QED with an external field: Hamiltonian treatment of Lorentz-non-invariant background as an anisotropic medium

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Nonlinear electrodynamics, QED included, is considered against the Lorentz-noninvariant external field background, treated as an anisotropic medium. Hamiltonian formalism is applied to electromagnetic excitations over the background, and entities of electrodynamics of media, such as field inductions and intensities, are made sense of in terms of canonical variables. Both conserved and nonconserved generators of space-time translations and rotations are defined on the phase space, and their Hamiltonian equations of motion and Dirac bracket relations, different from the Poincaré algebra, are established. Nonsymmetric, but—in return—gauge-invariant, energy-momentum tensor suggests a canonical momentum density other than the Poynting vector. A photon magnetic moment is found to govern the evolution of the photon angular momentum. It is determined by the antisymmetric part of the energy-momentum tensor.

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I. INTRODUCTION

While relativistic invariance as a symmetry under the Lorentz group $[SO(3,1)]$ is usually an obligatory requirement imposed on a theory, some classes of theories in which it is, at least, weakly broken are also of interest, especially when looking beyond the Standard Model. Quantum electrodynamics (QED) in an external classical electromagnetic field ($\mathcal{F}_{\mu\nu}$) is a clear example of a Lorentz-violating theory that may share principal features with other theories of that series. This is a motivation for studying it on a very general basis.

The self-coupling of electromagnetic fields by means of the creation and annihilation of virtual charged fermions makes QED a nonlinear electrodynamics. Like any other nonlinear electrodynamics—for instance, Born-Infeld electrodynamics—QED proposes an interaction between a strong classical external field and electromagnetic fields that live against its background even when these are small. The linearized approximation based on the smallness of these perturbations ("photons") will be dealt with in this paper. The most important object responsible for the interaction of photons with the background in the linearized theory is the vacuum polarization tensor, $\Pi_{\mu\nu}(x,x')$, calculated in the external field. Through this object, the gauge sector of QED is, in the first instance, provided with a dependence on the $\mathcal{F}_{\mu\nu}$ structure, and therefore on the reference frame. Consequently, the photon vacuum seems to behave like an (in general, moving) anisotropic material, in which light propagation is strongly modified. Perhaps the most remarkable property associated with this issue concerns the existence of photon degrees of freedom which are not in correspondence to the standard observable—helicity values—of the respective irreducible $SO(3,1)$ representations. Instead, the photon propagation modes turn out to be closely associated with birefringent states [1], and their speeds of propagation differ from the speed of light in an empty space-time. Some interesting features that occur in the linearized approximation, when the background is a constant and homogeneous magnetic field, have been predicted. These are cyclotron resonance in the vacuum [2] leading to photon capture [3], anisotropization, the short-ranging and the dimensional reduction of the potential [4] produced by a pointlike static charge in a supercritical magnetic field $|\boldsymbol{B}| \gg B_c$, $B_c = \mathrm{m}^2c^3/e\hbar = 4.42\times10^{13}$ G, where m and e are the electron mass and charge, respectively,[1] and also the production of a magnetic field by a static charge (the magnetoelectric effect) [5] that takes place in the external field, where electric and magnetic fields coexist in parallel. Beyond the linearized approximation, the important effect of photon splitting and merging in a magnetic field [6] has attracted much attention. Other consequences of the self-interaction of small electromagnetic fields are the magnetoelectric effect in QED with an external magnetic field [7] and in the nonlinear electrodynamics generated by the $U_*(1)$ noncommutative theory [8].

Despite the achievements reached in this area, a formal treatment of Lorentz Symmetry Breaking (LSB) in nonlinear electrodynamics has not yet been fully developed. A clear understanding of this theme has paramount

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[1]From now on, rationalized Heaviside-Lorentz units, $\epsilon_0 = \hbar = c = 1$, are used.

importance in theoretical physics, since the QED vacuum in an external field constitutes an ideal laboratory for studying the unconventional properties of other Lorentz-violating theories encompassed as possible extensions of the minimal Standard Model of the fundamental interactions. Among the candidates appear the Lorentz-violating electrodynamics [9–11] and the noncommutative field theories [12] with their phenomenologies closely related to those described in the preceding paragraph. Moreover, since the classification of particles is intimately related with the realization of the space-time symmetry, the employing of this analysis could lead to new insights on plausible phenomena in which other spin representations like axions (spin 0) [13] and gravitons (spin 2) [14] are coupled to $\mathcal{F}$.

Of course, an experimental confirmation of all these processes strongly depends on the external field strengths, whose current laboratory values are much lower than the critical field. As a consequence, all predicted phenomena remain elusive and far from being detectable. Even so, their studies still find a high motivation in light of upcoming laser facilities [15,16] that will achieve the unprecedented level of $|\boldsymbol{E}| \sim 0.01$–$0.1E_c$ with $E_c = \mathrm{m}^2/\mathrm{e} = 1.3 \times 10^{16}$ V/cm In addition, some evidence points to the possible existence of ultrahigh magnetic $|\boldsymbol{B}| \gg B_c$ and electric $|\boldsymbol{E}| \gg E_c$ fields in the surfaces of stellar objects identified as neutron stars [17] and strange stars [18], respectively. In such scenarios a pronounced LSB is expected, and most of the quantum processes described above could have significant astrophysical and cosmological interest.

Inspired by the importance associated with LSB, we make an attempt to fill some gaps in this topic. Our main purpose is to analyze how the vacuum polarization effects modify the Lorentz and Poincaré generators. The results presented in this work are based on the Poincaré invariance of the photon effective action as a functional of the background field. This implies that those Lorentz transformations which leave the external field invariant, together with the space-time translations that do not affect the external field either, as long as it is time and space independent, provide the residual symmetry subgroup of the anisotropic vacuum, while the full Poincaré group remains the group of broken symmetry. To define the space-time translation and rotation generators, although only a part of them is conserved, we appeal to the context of the Noether theorem and then introduce them into the framework of the Hamiltonian formalism, which requires us to impose constraints associated with the gauge invariance of the theory, and thus the Dirac brackets. The general aspects related to the constrained Hamiltonian dynamics were developed by Dirac [19] and have been applied to several problems in quantum field theory [20], including the analysis of Poincaré invariance in Yang-Mills theories quantized in noncovariant gauges [21] (for the Coulomb gauge, see Ref. [22]). We do not know whether the Hamiltonian formalism was ever applied to electrodynamics of an anisotropic medium or whether the characteristic entities of the latter are known in terms of canonical variables, but in any event the present exploitation of this formalism results in some interesting features. Among them is the distinction between the generating function of infinitesimal canonical transformations of spatial translations, that most naturally turns out to be parallel to the wave vector in each eigenmode, and the Poynting vector that points in the direction of the energy propagation and the group velocity. The Dirac commutation relations between the space-time generators leave intact the $O(3)$ algebra of the angular momentum, although this symmetry group is broken, but the rest of the Poincaré algebra commutation relations that include transformations of the time are distorted. The Hamiltonian equation of motion for the nonconserving angular momentum indicates that its time evolution is determined by the interaction of the external field with a certain magnetic moment (that also contributes to the Hamiltonian) emerging in the special frame in which a pure magnetic field is present. Such a quantity is interpreted as the magnetic moment of the photon; i.e., the magnetic moment of the anisotropic medium that is not only polarized, but also magnetized by the photon field. The possible connection of the photon magnetic moment with the same notion introduced previously by Peréz Rojas and one of the present authors (Villalba-Chavez) [23] (see also Refs. [24,25]) will be discussed in a due place below. It looks at an attribute of the photon interaction with an anisotropic medium.

