

New class of spin projection operators for 3D models

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A new set of projection operators for three-dimensional models are constructed. Using these operators, an uncomplicated and easy handling algorithm for analyzing the unitarity of the aforementioned systems is built up. Interestingly enough, this method converts the task of probing the unitarity of a given three-dimensional system, which is in general a time-consuming work, into a straightforward algebraic exercise; besides, it also greatly clarifies the physical interpretation of the propagating modes. To test the efficacy and quickness of the algorithm at hand, the unitarity of some important and timely higher-order electromagnetic (gravitational) systems augmented by both Chern-Simons and higher order Chern-Simons terms are investigated.

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I. INTRODUCTION

The well-known complexities of four-dimensional (4D) field theory have often forced theorists to test models in lower-dimensional spaces. In general, the foundations of such models have been obtained only by projection from the physical dimension, which, of course, cannot shed light on the subtleties inherent to a particular dimension. In this vein—as it was pointed out by Binegar [1] with good reason—an independent development of the theories in their native dimension is required since a theorist is not supposed to be omniscient.

In this sense, three-dimensional theories deserve special attention because of its closeness to reality. Fortunately, planar physics has undergone a remarkable development in the past few decades. A host of new experimental results coming mainly from condensed matter physics and the accompanying rapid convergence of theoretical ideas has brought to the subject a new coherence and has also raised new interests. Among the many and interesting planar models that have been investigated, it is worth mentioning

the graphene [2]. This genuinely planar carbon system seems likely to be a good framework for the verification of ideas and methods developed in quantum (gauge) field theories. Consequently, we hopefully expect that the techniques of QED₃ when applied to this low-dimensional condensed-matter model lead to new and relevant results [3,4]. As far as gravity is concerned, the reason for doing research on planar gravity is quite amazing: (2 + 1)-dimensional gravity has a direct physical relevance to modeling phenomena that are actually confined to lower dimensionality. In fact, gravitational physics in the presence of straight cosmic strings (infinitely long, perpendicular to a plane) is adequately described by three-dimensional gravity [5]. We remark that the causality puzzles raised by “Gott time machines” were solved with the help of this lower dimensional model [6].

On the other hand, three-dimensional (3D) electromagnetic (gravitational) models enlarged by a Chern-Simons term have been the object of much attention. In the vector case this term gives mass to the photon in a gauge-invariant way, while for planar gravity the Chern-Simons term is responsible for the presence of a propagating parity-breaking massive spin-2 mode in the spectrum of the model [7].

Recently, both higher-order electromagnetic and gravitational models have enjoyed a revival of interest. Indeed,

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the fourth-dimensional theory of quantum electrodynamics proposed by Lee and Wick (LW) with the purpose of understanding the finite electromagnetic mass splitting of mesons, prior to when QCD was established [8,9], has been explored as a kind of toy model for the more complex dynamics of the LW Standard Model; i.e., the model in which Grinstein, O’Connell, and Wise, building on the pioneering work of LW, introduced non-Abelian LW gauge theories [10–19]; whereas, just about three years ago, Bergshoeff, Hohm, and Townsend (BHT) [20–28] proposed a particular higher-derivative extension of the Einstein-Hilbert action in three spacetime dimensions whose linearized version is a rare example of a fourth-order system that is not pestered by ghosts [29]. Besides, a canonical analysis of the quadratic curvature part of the BHT system done by Deser [29] establishes its weak field limit as both ghost-free and power-counting UV finite, thus violating standard folklore in the extreme.

The preceding considerations naturally suggest that investigations into general 3D higher-order electromagnetic (gravitational) models with a Chern-Simons term are welcome. The introduction of higher derivatives, nevertheless, could in principle jeopardize the unitarity of the models. It would thus be very convenient, in the spirit of paragraph one, to devise an easy handling procedure, specific to planar models, which allowed, on physical grounds, a constructive and meaningful discussion of the unitarity of generic 3D electromagnetic (gravitational) models, in an uncomplicated way.

In this paper, the aforementioned procedure is constructed by means of a new basis of spin operators, specific to 3D models, which allows a Lagrangian decomposition into spin components.

The ideas underlying our theoretical framework are described in Sec. II. We start off by building up a new class of spin projectors for 3D models and then discuss how to obtain the propagator for these models via the mentioned operators. The procedure for probing the unitarity of the 3D models is constructed afterward. Two important and timely higher-derivative systems enlarged by both Chern-Simons and higher derivative Chern-Simons terms are employed in Sec. III to illustrate the level of generality and quickness of the method. In Sec. IV it is shown that the expressions “closure” and “completeness” cannot be used interchangeably, as far as projection operators are concerned. In other words, the use of a closed set that does not obey the axiom of completeness leads undoubtedly to false physical results. Consequently, the fact that the set constituted by our spin operators is indeed complete guarantees that the physical results obtained by means of them are reliable. We also discuss in this section whether the addition of a Chern-Simons term to a nonunitary higher-derivative model can convert it into a unitary system. Some of the more technical results are gathered in the Appendix.

In our conventions the Greek letters denote spacetime indices, the metric signature is $(+1, -1, -1)$, $\epsilon_{012} = +1$, where $\epsilon_{\mu\nu\rho}$ is the Levi-Civita symbol, and $\hbar = c = 1$.

II. PRESCRIPTION FOR PROBING THE UNITARITY OF 3D MODELS

In the analysis of quantum aspects of any field theory, considerable interest is devoted to the description of the particle spectrum and the relativistic quantum properties of scattering processes of the theory under investigation. Some of these issues may be understood by means of the analysis of the propagator of the theory, which is obtained by the inversion of the wave operator. Accordingly, it is of great and fundamental importance to perform this inversion judiciously. We shall begin by looking for a suitable basis for the linear operators acting on the fields of the model. Using this basis, a generic expression for the propagator will then be constructed. Finally, a procedure for analyzing the unitarity of the 3D models, based on the preceding ingredients, will be worked out.

A. A new set of spin projection operators for 3D models

We start off by searching for a basis for the vectorial space of the wave operators. The vector space where these operators act is formed by finite-dimensional representations of the Lorentz group. In 4D, for instance, it is always possible to decompose these vector spaces in a direct sum of subspaces with well-defined spin since locally the Lorentz group can be regarded as the tensor product $SU(2) \otimes SU(2)$. Besides, the only mapping operators that can be built among these projectors are those associated with the same spin. In fact, the existence of mapping operators implies a bijection between spaces that can be achieved if and only if they have the same dimension. However, the construction of operators that map a spin space into a subspace with a different spin can only be realized by decomposing the larger spin space into a direct sum of subspaces defined by preferential vectors. The explicit construction of the spin projectors for fields of arbitrary rank can be made by appealing to the tensor product of projectors of vector fields according to the rules of group representations. The spin projectors, in 4D, that decompose the vector fields can be explicitly constructed using the Minkowski metric and partial derivatives, as follows:

$$P(0)_{\mu\nu} \equiv \omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\square}, \quad \square = \partial_\mu \partial^\mu, \quad (1a)$$

$$P(1)_{\mu\nu} \equiv \theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}. \quad (1b)$$

A careful analysis of the preceding equations allows us to conclude that the spin projectors and mapping operators of spin subspaces with the same dimension should also be built solely with the metric and partial derivatives, which leads us to the well-known Barnes-Rivers operators

[30–32]. It is worth noticing that if extra vectors are used in the construction of the models, such as a background vector in Lorentz violating models, operators with well-defined spin will be insufficient to form a basis for the wave operators [33].

