Integrable vortex-type equations on the two-sphere

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We consider the Yang-Mills instanton equations on the four-dimensional manifold $S^2 \times \Sigma$, where Σ is a compact Riemann surface of genus $g > 1$ or its covering space $H^2 = \mathrm{SU}(1,1)/\mathrm{U}(1)$. Introducing a natural ansatz for the gauge potential, we reduce the instanton equations on $S^2 \times \Sigma$ to vortex-type equations on the sphere S^2. It is shown that when the scalar curvature of the manifold $S^2 \times \Sigma$ vanishes, the vortex-type equations are integrable, i.e., can be obtained as compatibility conditions of two linear equations (Lax pair), which are written down explicitly. Thus, the standard methods of integrable systems can be applied for constructing their solutions. However, even if the scalar curvature of $S^2 \times \Sigma$ does not vanish, the vortex equations are well defined and have solutions for any values of the topological charge N. We show that any solution to the vortex equations on S^2 with a fixed topological charge N corresponds to a Yang-Mills instanton on $S^2 \times \Sigma$ of charge $(g-1)N$.

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I. INTRODUCTION AND SUMMARY

The Abelian Higgs model on $\mathbb{R} \times \mathbb{R}^2$ at critical value of the coupling constant (the Bogomolny regime) admits static vortex solutions on $\mathbb{R}^2$ [1] that describe magnetic flux tubes (vortex strings) penetrating a two-dimensional superconductor. Vortices are important objects in modern field theory [2] since it is believed that (electric) vortex strings play an important role in the confinement of quarks. Their stability is ensured by topology [3]. Many results known for the Abelian Higgs model were generalized to Riemann surfaces, noncommutative spaces, and to the non-Abelian case (see e.g., Refs. [4–13] and references therein).

It was shown recently that the vortex equations on a Riemann surface Σ of genus g have a Lax pair representation if $g > 1$ and do not have it for $g = 0, 1$ [8]. This was done by using the correspondence between vortices on Σ and SU(2)-equivariant[1] instantons on the four-manifold $\Sigma \times S^2$—the invariance conditions reduce the instanton equations on $\Sigma \times S^2$ to vortex equations on Σ. Existence of a Lax pair for the reduced equations on Σ is related with vanishing of scalar curvature of $\Sigma \times S^2$ when this manifold becomes [15] a gravitational instanton. The nonexistence of a Lax pair for vortex equations on S^2, T^2, and $\mathbb{R}^2$ followed from the fact that the scalar curvature of $\Sigma \times S^2$ is nonvanishing for $\Sigma = S^2, T^2$, and $\mathbb{R}^2$.

In this paper, we introduce an ansatz reducing the instanton equations on $M = S^2 \times \Sigma$ to vortex-type equations not on Σ but on S^2, and we show that these equations are the compatibility conditions of two linear equations (Lax pair) if the scalar curvature of M vanishes, similar to the previous $g > 1$ cases [8]. Furthermore, the existence of solutions to the reduced equations on S^2 for any topological charge $N \geq 0$ demands noncompact initial gauge group for Yang-Mills theory on M and compact gauge group of reduced Yang-Mills-Higgs theory on S^2. This is similar to the case of the Hitchin equations on S^2 and T^2 obtainable as reduction of the instanton equations [16]—smooth solutions on S^2 (and T^2) exist only if one chooses noncompact gauge group[2] in four dimensions [19].

The organization of this paper is as follows. In Sec. II, we collect various facts concerning the geometry of the manifold $S^2 \times H^2$, where $H^2 = \mathrm{SU}(1,1)/\mathrm{U}(1)$ is the unit disk in the complex plane $\mathbb{C}$. The explicit form of the metric, Christoffel symbols, etc. are written down. Then, in Sec. III, we introduce an SU(1,1)-equivariant ansatz that reduces the instanton equations on $S^2 \times H^2$ to Abelian vortex-type equations on S^2. Solutions to these equations give solutions of the self-dual Yang-Mills equations on $S^2 \times H^2$ with the noncompact gauge group SU(1,1). Section IV deals with integrability properties of the introduced Abelian vortex equations. Finally, in Secs. V and VI, we generalize results of Secs. II, III, and IV to the case of non-Abelian vortex-type equations on S^2 and instantons on manifolds $S^2 \times \Sigma$ with compact Riemann surfaces Σ. Bogomolny transformations for the Yang-Mills-Higgs action functional is discussed and a relation between the instanton and vortex topological charges is derived.

II. MANIFOLD $S^2 \times H^2$

A. Riemann sphere

Consider the standard two-sphere $S^2 \cong \mathbb{C}P^1 = \mathrm{SU}(2)/\mathrm{U}(1)$ of constant radius R_1. In local coordinates $y = x^1 + \mathrm{i}x^2$, $\bar{y} = x^1 - \mathrm{i}x^2$ on $\mathbb{C}P^1$ the metric and the volume

[1] This means a generalized SU(2) invariance, i.e., invariance under space-time transformations up to gauge transformations [14].

[2] Yang-Mills fields with noncompact gauge groups were considered in many papers (see e.g., Refs. [17,18] and references therein).

form read

$$\mathrm{d}s^2_{S^2} = 2g_{y\bar{y}}\mathrm{d}y\mathrm{d}\bar{y} = \frac{4R_1^4}{(R_1^2 + y\bar{y})^2}\mathrm{d}y\mathrm{d}\bar{y} \tag{2.1}$$

and

$$\omega_{S^2} = \frac{2\mathrm{i}R_1^4}{(R_1^2 + y\bar{y})^2}\mathrm{d}y \wedge \mathrm{d}\bar{y} = \mathrm{i}g_{y\bar{y}}\mathrm{d}y \wedge \mathrm{d}\bar{y}, \tag{2.2}$$

respectively. For the nonvanishing components of the Christoffel symbols and the Ricci tensor we have

$$\Gamma^y_{yy} = 2\partial_y \log\rho_1 \quad \text{and} \quad \Gamma^{\bar{y}}_{\bar{y}\bar{y}} = 2\partial_{\bar{y}} \log\rho_1 \quad \text{with} \quad \rho_1^2 := g_{y\bar{y}}, \tag{2.3}$$

$$R_{y\bar{y}} = -2\partial_y\partial_{\bar{y}} \log\rho_1 = \frac{1}{R_1^2}g_{y\bar{y}} \Rightarrow R_{S^2} = 2g^{y\bar{y}}R_{y\bar{y}} = \frac{2}{R_1^2}, \tag{2.4}$$

where R_{S^2} is the scalar curvature of S^2.