We organize the manuscript as follows: In Sec. II A, we recall some basic features of the photon propagation in an external field based on the relativistic covariant formalism introduced in Ref. [1] that involves diagonalization of the polarization tensor and analysis of its eigenvalues and eigenvectors. In Sec. II B, we consider the contribution of the polarization operator to the effective nonlocal action (the generating functional of irreducible vertices [26]) to give it the form of the action of the equivalent linear anisotropic medium with time and space dispersion. The tensor decompositions and principal values of dielectric, ε_{ij}, and magnetic, μ_{ij}, permittivities—related to the special class of Lorentz frames where there is only a magnetic, or only an electric, external field—are presented in terms of the polarization operator eigenvalues in an approximation-independent way. We point out the uniaxial character of the vacuum in these frames. This statement holds true also for the most general combination of constant electric and magnetic fields in the Lorentz frames, where these fields are parallel. In Sec. II C, for the same general external field, we obtain the covariant decompositions of the global coordinate transformations that leave the external field intact and thus make the invariance subgroup of our problem.

In Sec. III and thenceforth, we confine ourselves to the local approximation of the effective action, when it does not include field derivatives as the functional arguments. This limitation will make the subsequent development of the Hamiltonian formalism straightforward. The local approximation corresponds to the small four-momentum (infrared) limit of the polarization operator and to the frequency- and momentum-independent tensors ε_{ij} and μ_{ij}. The latter are expressed in terms of derivatives of the effective Lagrangian over the field invariants for the cases reducible to a single field in a special frame (magneticlike and electriclike cases). In Sec. III A, based on the Noether theorem, we define a nonsymmetrical, but gauge-invariant, energy-momentum tensor of small perturbations of the vacuum that satisfies the continuity equation with respect to only one of its tensor indices. The antisymmetric part of this tensor will become, in what follows, responsible for the nonconservation of those Lorentz and spatial rotation generators of canonical transformations constructed using the energy-momentum tensor, which do not leave the external field invariant. We stress the difference between the momentum flux vector (canonical momentum), whose direction is parallel to the wave vector according to Appendix C, and the energy flux Poynting vector, whose direction coincides with that of the group velocity and the center-of-mass velocity of eigenmodes, as demonstrated in Appendix D. The constrained Hamiltonian formalism serving the dynamics of small perturbations over the background field is presented in Sec. III B following the procedure well elaborated in gauge theories. It is the electric induction of the perturbation, and not the field strength, that comes out as a variable, canonically conjugated to its three-vector potential. With the Coulomb gauge condition extended to the problem of the anisotropic vacuum under consideration, the Dirac brackets are defined as performing the infinitesimal canonical transformations in the phase space. In Sec. III C, referring again to the Noether transformations, we define the conserved and nonconserved components of the angular momentum and the Lorentz boost, express them in terms of the canonical variables, and find their Dirac brackets with the fields and inductions. In Sec. IV, the Hamiltonian equations of motion for these quantities are given (Sec. IV A), as well as the set of Dirac commutators for the generators of space-time translations and rotations, defined above, that substitute for the standard relations of the Poincaré algebra in the present case of the vacuum invariant not under all Lorentz and space rotations (Sec. IV B and Appendixes A and B). In Sec. IV C, we dwell on the algebra of the space-time invariance subgroup and define its conserved Casimir invariants.

In Sec. V, we deal with the magneticlike external field. The magnetic moment $\boldsymbol{\mathcal{M}}$ of the photon propagating over the magnetized vacuum is analyzed. It appears as an entity that governs the evolution of the photon angular momentum $\boldsymbol{\mathcal{J}}$ in the magnetic field $\boldsymbol{B}$ following the equation of motion $d\boldsymbol{\mathcal{J}}/dx^0 = 2\boldsymbol{\mathcal{M}} \times \boldsymbol{B}$ and contributes to the photon energy as $-\boldsymbol{\mathcal{M}} \cdot \boldsymbol{B}$. A further step in our understanding of this quantity is given by showing its connection with the optical tensors of our problem. In the large-$|\boldsymbol{B}|$ region, the photon magnetic moment treated following the one-loop approximation of quantum electrodynamics depends quadratically on the photon electric field alone. Its appearance may be understood as another manifestation of the magnetoelectric effect [5,7] in QED, known also in noncommutative electrodynamics [8].

In Sec. VI, we write down the coefficient tensor customarily used to serve the gauge sector in the general Lorenz-violation approach to a $U(1)$-invariant theory, as it follows from the general covariant decomposition of the polarization tensor in a magnetic field found in Ref. [1]. We establish that the coefficient tensor is not double traceless, contrary to what is assumed in the above approach. The double trace is physically meaningful as being connected with the magnetic and electric permeability of the magnetized vacuum. We express the condition of the absence of birefringence in terms of field derivatives of the effective Lagrangian. Finally, we estimate the values of the magnetic field likely to produce the Lorenz violation that would be equivalent to the Lorentz violations intrinsic in the vacuum and detectable using experimental devices of present-day sensitivity. These magnetic fields are too large to make a realistic cosmic background.

We present our concluding summary in Sec. VII, while the essential steps of many calculations have been deferred to the appendixes.

II. LORENTZ SYMMETRY BREAKING: GENERAL ASPECTS

A. The photon effective action

In the presence of an external field $\mathcal{A}_\mu(x) = -\frac{1}{2}\mathcal{F}_{\mu\nu}x^\nu$ with a constant field strength $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$, the action which describes small-amplitude electromagnetic waves $a_\mu(x)$ over a constant background field reads

$$S = -\frac{1}{4}\int f^{\mu\nu} f_{\mu\nu} d^4x + \Gamma, \tag{1}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, and Γ is called the effective action, connected as $\Gamma = \int \mathfrak{L} d^4x$ with the effective Lagrangian $\mathfrak{L}$. Effective action may be expanded in powers of the small field potential as

$$\Gamma = \frac{1}{2}\int d^4x d^4x' a^\mu(x) \Pi_{\mu\nu}(x, x'|\mathcal{A}) a^\nu(x') + \ldots, \tag{2}$$

where $+\ldots$ stands for higher-order terms in a_μ, whereas $\Pi_{\mu\nu}(x, x'|\mathcal{A})$ is the second-rank polarization tensor related as

$$\mathcal{D}^{-1}_{\mu\nu}(x, x'|\mathcal{A}) = [\Box \eta_{\mu\nu} - \partial_\mu\partial_\nu]\delta^{(4)}(x' - x) + \Pi_{\mu\nu}(x, x'|\mathcal{A}), \tag{3}$$

with the inverse photon Green function $\mathcal{D}^{-1}_{\mu\nu}(x, x'|\mathcal{A})$. Hereafter, the metric tensor $\eta^{\mu\nu}$ has the signature $+++-$ with $\eta^{11} = \eta^{22} = \eta^{33} = -\eta^{00} = 1$; the electric field is given by $E^i = \mathcal{F}^{0i}$, whereas the magnetic field is defined by $B^i = 1/2\epsilon^{ijk}\mathcal{F}_{jk}$. The tensor $\tilde{\mathcal{F}}^{\mu\nu} = 1/2\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\rho\sigma}$ (with $\epsilon^{\mu\nu\rho\sigma}$ being the fully antisymmetric unit tensor, $\epsilon^{1230} = 1$) represents the dual of $\mathcal{F}_{\mu\nu}$. The field invariants are $\mathfrak{F} = 1/4\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = (\boldsymbol{B}^2 - \boldsymbol{E}^2)/2$ and $\mathfrak{G} = 1/4\mathcal{F}^{\mu\nu}\tilde{\mathcal{F}}_{\mu\nu} = -\boldsymbol{E}\cdot\boldsymbol{B} = 0$. For the special case of an external field mainly dealt with in the present work, we shall choose $\mathfrak{G} = 0$. Then the corresponding external field will be referred to as magneticlike (if $\mathfrak{F} > 0$) or electriclike (if $\mathfrak{F} < 0$), because in either case a Lorentz frame exists wherein the field is purely magnetic or electric, respectively.