On the other hand, the issue of the attainment of the wave operator and, subsequently, that of the propagator, for 3D models, need to be dealt carefully. Why is this so? Because now we have both parity-conserving (PC) and parity-violating (PV) models [34]. Since the PC systems are defined by Lagrangians involving only the metric and partial derivatives, the appropriate basis for expanding the wave operator is, of course, that made up of the usual 3D Barnes-Rivers operators. The “mark” of the PV models, i.e., the characteristic feature that enables us to recognize them, is, in turn, the presence of the Levi-Civita tensor, which allows us to define another vector linear operator,

$$S^{\mu\nu} = \epsilon^{\mu\nu\rho} \partial_\rho. \quad (2)$$

Using this operator, we can enlarge the usual operator basis, $\{\theta, \omega\}$, in order to obtain a complete set of linear operators $\{\theta, \omega, S\}$ [35]. It is worth noticing, however, that θ is no longer a spin operator since in the massive case the 3D spin corresponds to unitary representations of $SO(2)$ that are one dimensional. In fact, the operators θ and ω divide the three-dimensional space into a direct sum of two subspaces with dimensions 2 and 1, in this order, which implies that θ does not project into a spin subspace. Consequently, it is impossible to put a transparent and accurate physical interpretation on the excitation modes related to the PV models if they are expressed in terms of the basis $\{\theta, \omega, S\}$. A successful way of dealing with this problem would be to opt for a basis associated with the spin of the particles in 3D. We discuss in the following how this basis can be constructed.

We begin by recalling that in quantum field theory particles are identified as unitary irreducible representations (irreps) of the Poincaré group. This identification provides two quantum numbers for particles: mass and spin. The spin is characterized by the unitary representations of the little group of the representative momentum of the particle, namely, the subgroup of the Lorentz group that leaves the representative momentum unchanged.

In 4D, the task of identifying the spin operators is easier than in 3D. The reason is because in 4D the spin of the massive particles is given by unitary irreps of the group $SO(3)$. Such representations are associated with the representations of the group $SU(2)$, which is the covering group of $SO(3)$. Furthermore, the fundamental representation of $SU(2)$ is equivalent to its complex conjugate. Therefore all the representations of $SU(2)$ are real and, consequently, representations of $SO(3)$ are univocally related to the $SU(2)$ representations. This implies that if the wave operator is decomposed into operators that project

into well-defined irreps of $SO(3)$, they are automatically identified as operators with well-defined spin.

In 3D, on the other hand, we have a distinct situation, since the unitary representations of $SO(2)$ are associated with $U(1)$. The fundamental representation of $U(1)$ is not equivalent to its complex conjugate. Since all representations of $SO(2)$ are real, they are not directly related to representations of $U(1)$, but rather they should be identified with the direct sum of the fundamental representation and its complex conjugate. However, all the irreps of $SO(2)$ are two-dimensional and can be associated with representations of $U(1)$ by the complexification of the fields. Consider, for instance, the vectorial representation of $SO(2)$. A transformation of $SO(2)$ acting on a vector $A = (A_1, A_2)$ yields

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \quad (3)$$

The normalized eigenvectors of this transformation are

$$\lambda_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \lambda_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \quad (4)$$

So, we can define a basis for the 3D Minkowski space, with the characteristic that each of its vectors spans one and only one one-dimensional (1D) subspace that is an eigenspace of the $U(1)$ transformations, i.e.,

$$e^{(0)}_\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (5)$$

$$e^{(+1)}_\mu \equiv (e_1)_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, \quad (6)$$

$$e^{(-1)}_\mu \equiv (e_2)_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}, \quad (7)$$

where $e^{(0)}$ is timelike, whereas e_1 and e_2 are spacelike vectors.

Thence, under a suitable unitary change of variables, a real vector field transforms as

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \rightarrow \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} A_1 + iA_2 \\ A_1 - iA_2 \end{pmatrix}. \quad (8)$$

In this way the transformation of the vector \bar{A} under the rotation (3) is given by

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix}. \quad (9)$$

We conclude, therefore, that it is possible to make the identification of the vectorial representation of $SO(2)$, Δ , as the direct sum of the $U(1)$ fundamental representation, \square , and its complex conjugation, \square^* , i.e.,

$$\Delta \sim \square \oplus \square^*. \quad (10)$$

The θ operator is the identity in the space of the representation Δ . We may split this space in the direct sum of one-dimensional subspaces. Keeping in mind the 3D spin representations, spin projection operators may be associated with the basic complex vectors e_1 and e_2 as follows:

$$\rho^{\mu\nu} = -e_1^\mu (e_1^\nu)^*, \quad (11)$$

$$\sigma^{\mu\nu} = -e_2^\mu (e_2^\nu)^*. \quad (12)$$

Here, ρ is the projection operator associated with \square , while σ is related to \square^* . Note that ρ and σ are related by complex conjugation, $\rho^* = \sigma$; in addition, they are Hermitian and nonsymmetric:

$$\rho_{\mu\nu} = (e_1)_\mu (e_1)_\nu^* = (e_2)_\nu (e_2)_\mu^* = \sigma_{\nu\mu}. \quad (13)$$

This completes the identification of the spin modes for vector fields. It is also important to express the Chern-Simons operator (2) in terms of the spin projection operators. To do this, we note that $\varepsilon^{\mu\nu\rho} e_\mu^1 e_\nu^2 \frac{k_\rho}{\sqrt{k^2}} = -i$, since e_1 , e_2 , and $\frac{k_\rho}{\sqrt{k^2}}$ are normalized, which allows us to write

$$S_{\mu\nu} = -\sqrt{k^2}(\rho_{\mu\nu} - \sigma_{\mu\nu}). \quad (14)$$

One could wonder about the possibility of building mapping operators between the subspaces defined by k and e_1 or k and e_2 . In reality, these mapping operators are unnecessary since they would explicitly depend on e_1 and e_2 . Actually, in the case of Lorentz preserving models, the wave operator is constructed using solely η 's, ∂ 's, and ε 's.

Before going on, it is important to discuss the meaning of parity as far as the 3D operators we have just analyzed are concerned. In 3D, the representation of the parity operator in the Minkowski vector space is given by

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (15)$$

As a result, we get from (11) and (12) that $P\rho P^{-1} = \sigma$ and $P\sigma P^{-1} = \rho$, which clearly shows that, unlike what occurs in 4D, we cannot assign a definite parity for a given spin. We remark that

$$P\omega P^{-1} = \omega, \quad P\theta P^{-1} = \theta. \quad (16)$$

In the special case of the PC models, the aforementioned relations allow us to conclude that the massive excitation modes for nontrivial spins are nothing but spin doublets.

After this little digression, let us build up the spin projection operator for rank-2 tensors. For these tensors, we have

$$\begin{aligned} & (\underline{1} \oplus -\underline{1} \oplus \underline{0}) \otimes (\underline{1} \oplus -\underline{1} \oplus \underline{0}) \\ & = (3 \times \underline{0} \oplus 2 \times \underline{1} \oplus 2 \times -\underline{1} \oplus \underline{2} \oplus -\underline{2}), \end{aligned} \quad (17)$$

where the underlined numbers denote the spin, and the remaining ones are related to the spin multiplicity.

Now, a general rank-2 tensor, $T_{\mu\nu}$, may be written as the product of two vectors, say, A_μ and B_ν . Therefore,

$$T_{\mu\nu} = A_\mu B_\nu. \quad (18)$$

Since a generic vector can be split in its spin components $A_\mu \supset (1 \oplus -1 \oplus 0)$, by means of the spin projection operators, ρ , σ , and ω , namely,

$$A_\mu = (\rho_{\mu\rho} + \sigma_{\mu\rho} + \omega_{\mu\rho})A^\rho, \quad (19)$$

a generic rank-two tensor may also be decomposed in its spin components as follows:

$$\begin{aligned} T_{\mu\nu} = & (\rho_{\mu\rho}\rho_{\nu\sigma} + \rho_{\mu\rho}\sigma_{\nu\sigma} + \rho_{\mu\rho}\omega_{\nu\sigma} + \sigma_{\mu\rho}\rho_{\nu\sigma} \\ & + \sigma_{\mu\rho}\sigma_{\nu\sigma} + \sigma_{\mu\rho}\omega_{\nu\sigma} + \omega_{\mu\rho}\rho_{\nu\sigma} \\ & + \omega_{\mu\rho}\sigma_{\nu\sigma} + \omega_{\mu\rho}\omega_{\nu\sigma})T^{\rho\sigma}. \end{aligned} \quad (20)$$

Note that ρ , σ , and ω are associated with spin $+1$, -1 , and 0 , respectively, which implies that $\rho\rho$, $\rho\omega$, $\omega\rho$, $\rho\sigma$, $\sigma\rho$, $\omega\omega$, $\sigma\omega$, $\omega\sigma$, and $\sigma\sigma$ (with the indices omitted) are associated with spin $+2$, $+1$, $+1$, 0 , 0 , 0 , -1 , -1 , and -2 , in this order.

