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Color Coulomb potential in Yang-Mills theory from Hamiltonian flows

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We consider the Hamiltonian formulation of Yang-Mills theory in the Coulomb gauge and apply the recently developed technique of Hamiltonian flows. We formulate a flow equation for the color Coulomb potential which allows for a scaling solution that results in an almost linearly rising confining potential.

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A profound understanding of the confinement mechanism in QCD still represents, after almost 40 years of intense research, one of the most important challenges in modern theoretical particle physics. We here report on a new approach to the subject which uses the recently developed technique of Hamiltonian flows [1]. The general setup is the Hamiltonian formulation of Yang-Mills theory in the Coulomb gauge [2]. Important progress has been made over the last decade in this formulation, mainly via the variational principle [3–12]. The horizon condition, in its simplest form as a condition for the infrared behavior of the ghost two-point function, is implemented in accord with the Gribov-Zwanziger confinement scenario [13,14], and scaling behavior of the equal-time two-point correlation functions results together with an infrared fixed point of an appropriately defined running coupling constant [10].

In Coulomb gauge Yang-Mills theory, after resolving the Gauss law, a potential between static color sources can be extracted, which is referred to as color Coulomb potential, and which represents an upper bound to the true static potential extracted from the Wilson loop. A confining Coulomb potential is a necessary condition for the potential between static color charges to be confining [15]. The color Coulomb potential is given by the vacuum expectation value

$$\langle(-\partial D)^{-1}(-\partial^2)(-\partial D)^{-1}\rangle \tag{1}$$

[see Eq. (5) below for our definition of the covariant derivative D] and is usually expressed as

$$\langle(-\partial D)^{-1}\rangle(-\partial^2)f(-\partial^2)\langle(-\partial D)^{-1}\rangle, \tag{2}$$

with the so-called Coulomb form factor f, which satisfies a Dyson-Schwinger equation (DSE). In order to calculate the color Coulomb potential, in Ref. [6] the Coulomb form factor was simply set equal to 1, while in Refs. [7,8,11] the DSE for this form factor was approximated by replacing in the loop integral the full ghost propagator $\langle(-\partial D)^{-1}\rangle$ with the bare one, which results in an infrared finite Coulomb form factor. In this way, a strictly linear growth of the color Coulomb potential with the distance between the color sources (for sufficiently large distances) has been found in Ref. [11]. It would now be natural to try to improve this approximation by using the full DSE for the Coulomb form factor. However, it turns out [7,16] that the one-loop DSE for the Coulomb form factor considered so far cannot be consistently solved together with the DSEs for the static (equal-time) gluon and ghost propagators with an infrared-divergent ghost form factor, i.e., implementing the horizon condition. In other words, a confining color Coulomb potential cannot be obtained within the present approximation if the one-loop DSE for the Coulomb form factor is used.

In the present report, we focus on the determination of the color Coulomb potential with the help of a different functional technique, the Hamiltonian flows [1]. Interestingly, a consistent solution which exhibits scaling behavior of the static propagators and the color Coulomb potential is readily found in this framework, without any additional approximation for the Coulomb form factor. The organization of this report is as follows: We start with a brief presentation of the Hamiltonian flow technique and summarize the results of Ref. [1]. We then derive the flow equation for the color Coulomb potential and, finally, present and discuss its solution.

The Hamiltonian flows constitute an adaptation of the functional renormalization group as put forward in Ref. [17] to the Hamiltonian formulation of the theory which seems more appropriate for the Coulomb gauge fixing. The construction of the Hamiltonian flows starts from the k-dependent generating functional for Green's functions at equal times,

$$Z_k[J,\sigma,\bar\sigma] = \int \mathcal{D}A\mathcal{D}\bar c\mathcal{D}c\, e^{-S-\Delta S_k+J\cdot A+\bar\sigma\cdot c+\bar c\cdot\sigma}, \tag{3}$$

with the "action"

$$S = -\ln|\psi[A]|^2 + \int d^3x\,\bar c^a(\mathbf{x})(-\partial D)^{ab}c^b(\mathbf{x}), \tag{4}$$

where $\psi[A]$ represents the vacuum wave functional and D denotes the covariant derivative

$$D_i^{ab} = \delta^{ab}\partial_i - gf^{abc}A_i^c. \quad (5)$$

The functional integral $\int \mathcal{D}A$ is taken over the transverse spatial gauge fields that fulfill the Coulomb gauge condition $\partial_i A_i^a = 0$ (we denote the contravariant spatial indices as subindices). The dot in, e.g., $J \cdot A$ stands for the contraction of color and spatial indices and the integral over position or momentum:

$$J \cdot A = \int d^3x J_i^a(\mathbf{x}) A_i^a(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} J_i^a(-\mathbf{p}) A_i^a(\mathbf{p}). \quad (6)$$