According to Eq. (2), a photon can interact with the external field through the vacuum polarization tensor $\Pi_{\mu\nu}(x, x'|\mathcal{A})$. In this context, the QED Schwinger-Dyson equation for the photon field $a_\mu(x)$ is given by

$$\int d^4x'\mathcal{D}^{-1}_{\mu\nu}(x, x'|\mathcal{A})a^\nu(x') = 0. \quad (4)$$

The external field strength is independent of the space-time coordinates; therefore, the polarization tensor, as a gauge-invariant quantity, should correspond to a spatially homogeneous optical medium whose properties do not change with time. This is provided by the translational invariance of $\Pi_{\mu\nu}$: it depends only on the coordinate difference $\Pi_{\mu\nu}(x, x'|\mathcal{A}) = \Pi_{\mu\nu}(x - x'|\mathcal{A})$ [27]. In this case, a Fourier transform converts Eq. (4) into a linear homogeneous algebraic equation given by

$$[k^2\eta_{\mu\nu} - k_\mu k_\nu - \Pi_{\mu\nu}(k|\mathcal{A})]a^\nu(k) = 0, \quad (5)$$

with

$$\Pi_{\mu\nu}(k|\mathcal{A}) = \int \Pi_{\mu\nu}(x - x'|\mathcal{A})\mathrm{e}^{-ik(x-x')}d^4(x - x'). \quad (6)$$

To understand what follows, it is necessary to recall some basic results developed in Refs. [1,27]. In the presence of a constant magneticlike or electriclike field, the four eigenvectors of the polarization operator $\flat^{(\lambda)}_\mu$ are known in a final, approximation-independent form. In addition to the photon momentum four-vector $\flat^{(4)}_\mu = k_\mu$ (its zeroth component k^0 being the frequency ω), the three other mutually orthogonal four-transverse eigenvectors $\flat^{(\lambda)}_\mu$ are

$$\flat^{(1)}_\mu = k^2\mathcal{F}^2_{\mu\lambda}k^\lambda - k_\mu(k\mathcal{F}^2k), \qquad \flat^{(2)}_\mu = \frac{\tilde{\mathcal{F}}_{\mu\lambda}k^\lambda}{(k\tilde{\mathcal{F}}^2k)^{1/2}},$$
$$\flat^{(3)}_\mu = \frac{\mathcal{F}_{\mu\lambda}k^\lambda}{(-k\mathcal{F}^2k)^{1/2}}, \quad (7)$$

$k^\mu\flat^{(\lambda)}_\mu = 0$ for $\lambda = 1, 2, 3$. (The eigenvectors relating to the most general case, $\mathfrak{F} \neq 0$, $\mathfrak{G} \neq 0$, are written in Refs. [1,27–29].) We remark that $\flat^{(\lambda)}_\mu$ fulfills both the orthogonality condition, $\flat^{(\lambda)}_\mu\flat^{\mu(\lambda')} = \delta^{\lambda\lambda'}(\flat^{(\lambda)})^2$, and the completeness relation,

$$\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} = \sum_{\lambda=1}^{3}\frac{\flat^{(\lambda)}_\mu\flat^{(\lambda)}_\nu}{(\flat^{(\lambda)})^2}. \quad (8)$$

Note that from $\flat^{(\lambda)}$, one obtains the fundamental scalars

$$k^2 = z_1 + z_2, \qquad z_1 = \frac{k\tilde{\mathcal{F}}^2k}{2\mathfrak{F}} \quad \text{and} \quad z_2 = -\frac{k\mathcal{F}^2k}{2\mathfrak{F}}. \quad (9)$$

The last two scalars acquire simple forms in a special reference frame where the external field is purely magnetic (if $\mathfrak{F} > 0$) or purely electric (when the opposite inequality holds). The same equations hold in reference frames that are moving parallel to the external field. For the magnetic background $\mathfrak{F} > 0$, one finds that $z_2 = k_\perp^2$ and $z_1 = k_\parallel^2 - \omega^2$. On the contrary, if the electric field is considered, $z_2 = k_\parallel^2 - \omega^2$, whereas $z_1 = k_\perp^2$. The previous relations involve the vectors $\boldsymbol{k}_\perp$ and $\boldsymbol{k}_\parallel$, which denote the components of $\boldsymbol{k}$ perpendicular to and along the external field, respectively. Henceforth, boldface letters will designate the spatial part of our four-vectors.

Besides the creation of the fundamental scalars, i.e., Eq. (9), the vectorial basis $\flat^{(\lambda)}$ is suitable to express the vacuum polarization tensor in a diagonal form:

$$\Pi_{\mu\nu} = \sum_{\lambda=0}^{4}\varkappa_\lambda(z_1, z_2, \mathfrak{F})\frac{\flat^{(\lambda)}_\mu\flat^{(\lambda)}_\nu}{(\flat^{(\lambda)})^2}. \quad (10)$$

Here $\varkappa_\lambda$ denotes the eigenvalues of the vacuum polarization tensor

$$\Pi_\mu{}^\tau\flat^{(a)}_\tau = \varkappa_a(k)\flat^{(a)}_\mu, \qquad a = 1, 2, 3, 4,$$

which define the energy spectrum of the electromagnetic waves and poles of the photon propagator. Owing to the transversality property ($k^\mu\Pi_{\mu\nu} = 0$), the eigenvalue corresponding to the fourth eigenvector vanishes identically [$\varkappa^{(4)} = 0$]. Substituting Eq. (10) into Eq. (5) and using the orthogonality condition, we find its solutions in the form of a superposition of eigenmodes given by $a_\mu(k) = \sum_{\lambda=1}^{3}F_\lambda\delta(k^2 - \varkappa_\lambda)\flat^{(\lambda)}_\mu$, where F_λ are arbitrary functions of k. According to the latter, three nontrivial dispersion relations arise:

$$k^2 = \varkappa_\lambda(z_2, z_1, \mathfrak{F}), \qquad \lambda = 1, 2, 3, \quad (11)$$

whose solutions can be written as

$$\omega_\lambda^2 = k_\parallel^2 + f_\lambda(k_\perp^2, \mathfrak{F}). \quad (12)$$

The term $f_\lambda(k_\perp^2, \mathfrak{F})$ arises as a sort of dynamical mass. Due to the gauge invariance condition $\varkappa_\lambda(0, 0, \mathfrak{F}) = 0$, there always exist two (out of three) solutions with $f_\lambda(0, \mathfrak{F}) = 0$ that correspond to photons whose rest energy is zero, and the number of polarization degrees of freedom is two. Massive branches of all the three polarizations, $f_\lambda(0, \mathfrak{F}) \neq 0$, $\lambda = 1, 2, 3$, may also exist. For more details, we refer the reader to Refs. [5,28].