In the case of the graviton field $h_{\mu\nu}$, which is a symmetric rank-2 tensor, the symmetrization of the operators above yields the following spin projection operators:

$$P^{hh}(+2)_{\mu\nu;\rho\sigma} = \rho_{\mu\rho}\rho_{\nu\sigma}, \quad (21a)$$

$$P^{hh}(-2)_{\mu\nu;\rho\sigma} = \sigma_{\mu\rho}\sigma_{\nu\sigma}, \quad (21b)$$

$$\begin{aligned} P^{hh}(+1)_{\mu\nu;\rho\sigma} = & \frac{1}{2}(\rho_{\mu\rho}\omega_{\nu\sigma} + \rho_{\nu\rho}\omega_{\mu\sigma} \\ & + \rho_{\mu\sigma}\omega_{\nu\rho} + \rho_{\nu\sigma}\omega_{\mu\rho}), \end{aligned} \quad (21c)$$

$$\begin{aligned} P^{hh}(-1)_{\mu\nu;\rho\sigma} = & \frac{1}{2}(\sigma_{\mu\rho}\omega_{\nu\sigma} + \sigma_{\nu\rho}\omega_{\mu\sigma} \\ & + \sigma_{\mu\sigma}\omega_{\nu\rho} + \sigma_{\nu\sigma}\omega_{\mu\rho}), \end{aligned} \quad (21d)$$

$$P_{11}^{hh}(0)_{\mu\nu;\rho\sigma} = \omega_{\mu\rho}\omega_{\nu\sigma}, \quad (21e)$$

$$\begin{aligned} P_{22}^{hh}(0)_{\mu\nu;\rho\sigma} = & \frac{1}{2}(\rho_{\mu\rho}\sigma_{\nu\sigma} + \rho_{\nu\rho}\sigma_{\mu\sigma} \\ & + \rho_{\mu\sigma}\sigma_{\nu\rho} + \rho_{\nu\sigma}\sigma_{\mu\rho}). \end{aligned} \quad (21f)$$

The preceding operators, of course, are Hermitian. In addition, the projection operators associated with nontrivial spins are complex, whereas those related to spin-0 are real because nontrivial spins are nontrivial representations of $U(1)$, which are complex. For a real Lagrangian, the complex structures (21a)–(21d) alone cannot appear in the

wave operator decomposition in terms of the spin projection operators (this decomposition will be clarified later). We can ensure, however, because of the Lorentz invariance of the model, that projectors of the irreps of $SO(2)$ will be present in the wave operator. Such operators, usually known as Barnes-Rivers operators, are written in terms of θ and ω . Using the identity $\theta = \rho + \sigma$, we may split the real Barnes-Rivers operators into spin projection operators. Since $\rho^* = \sigma$, the wave operator is obviously real.

Consider, for instance, the projector associated with a symmetric and traceless rank-2 tensor. This operator projects into a nontrivial and irrep of $SO(2)$ and, therefore, it is two dimensional and can be written as

$$P^{hh}(2)_{\mu\nu;\rho\sigma} = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{2}\theta_{\mu\nu}\theta_{\rho\sigma}. \quad (22)$$

Now, taking into account that $\theta = \rho + \sigma$, we obtain two projectors in terms of ρ and σ , one for each degree of freedom of spin, i.e.,

$$P^{hh}(2)_{\mu\nu;\rho\sigma} = \rho_{\mu\rho}\rho_{\nu\sigma} + \sigma_{\mu\rho}\sigma_{\nu\sigma}, \quad (23)$$

clearly showing that the Barnes-Rivers operator $P^{hh}(2)$ is nothing but a sum of spin +2 and spin -2 operators. This process of decomposition can be repeated for all operators needed to exhaust all the possibilities of contraction of the fields present in the free Lagrangian. With this decomposition, the gravitational Chern-Simons operator

$$S_{\mu\nu;\rho\sigma} = \theta_{\mu\rho}S_{\nu\sigma} + \theta_{\mu\sigma}S_{\nu\rho} + \theta_{\nu\rho}S_{\mu\sigma} + \theta_{\nu\sigma}S_{\mu\rho} \quad (24)$$

can be expressed as

$$S_{\mu\nu;\rho\sigma} = -4i\sqrt{k^2}(P(+2)_{\mu\nu;\rho\sigma} - P(-2)_{\mu\nu;\rho\sigma}). \quad (25)$$

The other relations among the operators are listed in the Appendix.

B. The propagator

We are now ready to find the propagator and present afterward the algorithm for probing the unitarity of 3D electromagnetic (gravitational) models. Consider, in this vein, a 3D Lagrangian \mathcal{L} , which is a function either of a vector field A_a or of a symmetric rank-2 field h_{ab} . To compute the propagator for the model, we need beforehand the quadratic part of \mathcal{L} , i.e.,

$$(\mathcal{L})_2 = \frac{1}{2} \sum_{\alpha,\beta} \varphi_\alpha \mathcal{O}_{\alpha\beta} \varphi_\beta, \quad (26)$$

where α, β represent vectorial or tensorial indices, $\mathcal{O}_{\alpha\beta}$ is a local differential operator (the wave operator), and φ_α encompasses the fundamental quantum fields of the model. For gravity models, for instance, this is accomplished by means of the weak field approximation of the metric, i.e., $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

Using the identities of the Appendix, we then expand the wave operator in the basis of the spin operators, namely,

$$\mathcal{O}_{\alpha\beta} = \sum_{ij,J} a(J)_{ij} P_{ij}^{\varphi\varphi}(J)_{\alpha\beta}. \quad (27)$$

Here, $a(J)_{ij}$ are the coefficients of the expansion of the wave operators. The diagonal operators, $P_{ii}^{\varphi\varphi}(J)$, are operators that project the field φ into its spin J , whereas the off-diagonal operators ($i \neq j$) implement mappings into the corresponding spin doublet subspace. The resulting spin operators do obey the orthonormal multiplicative rules and the decomposition of unity, i.e.,

$$\sum_{\beta} P_{ij}(I)_{\alpha\beta} P_{kl}(J)_{\beta\gamma} = \delta_{jk} \delta^{IJ} P_{il}(I)_{\alpha\gamma}, \quad (28)$$

$$\sum_{i,J} P_{ii}(J)_{\alpha\beta} = \delta_{\alpha\beta}. \quad (29)$$

This converts the task of inverting the wave operator (27) into a straightforward algebraic exercise. Indeed, all we have to do is to invert the matrix of coefficients $a(J)_{ij}$. Nevertheless, $a(J)_{ij}$ may be degenerate because of the gauge symmetries of the model since the physical sources actually may satisfy some constraints. These consistently appear in order to inhibit the propagation of nonphysical modes. The explicit expressions for these constraints are given in terms of the left null eigenvectors $V_j^{(L,n)}$ of the degenerate coefficient matrices [36]

$$\sum_{\beta} V_j^{(L,n)}(J) P_{kj}(J)_{\alpha\beta} S_{\beta} = 0. \quad (30)$$

Nonetheless, since the propagator is saturated with the physical sources, the correct procedure for the attainment of the propagator is to invert any largest nondegenerate (for general values of momenta k) submatrix of $a(J)_{ij}$. Accordingly, to obtain the propagator, it suffices, in practice, as follows: (i) to delete rows and columns of $a(J)_{ij}$ according to the number of gauge symmetries, which gives rise to a matrix that we shall call $A(J)_{ij}$, and (ii) to invert $A(J)_{ij}$ and subsequently saturate this matrix with physical sources. As a result, the saturated propagator (Π) assumes the form [36]

$$\Pi = i \sum_{J,ij} S_{\alpha}^* A(J)_{ij}^{-1} P_{ij}(J)_{\alpha\beta} S'_{\beta}. \quad (31)$$

It must be emphasized that since the physical sources satisfy the constraints (30), the propagator is gauge independent. That is a great virtue of our method in comparison with the methods that do not use orthonormal projection operators. Indeed, in our procedure no gauge fixing is required.