The regulator term in Eq. (3) is given by

$$\Delta S_k[A, c, \bar{c}] = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} A_i^a(-\mathbf{p}) R_{A,k}(p) A_i^a(\mathbf{p}) + \int \frac{d^3p}{(2\pi)^3} \bar{c}^a(-\mathbf{p}) g \bar{R}_{c,k}(p) c^a(\mathbf{p}), \quad (7)$$

with the regulator functions chosen in the present work as

$$R_{A,k}(p) = 2p \exp\left(\frac{k^2}{p^2} - \frac{p^2}{k^2}\right),$$
$$\bar{R}_{c,k}(p) = p^2 \exp\left(\frac{k^2}{p^2} - \frac{p^2}{k^2}\right). \quad (8)$$

Here and in the following, we use the notation $p = |\mathbf{p}|$. The change of Z_k as defined in Eq. (3) under a change of k constitutes a (functional) renormalization group transformation.

We parameterize the static propagators as

$$(2\pi)^6 \frac{\delta^2 \ln Z_k}{\delta J_i^a(-\mathbf{p}) \delta J_j^b(\mathbf{q})}\bigg|_{J=\sigma=\bar{\sigma}=0} = G_{A,k}(p)\delta^{ab} t_{ij}(\mathbf{p})(2\pi)^3 \delta(\mathbf{p}-\mathbf{q}),$$
$$-(2\pi)^6 \frac{\delta^2 \ln Z_k}{\delta \bar{\sigma}^a(-\mathbf{p}) \delta \sigma^b(\mathbf{q})}\bigg|_{J=\sigma=\bar{\sigma}=0} = \frac{1}{g}\bar{G}_{c,k}(p)\delta^{ab}(2\pi)^3 \delta(\mathbf{p}-\mathbf{q}), \quad (9)$$

with the functions

$$G_{A,k}(p) = \frac{1}{2\omega_k(p) + R_{A,k}(p)},$$
$$\bar{G}_{c,k}(p) = \frac{1}{p^2/d_k(p) + \bar{R}_{c,k}(p)}. \quad (10)$$

In the first part of Eq. (9), $t_{ij}(\mathbf{p})$ denotes the transverse projector or spatially transverse Kronecker delta.

Retaining only the contributions that are relevant to the infrared behavior, the renormalization group equation for Z_k induces the following flow equations for the static propagators [1]:

$$\frac{\partial}{\partial k}\omega_k(p) = -\frac{N_c}{2}\int \frac{d^3q}{(2\pi)^3}\left(\bar{G}_{c,k}\frac{\partial \bar{R}_{c,k}}{\partial k}\bar{G}_{c,k}\right)(q) \times \bar{G}_{c,k}(|\mathbf{p}+\mathbf{q}|) q^2 (1 - (\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2),$$
$$\frac{\partial}{\partial k} d_k^{-1}(p) = N_c \int \frac{d^3q}{(2\pi)^3}\left[\left(G_{A,k}\frac{\partial R_{A,k}}{\partial k}G_{A,k}\right)(q) \times \bar{G}_{c,k}(|\mathbf{p}+\mathbf{q}|) + \left(\bar{G}_{c,k}\frac{\partial \bar{R}_{c,k}}{\partial k}\bar{G}_{c,k}\right)(q) \times G_{A,k}(|\mathbf{p}+\mathbf{q}|)\frac{q^2}{(\mathbf{p}+\mathbf{q})^2}\right](1 - (\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2), \quad (11)$$

where we have neglected in each equation the contribution of a tadpole term. For details of the derivation of these equations and a diagrammatic representation, we refer the reader to Ref. [1].