For the components $g_{y\bar{y}}$ and $g^{y\bar{y}} = 1/g_{y\bar{y}}$ we have

$$g_{y\bar{y}} = e^1_y e^{\bar{1}}_{\bar{y}} \quad \text{and} \quad g^{y\bar{y}} = e^y_1 e^{\bar{y}}_{\bar{1}}, \tag{2.5}$$

where e^y_1 and $e^{\bar{y}}_{\bar{1}}$ are unitary (local) frame. We introduce a basis of type (1,0) and (0,1) vector fields

$$e_1 := e^y_1\partial_y \quad \text{and} \quad e_{\bar{1}} := e^{\bar{y}}_{\bar{1}}\partial_{\bar{y}} \tag{2.6}$$

on $S^2 \cong \mathbb{C}P^1$. The dual basis of type (1,0) and (0,1) forms is $e^1_y\mathrm{d}y$ and $e^{\bar{1}}_{\bar{y}}\mathrm{d}\bar{y}$.

B. Coset space H^2

Consider the symmetric space (unit disk)

$$H^2 = \mathrm{SU}(1,1)/\mathrm{U}(1), \tag{2.7}$$

where SU(1,1) is a noncompact real form of the group $\mathrm{SL}(2,\mathbb{C})$ with elements h defined by

$$h^\dagger\eta h = \eta \quad \text{for } \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.8}$$

The metric and the Kähler form in the coordinates $z = x^3 - \mathrm{i}x^4$, $\bar{z} = x^3 + \mathrm{i}x^4$ on H^2 are given by

$$\mathrm{d}s^2_{H^2} = 2g_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z} = \frac{4R_2^4}{(R_2^2 - z\bar{z})^2}\mathrm{d}z\mathrm{d}\bar{z}, \tag{2.9}$$

and

$$\omega_{H^2} = -\frac{2\mathrm{i}R_2^4}{(R_2^2 - z\bar{z})^2}\mathrm{d}z \wedge \mathrm{d}\bar{z} = -\mathrm{i}\beta \wedge \bar{\beta}, \tag{2.10}$$

where

$$\beta := \frac{\sqrt{2}R_2^2\mathrm{d}z}{R_2^2 - z\bar{z}} \quad \text{and} \quad \bar{\beta} := \frac{\sqrt{2}R_2^2\mathrm{d}\bar{z}}{R_2^2 - z\bar{z}} \tag{2.11}$$

are forms on H^2 of type (1,0) and (0,1). These forms satisfy the equations

$$\mathrm{d}\beta = -2a \wedge \beta, \qquad \mathrm{d}\bar{\beta} = 2a \wedge \bar{\beta}, \quad \text{and} \quad \mathrm{d}a = -\frac{1}{2R_2^2}\beta \wedge \bar{\beta} = -\frac{\mathrm{i}}{2R_2^2}\omega_{H^2}. \tag{2.12}$$

The anti-Hermitian connection oneform

$$a = \frac{1}{2(R_2^2 - z\bar{z})}(\bar{z}\mathrm{d}z - z\mathrm{d}\bar{z}) \tag{2.13}$$

with the curvature form $\mathrm{d}a$ given in (2.12) is an H^2-analog of the monopole connection on $\mathbb{C}P^1$. Note that $2a$ is the Levi-Civita connection on the tangent bundle TH^2. The one-form a is a connection on the square root L of the holomorphic bundle $T^{1,0}H^2$.

The Christoffel symbols, Ricci tensor, and scalar curvature for H^2 are

$$\Gamma^z_{zz} = 2\partial_z \log\rho_2 \quad \text{and} \quad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = 2\partial_{\bar{z}} \log\rho_2 \quad \text{with} \quad \rho_2^2 := g_{z\bar{z}}, \tag{2.14}$$

$$R_{z\bar{z}} = -2\partial_z\partial_{\bar{z}} \log\rho_2 = -\frac{1}{R_2^2}g_{z\bar{z}} \Rightarrow R_{H^2} = 2g^{z\bar{z}}R_{z\bar{z}} = -\frac{2}{R_2^2}. \tag{2.15}$$

For (1,0) and (0,1) vector fields on H^2 dual to forms (2.11) we have

$$e_2 := e^z_2\partial_z = \rho_2^{-1}\partial_z \quad \text{and} \quad e_{\bar{2}} := e^{\bar{z}}_{\bar{2}}\partial_{\bar{z}} = \rho_2^{-1}\partial_{\bar{z}} \tag{2.16}$$

with ρ_2 given in (2.14) and (2.9).

We also consider a four-manifold M given by a product of S^2 and H^2 with the product metric

$$\mathrm{d}\, s^2_M = \mathrm{d}s^2_{S^2} + \mathrm{d}s^2_{H^2}. \tag{2.17}$$

For the scalar curvature of $M = S^2 \times H^2$ we have

$$R_M = R_{S^2} + R_{H^2} = 2\Big(\frac{1}{R_1^2} - \frac{1}{R_2^2}\Big). \tag{2.18}$$

III. VORTICES ON S^2 AS YANG-MILLS CONFIGURATIONS ON $S^2 \times H^2$

A. SU(1,1)-equivariant gauge potential

Consider the manifold $M = S^2 \times H^2$. Let $\mathcal{E} \to M$ be an SU(1,1)-equivariant complex vector bundle of rank 2 over M with the group SU(1,1) acting trivially on S^2 and in the standard way by SU(1,1)-isometry on $H^2 = \mathrm{SU}(1,1)/\mathrm{U}(1)$. Let $\mathcal{A}$ be an su(1,1)-valued local form of SU(1,1)-equivariant connection on $\mathcal{E}$ (cf. Refs. [7,8]); it can be chosen in the form

$$\mathcal{A} = \begin{pmatrix} \frac{1}{2}A \otimes 1 + 1 \otimes a & \frac{1}{\sqrt{2}}\phi \otimes \beta \\ \frac{1}{\sqrt{2}}\bar{\phi} \otimes \bar{\beta} & -\frac{1}{2}A \otimes 1 - 1 \otimes a \end{pmatrix} = \Big(\frac{1}{2}A + a\Big)\sigma_3 + \frac{1}{\sqrt{2}}\phi\beta\sigma_+ + \frac{1}{\sqrt{2}}\bar{\phi}\,\bar{\beta}\,\sigma_-, \tag{3.1}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.2}$$