Moreover, by considering $a_\mu \sim \flat_\mu^{(\lambda)}$ as the electromagnetic four-vector describing the eigenmodes, we obtain the corresponding electric and magnetic fields of each mode in the special frame, provided that $\mathfrak{F} > 0$:

$$\boldsymbol{e}^{(\lambda)} \simeq i(\omega^{(\lambda)} \flat^{(\lambda)} - \boldsymbol{k}^0 \flat^{(\lambda)}) \quad \text{and} \quad \boldsymbol{b}^{(\lambda)} \simeq i\boldsymbol{k} \times \flat^{(\lambda)}. \tag{13}$$

Up to a nonessential proportionality factor, they are explicitly given by

$$\boldsymbol{e}^{(1)} \simeq -i\boldsymbol{n}_\perp \omega, \quad \boldsymbol{b}^{(1)} \simeq -i\boldsymbol{k}_\| \times \boldsymbol{n}_\perp,$$
$$\boldsymbol{e}_\perp^{(2)} \simeq -i\boldsymbol{k}_\perp k_\| /(k_\|^2 - \omega^2)^{1/2}, \quad \boldsymbol{e}_\|^{(2)} \simeq -i\boldsymbol{n}_\|(k_\|^2 - \omega^2)^{1/2},$$
$$\boldsymbol{b}^{(2)} \simeq i\omega(\boldsymbol{k}_\perp \times \boldsymbol{n}_\|)/(k_\|^2 - \omega^2)^{1/2}, \quad \boldsymbol{e}^{(3)} \simeq i\omega(\boldsymbol{n}_\perp \times \boldsymbol{n}_\|),$$
$$\boldsymbol{b}_\|^{(3)} \simeq -i\boldsymbol{n}_\| k_\perp, \quad \boldsymbol{b}_\perp^{(3)} \simeq i\boldsymbol{n}_\perp k_\|.$$

Here, $\boldsymbol{n}_\| = \boldsymbol{k}_\|/|\boldsymbol{k}_\||$ and $\boldsymbol{n}_\perp = \boldsymbol{k}_\perp/|\boldsymbol{k}_\perp|$ are the unit vectors associated with the parallel and perpendicular directions with respect to the magnetic field $\boldsymbol{B}$ in the special frame in which the external electric field $\boldsymbol{E}$ vanishes identically.

B. The vacuum as an anisotropic medium within a linear optics approximation

In classical electrodynamics, the Maxwell action of a linear continuous medium with the dielectric tensor $\varepsilon_{ij}(\boldsymbol{k}, \omega)$ and the magnetic permeability tensor $\mu_{ij}(\boldsymbol{k}, \omega)$ is given by the expression (quadratic in the fields)

$$\mathcal{S} = \int L = \tfrac{1}{2}\boldsymbol{d}(-\boldsymbol{k}, -\omega) \cdot \boldsymbol{e}(\boldsymbol{k}, \omega) - \tfrac{1}{2}\boldsymbol{h}(-\boldsymbol{k}, -\omega) \cdot \boldsymbol{b}(\boldsymbol{k}, \omega). \tag{14}$$

Here $\boldsymbol{e}$, $\boldsymbol{b}$ are connected by means of the relations

$$\boldsymbol{d}(\boldsymbol{k}, \omega) = \overleftrightarrow{\boldsymbol{\varepsilon}}(\boldsymbol{k}, \omega) \cdot \boldsymbol{e}(\boldsymbol{k}, \omega),$$
$$\boldsymbol{h}(\boldsymbol{k}, \omega) = \overleftrightarrow{\boldsymbol{\mu}}^{-1}(\boldsymbol{k}, \omega) \cdot \boldsymbol{b}(\boldsymbol{k}, \omega), \tag{15}$$

where the double-sided arrow denotes a tensorial quantity. In this context, $\overleftrightarrow{\boldsymbol{\varepsilon}} \cdot \boldsymbol{e} = \varepsilon_{ij} e_j$ and $\overleftrightarrow{\boldsymbol{\mu}}^{-1} \cdot \boldsymbol{b} = \mu_{ij}^{-1} b_j$.

The optical properties of an anisotropic medium depend primarily on the symmetry of its tensors ε_{ij} and μ_{ij}. In an uniaxial medium, one of the principal axes of ε_{ij} and μ_{ij} forms the "optical axis." In what follows, we denote the principal values of ε_{ij} and μ_{ij} relating to this axis as $\varepsilon_\|$ and $\mu_\|$, respectively, and the values relating to the plane perpendicular to the optical axis as $\varepsilon_\perp$ and $\mu_\perp$, respectively. With this in mind, the Maxwell Lagrangian [Eq. (14)] acquires the following form:

$$L = \frac{1}{2}\left\{\varepsilon_\perp |\boldsymbol{e}_\perp|^2 - \frac{1}{\mu_\perp}|\boldsymbol{b}_\perp|^2 + \varepsilon_\| |\boldsymbol{e}_\||^2 - \frac{1}{\mu_\|}|\boldsymbol{b}_\||^2\right\}, \tag{16}$$

where we have decomposed $\boldsymbol{e} = (\boldsymbol{e}_\perp, \boldsymbol{e}_\|)$ and $\boldsymbol{b} = (\boldsymbol{b}_\perp, \boldsymbol{b}_\|)$. Here the symbols $\perp$ and $\|$ refer to the optical axis as well.

Let us consider now the quadratic part of the effective action corresponding to the dynamical gauge field sector of QED in an external field. Substituting Eq. (10) into Eq. (2) and making use of Eq. (8), we find for the integrand of S [Eq. (1)] in momentum space $\mathcal{L} = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} + \mathfrak{L}$,

$$\mathcal{L} = -\tfrac{1}{4}\mathcal{O}^{\mu\nu} f_{\mu\nu}. \tag{17}$$

Here the second-rank antisymmetric tensor $\mathcal{O}^{\mu\nu}$ reads

$$\mathcal{O}^{\mu\nu} = \left(1 - \frac{\varkappa_1}{k^2}\right) f^{\mu\nu} - \frac{1}{2}\frac{\varkappa_1 - \varkappa_2}{k\tilde{\mathcal{F}}^2 k}(f^{\lambda\sigma}\tilde{\mathcal{F}}_{\lambda\sigma})\tilde{\mathcal{F}}^{\mu\nu}$$
$$- \frac{1}{2}\frac{\varkappa_1 - \varkappa_3}{k\mathcal{F}^2 k}(f^{\lambda\sigma}\mathcal{F}_{\lambda\sigma})\mathcal{F}^{\mu\nu}. \tag{18}$$

The Lagrangian $\mathcal{L}$ was written in Ref. [28]; its small-momentum form [corresponding to Eq. (47) in the next subsection] is present in an earlier paper [30].