We have found in Ref. [1] numerical solutions of Eq. (11) that show a power behavior for small momenta of the functions $\omega_{k=0}(p)$ and $d_{k=0}(p)$ (in the physical limit of vanishing infrared regulators):

$$\omega_0(p \to 0) \propto p^{-\alpha}, \qquad d_0(p \to 0) \propto p^{-\beta}, \quad (12)$$

with the numerical values for the exponents

$$\alpha = 0.28, \qquad \beta = 0.64. \quad (13)$$

We have argued in Ref. [1] that replacing the functions $\omega_k(p)$ and $d_k(p)$ on the right-hand sides of Eq. (11) with $\omega_0(p)$ and $d_0(p)$ effectively takes into account part of the tadpole terms that we have omitted so far. The related technical derivation is detailed in Ref. [18], Chap. V. With $\omega_0(p)$ and $d_0(p)$ on the right-hand sides of Eq. (11), we can perform the integration over k analytically and end up with equations very similar to those of the variational approach. We have shown in Ref. [1] that the numerical solution of these equations matches almost perfectly that of Refs. [7,8], with

$$\alpha = 0.60, \qquad \beta = 0.80. \quad (14)$$

We now come to the calculation of the color Coulomb potential in the functional renormalization group approach. For its definition, one considers the theory in the presence of an external static color charge density. Then the color Coulomb potential is the vacuum expectation value of the part of the Hamiltonian that depends on the external color charges, explicitly in momentum space (in an integral kernel notation):

$$F^{ab}(\mathbf{p}, -\mathbf{q}) = \langle\langle \mathbf{p}, a|(-\partial D)^{-1}(-\partial^2)(-\partial D)^{-1}|\mathbf{q}, b\rangle\rangle = V_c(p)\delta^{ab}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}). \quad (15)$$

Introducing the composite operator

$$K = \int \frac{d^3k}{(2\pi)^3}\bar{c}^d(-\mathbf{k}) k^2 c^d(\mathbf{k}), \quad (16)$$

one can write

$$F^{ab}(\mathbf{p},-\mathbf{q}) = \langle c^a(\mathbf{p}) K \bar{c}^b(-\mathbf{q})\rangle_{\mathrm{GC}}. \tag{17}$$

The label GC on the vacuum expectation value stands for "ghost-connected," meaning that one has to restrict the contributing diagrams to those where the operator K is connected to the external points via ghost lines.

The k-dependent color Coulomb potential $F_k^{ab}(\mathbf{p},-\mathbf{q})$ is then naturally defined by including the cutoff term ΔS_k in the functional integral representation of the vacuum expectation value [Eq. (17)] as in Eq. (3), and a flow equation for F_k can be derived in the standard way. For reasons of space, however, here we present a much quicker and equivalent derivation of the flow equation for F_k which is based on the identity [19]

$$\frac{\partial}{\partial g}[g(-\partial D)^{-1}] = (-\partial D)^{-1}(-\partial^2)(-\partial D)^{-1} \tag{18}$$

for the operators. We generalize this identity to

$$\frac{\partial}{\partial g}[g(-\partial D + g\bar{R}_{c,k})^{-1}] = (-\partial D + g\bar{R}_{c,k})^{-1}(-\partial^2)(-\partial D + g\bar{R}_{c,k})^{-1} \tag{19}$$

for our present purposes, so that

$$F_k^{ab}(\mathbf{p},-\mathbf{q}) = \left\langle \langle \mathbf{p}, a| \frac{\partial}{\partial g}[g(-\partial D + g\bar{R}_{c,k})^{-1}]|\mathbf{q}, b\rangle \right\rangle_k = V_{c,k}(p)\delta^{ab}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}). \tag{20}$$

Note that the rescaling of the ghost regulator function with a factor of g is essential to achieve a form equivalent to the definition of F_k described above.