Here, $A = A_y \mathrm{d}y + A_{\bar{y}} \mathrm{d}\bar{y}$ is an Abelian connection on a (Hermitian) complex line bundle E over $\mathbb{C}P^1 \cong S^2$, a is the connection (2.13) on the complex line bundle L over H^2, ϕ is a section of the bundle E, $\bar{\phi}$ is its complex conjugate, and forms β, $\bar{\beta}$ on H^2 are given in (2.11). In local complex coordinates y, $\bar{y}$ on $\mathbb{C}P^1$ we have $A = A(y, \bar{y})$ and $\phi = \phi(y, \bar{y})$.

B. Field strength tensor

In local coordinates on $S^2 \times H^2$ the calculation of the curvature $\mathcal{F}$ for $\mathcal{A}$ of the form (3.1) yields

$$\begin{aligned} \mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} &= \begin{pmatrix} \frac{1}{2} F - \frac{1}{2}\left(\frac{1}{R_2^2} - \phi\bar{\phi}\right)\beta \wedge \bar{\beta} & \frac{1}{\sqrt{2}}(\mathrm{d}\phi + A\phi) \wedge \beta \\ \frac{1}{\sqrt{2}}(\mathrm{d}\bar{\phi} - A\bar{\phi}) \wedge \bar{\beta} & -\frac{1}{2} F + \frac{1}{2}\left(\frac{1}{R_2^2} - \phi\bar{\phi}\right)\beta \wedge \bar{\beta} \end{pmatrix} \\ &= \mathcal{F}_{y\bar{y}} \mathrm{d}y \wedge \mathrm{d}\bar{y} + \mathcal{F}_{yz} \mathrm{d}y \wedge \mathrm{d}z + \mathcal{F}_{y\bar{z}} \mathrm{d}y \wedge \mathrm{d}\bar{z} + \mathcal{F}_{\bar{y}z} \mathrm{d}\bar{y} \wedge \mathrm{d}z + \mathcal{F}_{\bar{y}\bar{z}} \mathrm{d}\bar{y} \wedge \mathrm{d}\bar{z} + \mathcal{F}_{z\bar{z}} \mathrm{d}z \wedge \mathrm{d}\bar{z} \end{aligned} \tag{3.3}$$

with the nonvanishing field strength components

$$\mathcal{F}_{y\bar{y}} = \frac{1}{2} F_{y\bar{y}} \sigma_3, \qquad \mathcal{F}_{z\bar{z}} = -\frac{1}{2} g_{z\bar{z}} \left(\frac{1}{R_2^2} - \phi\bar{\phi}\right)\sigma_3, \tag{3.4}$$

$$\begin{aligned} \mathcal{F}_{\bar{y}z} &= \frac{\rho_2}{\sqrt{2}}(\partial_{\bar{y}}\phi + A_{\bar{y}}\phi)\sigma_+, \\ \mathcal{F}_{yz} &= \frac{\rho_2}{\sqrt{2}}(\partial_y\phi + A_y\phi)\sigma_+, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \mathcal{F}_{y\bar{z}} &= \frac{\rho_2}{\sqrt{2}}(\partial_y\bar{\phi} - A_y\bar{\phi})\sigma_-, \\ \mathcal{F}_{\bar{y}\bar{z}} &= \frac{\rho_2}{\sqrt{2}}(\partial_{\bar{y}}\bar{\phi} - A_{\bar{y}}\bar{\phi})\sigma_-. \end{aligned} \tag{3.6}$$

In (3.4) we have defined $F = \mathrm{d}A = F_{y\bar{y}} \mathrm{d}y \wedge \mathrm{d}\bar{y} = (\partial_y A_{\bar{y}} - \partial_{\bar{y}} A_y) \mathrm{d}y \wedge \mathrm{d}\bar{y}$ for $A = A_y \mathrm{d}y + A_{\bar{y}} \mathrm{d}\bar{y}$.

C. Vortex equations on S^2

Let us consider the self-dual Yang-Mills equations $*\mathcal{F} = \mathcal{F}$ on $S^2 \times H^2$, where $*$ is the Hodge operator. In local coordinates these equations have the form

$$\mathcal{F}_{\bar{y}\bar{z}} = 0 = (\mathcal{F}_{yz})^\dagger \quad \text{and} \quad g^{y\bar{y}} \mathcal{F}_{y\bar{y}} + g^{z\bar{z}} \mathcal{F}_{z\bar{z}} = 0. \tag{3.7}$$

Substitution of (3.4), (3.5), and (3.6) into (3.7) shows that the self-dual Yang-Mills equations (3.7) on $S^2 \times H^2$ are equivalent to the Bogomolny-Prasad-Sommerfield (BPS) vortex-type equations on S^2:

$$F_{y\bar{y}} = g_{y\bar{y}} \left(\frac{1}{R_2^2} - \phi\bar{\phi}\right) \Leftrightarrow \mathrm{i}F = \left(\frac{1}{R_2^2} - \phi\bar{\phi}\right)\omega_{S^2}, \tag{3.8}$$

$$\partial_y\phi + A_y\phi = 0 \quad \Leftrightarrow \quad \partial_A\phi = 0, \tag{3.9}$$

where $\partial_A = \mathrm{d}y(\partial_y + A_y)$. Note that for the standard vortex equations instead of Eqs. (3.9) one has $\partial_{\bar{y}}\phi + A_{\bar{y}}\phi = 0$. This equation can be obtained if in (3.1) one chooses $\bar{\beta}$ in the upper right corner and $-\beta$ in the lower left corner [compact gauge group SU(2)] but then in (3.8) one will have $-1/R_2^2$ and such vortex-type equations will not have solutions due to the Kazdan-Warner theorem [20].