C. The prescription

For the sake of simplicity, we shall divide our discussion about the unitarity of the 3D models into two parts: one of

them related to the massive poles, the other concerning the massless ones.

(i) *Massive poles*

To ensure that there are neither ghosts nor tachyons in the propagation mode of a given 3D model, we must require that at each simple pole of the propagator ($k^2 = m^2$),

$$\Im m \text{Res}(\Pi|_{k^2=m^2}) > 0, \quad \text{and} \quad m^2 \geq 0. \quad (32)$$

In the light of (31), we come to the conclusion that the condition for the absence of ghosts for each spin and for arbitrary sources is directly related to the positivity of the matrices $(\sum A(J, m^2)_{ij}^{-1} P_{ij}(J))_{\alpha\beta}$, where $A(J, m^2)_{ij}^{-1} = \text{Res} A(J, m^2)_{ij}^{-1}|_{k^2=m^2}$ is the matrix $A(J)_{ij}^{-1}$ with the pole extracted. Furthermore, it can be shown that these matrices have only one nonvanishing eigenvalue at the pole, which is equal to the trace of $A^{-1}(J, m^2)|_{k^2=m^2}$. Besides, the operators $P_{ij}(J)$ contribute only with a sign $(-1)^N$, whenever calculated at the pole, where N is the sum of the number of ρ 's and σ 's in each term of the projector. This sign $(-1)^N$ can be understood by noting that in the rest frame of the particle, ρ and σ contribute with a minus sign, whereas ω contributes with a positive sign. In summary, we may say that the conditions for the absence of ghosts and tachyons are such that for each massive single pole: (i) $m^2 > 0$, and (ii) $(-1)^N \text{tr} A^{-1}(J, m^2) > 0$.

(ii) *Massless poles*

The massless modes have some subtleties that require extra care. Indeed, at first sight it seems that the basis of operators is not well defined for lightlike momenta. However, the physical sources constraints that have their origin in the gauge symmetries of the model allow us to find a well-defined expression for the saturated propagator, even for lightlike momenta. These physical constraints (30) take the form $k_\mu \mathcal{S}^\mu = 0$ for the electromagnetic models and $k_\mu \mathcal{S}^{\mu\nu} = 0$ for the gravitational ones. Consequently, a convenient way of avoiding ghosts in the massless modes is to rewrite the inverse of the wave operator in terms of the following structures:

$$\omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}, \quad \theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}, \quad \epsilon_{\mu\nu\rho}, \quad k_\mu. \quad (33)$$

This task can be greatly facilitated by appealing to the relations of the Appendix; in addition, the sources must be expanded in a suitable momentum basis,

$$\mathcal{S}_\mu = c_1 k_\mu + c_2 q_\mu + c_3 \epsilon_\mu, \quad (34)$$

$$\begin{aligned} \mathcal{S}_{\mu\nu} = & c_1 k_\mu k_\nu + c_2 (k_\mu \epsilon_\nu + k_\nu \epsilon_\mu) \\ & + c_3 (k_\mu q_\nu + k_\nu q_\mu) + c_4 q_\mu q_\nu \\ & + c_5 (q_\mu \epsilon_\nu + q_\nu \epsilon_\mu) + c_6 \epsilon_\mu \epsilon_\nu, \end{aligned} \quad (35)$$

where the c_i 's are complex coefficients, and

$$k_\mu = (k_0, \vec{k}), \quad (36a)$$

$$q_\mu = (k_0, -\vec{k}), \quad (36b)$$

with

$$k^2 = q^2 = 0, \quad (37a)$$

$$k \cdot q = (k_0)^2 + (\vec{k})^2, \quad (37b)$$

$$k \cdot \epsilon = q \cdot \epsilon = 0, \quad (37c)$$

$$\epsilon^2 = -1. \quad (37d)$$

The expansion (34) and (35) is the most general one for both vectors and symmetric rank-2 tensors and must be supplemented by the sources constraints (30). Accordingly, the positivity of the residue of the propagator is assured if

$$\Im m \text{Res}(\Pi|_{k^2=0}) \geq 0. \quad (38)$$

III. TESTING THE EFFICACY AND QUICKNESS OF THE PRESCRIPTION

To explicitly illustrate the generality and simplicity of the proposed method, we analyze in the following the unitarity of some higher-derivative electromagnetic (gravitational) models enlarged by both Chern-Simons and higher-order Chern-Simons terms. We also comment, in passing, on some interesting and remarkable properties of these systems.

A. Higher-derivative electromagnetic models

We begin our study by checking the unitarity of the Lee-Wick-Chern-Simons model, which is defined by the Lagrangian,

$$\begin{aligned} \mathcal{L}_{\text{LWCS}} = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4m^2} F_{\mu\nu} \square F^{\mu\nu} \\ & + \frac{\mu}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \end{aligned} \quad (39)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and m (μ) is a parameter with dimension of mass.

Now, writing the Lagrangian above in the form (26), we promptly obtain the expression for the wave operator in momentum space, namely,

$$\mathcal{O}_{\mu\nu} = \left(-k^2 + \frac{k^4}{m^2}\right) \theta_{\mu\nu} - i\mu \epsilon_{\mu\nu\rho} k^\rho. \quad (40)$$

With the help of the identities listed in the Appendix, the wave operator (40) may be expanded in the 3D spin projection operators basis as follows:

$$\mathcal{O}_{\mu\nu} = \sum_{ij,J} a(J)_{ij} P_{ij}^{AA}(J)_{\mu\nu}, \quad (41)$$

where

$$a(0) = 0, \quad (42)$$

$$a(1) = \begin{pmatrix} -k^2 + \frac{k^4}{m^2} + \mu\sqrt{k^2} & 0 \\ 0 & -k^2 + \frac{k^4}{m^2} - \mu\sqrt{k^2} \end{pmatrix}. \quad (43)$$

It is worth noticing that the spin-0 sector is completely degenerate, which is fully expected since the model (39) has a gauge symmetry

$$A'_\mu = A_\mu + \delta A_\mu. \quad (44)$$

The term δA_μ can easily be obtained by noticing that it can be associated with the right null eigenvalues of the matrices of the coefficients $V_i^{(R,n)}$ [36],

$$\delta\Psi_\alpha = \sum_{i,J,\beta} V_i^{(R,n)\psi}(J) P_{ij}^{\Psi\lambda}(J)_{\alpha\beta} f_\beta(J). \quad (45)$$

This result applies to every independent value of j and n . For this example, we get

$$\delta A_\mu = \partial_\mu(\partial_\nu f^\nu), \quad (46)$$

where f^ν is an arbitrary function, as it should.

Interestingly, the gauge symmetry of the model inhibits the propagation of the spinless mode; as a consequence of this symmetry, there appears a source constraint that prevents the propagation of this unphysical state. It is trivial to see that the general expression (30) reduces now to

$$k_\mu \mathcal{S}^\mu = 0, \quad (47)$$

which is nothing but the familiar source conservation relation.