In order to put the identity of Eq. (19) to use inside the vacuum expectation values we are interested in, we define a g derivative "at fixed integration measure,"

$$\partial_g|_{\mathrm{fm}}\langle \mathcal{O}\rangle_k = \int \mathcal{D}A \det(-\partial D + g\bar{R}_{c,k})\left(\frac{\partial}{\partial g}\mathcal{O}[A]\right)|\psi[A]|^2 \times \exp\left(-\frac{1}{2}A\cdot R_{A,k}\cdot A\right), \tag{21}$$

for an arbitrary operator $\mathcal{O}[A]$. This definition immediately implies that

$$\partial_g|_{\mathrm{fm}} G_{A,k} = 0. \tag{22}$$

For the application of the g derivative to the static ghost propagator, we use the identity

$$\langle c^a(\mathbf{p})\bar{c}^b(-\mathbf{q})\rangle_k = \langle\langle \mathbf{p}, a|(-\partial D + g\bar{R}_{c,k})^{-1}|\mathbf{q}, b\rangle\rangle_k. \tag{23}$$

With the help of the definitions in Eqs. (9) and (20), we then find

$$\partial_g|_{\mathrm{fm}}\bar{G}_{c,k}(p) = V_{c,k}(p). \tag{24}$$

Introducing the *Coulomb form factor* $f_k(p)$ by

$$V_{c,k}(p) = \frac{1}{g^2}\bar{G}_{c,k}(p)p^2 f_k(p)\bar{G}_{c,k}(p), \tag{25}$$

we may rewrite the latter identity as

$$\partial_g|_{\mathrm{fm}} d_k^{-1}(p) = \frac{1}{p^2}\partial_g|_{\mathrm{fm}}\bar{G}_{c,k}^{-1}(p) = -\frac{1}{g^2}f_k(p). \tag{26}$$

As a consequence of these relations, we can derive a flow equation for the Coulomb form factor by simply differentiating the flow equation [Eq. (11)] for d_k^{-1} with respect to g, with the result

$$\begin{aligned}\frac{\partial}{\partial k}f_k(p) = -N_c\int\frac{d^3q}{(2\pi)^3}\Bigg[&\left(G_{A,k}\frac{\partial R_{A,k}}{\partial k}G_{A,k}\right)(q) \\ &\times\bar{G}_{c,k}^2(|\mathbf{p}+\mathbf{q}|)(\mathbf{p}+\mathbf{q})^2 f_k(|\mathbf{p}+\mathbf{q}|) \\ &+2\left(\bar{G}_{c,k}\frac{\partial\bar{R}_{c,k}}{\partial k}\bar{G}_{c,k}^2\right)(q)q^2 f_k(q) \\ &\times G_{A,k}(|\mathbf{p}+\mathbf{q}|)\frac{q^2}{(\mathbf{p}+\mathbf{q})^2}\Bigg](1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2),\end{aligned} \tag{27}$$

where we have used the fact that $\partial_g|_{\mathrm{fm}}$ and the k derivative commute. Since we have derived Eq. (27) from Eq. (11), several approximations are implicit in Eq. (27), corresponding to those employed before in the derivation of the flow equation for d_k^{-1}.

The standard derivation of the flow equation that makes use of the composite operator K [defined in Eq. (16)] and its equivalence with the argument presented above, as well as an algebraic construction that avoids reference to ghost-connected diagrams as in Eq. (17), will be detailed in a future publication.

Since the flow equations [Eq. (11)] for $\omega_k(p)$ and $d_k(p)$ do not involve $f_k(p)$, we can insert the solutions of the latter equations found in Ref. [1] into Eq. (27) and integrate this flow equation applying the same techniques used in the solution of Eq. (11); i.e., we convert Eq. (27) into an integral equation and solve it numerically by an iterative procedure. Equation (27) is linear and homogeneous in $f_k(p)$, and we decided to normalize $f_k(p)$ to 1 in the ultraviolet (below the initial scale Λ) by appropriately adjusting the initial condition $f_\Lambda(p) \equiv f_\Lambda$. It is clear from perturbation theory that $f_k(p)$ should be constant in the ultraviolet (as long as $k \ll p$) except for logarithmic corrections.