Vortex number N is defined as the first Chern number $c_1(E)$ of the bundle $E \to \mathbb{C}P^1$,

$$N = c_1(E) = \frac{\mathrm{i}}{2\pi} \int_{S^2} F. \tag{3.10}$$

From (3.8) it follows that

$$\frac{\mathrm{i}}{2\pi} \int_{S^2} F + \frac{1}{2\pi} \int_{S^2} \phi\bar{\phi}\,\omega_{S^2} = \frac{1}{2\pi R_2^2} \int_{S^2} \omega_{S^2} = 2\left(\frac{R_1}{R_2}\right)^2, \tag{3.11}$$

and we obtain (cf. Ref. [4]) the inequality

$$N \le 2\left(\frac{R_1}{R_2}\right)^2. \tag{3.12}$$

For any $N \ge 0$ the condition (3.12) can be satisfied for a sufficiently large ratio R_1/R_2 and then the moduli space of vortices on S^2 will be nonempty.

D. Liouville-type equations on S^2

Consider the N-vortex solution $\phi = \exp(\frac{1}{2}(u + \mathrm{i}\theta))$, where u and θ are real-valued functions. Since ϕ can have zeros at $y_i \in \mathbb{C}P^1$, then $u(y) \to -\infty$ as $y \to y_i$ and $\theta(y)$ is a multivalued function with ramification points at y_i. Equation (3.9) implies that

$$\begin{aligned} A_y &= -\partial_y \log\phi = -\frac{1}{2}\partial_y(u + \mathrm{i}\theta) \quad \text{and} \\ A_{\bar{y}} &= \partial_{\bar{y}} \log\bar{\phi} = \frac{1}{2}\partial_{\bar{y}}(u - \mathrm{i}\theta). \end{aligned} \tag{3.13}$$

Plugging (3.13) into (3.8), we obtain the Liouville-type equations on S^2,

$$\partial_y \partial_{\bar{y}} u = g_{y\bar{y}}\left(\frac{1}{R_2^2} - e^u\right), \tag{3.14}$$

away from the singularities of u.

Note that the sign on the right-hand side of Eq. (3.14) with $g_{y\bar{y}}$ given in (2.1) is opposite to the sign in the standard vortex equations on S^2. However, equations of type (3.14) on compact Riemann surfaces (including S^2) were considered by Kazdan and Warner [20]. They have shown, in particular, that equations

$$\partial_y \partial_{\bar{y}} u = \pm g_{y\bar{y}}\left(\frac{1}{R_2^2} - e^u\right) \tag{3.15}$$

have solutions for *both* signs in (3.15) and equations

$$\partial_y \partial_{\bar{y}} u = \mp g_{y\bar{y}}\left(\frac{1}{R_2^2} + e^u\right) \tag{3.16}$$

have no solutions. These four cases exhaust possible Liouville-type equations on S^2 with $R_2^2 \neq \infty$.

Recall that Eq. (3.15) can be obtained by the reduction of the self-dual Yang-Mills (SDYM) equations from $S^2 \times S^2$ to S^2 with gauge group SU(2) (lower sign) and from $S^2 \times H^2$ to S^2 with gauge group SU(1,1) (upper sign). Similarly, Eqs. (3.16) correspond to the reduction of the SDYM equations from $S^2 \times H^2$ to S^2 with gauge group SU(2) (lower sign) and from $S^2 \times S^2$ to S^2 with gauge group SU(1,1) (upper sign). Thus, only the gauge group SU (1,1) is allowed for the considered case of the reduction $S^2 \times H^2 \to S^2$, and solutions of (3.14) exist for any $N \geq 0$.

If one considers the reduction of the SDYM equations from $S^2 \times H^2$ to H^2, the allowed gauge group is SU(2) [8]. In other words, depending on a symmetry [SU(2) or SU (1,1) equivariance] imposed on gauge fields, on $S^2 \times H^2$ there exist solutions of the SDYM equations with gauge groups as SU(2) and SU(1,1).

IV. INTEGRABILITY OF VORTEX EQUATIONS ON S^2

A. Integrable case

We considered the BPS vortex-type equations (3.8) and (3.9) and showed their equivalence to the self-dual Yang-Mills equations (3.7) on the manifold $M = S^2 \times H^2$. Note that for equal radii $R_1 = R_2$ of S^2 and H^2 the scalar curvature (2.18) of M vanishes. In this case the Weyl tensor for the manifold M is self-dual [15].

An important feature of Kähler manifolds M with scalar curvature R_M is that the so-called twistor space $\mathcal{Z}$ of M becomes a complex manifold if $R_M = 0$. Let us consider an open subset $\mathcal{U}$ of $M = S^2 \times H^2$ with complex coordinates y, z. Then the twistor space of $\mathcal{U}$ (i.e., the restriction of $\mathcal{Z}$ to $\mathcal{U}$) is diffeomorphic to $\mathcal{U} \times \mathbb{C}P^1$, $\mathcal{Z}|_{\mathcal{U}} \simeq \mathcal{U} \times \mathbb{C}P^1$, with a local complex coordinate $\lambda \in \mathbb{C}P^1 \backslash \{\infty\}$ on the last factor. On $\mathcal{Z}$ there is a distribution generated by three vector fields of type (0,1) closed under the Lie bracket. They hazve the form (cf. Ref. [8])

$$V_{\bar{1}} := \tilde{e}_{\bar{1}} - \lambda \tilde{e}_2, \qquad V_{\bar{2}} := \tilde{e}_{\bar{2}} + \lambda \tilde{e}_1, \quad \text{and} \quad V_{\bar{3}} = \partial_{\bar{\lambda}}, \tag{4.1}$$

where

$$\begin{aligned} \tilde{e}_1 &= \rho_1^{-1}(\partial_y - (\partial_y \log\rho_1)\lambda\partial_\lambda), \\ \tilde{e}_{\bar{1}} &= \rho_1^{-1}(\partial_{\bar{y}} + (\partial_{\bar{y}} \log\rho_1)\lambda\partial_\lambda), \end{aligned} \tag{4.2}$$

$$\begin{aligned} \tilde{e}_2 &= \rho_2^{-1}(\partial_z - (\partial_z \log\rho_2)\lambda\partial_\lambda), \\ \tilde{e}_{\bar{2}} &= \rho_2^{-1}(\partial_{\bar{z}} + (\partial_{\bar{z}} \log\rho_2)\lambda\partial_\lambda). \end{aligned} \tag{4.3}$$

Recall that $\rho_1^2 = g_{y\bar{y}}$ and $\rho_2^2 = g_{z\bar{z}}$ are components of metrics on S^2 and H^2; their explicit forms are given in Sec. II.

