1^2}\right]. \tag{115}$$

We can compute the relevant traces in the DNTB$_0$ pattern to find

$$\mathrm{Tr}(\lambda^1\lambda^{2\dagger}) = 2\left(A_1 - \frac{49}{47}B_1\right)\left(A_2^* - \frac{49}{47}B_2^*\right) + \left(A_1 + \frac{51}{47}B_1\right)\left(A_2^* + \frac{51}{47}B_2^*\right) \quad (116)$$

$$\mathrm{Tr}(\lambda^1\lambda^{1\dagger}) = 2\left|A_1 - \frac{49}{47}B_1\right|^2 + \left|A_1 + \frac{51}{47}B_1\right|^2. \quad (117)$$

If we now denote the phases of $A_a - \frac{49}{47}B_a$, $A_a + \frac{51}{47}B_a$, $\mu_a$ by $\alpha_a$, $\beta_a$, $\phi_a$ ($a = 1, 2$), respectively, then by rewriting the $A_a$ and $B_a$ coefficients of the $\lambda^a$'s in the Yukawa term ($\lambda^a_{\alpha\beta}L^T_\alpha C\Sigma_a i\tau_2 L_\beta$) in a function of the combinations $(A_a - \frac{49}{47}B_a)$ and $(A_a + \frac{51}{47}B_a)$, we find, say, the first combinations $(A_a - \frac{49}{47}B_a)$ multiplied always with the field $\Sigma_a$. This means that shifting the latter fields by a phase $(-\alpha_a)$ would put the phases $\alpha_a$ equal to zero. For $\mu_a \approx M_{\Sigma_a} \sim 10^{13}$ GeV, $a = 1, 2$ (which give a neutrino mass in the sub-eV range) we get

$$\epsilon_{CP} \approx -\frac{1}{\pi}\frac{2|A_1 - \frac{49}{47}B_1||A_2 - \frac{49}{47}B_2|\sin(\phi_1 - \phi_2) + |A_1 + \frac{51}{47}B_1||A_2 + \frac{51}{47}B_2|\sin(\phi_1 - \phi_2 + \beta_1 - \beta_2)}{1 + 2|A_1 - \frac{49}{47}B_1|^2 + |A_1 + \frac{51}{47}B_1|^2}. \quad (118)$$

The baryon to photon density is approximately given by

$$\eta_B \equiv \frac{n_B}{s} = \frac{1}{3}\eta_L \simeq \frac{1}{3}\frac{1}{g_*}\kappa\epsilon_{CP}, \quad (119)$$

where $g_* \sim 100$ is the number of relativistic degrees of freedom at the time when the Higgs triplet decouples from the thermal bath and $\kappa$ is the efficiency factor which takes into account the fraction of out-of-equilibrium decays and the washout effect. In the case of a strong washout, the efficiency factor can be approximated by ($H$ is the Hubble parameter)

$$\kappa \simeq \frac{H}{\Gamma_1}(T = M_1). \quad (120)$$

With the above numerical values and with an efficiency factor of order $10^{-4}$, we get, for $\beta_1 = \beta_2$, a baryon asymmetry:

$$\eta_B \approx 10^{-7}\frac{\mathrm{Tr}(\lambda^1\lambda^{2\dagger})}{\mathrm{Tr}(\lambda^1\lambda^{1\dagger}) + 1}\sin(\phi_2 - \phi_1). \quad (121)$$

Thus, one can bring about the correct baryon-to-photon ratio of $\eta_B \simeq 10^{-10}$ by choosing $\lambda$'s of order 0.1 and not too small relative phase between $\mu_1$ and $\mu_2$.

## V. SUMMARY AND CONCLUSION

We have derived an explicit realization of the $U(1)$ symmetry underlying the non-tribimaximal pattern of the neutrino mass matrix in the degenerate mass spectrum case. We deduced a bipartite form which uniquely characterizes this pattern. The departure from the tribimaximal pattern is suggested by recent oscillation data, whereas the degenerate mass spectrum case is a good approximation motivated by experimental data and numerical studies. One can consider it as a first step to be perturbed by a term proportional to the mass splitting $(m_2 - m_1)$, leading to a modified tripartite model without degeneracy. We have implemented this symmetry in a setup including charged leptons supplemented either with many Higgs doublets or with many SM-singlet scalars. In both cases, one could accommodate the observed charged lepton mass hierarchies. Similarly, the $U(1)$ symmetry can generate all sorts of neutrino mass hierarchies. We showed this explicitly in type-I seesaw scenarios, where we found that no lepton/baryon asymmetry can be generated. However, in type-II seesaw mechanisms, one can account for the photon/baryon density observed in the Universe.

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