The expression above is valid for both magneticlike and electriclike cases ($\mathfrak{F} \lessgtr 0$, $\mathfrak{G} = 0$) and defines the corresponding induction vectors according to the following rule:

$$d^i = \mathcal{O}^{0i} = \partial\mathcal{L}/\partial e^i, \quad h^i = \frac{1}{2}\epsilon^{ijk}\mathcal{O}_{jk} = -\partial\mathcal{L}/\partial b^i, \tag{19}$$

where $\boldsymbol{e}(x) = \boldsymbol{\nabla} a_0(x) - \partial_0 \boldsymbol{a}(x)$ and $\boldsymbol{b} = \boldsymbol{\nabla} \times \boldsymbol{a}$ are the averaged (classical) electric and magnetic fields associated with the electromagnetic wave. With these definitions, the Maxwell equations have the recognizable form

$$\boldsymbol{\nabla} \cdot \boldsymbol{d} = 0, \quad \boldsymbol{\nabla} \cdot \boldsymbol{b} = 0;$$
$$\boldsymbol{\nabla} \times \boldsymbol{e} = -\frac{\partial \boldsymbol{b}}{\partial x^0}, \quad \boldsymbol{\nabla} \times \boldsymbol{h} = \frac{\partial \boldsymbol{d}}{\partial x^0}. \tag{20}$$

The above expressions allow us to obtain the most general structure of the dielectric and magnetic permeability tensor. To derive it, we first note that our effective Lagrangian $\mathcal{L}$ in Eq. (17) acquires the structure of Eq. (14) as long as it is expressed in terms of the induction vectors $\boldsymbol{d}$, $\boldsymbol{h}$ and the electric $\boldsymbol{e}$ and magnetic $\boldsymbol{b}$ fields of the small electromagnetic waves. Likewise, the optical tensors can be defined as in Eq. (15). In fact, by considering the following relations, valid in the special frames,

$$-\frac{1}{4}f^{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu} = \frac{1}{2}\boldsymbol{e}\cdot\boldsymbol{B}, \quad \frac{1}{4}f^{\mu\nu}\mathcal{F}_{\mu\nu} = \frac{1}{2}\boldsymbol{b}\cdot\boldsymbol{B}, \quad \mathfrak{F} > 0,$$
$$-\frac{1}{4}f^{\mu\nu}\tilde{\mathcal{F}}_{\mu\nu} = \frac{1}{2}\boldsymbol{b}\cdot\boldsymbol{E}, \quad \frac{1}{4}f^{\mu\nu}\mathcal{F}_{\mu\nu} = -\frac{1}{2}\boldsymbol{e}\cdot\boldsymbol{E}, \quad \mathfrak{F} < 0, \tag{21}$$

one can express them for the magnetic external field:

$$\varepsilon_{ij}(\boldsymbol{k}, \omega) = \left(1 - \frac{\varkappa_1}{k^2}\right)\delta_{ij} + \frac{\varkappa_1 - \varkappa_2}{k_\|^2 - \omega^2}\frac{B_i B_j}{B^2},$$
$$\mu_{ij}^{-1}(\boldsymbol{k}, \omega) = \left(1 - \frac{\varkappa_1}{k^2}\right)\delta_{ij} + \frac{\varkappa_1 - \varkappa_3}{k_\perp^2}\frac{B_i B_j}{B^2}. \tag{22}$$

It is notable that the components of the three-momentum vector $\boldsymbol{k}$ do not take part in forming these tensors; only components of $\boldsymbol{B}$ do. This feature is not typical of crystal optics with spatial dispersion and can be attributed to the explicit exploitation of the gauge invariance laid in Eq. (18). The eigenvalues of matrices in Eq. (22) [the principal values of the electric and (inverse) magnetic permittivities] are

$$\varepsilon_\perp = \mu_\perp^{-1} = 1 - \frac{\varkappa_1}{k^2}, \qquad \varepsilon_\parallel = 1 - \frac{\varkappa_1}{k^2} + \frac{\varkappa_1 - \varkappa_2}{k_\parallel^2 - \omega^2}, \\ \mu_\parallel^{-1} = 1 - \frac{\varkappa_1}{k^2} + \frac{\varkappa_1 - \varkappa_3}{k_\perp^2}. \tag{23}$$

The values $\varepsilon_\parallel$ and $\mu_\parallel^{-1}$ correspond to the eigenvector directed along the external magnetic field $\boldsymbol{B}$, which therefore makes it the direction of the principal optical axis. The values $\varepsilon_\perp = \mu_\perp^{-1}$ correspond to the eigenvectors directed transverse to the external magnetic field. The principal values of Eq. (23) are rotational scalars and depend upon direction: their arguments are the frequency ω and the scalar product $\boldsymbol{B}\cdot\boldsymbol{k}$, combined into $z_1 = \omega^2 - k_\parallel^2$, $z_2 = k_\perp^2$. We also find a similar result for an electriclike background ($\mathfrak{F} < 0$). In this case, the optical axis is determined by $\boldsymbol{E}$ and

$$\varepsilon_{ij}(\boldsymbol{k},\omega) = \left(1 - \frac{\varkappa_1}{k^2}\right)\delta_{ij} + \frac{\varkappa_1 - \varkappa_3}{k_\parallel^2 - \omega^2}\frac{E_iE_j}{E^2}, \\ \mu_{ij}^{-1}(\boldsymbol{k},\omega) = \left(1 - \frac{\varkappa_1}{k^2}\right)\delta_{ij} + \frac{\varkappa_1 - \varkappa_2}{k_\perp^2}\frac{E_iE_j}{E^2}. \tag{24}$$

From these tensors, we obtain that $\varepsilon_\perp$ and $\mu_\perp^{-1}$ have the same structure as in Eq. (23), whereas

$$\varepsilon_\parallel = \varepsilon_\perp + \frac{\varkappa_1 - \varkappa_3}{k_\parallel^2 - \omega^2}, \qquad \mu_\parallel^{-1} = \mu_\perp^{-1} + \frac{\varkappa_1 - \varkappa_2}{k_\perp^2}. \tag{25}$$

The results given in Eqs. (22)–(25) point out that in the presence of an external magnetic or electric field, the vacuum behaves like a uniaxial anisotropic material. Note that the procedure shown in this section is independent of any approximation made in the calculation of the vacuum polarization tensor. However, it is only valid in a class of special frames, in which the external field is purely magnetic (when $\mathfrak{G} = 0$, $\mathfrak{F} > 0$) or purely electric (when $\mathfrak{G} = 0$, $\mathfrak{F} < 0$). In a general Lorentz frame, an electric (magnetic) component is added to the primarily purely magnetic (electric) field, as produced by the Lorentz boost. Hence, the statement that the vacuum is uniaxial is no longer true in that frame, because the second axis is specialized by the direction of the added component—or, in other words, by the direction of the motion of the reference frame with respect to the special frame. Therefore, the vacuum is a biaxial medium that can be rendered uniaxial by an appropriate Lorentz transformation. (In the case of a material anisotropic medium, which is uniaxial in its rest frame, we can also state that it becomes biaxial if it moves with respect to an observer, the direction of motion specializing additional direction in the frame of the observer.)

The same statements are readily extended to the case of a general external field with both the field invariants different from zero: $\mathfrak{G} \neq 0$, $\mathfrak{F} \neq 0$. In this general case, (a class of) special frames exist, where the external electric and magnetic fields are mutually parallel, their common direction specializing the principal optical axis in such frames. The point is that the diagonal representation for the polarization operator [Eq. (10)] remains valid, the only reservation being that now the eigenvectors $\flat_\nu^{(\lambda)}$ in it are not just the vectors of Eq. (7), but linear combinations of them [29]. A representation analogous to Eq. (18) can be written in that case, and the principal values substituting for Eq. (23) again depend on the same combinations of momenta $\omega^2 - k_\parallel^2$ and $k_\perp^2$, where now the designations $\parallel$ and $\perp$ mark the directions parallel and orthogonal to the common direction of the external fields.