The vector fields (4.1) define an almost complex structure $\mathcal{J}$ on $\mathcal{Z}$ such that

$$\mathcal{J}(V_{\bar{k}}) = -\mathrm{i}V_{\bar{k}} \tag{4.4}$$

for $k = 1, 2, 3$. For commutators of type (0,1) vector fields (4.1) we have

$$\begin{aligned} [V_{\bar{1}}, V_{\bar{2}}] &= \lambda\rho_1^{-2}(\partial_y\rho_1)V_{\bar{1}} + \lambda\rho_2^{-2}(\partial_z\rho_2)V_{\bar{2}} \\ &\quad + 2\lambda^2\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right)V_3, \\ [V_{\bar{1}}, V_{\bar{3}}] &= 0 = [V_{\bar{2}}, V_{\bar{3}}], \end{aligned} \tag{4.5}$$

where $V_3 = \partial_\lambda$ is the (1,0) vector field on $\mathcal{Z}$. Recall that for integrability of an almost complex structure $\mathcal{J}$ on $\mathcal{Z}$ it is necessary and sufficient that the commutator of any two vector fields of type (0,1) with respect to $\mathcal{J}$ is of type (0,1). For our case we see from (4.5) that $\mathcal{J}$ is integrable—and $\mathcal{Z}$ is a complex manifold—if and only if

$$R_1 = R_2, \tag{4.6}$$

i.e., when the scalar curvature R_M of the manifold $M = S^2 \times H^2$ vanishes. In this case the bundle $\mathcal{E} \to M$ pulled back to the bundle $\hat{\mathcal{E}}$ over the twistor space $\mathcal{Z}$ allows an integrable holomorphic structure defined by a (0,1)-type connection along the vector fields (4.1). The integrability of this structure, $\mathcal{F}^{0,2} = 0$, is equivalent [21] to the self-duality equations on M.

B. Lax pair

For the case (4.2) from (3.12) we obtain the inequality

$$N \leq 2, \tag{4.7}$$

i.e., bundles $\hat{\mathcal{E}}$ over $\mathcal{Z}$ with integrable holomorphic structures describe configurations of $N = 1$ and $N = 2$ vortices on S^2. We emphasize that vortices exist for any $N > 0$ but only for $N \leq 2$ the vortex equations (3.8) and (3.9) appear from a Lax pair.

For presenting vortex equations on S^2 as an integrable system (for $R_1 = R_2$) one should introduce two linear equations (Lax pair) whose compatibility conditions will produce the vortex equations. For that we introduce a (0,1) part $\hat{\nabla}^{0,1}$ of the covariant derivative $\hat{\nabla}$ on $\hat{\mathcal{E}}$ by the formulas

$$\hat{\nabla}_{V_{\bar 1}} \equiv V_{\bar 1} + \hat{\mathcal{A}}_{V_{\bar 1}} := \tilde{e}_{\bar 1} - \lambda \tilde{e}_2 + \mathcal{A}_{\bar 1} - \lambda \mathcal{A}_2, \tag{4.8a}$$

$$\hat{\nabla}_{V_{\bar 2}} \equiv V_{\bar 2} + \hat{\mathcal{A}}_{V_{\bar 2}} := \tilde{e}_{\bar 2} + \lambda \tilde{e}_1 + \mathcal{A}_{\bar 2} + \lambda \mathcal{A}_1, \tag{4.8b}$$

$$\hat{\nabla}_{V_{\bar 3}} \equiv V_{\bar 3} + \hat{\mathcal{A}}_{V_{\bar 3}} := \partial_{\bar\lambda}, \tag{4.8c}$$

where $\hat{\nabla}_X$ denotes the covariant derivative along the vector field X. The components

$$\mathcal{A}_1 = e_1^y \mathcal{A}_y, \qquad \mathcal{A}_{\bar 1} = e_{\bar 1}^{\bar y} \mathcal{A}_{\bar y}, \qquad \mathcal{A}_2 = e_2^z \mathcal{A}_z, \qquad \mathcal{A}_{\bar 2} = e_{\bar 2}^{\bar z} \mathcal{A}_{\bar z} \tag{4.9}$$

are easily extracted from (3.1).

Let us now introduce a 2×2 matrix $\psi = \psi(y, \bar y, z, \bar z, \lambda)$ that does not depend on $\bar\lambda$ and consider two linear equations

$$\hat{\nabla}_{V_{\bar 1}} \psi := [\tilde{e}_{\bar 1} + \mathcal{A}_{\bar 1} - \lambda(\tilde{e}_2 + \mathcal{A}_2)]\psi = 0, \tag{4.10a}$$

$$\hat{\nabla}_{V_{\bar 2}} \psi := [\lambda(\tilde{e}_1 + \mathcal{A}_1) + \tilde{e}_{\bar 2} + \mathcal{A}_{\bar 2})]\psi = 0. \tag{4.10b}$$

It is not difficult to check that the compatibility conditions of the linear equations (4.10),

$$([\hat{\nabla}_{V_{\bar 1}}, \hat{\nabla}_{V_{\bar 2}}] - \hat{\nabla}_{[V_{\bar 1}, V_{\bar 2}]})\psi = 0, \tag{4.11}$$

are equivalent to the vortex equations (3.8) and (3.9) for $\mathcal{A}$ given in (3.1).