The inverse matrix of the spin-1 sector, on the other hand, reads

$$a(1)^{-1} = \frac{1}{\Delta} \begin{pmatrix} -k^2 + \frac{k^4}{m^2} - \mu\sqrt{k^2} & 0 \\ 0 & -k^2 + \frac{k^4}{m^2} + \mu\sqrt{k^2} \end{pmatrix}, \quad (48)$$

where $\Delta = [(k^2 - m^2)^2 \frac{k^2}{m^4} - \mu^2]k^2 = (\frac{k^6}{m^4} - 2\frac{k^4}{m^2} + k^2 - \mu^2)k^2$ is a quartic polynomial in k^2 . As a result, it has four roots: one massless pole, and three massive ones, which we shall call m_1 , m_2 , and m_3 , respectively. Therefore, as far as the nature of the roots are concerned, there exist precisely four distinct cases to be dealt with (see Fig. 1): (1) $\mu = 0$ (Lee-Wick electrodynamics),

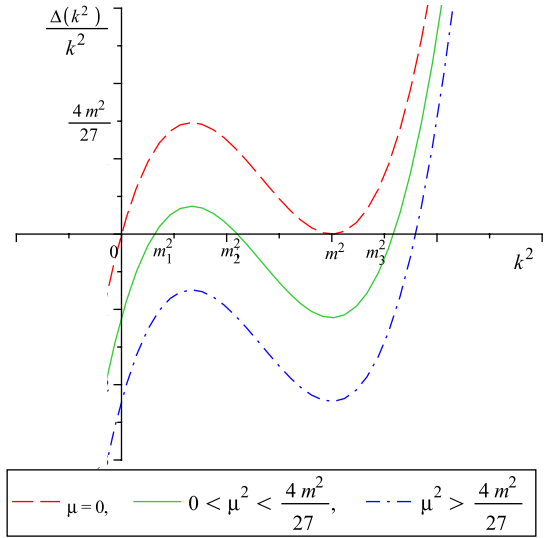


FIG. 1 (color online). Polynomial function $\frac{\Delta(k^2)}{k^2}$ versus k^2 , where $\Delta(k^2)$ refers to the denominator of the propagator (48).

(2) $0 < \mu^2 < \frac{4m^2}{27}$ (In this case there are three real positive masses.), (3) $\mu^2 > \frac{4m^2}{27}$ (Here there are necessarily two complex roots, implying the existence of tachyonic excitations.), and (4) $\mu^2 = \frac{4m^2}{27}$ (In this case there appears a double pole; as a result, the model is nonunitary.). We discuss in the following only the physical models, i.e., cases (1) and (2).

1. $\mu = 0$ (Lee-Wick Electrodynamics)

The matrix of the coefficients is now given by

$$a(1) = \begin{pmatrix} -k^2 + \frac{k^4}{m^2} & 0 \\ 0 & -k^2 + \frac{k^4}{m^2} \end{pmatrix}, \quad (49)$$

while its inverse can be written as

$$a(1)^{-1} = \frac{1}{(k^2 - m^2)k^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (50)$$

Consequently, the absence of tachyons and ghosts in the model is subordinated, respectively, to the following conditions:

$$m^2 > 0, \quad (51)$$

$$(-1) \text{tr}A(1, m^2)^{-1}|_{k^2=m^2} = -1, \quad (52)$$

which clearly shows the presence of a nontachyonic massive ghost in the system.

For the massless pole, the constraint $k_\mu \mathcal{S}^\mu = 0$ allows us to write the saturated propagator as

$$\Pi = \frac{1}{(k^2 - m^2)k^2} i\mathcal{S}^{*\mu} \mathcal{S}_\mu. \quad (53)$$

Expanding now the current S^μ in the momentum basis (34) yields

$$\Im m \text{Res}(\Pi|_{k^2=0}) = |c_1|^2 > 0, \quad (54)$$

which allows us to conclude that the massless mode does not violate the unitarity.

The wrong sign of Eq. (53) indicates an instability of the theory at the classical level. From the quantum point of view it means that the theory is nonunitary. Luckily, these difficulties can be circumvented. Indeed, the classical instability can be removed at least in 4D by imposing a future boundary condition in order to prevent exponential growth of certain modes. However, this procedure leads to causality violations in the theory [37]; fortunately, this acausality is suppressed below the scales associated with the Lee-Wick particles. On the other hand, Lee and Wick argued that despite the presence of the aforementioned degrees of freedom associated with a nonpositive definite norm on the Hilbert space, the theory could nonetheless be unitary as long as the new Lee-Wick particles obtain decay widths. There is no general proof of unitarity at arbitrary loop order for the Lee-Wick electrodynamics; nevertheless, there is no known example of unitarity violation. Accordingly, the Lee-Wick electrodynamics is finite. Therefore, we need not be afraid of the massive spin-1 ghost. It is worth mentioning that recently a quantum bound on the Lee-Wick heavy particle mass was found that is of the order of the mass of the Z^0 boson [38]. Actually, this is a very important result. Indeed, on the one hand, the knowledge of this parameter converts the aforementioned model into a predictable one; on the other, it introduces a natural scale for the model. As a result, we can estimate in the limit of static charge, for instance, the distance in which the Lee-Wick potential departs from the usual Coulomb one. In a sense, this result allows us to ascertain that only for small distances does the higher-derivative term coming from the Lee-Wick model affect the well-established results obtained within the context of the usual QED, as expected [38].

$$2. \mathbf{0} < \mu^2 < \frac{4m^2}{27}$$

Here $\Delta(k^2) = (k^2 - m_1^2)(k^2 - m_2^2)(k^2 - m_3^2)k^2/m^4$, where m_1 , m_2 , and m_3 are the three real positive roots of Δ . We assume without any loss of generality that $m_1 < m_2 < m_3$ (see Fig. 1). On the other hand, the relations $(-1) \text{tr} A(1, m_i^2)|_{k^2=m_i^2} > 0$ ($i = 1, 2, 3$), lead to the following inequalities:

$$m_1: \frac{(m_1^2 - m^2)}{(m_1^2 - m_2^2)(m_1^2 - m_3^2)} < 0, \quad (55)$$

$$m_2: \frac{(m_2^2 - m^2)}{(m_2^2 - m_1^2)(m_2^2 - m_3^2)} < 0, \quad (56)$$

$$m_3: \frac{(m_3^2 - m^2)}{(m_3^2 - m_1^2)(m_3^2 - m_2^2)} < 0, \quad (57)$$

implying that $m_1^2 < m^2$, $m_2^2 > m^2$, and $m_3^2 < m^2$, which, of course, contradicts the original assumption that $m_1 < m_2 < m_3$. Thence, we come to the conclusion that this model is plagued by ghosts.

3. Lee-Wick-Chern-Simons model enlarged by a higher derivative Chern-Simons extension

Another interesting model can be built up from the Lee-Wick-Chern-Simons system by adding to the Lagrangian (39) the higher-derivative Chern-Simons extension proposed by Deser and Jackiw [39]

$$\mathcal{L}_{\text{ECS}} = \frac{\lambda}{2} \epsilon^{\mu\nu\rho} \square A_\mu \partial_\nu A_\rho. \quad (58)$$

Let us then check the unitarity of this curious model. Starting from the wave operator in momentum space,

$$\mathcal{O}_{\mu\nu} = \left(-k^2 + \frac{k^4}{m^2} \right) \theta_{\mu\nu} - i(\mu + \lambda k^2) \epsilon_{\mu\nu\rho} k^\rho, \quad (59)$$

it is straightforward to show that the spin-1 matrix of the coefficients and its inverse are, respectively, given by

$$a(1) = \begin{pmatrix} -k^2 + \frac{k^4}{m^2} + (\mu + \lambda k^2)\sqrt{k^2} & 0 \\ 0 & -k^2 + \frac{k^4}{m^2} - (\mu + \lambda k^2)\sqrt{k^2} \end{pmatrix}, \quad (60)$$

$$a^{-1}(1) = \frac{1}{\Delta} \begin{pmatrix} -k^2 + \frac{k^4}{m^2} - (\mu + \lambda k^2)\sqrt{k^2} & 0 \\ 0 & -k^2 + \frac{k^4}{m^2} + (\mu + \lambda k^2)\sqrt{k^2} \end{pmatrix}, \quad (61)$$

where

$$\Delta = \left[\frac{k^6}{m^4} - (1 + \lambda^2) \frac{k^4}{m^2} + (1 - 2\lambda\mu)k^2 - \mu^2 \right] k^2. \quad (62)$$

An analysis similar to that done in Sec. III A tells us that the addition of the higher-derivative Chern-Simons extension does not improve the nonunitarity of the Lee-Wick-Chern-Simons model. Neither does it cure the nonunitarity of the Lee-Wick system.