C. The vacuum symmetry subgroup $ISO_A(3,1)$: Covariant decomposition of transformations

Actually, the anisotropic character of the medium, equivalent to that of the vacuum with an external field, arises due to the Lorentz and rotational symmetry breakdown [which is not manifest in the Maxwell Lagrangian of Eq. (14), because it relates only to the rest frame of the medium and does not reflect its spatial symmetry]. The Lagrangian [Eq. (17)] is not Lorentz and rotational invariant, because it contains an external tensor responsible for the external field. So, to keep it invariant, one should transform the external field together with the photon field. Correspondingly, the explicit forms of the scalars z_2 and z_1, as well as $\varkappa_i$, when these are expressed through the photon momentum components, depend on the reference frame. However, the Lagrangian [Eq. (17)] turns out to be invariant under those space-time transformations which leave the external field intact. Thus, bearing in mind the translational invariance of our problem, the proper inhomogeneous orthochronous Lorentz transformations relating to the symmetry group of an anisotropic homogeneous vacuum occupied by an external space- and time-independent classical field must fulfill the conditions

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + \epsilon^\mu, \qquad \eta_{\lambda\rho} = \Lambda^\mu{}_\lambda \Lambda^\nu{}_\sigma \eta_{\mu\nu}, \\ \mathcal{F}_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \mathcal{F}_{\rho\sigma}, \qquad \det\Lambda = 1, \qquad \Lambda^0_0 > 0. \tag{26}$$

The set of pairs $\{\epsilon, \Lambda\}$ satisfying Eq. (26) form a subgroup of the Poincaré group [$(ISO(3,1))$] which will be referred to as the "Amputated Poincaré Group," $ISO_A(3,1)$. Also, $\Lambda \in SO_A(3,1)$, where $SO_A(3,1)$ is called the "Amputated Lorentz Group." Due to Eq. (26), the infinitesimal Lorentz transformation associated with our problem can be written as [31]

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \ \text{with} \ \omega^\mu{}_\nu = \vartheta' \mathcal{F}^\mu{}_\nu + \xi' \tilde{\mathcal{F}}^\mu{}_\nu, \tag{27}$$

where ϑ' and ξ' are real infinitesimal parameters. Note that the description above is not restricted to a magneticlike or electriclike field. On the contrary, it holds for any other constant and homogeneous external field configuration with the second field invariant, $\mathfrak{G} \neq 0$, and it emerges in any other nonlinear electrodynamics different from QED.

We exploit Eq. (27) to obtain explicitly the structure of those Lorentz transformations that are associated with our problem, i.e., $\Lambda \in SO_A(3,1)$. Before doing this, we redefine the group parameter in Eq. (27), $\vartheta' = \vartheta/(\mathcal{N}_+ + \mathcal{N}_-)$ and $\xi' = \xi/(\mathcal{N}_+ + \mathcal{N}_-)$, where $\mathcal{N}_\pm = [(\mathfrak{F}^2 + \mathfrak{G}^2)^{1/2} \pm \mathfrak{F}]^{1/2}$ are eigenvalues of the external field tensor $\mathcal{F}$. In addition, we express the finite Lorentz transformation as an exponential of a matrix argument

$$\Lambda = \exp\left(\frac{\vartheta \mathcal{F}}{\mathcal{N}_+ + \mathcal{N}_-} + \frac{\xi \tilde{\mathcal{F}}}{\mathcal{N}_+ + \mathcal{N}_-}\right), \tag{28}$$

defined by its series expansion.

Substantial simplifications can be achieved by the introduction of the matrix basis

$$(Z_\pm)^\mu{}_\nu = \frac{\mathcal{N}_\pm \tilde{\mathcal{F}}^\mu{}_\nu \mp \mathcal{N}_\mp \mathcal{F}^\mu{}_\nu}{\mathcal{N}_+^2 + \mathcal{N}_-^2}, \tag{29}$$

$$(Z_\pm^2)^\mu{}_\nu = \frac{\mathcal{F}^{\mu\lambda}\mathcal{F}_{\lambda\nu} \pm \mathcal{N}_\pm^2 \delta^\mu{}_\nu}{\mathcal{N}_+^2 + \mathcal{N}_-^2}, \tag{30}$$

whose elements fulfill the following properties:

$$\begin{aligned} &Z_+ Z_- = 0, \qquad Z_+^{2n} = Z_+^2, \qquad Z_+^{2n+1} = Z_+, \\ &Z_-^{2n} = (-1)^{n-1} Z_-^2, \qquad Z_-^{2n+1} = (-1)^n Z_-. \end{aligned} \tag{31}$$

With these details in mind, a finite Lorentz transformation belonging to $SO_A(3,1)$ decomposes according to

$$\Lambda = Z_- \sin\varphi + Z_+ \sinh\zeta - Z_-^2 \cos\varphi + Z_+^2 \cosh\zeta, \tag{32}$$

where the arguments of the trigonometric and hyperbolic functions are

$$\zeta = \frac{\mathcal{N}_+ \xi}{\mathcal{N}_+ + \mathcal{N}_-} - \frac{\mathcal{N}_- \vartheta}{\mathcal{N}_+ + \mathcal{N}_-}, \tag{33}$$

$$\varphi = \frac{\mathcal{N}_- \xi}{\mathcal{N}_+ + \mathcal{N}_-} + \frac{\mathcal{N}_+ \vartheta}{\mathcal{N}_+ + \mathcal{N}_-}. \tag{34}$$

In contrast to the cases considered in Secs. II A and II B, the covariant decomposition of the Lorentz transformation [Eq. (32)] deduced here relates to the most general case of a constant and homogeneous external field with both its invariants $\mathfrak{G}$, $\mathfrak{F}$ different from zero. (Therefore, it remains valid in the crossed field system $\mathfrak{G} = \mathfrak{F} = 0$ and $|\boldsymbol{E}| = |\boldsymbol{B}|$.) For the magnetized vacuum ($\mathfrak{G} = 0$, $\mathfrak{F} > 0$), the variables in Eq. (32) become $\zeta = \xi$ and $\varphi = \vartheta$, whereas in an electric background ($\mathfrak{G} = 0$, $\mathfrak{F} < 0$) they turn out to be $\zeta = -\vartheta$ and $\varphi = \xi$.

Now, the explicit structure of Λ in terms of the external electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields is rather complicated in a general Lorentz frame. However, it becomes simpler if one considers the external field configurations inherent to special frames, where the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ are parallel, and directed, say, along the z axis. In these special frames, the transformations of Eq. (32) have a simple sense of rotation about the axis z by the angle φ, and the Lorentz boost along this axis is parametrized by ζ.