Note that equations

$$\hat{\mathcal{F}}^{0,2} = 0 \Leftrightarrow \hat{\mathcal{F}}(V_{\bar i}, V_{\bar j}) = [\hat{\nabla}_{V_{\bar i}}, \hat{\nabla}_{V_{\bar j}}] - \hat{\nabla}_{[V_{\bar i}, V_{\bar j}]} = 0 \tag{4.12}$$

for $\hat{\nabla}_{V_{\bar j}}$ given in the first two formulas from (4.8) can be imposed even if an almost complex structure $\mathcal{J}$ on $\mathcal{Z}$ is not integrable, that is, the case when $R_1 \neq R_2$. Then Eqs. (4.12) define a pseudoholomorphic structure [22] on the bundle $\hat{\mathcal{E}} \to \mathcal{Z}$. These equations are again equivalent to the self-duality equations on $S^2 \times H^2$ since

$$\hat{\mathcal{F}}(V_{\bar 1}, V_{\bar 2}) = \mathcal{F}_{\bar 1 \bar 2} - \lambda(\mathcal{F}_{1\bar 1} + \mathcal{F}_{2\bar 2}) + \lambda^2 \mathcal{F}_{12},$$
$$\hat{\mathcal{F}}(V_{\bar 1}, V_{\bar 3}) = 0 = \hat{\mathcal{F}}(V_{\bar 2}, V_{\bar 3}), \tag{4.13}$$

where

$$\mathcal{F}_{\bar 1 \bar 2} = e_{\bar 1}\mathcal{A}_{\bar 2} - e_{\bar 2}\mathcal{A}_{\bar 1} + [\mathcal{A}_{\bar 1}, \mathcal{A}_{\bar 2}] = e_{\bar 1}^{\bar y} e_{\bar 2}^{\bar z} \mathcal{F}_{\bar y \bar z}, \tag{4.14a}$$

$$\mathcal{F}_{12} = e_1\mathcal{A}_2 - e_2\mathcal{A}_1 + [\mathcal{A}_1, \mathcal{A}_2] = e_1^y e_2^z \mathcal{F}_{yz}, \tag{4.14b}$$

$$\mathcal{F}_{1\bar 1} = e_1\mathcal{A}_{\bar 1} - e_{\bar 1}\mathcal{A}_1 + [\mathcal{A}_1, \mathcal{A}_{\bar 1}] - \rho_1^{-1}(e_{\bar 1}\rho_1)\mathcal{A}_1 + \rho_1^{-1}(e_1\rho_1)\mathcal{A}_{\bar 1} = g^{y\bar y}\mathcal{F}_{y\bar y}, \tag{4.14c}$$

$$\mathcal{F}_{2\bar 2} = e_2\mathcal{A}_{\bar 2} - e_{\bar 2}\mathcal{A}_2 + [\mathcal{A}_2, \mathcal{A}_{\bar 2}] - \rho_2^{-1}(e_{\bar 2}\rho_2)\mathcal{A}_2 + \rho_2^{-1}(e_2\rho_2)\mathcal{A}_{\bar 2} = g^{z\bar z}\mathcal{F}_{z\bar z}. \tag{4.14d}$$

After substituting SU(1,1)-equivariant gauge potential (3.1), Eqs. (4.13) reduce to the vortex equations on S^2 having solutions with $N > 2$. So, for $N > 2$ vortex equations on S^2 do not appear as a compatibility condition of a Lax pair but are derivable nevertheless from the self-dual Yang-Mills equations similarly to vortex equations on Riemann surfaces with genus $g > 1$, where vortex equations were integrable only for $N \leq 2(g - 1)$ [8].

V. QUIVER VORTEX EQUATIONS

Here and in Sec. VI, we generalize Eqs. (3.8) and (3.9) to the non-Abelian case and describe relations between vortices on S^2 and instantons on the manifold $S^2 \times \Sigma$, where Σ is a compact Riemann surface of genus $g > 1$.

A. Equivariant vector bundle

Consider the manifold $M = \mathbb{C}P^1 \times H^2$, where $\mathbb{C}P^1 \cong S^2$ is the Riemann sphere and H^2 is the unit disk described in Sec. II. Let $\mathcal{E} \to M$ be an SU(1,1)-equivariant rank-k complex vector bundle, with the group SU(1,1) acting trivially on $\mathbb{C}P^1$ and by isometry on $H^2 = \mathrm{SU}(1,1)/\mathrm{U}(1)$. Let $\mathcal{A}$ be a connection on $\mathcal{E}$. Imposing the condition of SU(1,1) equivariance means that we should look for representations of the group SU(1,1) on $\mathbb{C}^k$. Notice that for each positive integer m, the module

$$\mathbb{C}^k = \bigoplus_{i=0}^{m} \mathbb{C}^{k_i} \quad \text{with} \quad \sum_{i=0}^{m} k_i = k \tag{5.1}$$

gives such a representation if $\mathbb{C}^{m+1}$ is an irreducible representation of SU(1,1). Let

$$E = \bigoplus_{i=0}^{m} E_i \to \mathbb{C}P^1 \tag{5.2}$$

be a rank-k $\mathbb{Z}_{m+1}$-graded complex vector bundle over $\mathbb{C}P^1$ and A^i's are connection forms on the bundles $E_i \to \mathbb{C}P^1$. Then

$$\mathcal{E} = \bigoplus_{i=0}^{m} \mathcal{E}_i \quad \text{with} \quad \mathcal{E}_i = E_i \otimes L^{m-2i}, \tag{5.3}$$

where $L^{m-2i} = (L)^{\otimes(m-2i)}$ and the bundle $L \to H^2$ with a connection a given in (2.13) was introduced in Sec. II.