Somewhat surprisingly, and contrary to the negative result of the search for a scaling solution in the variational approach complemented with DSEs [16], a solution of Eq. (27) is readily found in the way described and is here represented in Fig. 1. With the propagators obtained from the flow equations [Eq. (11)], we get a power behavior

$$f_k(p) \propto p^{-\gamma} \tag{28}$$

in the infrared, with $\gamma = 0.57$ for $p \gtrsim k_{\min}$; see Fig. 1 (left). Note that in the numerical solution of the equation, as in the numerical solution of Eq. (11) before, we have to introduce a minimal cutoff value $k_{\min}$ for technical reasons. The equation can then be integrated over k down to

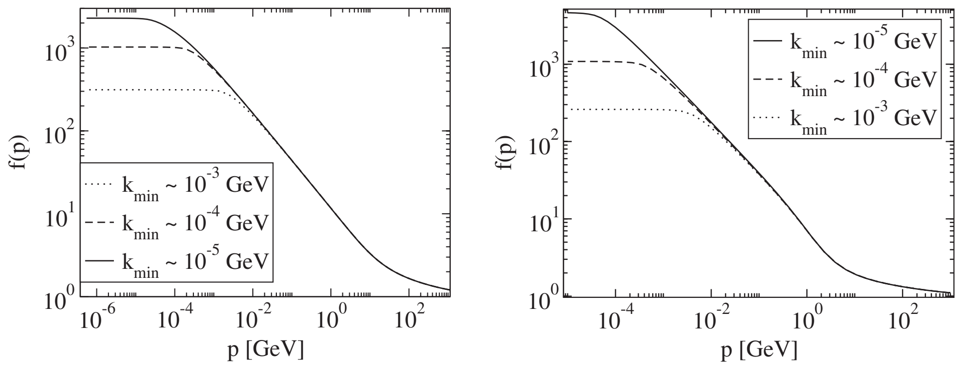

FIG. 1. The Coulomb form factor $f_k(p)$ as obtained from Eq. (27) for different minimal cutoffs $k = k_{\rm min}$, calculated with the propagators from the flow equations [Eq. (11)] (left) and the improved propagators (right).

$k = k_{\rm min}$. It is clear from the figure that the scaling or power behavior of the solution extends deeper and deeper into the infrared as the value of $k_{\rm min}$ is lowered.

Consequently, the color Coulomb potential behaves as

$$V_{c,k}(p) \propto p^{-\delta}, \qquad \delta = 2 + 2\beta + \gamma \tag{29}$$

for $p \gtrsim k_{\rm min}$; see Eqs. (10) and (25). Making use of our result in Eq. (13) for β, we extrapolate $V_{c,k}(p)$ to

$$V_c(p \to 0) \propto p^{-3.85} \tag{30}$$

for $k = 0$. We thus come quite close to a p^{-4} behavior which would correspond to a potential that rises linearly with distance (for sufficiently large distances). Also note that our result in Eq. (13) for β is supposed to be smaller than the correct value (see Ref. [1]), hence an improvement of the current approximation is expected to enhance the infrared exponent of $V_c(p)$.

The result of Eq. (30) has been obtained with the propagators taken from the flow equations [Eq. (11)] that do not include the tadpole diagrams. As argued below Eq. (13), we can easily take a part of the tadpole contributions into account in an effective way by replacing $\omega_k(p)$ and $d_k(p)$ on the right-hand sides of Eq. (11) with $\omega_0(p)$ and $d_0(p)$. In Ref. [1], we have also computed the flow of this improved truncation leading to $\beta = 0.80$. Inserting the latter solution into the flow equation for the Coulomb form factor [Eq. (27)], we are led to Fig. 1 (right). The resulting infrared potential reads

$$V_c(p \to 0) \propto p^{-4.25}. \tag{31}$$

The exponents in Eqs. (30) and (31) provide us with an estimate for the systematic error of the present approximation:

$$V_c(p \to 0) \propto p^{-\delta} \quad \text{with} \quad \delta \in [3.85, 4.25], \tag{32}$$

including $\delta = 4$.

In summary, the method of Hamiltonian flows allows for scaling solutions for the static two-point functions and the color Coulomb potential without additional approximations for the Coulomb form factor, contrary to variational approaches. In particular, we find an infrared-divergent Coulomb form factor and an almost linearly confining potential $V_c(p) \propto p^{-\delta}$ with $\delta \in [3.85, 4.25]$. We expect that an improvement of the approximation employed narrows the above interval for δ while still including $\delta = 4$.

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