Considering the general case $\mathfrak{G} \neq 0$, $\mathfrak{F} \neq 0$ in the special frame where $\boldsymbol{B} = (0, 0, B_z)$, $\boldsymbol{E} = (0, 0, E_z)$, we found that

$$\Lambda^\mu{}_\nu = \mathcal{R}^\mu{}_\lambda(\varphi)\mathcal{B}^\lambda{}_\nu(\zeta) = \mathbb{D}^{SO(2)} \oplus \mathbb{D}^{SO(1,1)}. \tag{35}$$

The expression above involves the matrices

$$\mathcal{R}(\varphi) = \mathbb{D}^{SO(2)} \oplus \mathbb{1} \quad \text{and} \quad \mathcal{B}(\zeta) = \mathbb{1} \oplus \mathbb{D}^{SO(1,1)}, \tag{36}$$

where the $\oplus$ denotes the usual direct sum of matrices, and $\mathbb{1}$ denotes a 2×2 identity matrix. Here both $\mathbb{D}^{SO(2)}$ and $\mathbb{D}^{SO(1,1)}$ are two-dimensional representations of $SO(2)$ and $SO(1,1)$, respectively. Explicitly, they read

$$\mathbb{D}^{SO(2)} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}, \qquad \mathbb{D}^{SO(1,1)} = \begin{bmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{bmatrix}.$$

Both matrices conform two independent Abelian-invariant subgroups of $SO_A(3,1)$, which emphasizes their nonsemisimple structure. Moreover, the group parameter space is the product manifold $S^{(1)} \times \mathfrak{R}^{(1)}$, where $S^{(1)}$ is the circle and $\mathfrak{R}^{(1)}$ the real line. Therefore, the topology of $SO_A(3,1)$ is the surface of a circular cylinder, a manifold which is neither compact nor simply connected. (Indeed, it is infinitely connected.)

III. GENERATORS OF SPACE-TIME TRANSFORMATIONS

The constrained Hamiltonian formulation of the electromagnetic field in an anisotropic material remains poorly developed in the literature; we are not aware of previous Hamiltonian treatment for these optical media. In this section, we shall see how this method can be implemented in QED with an external constant electromagnetic field. To pursue this analysis, we first note that the effective Lagrangian $\mathcal{L}$ in Eq. (17) is not a local function, since it depends on momenta in a very cumbersome way. The canonical formalism in its standard form is not applicable to nonlocal situations. For this reason, we will restrict ourselves to the infrared approximation, $k_\mu \to 0$, in which case the effective Lagrangian $\mathcal{L}$ becomes a local function on the photon field $a_\mu(x)$. This approximation corresponds to anisotropic media with no spatial or frequency dispersion. In our case, it becomes actual in the region of very strong external fields, where the external field dominates over the photon momentum [32].

Consider the static ($\omega = 0$) case. When $k_\perp = 0$, we have from Eq. (23)

$$\varepsilon_\parallel(k_\parallel, 0) = 1 - \frac{\varkappa_2}{k_\parallel^2}\bigg|_{\omega, k_\perp = 0}.$$

When $\omega = 0$, $k_\parallel = 0$, we have that

$$\mu_\parallel^{-1}(k_\perp, 0) = 1 - \frac{\varkappa_3}{k_\perp^2}. \tag{37}$$

Owing to the degeneracy property [5], $\varkappa_1 = \varkappa_2$ at $\omega^2 - k_\parallel^2 = 0$, we also have that

$$\varepsilon_\perp(k_\perp, 0) = \mu_\perp^{-1}(k_\perp, 0) = 1 - \frac{\varkappa_1}{k_\perp^2}\bigg|_{\omega, k_\parallel = 0}.$$

So, in the infrared limit $k_\mu \to 0$, the quantities in Eq. (23) coincide with the permittivities defined for that case in Ref. [28]—according to the correspondences $\varepsilon_\perp \Leftrightarrow \varepsilon_{\rm tr}$, $\mu_\perp \Leftrightarrow \mu_{\rm tr}^{\rm w}$, $\varepsilon_\parallel \Leftrightarrow \varepsilon_{\rm long}$, $\mu_\parallel^{-1} \Leftrightarrow \mu_{\rm tr}^{\rm pl}$—and responsible for screening charges and stationary currents of special configurations.

In the infrared limit, the eigenvalues of the vacuum polarization tensor can be expressed in terms of the first and second derivatives of an effective Lagrangian $\mathfrak{L}$ [connected with the generating functional Γ of irreducible many-photon vertices in an external field, as pointed out below Eq. (1)] over the constant field with respect to the corresponding external field invariants $\mathfrak{F}$ and $\mathfrak{G}$ (see details in Ref. [28]):

$$\varkappa_1 = k^2\mathfrak{L}_{\mathfrak{F}}, \qquad \varkappa_2 = \varkappa_1 - 2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}z_1, \qquad \varkappa_3 = \varkappa_1 + 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}z_2, \tag{38}$$

where $\mathfrak{L}_{\mathfrak{F}} = \partial\mathfrak{L}/\partial\mathfrak{F}$, $\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} = \partial^2\mathfrak{L}/\partial\mathfrak{G}^2$, and $\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} = \partial^2\mathfrak{L}/\partial\mathfrak{F}^2$, with $\mathfrak{G}$ set equal to zero after differentiation. It follows from Eqs. (38), (22), and (24) that

$$\left.\begin{aligned}\varepsilon_{ij} &= (1-\mathfrak{L}_{\mathfrak{F}})\delta_{ij} + \mathfrak{L}_{\mathfrak{G}\mathfrak{G}}B_iB_j\\ \mu_{ij}^{-1} &= (1-\mathfrak{L}_{\mathfrak{F}})\delta_{ij} - \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}B_iB_j\end{aligned}\right\}\mathfrak{F} > 0,$$
$$\left.\begin{aligned}\varepsilon_{ij} &= (1-\mathfrak{L}_{\mathfrak{F}})\delta_{ij} + \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}E_iE_j\\ \mu_{ij}^{-1} &= (1-\mathfrak{L}_{\mathfrak{F}})\delta_{ij} - \mathfrak{L}_{\mathfrak{G}\mathfrak{G}}E_iE_j\end{aligned}\right\}\mathfrak{F} < 0. \tag{39}$$

For the magneticlike case $\mathfrak{F} > 0$, this is equivalent to

$$\varepsilon_\perp = \mu_\perp^{-1} = 1 - \mathfrak{L}_{\mathfrak{F}}, \qquad \varepsilon_\parallel = 1 - \mathfrak{L}_{\mathfrak{F}} + 2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}, \qquad \mu_\parallel^{-1} = 1 - \mathfrak{L}_{\mathfrak{F}} - 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}, \tag{40}$$

The following relations were established in Ref. [28] on the basis of causality and unitarity principles, valid for $\mathfrak{F} \lessgtr 0$:

$$1 - \mathfrak{L}_{\mathfrak{F}} \ge 0, \qquad 1 - \mathfrak{L}_{\mathfrak{F}} + 2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} \ge 0, \mathfrak{L}_{\mathfrak{G}\mathfrak{G}} \ge 0, \qquad 1 - \mathfrak{L}_{\mathfrak{F}} - 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} \ge 0, \mathfrak{L}_{\mathfrak{F}\mathfrak{F}} \ge 0, \tag{41}$$

which guarantees the consistency of the theory.