B. Symmetric gauge potential and field strength tensor

Similar to the compact SU(2) case [7], the SU(1,1)-equivariant gauge potential $\mathcal{A}$ with values in End $(\mathbb{C}^k)$ decomposes into connections $A^i \in u(k_i)$ on the complex rank-k_i vector bundles $E_i \to \mathbb{C}P^1$ with $i = 0, 1, \ldots, m$ and a multiplet of scalar fields ϕ_{i+1} on $\mathbb{C}P^1$ with $i = 0, 1, \ldots, m - 1$ transforming in the bifundamental representation $\mathbb{C}^{k_i} \otimes (\mathbb{C}^\vee)^{k_i+1}$ of the group $\mathrm{U}(k_i) \times \mathrm{U}(k_{i+1})$, i.e., $\phi \epsilon Hom(E_i, E_{i+1})$. Collecting these Higgs fields into the upper triangular $k \times k$ complex matrix

$$\phi_{(m)} := \begin{pmatrix} 0 & \phi_1 & 0 & \ldots & 0 \\ 0 & 0 & \phi_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \phi_m \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \tag{5.4}$$

we get

$$\mathcal{A} = A^{(m)} \otimes 1 + \Upsilon_{(m)} \otimes a + \frac{1}{\sqrt{2}} \phi_{(m)} \otimes \beta + \frac{1}{\sqrt{2}} \phi^{\dagger}_{(m)} \otimes \bar{\beta}, \tag{5.5}$$

where

$$A^{(m)} := \sum_{i=0}^{m} A^i \otimes \Pi_i, \qquad \Upsilon_{(m)} := \sum_{i=0}^{m} (m-2i) 1_{k_i} \otimes \Pi_i, \tag{5.6}$$

and $\Pi_i \colon E \to E_i$ are the canonical orthogonal projectors of rank 1, $\Pi_i \Pi_j = \delta_{ij} \Pi_j$, which may be represented by diagonal $(m+1) \times (m+1)$ matrices $\Pi_i = (\delta_{ji}\delta_{li})_{j,l=0,1,\ldots,m}$ of unit trace. Here β and $\bar{\beta}$ are forms on H^2 of type (1,0) and (0,1) defined in Sec. II.

The calculation of the curvature $\mathcal{F}$ for $\mathcal{A}$ of the form (5.5) yields

$$\begin{aligned} \mathcal{F} &= \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \\ &= F^{(m)} \otimes 1 - \frac{1}{2}\Big(\frac{1}{R_2^2} \Upsilon_{(m)} - [\phi_{(m)}, \phi^{\dagger}_{(m)}]\Big) \beta \wedge \bar{\beta} \\ &\quad + \frac{1}{\sqrt{2}} (\mathrm{d}\phi_{(m)} + [A^{(m)}, \phi_{(m)}]) \wedge \beta + \frac{1}{\sqrt{2}} (\mathrm{d}\phi^{\dagger}_{(m)} \\ &\quad + [A^{(m)}, \phi^{\dagger}_{(m)}]) \wedge \bar{\beta}, \end{aligned} \tag{5.7}$$

where $F^{(m)} = \mathrm{d}A^{(m)} + [A^{(m)}, A^{(m)}]$. The derivation of (5.7) uses formulas (2.12).

C. Non-Abelian vortex equations on $\mathbb{C}P^1$

Let us consider the self-dual Yang-Mills equations $*\mathcal{F} = \mathcal{F}$ on M. In local coordinates on M these equations have the form (3.7). Substitution of (5.7) into the instanton equations on $M = S^2 \times H^2$ reduce them to non-Abelian quiver vortex equations (cf. Refs. [7,8])

$$\mathrm{i}\, F^{(m)} = \frac{1}{2}\Big(\frac{1}{R^2} \Upsilon_{(m)} - [\phi_{(m)}, \phi^{\dagger}_{(m)}]\Big) \omega_{S^2}, \tag{5.8}$$

$$\partial \phi_{(m)} + [A^{(m)}_{(1,0)}, \phi_{(m)}] = 0, \tag{5.9}$$

where $\partial = \mathrm{d}y \partial_y$ and ω_{S^2} is given in (2.2). In terms of (A^i, ϕ_i) these equations have the form

$$2\mathrm{i} F^i = \Big(\frac{m-2\mathrm{i}}{R^2} 1_{k_i} + \phi_i^{\dagger}\phi_i - \phi_{i+1}\phi_{i+1}^{\dagger}\Big) \omega_{S^2}, \tag{5.10}$$

$$\partial \phi_{i+1} + A^i_{(1,0)} \phi_{i+1} - \phi_{i+1} A^{i+1}_{(1,0)} = 0, \tag{5.11}$$

where $A^i_{(1,0)} = A^i_y \mathrm{d}y$, $i = 0, \ldots, m$ and $\phi_0 := 0 =: \phi_{m+1}$. Finally, in local complex coordinates on $\mathbb{C}P^1$ we get

$$2 g^{y\bar{y}} F^i_{y\bar{y}} = \frac{m-2\mathrm{i}}{R^2} 1_{k_i} + \phi_i^{\dagger}\phi_i - \phi_{i+1}\phi_{i+1}^{\dagger}, \tag{5.12}$$

$$\partial_y \phi_{i+1} + A^i_y \phi_{i+1} - \phi_{i+1} A^{i+1}_y = 0. \tag{5.13}$$

VI. INSTANTONS WITH NONCOMPACT GAUGE GROUPS

A. Riemann surfaces

Recall that any simply connected Riemann surface Σ of genus $g > 1$ is conformally equivalent to the unit disk $H^2 = \mathrm{SU}(1,1)/\mathrm{U}(1)$. In other words, H^2 is a universal cover of $\Sigma = H^2/\Gamma$, where Γ is a Fuchsian group.[3] Here, we will show that the ansatz (5.5) can also be used on the manifold $M = S^2 \times \Sigma$, where Σ is a compact Riemann surface of genus $g > 1$.