B. Higher-derivative gravitational models

Higher-derivative gravity augmented by a Chern-Simons term is defined by the Lagrangian

$$\mathcal{L} = \sqrt{g}(\alpha R + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R^2) + \frac{\mu}{2} \mathcal{L}_{\text{CS}}, \quad (63)$$

$$\begin{aligned} \mathcal{L}_{(2)} = & \frac{\alpha}{2} \left(-\frac{1}{2} h^{\mu\nu} \square h_{\mu\nu} + \frac{1}{2} h \square h - h \partial_\mu \partial_\nu h^{\mu\nu} + h^{\mu\nu} \partial_\mu \partial_\rho h^\rho{}_\nu \right) + \frac{\beta}{4} (h^{\mu\nu} \square^2 h_{\mu\nu} + h \square^2 h - 2h \square \partial_\mu \partial_\nu h^{\mu\nu} \\ & - 2h^{\mu\nu} \square \partial_\mu \partial_\rho h_\nu{}^\rho + 2h^{\mu\nu} \partial_\mu \partial_\nu \partial_\rho \partial_\sigma h^{\rho\sigma}) + \gamma (h \square^2 h - 2h \square \partial_\mu \partial_\nu h^{\mu\nu} + h^{\mu\nu} \partial_\mu \partial_\nu \partial_\rho \partial_\sigma h^{\rho\sigma}) \\ & + \frac{\mu}{4} h_\mu{}^\nu \varepsilon^{\mu\lambda\rho} \partial_\lambda (\square h_{\nu\rho} - \partial_\nu \partial_\sigma h_\rho{}^\sigma). \end{aligned} \quad (65)$$

We have now all the ingredients to compute the wave operator $\mathcal{O}_{\mu\nu,\rho\sigma}$ and expand it in the appropriate degree of freedom basis with the aid of the identities collected in the Appendix. The resulting matrices of the coefficients concerning this expansion are

$$a(0) = \begin{pmatrix} ((3\beta + 8\gamma)k^2 - \alpha)k^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad (66)$$

$$a(2) = \begin{pmatrix} (\alpha + \beta k^2 + \mu\sqrt{k^2})k^2 & 0 \\ 0 & (\alpha + \beta k^2 - \mu\sqrt{k^2})k^2 \end{pmatrix}. \quad (67)$$

Because of the gauge symmetry of the model the spin-0 matrix above is evidently noninvertible, which is translated into the usual source conservation constraint on the gravitational sources $k_\mu \mathcal{S}^{\mu\nu} = 0$.

On the other hand, the inverse $[A(0)_{ij}^{-1}]$ of the largest nondegenerate matrix extracted from $a(0)$, as well as the inverse of $a(2)_{ij}^{-1}$, can be written as

$$A(0)^{-1} = \frac{1}{[(3\beta + 8\gamma)k^2 - \alpha]k^2}, \quad (68)$$

$$\begin{aligned} a(2)^{-1} = & \frac{1}{[(\alpha + \beta k^2)^2 - \mu^2 k^2]k^2} \\ & \times \begin{pmatrix} \alpha + \beta k^2 - \mu\sqrt{k^2} & 0 \\ 0 & \alpha + \beta k^2 + \mu\sqrt{k^2} \end{pmatrix}. \end{aligned} \quad (69)$$

Using the constraints (32) on the matrices (68) and (69), the following relations for the parameters are obtained:

where

$$\mathcal{L}_{\text{CS}} = \varepsilon^{\mu\nu\rho} \Gamma_{\mu\sigma}^\lambda \left(\partial_\nu \Gamma_{\rho\lambda}^\sigma + \frac{2}{3} \Gamma_{\nu\lambda}^\kappa \Gamma_{\rho\kappa}^\sigma \right) \quad (64)$$

is the Chern-Simons term and α , β , γ , and μ are arbitrary coefficients.

In the weak field approximation ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$), this Lagrangian reduces to

$$\text{Spin-2: } \alpha < 0, \quad \beta > 0; \quad (70)$$

$$\text{Spin-0: } \alpha > 0, \quad 3\beta + 8\gamma > 0. \quad (71)$$

Accordingly, for arbitrary values of the parameters the model is nonunitary. Nevertheless, there exists a way of circumventing this difficulty: all we have to do is to prevent the propagation of the massive spin-0 mode by choosing $3\beta + 8\gamma = 0$. Remarkably, this is precisely the constraint utilized by BHT in the construction of their model [19]. Another alternative is to inhibit the propagation of the massive spin-2 mode by setting $\beta = 0$. As a consequence,

$$A(0)^{-1} = \frac{1}{8\gamma k^2 (k^2 - \frac{\alpha}{8\gamma})}. \quad (72)$$

If we take into account that the absence of tachyons and ghosts requires, respectively, that $\frac{\alpha}{\gamma} > 0$ and $\gamma > 0$, we come to the conclusion that the model $\alpha R + \gamma R^2$ is unitary if α and γ are positive.

For the massless poles, one must use the original expression (31) for the propagator in order to compute the residue. The constraints satisfied by the sources allow us to handle correctly the singularities. Using such constraints and discarding terms that do not contribute to the residue yields

$$\begin{aligned} \Pi = & \frac{1}{\alpha k^2} i \mathcal{S}^{*\mu\nu} \left[\frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\nu} \eta_{\rho\sigma}) \right. \\ & \left. - i \mu \varepsilon_{\mu\rho\lambda} \eta_{\nu\sigma} k^\lambda \right] \mathcal{S}^{\rho\sigma}. \end{aligned} \quad (73)$$

Using a suitable basis for the expansion of the sources in momentum space, we arrive at the conclusion that this expression vanishes identically, which clearly shows that the massless mode is nonpropagating.

The preceding analysis seems to indicate the existence of two unitary higher-derivative gravity models in 3D: the BHT and the $\alpha R + \gamma R^2$ systems. Actually, only the BHT model can be really considered a higher-derivative gravity system. Indeed, this model contains fourth derivatives of the metric, while the pure scalar curvature system is conformally equivalent to Einstein gravity with a scalar field [40], which means that despite having fourth derivatives at the metric level the $\alpha R + \gamma R^2$ model is ultimately second order in its scalar-tensor version.

We point out that the description of gravitational phenomena via the BHT model does not lead to some really bizarre results as in the usual 3D gravity (lack of a gravity force in the nonrelativistic limit, gravitational deflection independent of the impact parameter, complete absence of gravitational time dilation, and no time delay). Actually, in the framework of new massive gravity, short-range gravitational forces are exerted on slowly moving particles; besides, the light bending depends on the impact parameter, as it should [41]. And additionally, both time delay and spectral shift do take place in the context of the alluded model [42].

Until very recently it was believed that “new massive gravity” [20,21] was the only higher-derivative gravity model in 3D that was simultaneously perturbatively renormalizable and unitary in flat space [43]. In a sense, it was expected that most likely the full model would be non-renormalizable since it only improved the spin-2 projections of the propagator but not the spin-0 projection. We remark, however, that it was recently shown that the general theory of higher-derivative gravity in 3D is renormalizable, with two notable exceptions [44,45]: the models in which the coefficients are restricted to the special values $3\beta + 8\gamma = 0$ or $\beta = 0$. Unfortunately, those are precisely

the systems that are tree-level unitary. Consequently, despite being unitary at the tree level, neither the BHT system nor the $\alpha R + \gamma R^2$ model are renormalizable. Interestingly, this result seems to indicate that the conjecture that unitarity and renormalizability cannot coexist simultaneously in the framework of one and the same higher-derivative theory is correct.

It is worth noting that depending on the choice of the parameters in the action concerning higher-derivative gravity (HDG) in $2 + 1$ dimensions, one obtains gravity, anti-gravity, or gravitational shielding; in addition, we can analyze the gravitational properties of HDG using a model somewhat analogous to this one: a plane orthogonal to a straight cosmic string described by higher-derivative gravity in $3 + 1$ dimensions [46,47].