The metric and the volume form on Σ in local (conformal) coordinates z, $\bar{z}$ are given by

$$\mathrm{d}s^2_{\Sigma} = 2 g_{z\bar{z}} \mathrm{d}z \mathrm{d}\bar{z} \quad \text{and} \quad \omega_{\Sigma} = \mathrm{i} g_{z\bar{z}} \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{6.1}$$

Furthermore, for the nonvanishing components of the Christoffel symbols and the Ricci tensor we have

$$\Gamma^z_{zz} = 2\partial_z \log\rho \quad \text{and} \quad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = 2\partial_{\bar{z}} \log\rho \quad \text{with} \quad \rho^2 := g_{z\bar{z}}, \tag{6.2}$$

$$R_{z\bar{z}} = -2\partial_z\partial_{\bar{z}} \log\rho = \varkappa g_{z\bar{z}} \Rightarrow R_{\Sigma} = 2 g^{z\bar{z}} R_{z\bar{z}} = 2\varkappa, \tag{6.3}$$

where R_{Σ} is the constant scalar curvature of Σ. The area of the Riemann surface with genus $g \neq 1$ is

$$\mathrm{Vol}(\Sigma) = \int_{\Sigma} \omega_{\Sigma} = \frac{4\pi}{\varkappa}(1-g). \tag{6.4}$$

Introducing forms β and $\bar{\beta}$ of type (1,0) and (0,1) on Σ,

$$\beta := \rho \mathrm{d}z \quad \text{and} \quad \bar{\beta} := \rho \mathrm{d}\bar{z} \Rightarrow \mathrm{d}s^2_{\Sigma} = 2\beta\bar{\beta}, \tag{6.5}$$

we obtain that

$$\mathrm{d}\beta = -2a \wedge \beta, \qquad \mathrm{d}\bar{\beta} = 2a \wedge \bar{\beta}, \qquad \mathrm{d}a = \frac{1}{2}\varkappa \beta \wedge \bar{\beta}, \tag{6.6}$$

where

$$2a = (\partial_z \log\rho)\mathrm{d}z - (\partial_{\bar{z}} \log\rho)\mathrm{d}\bar{z} \tag{6.7}$$

is the Levi-Civita $u(1)$ connection on the tangent bundle $T\Sigma$ of Σ. Denoting the holomorphic part $T^{1,0}\Sigma$ of $T\Sigma \otimes \mathbb{C}$ by L^2, we obtain the complex line bundle $L \to \Sigma$ with the connection a. Finally, after choosing

$$\varkappa = -\frac{1}{R^2}, \tag{6.8}$$

we see that a, β, and $\bar{\beta}$ in (6.6) satisfy the same equations as forms in (2.12) and therefore the ansatz (5.5) on the manifold $\mathbb{C}P^1 \times \Sigma$ yields to the curvature (5.7) and to the quiver vortex equations (5.8), (5.9), (5.10), (5.11), (5.12), and (5.13). That is why, in what follows, we will consider our gauge theory on the compact spaces $M = \mathbb{C}P^1 \times \Sigma$.

[3]It is a discrete subgroup of the group $SU(1,1) \cong SL(2,\mathbb{R})$.

B. Reduction of the Yang-Mills functional

The dimensional reduction of the Yang-Mills equations from $\mathbb{C}P^1 \times \Sigma$ to $\mathbb{C}P^1$ can also be seen at the level of the Yang-Mills Lagrangian. For simplicity, we consider the case $m = 1$ for which the instanton equations on $\mathbb{C}P^1 \times \Sigma$ are equivalent to Eqs. (3.8) and (3.9) with $A = 2A^0$, $\phi = \phi_1$, and $R = R_2$. Substituting (5.5), (5.6), and (5.7) with $m = 1$ into the standard Yang-Mills functional and performing the integral over Σ, we arrive at the action

$$S = -\frac{1}{8\pi^2}\int_{S^2\times\Sigma} \mathrm{tr}(\mathcal{F}\wedge *\mathcal{F}) = -\frac{1}{16\pi^2}\int_{S^2\times\Sigma} \mathrm{d}^4x\sqrt{\det(g_{\rho\sigma})}\,\mathrm{tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})$$
$$= (g-1)\frac{R^2}{4\pi}\int_{S^2}\omega_{S^2}\Big\{(g^{y\bar y})^2(F_{y\bar y})^2 - 2g^{y\bar y}(D_y\phi\overline{D_y\phi} + D_{\bar y}\phi\overline{D_{\bar y}\phi}) + \Big(\frac{1}{R^2} - \phi\bar\phi\Big)^2\Big\}$$
$$= (g-1)\frac{R^2}{4\pi}\int_{S^2}\mathrm{i}\mathrm{d}y\wedge\mathrm{d}\bar y\Big\{g^{y\bar y}\Big(F_{y\bar y} + g_{y\bar y}\Big(\phi\bar\phi - \frac{1}{R^2}\Big)\Big)^2 - 4D_y\phi\overline{D_y\phi}\Big\} + (g-1)\frac{\mathrm{i}}{2\pi}\int_{S^2}F, \tag{6.9}$$

where $\mu, \nu, \ldots = 1, \ldots, 4$, $D_y = \partial_y + A_y$, and $D_{\bar y} = \partial_{\bar y} + A_{\bar y}$. On solutions (A, ϕ) of vortex equations (3.8) and (3.9) this action coincides with $(g-1)N$, where

$$N = \frac{\mathrm{i}}{2\pi}\int_{S^2} F = c_1(E) \tag{6.10}$$

is the vortex number.

C. Topological charges

For self-dual gauge fields we have

$$N_{\text{inst}} = -c_2(\mathcal{E}) = -\frac{1}{8\pi^2}\int_{S^2\times\Sigma}\mathrm{tr}(\mathcal{F}\wedge\mathcal{F})$$
$$= (g-1)\frac{\mathrm{i}}{2\pi}\int_{S^2}F = (g-1)N, \tag{6.11}$$

i.e., the instanton number N_{inst} is proportional to the vortex number N. In the derivation of (6.9), (6.10), and (6.11) it is assumed that $N \geq 0$.[4] From (6.9) we see that due to noncompactness of the gauge group SU(1,1) the energy density for vortices is not positive definite but for (A, ϕ) satisfying the BPS vortex equations (3.8) and (3.9) the action S coincides with the topological invariant $(g-1)c_1(E) = -c_2(\mathcal{E})$. Thus, by solving Eqs. (5.8) and (5.9) on $\mathbb{C}P^1$ one can obtain instantons on $\mathbb{C}P^1 \times \Sigma$ with noncompact gauge group and the topological charge $N_{\text{inst}} = (g-1)N$.

[4]For $N \leq 0$ one should consider the anti-self-dual Yang-Mills equations $*\mathcal{F} = -\mathcal{F}$ that reduce to antivortex equations.

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