1. Higher-derivative Chern-Simons gravity enlarged by the Ricci-Cotton tensor

For gravity theories there is also the possibility of the construction of a higher-derivative Chern-Simons extension, the so-called Ricci-Cotton term, which is defined by the Lagrangian

$$\mathcal{L}_{\text{RC}} = \lambda \varepsilon^{\mu\nu\rho} R_{\mu\sigma} D_\nu R_\rho{}^\sigma. \quad (74)$$

We remark that models including this term were recently investigated by Bergshoeff, Hohm, and Townsend in their research on higher derivatives in 3D gravity and higher-spin gauge theories [48].

Let us then probe the unitarity of higher-derivative Chern-Simons gravity augmented via the Ricci-Cotton term. It is curious that this term only alters the spin-2 sector of this model. The matrix of the coefficients and its inverse are now given by

$$a^{(2)} = \begin{pmatrix} [\alpha + \beta k^2 - (\mu + \lambda k^2)\sqrt{k^2}]k^2 & 0 \\ 0 & [\alpha + \beta k^2 + (\mu + \lambda k^2)\sqrt{k^2}]k^2 \end{pmatrix}, \quad (75)$$

$$a^{(2)-1} = \frac{1}{\Delta} \begin{pmatrix} \alpha + \beta k^2 + (\mu + \lambda k^2)\sqrt{k^2} & 0 \\ 0 & \alpha + \beta k^2 - (\mu + \lambda k^2)\sqrt{k^2} \end{pmatrix}, \quad (76)$$

where

$$\Delta = [-\lambda^2 k^6 + (\beta^2 - 2\mu\lambda)k^4 + (2\alpha\beta - \mu^2)k^2 + \alpha^2]k^2.$$

A cursory glance at the equation above is sufficient to convince us that the model at hand can describe at most three massive particles. Proceeding in the same way as we have done in Sec. III A, we conclude that the addition of the Ricci-Cotton term is not a good therapy for curing the nonunitarity of higher-derivative Chern-Simons gravity.

IV. CONCLUDING REMARKS AND COMMENTS

We have devised an easy procedure for checking the unitarity of 3D models based on a new class of spin projection operators. The great importance of these operators resides precisely in the fact that they form a complete (and, of course, closed) set for Lorentz preserving (PV and PC) models. In other words, they obey a completeness

relation. However, it may happen that a subset of a given complete set of operators is closed. A natural and important point to be discussed in this case is whether this “incomplete” set is appropriated for expanding the propagator. Consider, in this vein, three-dimensional PC gravity models. Now, as we have already commented, the suitable basis for computing the propagator is that those elements are the well-known Barnes-Rivers operators. Expanding the wave operator in this basis, we obtain in momentum space

$$\begin{aligned} \mathcal{O} = & x_1 P(1) + x_2 P(2) + x_s P(0^s) + x_\omega P(0^\omega) \\ & + x_{s\omega} P(0^{s\omega}) + x_{\omega s} P(0^{\omega s}). \end{aligned} \quad (77)$$

Consequently, the corresponding propagator is given by

$$\begin{aligned} \mathcal{O}^{-1} = & \frac{1}{x_2} P(2) + \frac{1}{x_1} P(1) + \frac{1}{x_s x_\omega - x_{s\omega} x_{\omega s}} \\ & \times [x_\omega P(0^s) + x_s P(0^\omega) - x_{s\omega} P(0^{s\omega}) - x_{\omega s} P(0^{\omega s})]. \end{aligned} \quad (78)$$

Now, if $S \equiv \{P(1), P(2), \dots, P(0^{\omega s})\}$, then $S' \equiv \{P(1), P(2), P(0^s), P(0^\omega)\}$ is a subset of S , which is closed under the same operation of multiplication as that concerning S ; in addition, the elements of S' obey the relation

$$P(1) + P(2) + P(0^s) + P(0^\omega) = \delta, \quad (79)$$

which is nothing but the decomposition of unity. The relevant question, nonetheless, is that (79) is not a completeness relation for the operators at hand. If S' were a complete set, we would arrive at the wrong conclusion that it should necessarily be a basis for performing our computations; as a consequence, the propagator for the PC gravity models would assume the form

$$\mathcal{O}_{\text{wrong}}^{-1} = \frac{1}{x_1} P(1) + \frac{1}{x_2} P(2) + \frac{1}{x_s} P(0^s) + \frac{1}{x_\omega} P(0^\omega). \quad (80)$$

Comparing (78) and (80), we come to the conclusion that these expressions coincide only and if only $x_{s\omega} = x_{\omega s} = 0$. Nevertheless, these coefficients cannot be zero. Indeed, since the PC gravity models are gauge invariant, we have to add to the Lagrangian of the model a gauge-fixing Lagrangian (\mathcal{L}_{gf}) so that the resulting wave operator can be inverted. Choosing for this purpose, without any loss of generality, the de Donder gauge and taking into account that its linearized version can be written as follows:

$$\mathcal{L}_{\text{gf}} = \frac{1}{2\lambda} (\partial_\mu \gamma^\mu)^2, \quad (81)$$

where $\gamma^\mu = \partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h$, we promptly obtain in momentum space

$$\begin{aligned} \mathcal{O}_{\text{gf}}(k) = & \frac{k^2}{2} \left[\frac{1}{2} P(1) + \frac{1}{2} P(0^s) + \frac{1}{4} P(0^\omega) \right. \\ & \left. - \frac{\sqrt{2}}{4} P(0^{s\omega}) - \frac{\sqrt{2}}{4} P(0^{\omega s}) \right], \end{aligned} \quad (82)$$

which clearly shows that both $x_{s\omega}$ and $x_{\omega s}$ are different from zero. In other words, the operator $\mathcal{O}_{\text{wrong}} \equiv x_1 P(1) + x_2 P(2) + x_s P(0^s) + x_\omega P(0^\omega)$ is obviously not invertible. Suppose, however, that we argue that both expressions for the propagator are correct owing to the fact that for physical problems in which the propagator (78) is contracted with conserved external currents ($S\mathcal{O}^{-1}S$, $kS = 0$), both operators $P(0^{s\omega})$ and $P(0^{\omega s})$ do not contribute for the final result of the calculations. Again, it is trivial to show that this argument is fallacious. In fact, a straightforward computation leads to the following results:

$$\begin{aligned} S\mathcal{O}^{-1}S = & S \left[\frac{1}{x_2} P(2) + \frac{x_\omega}{x_s x_\omega - x_{s\omega} x_{\omega s}} P(0^s) \right] S, \\ S\mathcal{O}_{\text{wrong}}^{-1}S = & S \left[\frac{1}{x_2} P(2) + \frac{1}{x_s} P(0^s) \right] S. \end{aligned}$$

In summary, only operators that form a complete set can be used to attain the propagator. In other words, the term “closure” cannot be used as a synonym for “completeness.” The fact that our three-dimensional set of operators is indeed complete thus guarantees that the physical results obtained through them can be trusted.

Another point that deserves to be discussed is whether the nonunitary disease that affects some three-dimensional models could be cured by the addition of a Chern-Simons term to the system; of course, we are not excluding from our considerations the possibility of enlarging the model via a higher-derivative Chern-Simons extension or even through the simultaneous addition of Chern-Simons and higher-derivative Chern-Simons terms. For the sake of brevity, we restrict our analysis to three-dimensional gravitational models. Everything started when it was found that the solution to the triviality problem of general relativity in (2 + 1)D could be cured by simply adding a topological Chern-Simons term to the system. The resulting model describes a nontrivial gravity theory with a propagating, massive, spin-2 mode [7]. Later on it was considered another way out of the triviality problem of 3D gravity: the addition of higher-derivative terms to the system [49]; unfortunately the resulting models are nonunitary [50]. On the other hand, it was claimed that the addition of a Chern-Simons term to the previous model would cure its nonunitarity [51]. This was proved afterward to be incorrect [52]. After this digression, let us respond to the question we have raised above. As we have seen in Sec. III, nonunitary higher-derivative electromagnetic (gravitational) models do not become unitary systems by simply augmenting them through Chern-Simons terms. Neither do they become unitary by enlarging them via a higher-derivative Chern-Simons extension. It is amazing, nonetheless, that there

are some examples in the literature of unitary systems whose unitarity is spoiled by the addition of Chern-Simons terms [53,54]. Therefore, in some cases, the coexistence between the topological term and higher-derivative theories is conflicting. Consequently, the addition of a Chern-Simons term to a given model should be based on transparent physical results. This was precisely the most important criterion we have adopted for choosing the models discussed in the text.

To conclude, we would like to point out that the results of this work are actually relevant to graphene. In fact, recently quantum field theory methods have been applied to analyze the properties of this interesting system. As a consequence, the Dirac model from the tight binding model was derived and calculations of the polarization operator (conductivity) were described. Subsequently, this polarization operator was used to describe the quantum Hall effect, light absorption by graphene, the Faraday effect, and the Casimir interaction [55]. There are also interesting studies of the graphene with emphasis on Chern-Simons terms [56,57]. The use of

our three-dimensional operators will certainly benefit the computations involving the polarization operator, as well as those related to models augmented by Chern-Simons terms.

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APPENDIX A: PROJECTION OPERATORS AND TENSOR RELATIONS

In this appendix, we gather the spin operators constructed in Sec. II A and some useful identities satisfied by them.

1. Vector field operators: $A - A$

(1) Spin-0 sector

$$(a) P^{AA}(0)_{\mu\nu} = \omega_{\mu\nu}$$

(2) Spin-1 sector

$$(a) P_{11}^{AA}(+1)_{\mu\nu} = \rho_{\mu\nu}$$

$$(b) P_{22}^{AA}(-1)_{\mu\nu} = \sigma_{\mu\nu}$$

(3) Identities among the operators

$$(a) P^{AA}(1)_{\mu\nu} = \theta_{\mu\nu} = P_{11}^{AA}(+1)_{\mu\nu} + P_{22}^{AA}(-1)_{\mu\nu}$$

(4) Tensorial identities

$$\eta_{\mu\nu} = P^{AA}(0)_{\mu\nu} + P^{AA}(1)_{\mu\nu}, \quad k_\mu k_\nu = k^2 P^{AA}(0)_{\mu\nu}, \quad \varepsilon_{\mu\nu\rho} k^\rho = i\sqrt{k^2}(P_{11}^{AA}(+1)_{\mu\nu} - P_{22}^{AA}(-1)_{\mu\nu}).$$

2. Rank-2 symmetric field operators: $h - h$

(1) Spin-0 sector

$$(a) P_{11}^{hh}(0^s)_{\mu\nu;\rho\sigma} = \frac{1}{2}\theta_{\mu\nu}\theta_{\rho\sigma},$$

$$(b) P_{22}^{hh}(0^\omega)_{\mu\nu;\rho\sigma} = \omega_{\mu\nu}\omega_{\rho\sigma},$$

$$(c) P_{12}^{hh}(0^{s\omega})_{\mu\nu;\rho\sigma} = \frac{1}{\sqrt{2}}\theta_{\mu\nu}\omega_{\rho\sigma},$$

$$(d) P_{21}^{hh}(0^{\omega s})_{\mu\nu;\rho\sigma} = \frac{1}{\sqrt{2}}\omega_{\mu\nu}\theta_{\rho\sigma}.$$

(2) Spin-1 sector

$$(a) P_{11}^{hh}(+1)_{\mu\nu;\rho\sigma} = \frac{1}{2}(\rho_{\mu\rho}\omega_{\nu\sigma} + \rho_{\nu\rho}\omega_{\mu\sigma} + \rho_{\mu\sigma}\omega_{\nu\rho} + \rho_{\nu\sigma}\omega_{\mu\rho}),$$

$$(b) P_{22}^{hh}(-1)_{\mu\nu;\rho\sigma} = \frac{1}{2}(\sigma_{\mu\rho}\omega_{\nu\sigma} + \sigma_{\nu\rho}\omega_{\mu\sigma} + \sigma_{\mu\sigma}\omega_{\nu\rho} + \sigma_{\nu\sigma}\omega_{\mu\rho}),$$

$$(c) P_{12}^{hh}(\pm 1)_{\mu\nu;\rho\sigma} = \frac{1}{2}\varepsilon_{\tau\eta\kappa}(\rho_\mu^\tau\sigma_\rho^\eta\omega_{\nu\sigma} + \rho_\nu^\tau\sigma_\rho^\eta\omega_{\mu\sigma} + \rho_\mu^\tau\sigma_\sigma^\eta\omega_{\nu\rho} + \rho_\nu^\tau\sigma_\sigma^\eta\omega_{\mu\rho})\frac{k^\kappa}{\sqrt{k^2}},$$

$$(d) P_{21}^{hh}(\mp 1)_{\mu\nu;\rho\sigma} = \frac{1}{2}\varepsilon_{\tau\eta\kappa}(\sigma_\mu^\eta\rho_\rho^\tau\omega_{\sigma\nu} + \sigma_\nu^\eta\rho_\rho^\tau\omega_{\sigma\rho} + \sigma_\nu^\eta\rho_\rho^\tau\omega_{\sigma\mu} + \sigma_\nu^\eta\rho_\sigma^\tau\omega_{\rho\mu})\frac{k^\kappa}{\sqrt{k^2}}.$$

(3) Spin-2 sector

$$(a) P_{11}^{hh}(+2)_{\mu\nu;\rho\sigma} = \rho_{\mu\rho}\rho_{\nu\sigma},$$

$$(b) P_{22}^{hh}(-2)_{\mu\nu;\rho\sigma} = \sigma_{\mu\rho}\sigma_{\nu\sigma}.$$

(4) Identities among the operators

$$(a) P^{hh}(1)_{\mu\nu;\rho\sigma} = \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\sigma}\omega_{\mu\rho}) = P_{11}^{hh}(+1) + P_{22}^{hh}(-1)$$

$$(b) P^{hh}(2)_{\mu\nu;\rho\sigma} = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho} - \theta_{\mu\nu}\theta_{\rho\sigma}) = P_{11}^{hh}(+2) + P_{22}^{hh}(-2)$$

(5) Tensorial identities

$$\begin{aligned} \delta_{\mu\nu,\rho\sigma} &= \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) = P^{hh}(2) + P^{hh}(1) + P_{11}^{hh}(0) + P_{22}^{hh}(0), \\ \eta_{\mu\nu}\eta_{\rho\sigma} &= 2P_{11}^{hh}(0^s) + \sqrt{2}P_{12}^{hh}(0) + \sqrt{2}P_{21}^{hh}(0) + P_{22}^{hh}(0), \\ k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu} &= \sqrt{2}k^2(P_{12}^{hh}(0) + P_{21}^{hh}(0)) + 2k^2 P_{22}^{hh}(0), \\ k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho} &= 2k^2 P^{hh}(1) + 4k^2 P_{22}^{hh}(0), \\ k_\mu k_\nu k_\rho k_\sigma &= k^4 P_{22}^{hh}(0), \\ (\varepsilon_{\kappa\rho\mu}\eta_{\nu\sigma} + \varepsilon_{\kappa\rho\nu}\eta_{\mu\sigma} + \varepsilon_{\kappa\sigma\mu}\eta_{\nu\rho} + \varepsilon_{\kappa\sigma\nu}\eta_{\mu\rho})k^\kappa &= 2\sqrt{k^2}(P_{11}^{hh}(+2) - P_{22}^{hh}(-2) - P_{12}^{hh}(\pm 1) + P_{21}^{hh}(\mp 1)), \\ (\varepsilon_{\kappa\rho\mu}k_\nu k_\sigma + \varepsilon_{\kappa\rho\nu}k_\mu k_\sigma + \varepsilon_{\kappa\sigma\mu}k_\nu k_\rho + \varepsilon_{\kappa\sigma\nu}k_\mu k_\rho)k^\kappa &= 2k^2\sqrt{k^2}(-P_{12}^{hh}(\pm 1) + P_{21}^{hh}(\mp 1)). \end{aligned}$$

Here the $\mu\nu; \rho\sigma$ indices of the operators $P_{ij}^{hh}(J)_{\mu\nu;\rho\sigma}$ were omitted.

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