A. The energy-momentum tensor

The symmetry reduction by the external magnetic field does not alter the translational group embedded in $ISO(3,1)$. Therefore, for a photon, the space-time configuration with an external classical field is translation invariant. To find in the local approximation the associated Noether current of the electromagnetic radiation, let us first insert Eq. (38) into Eq. (18) to find

$$\mathcal{O}^{\mu\nu} = (1 - \mathfrak{L}_{\mathfrak{F}})f^{\mu\nu} - \frac{1}{2}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}(f^{\rho\lambda}\tilde{\mathcal{F}}_{\rho\lambda})\tilde{\mathcal{F}}^{\mu\nu} - \frac{1}{2}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}(f^{\rho\lambda}\mathcal{F}_{\rho\lambda})\mathcal{F}^{\mu\nu}. \tag{42}$$

The substitution of this tensor into $\mathcal{L}$ [Eq. (17)] defines the Lagrangian [28,30] of the small-amplitude, low-frequency, long-wave electromagnetic field $a_\mu(x)$:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mathfrak{L}\\ &= -\frac{1}{4}(1-\mathfrak{L}_{\mathfrak{F}})f^{\mu\nu}f_{\mu\nu} + \frac{1}{8}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}(f^{\mu\nu}\tilde{\mathcal{F}}_{\mu\nu})^2\\ &\quad + \frac{1}{8}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}(f^{\mu\nu}\mathcal{F}_{\mu\nu})^2.\end{aligned} \tag{43}$$

The corresponding conserved stress-energy tensor is obtained from it by following the Noether theorem in the standard way:

$$T^{\mu\nu} = \eta^{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu a_\lambda)}\partial^\nu a_\lambda. \tag{44}$$

Here

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu a_\nu)} = -\frac{\partial\mathcal{L}}{\partial(\partial_\nu a_\mu)} = -\mathcal{O}^{\mu\nu}, \tag{45}$$

in accordance with Eqs. (17)–(19). The antisymmetricity of this tensor is owing to the gauge invariance manifesting, in that the Lagrangian contains only the field tensor.

Substituting Eq. (43) into Eq. (44), we obtain

$$\begin{aligned}T^{\mu\nu} &= (1-\mathfrak{L}_{\mathfrak{F}})f^{\mu\lambda}\partial^\nu a_\lambda - \frac{1}{2}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}(f^{\varrho\sigma}\tilde{\mathcal{F}}_{\varrho\sigma})\tilde{\mathcal{F}}^{\mu\lambda}\partial^\nu a_\lambda\\ &\quad - \frac{1}{2}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}(f^{\varrho\sigma}\mathcal{F}_{\varrho\sigma})\mathcal{F}^{\mu\lambda}\partial^\nu a_\lambda\\ &\quad + \eta^{\mu\nu}\bigg\{-\frac{1}{4}(1-\mathfrak{L}_{\mathfrak{F}})f_{\varrho\sigma}f^{\varrho\sigma} + \frac{1}{8}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}(f^{\sigma\varrho}\tilde{\mathcal{F}}_{\sigma\varrho})^2\\ &\quad + \frac{1}{8}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}(f^{\sigma\varrho}\mathcal{F}_{\sigma\varrho})^2\bigg\}.\end{aligned} \tag{46}$$

Let us define the tensor

$$\Theta^{\mu\nu} = \mathcal{L}\eta^{\mu\nu} + \mathcal{O}^{\mu\lambda}f^\nu{}_\lambda - j^\mu a^\nu \tag{47}$$

related to Eq. (44) as

$$T^{\mu\nu} = \Theta^{\mu\nu} - \partial_\lambda K^{\lambda\mu\nu}, \tag{48}$$

where $K^{\lambda\mu\nu} = -\mathcal{O}^{\mu\lambda}a^\nu$ is antisymmetric in its first two indices, while the electric current

$$j^{\nu} = \partial_{\mu} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} a_{\nu})} = -\partial_{\mu} \mathcal{O}^{\mu\nu} \tag{49}$$

disappears on equations of motion, $j_{\mu} \cong 0$. It conserves $\partial^{\mu} j_{\mu} = 0$ due to antisymmetricity of the tensor $\mathcal{O}^{\mu\lambda}$ [Eq. (45)]. When taken on the equations of motion, the tensors $T^{\mu\nu}$ and $\Theta^{\mu\nu}$ coincide up to a full derivative. However, the latter depends only on the field strengths of the small electromagnetic field $f_{\alpha\beta}$, and not on its four-vector potential a_{μ}, as was the case with the tensor of Eqs. (44) and (46). The achievement of this gauge-invariance property of the tensor $\Theta^{\mu\nu}$ [Eq. (47)] was the motivation [28] for taking it as the stress-energy tensor (on equations of motion). It remains, however, not symmetric. Substituting Eq. (42) into Eq. (47) results in

$$\begin{aligned}
\Theta^{\mu\nu} = {} & (1 - \mathfrak{L}_{\mathfrak{F}}) f^{\mu\lambda} f^{\nu}{}_{\lambda} - \frac{1}{2} \mathfrak{L}_{\mathfrak{G}\mathfrak{G}} (f^{\varrho\sigma} \tilde{\mathcal{F}}_{\varrho\sigma}) \tilde{\mathcal{F}}^{\mu\lambda} f^{\nu}{}_{\lambda} \\
& - \frac{1}{2} \mathfrak{L}_{\mathfrak{F}\mathfrak{F}} (f^{\varrho\sigma} \mathcal{F}_{\varrho\sigma}) \mathcal{F}^{\mu\lambda} f^{\nu}{}_{\lambda} \\
& + \eta^{\mu\nu} \Big\{ -\frac{1}{4} (1 - \mathfrak{L}_{\mathfrak{F}}) f_{\varrho\sigma} f^{\varrho\sigma} + \frac{1}{8} \mathfrak{L}_{\mathfrak{G}\mathfrak{G}} (f^{\sigma\varrho} \tilde{\mathcal{F}}_{\sigma\varrho})^2 \\
& + \frac{1}{8} \mathfrak{L}_{\mathfrak{F}\mathfrak{F}} (f^{\sigma\varrho} \mathcal{F}_{\sigma\varrho})^2 \Big\} - j^{\mu} a^{\nu}.
\end{aligned} \tag{50}$$

In an empty Minkowski space ($\mathcal{F} = 0$), the possibility of symmetrization of the canonical stress-energy tensor by adding a full derivative is provided by the conservation of $SO(3,1)$ generators. (For details, we refer the reader to Ref. [26] and references therein.) In our case, there is a lack of isotropy due to the external field which implies that only a subset of the Lorentz generators are conserved (see Secs. III C and IV A). This fact, therefore, prevents us from obtaining an equivalently symmetrized version of Eq. (44), and some dramatic differences arise in comparison with the case of an empty vacuum. The relation obeyed by Eq. (50) to substitute for the symmetricity property is $\check{f}\check{\Theta} = \check{\Theta}^T \check{f}$, where $\check{\Theta}$ and $\check{f}$ are the matrices $\Theta_{\mu\nu}$ and $f_{\mu\nu}$, while $\check{\Theta}^T$ is the transposed matrix. When there is no Lorentz breaking, the matrix $\check{\Theta}$ commutes with the matrix $\check{f}$, because in this case the stress-energy tensor is built only of the field tensor and the unit metric tensor, its explicit dependence on the coordinate vector not being admitted. In this case, the symmetricity of Θ is in agreement with the above relation.

The stress-energy tensor $\Theta^{\mu\nu}$, as well as $T^{\mu\nu}$, satisfies the continuity equation with respect to the first index on equations of motion,

$$\partial_{\mu} \Theta^{\mu\nu} = 0, \tag{51}$$

with their difference satisfying the same property $\partial_{\mu} \partial_{\lambda} K^{\lambda\mu\nu} = 0$. Its components, explicitly, are

$$\Theta^{00} = \frac{1}{2} \boldsymbol{d} \cdot \boldsymbol{e} + \frac{1}{2} \boldsymbol{h} \cdot \boldsymbol{b} + a_0 \boldsymbol{\nabla} \cdot \boldsymbol{d}, \tag{52}$$

$$\Theta^{0i} = [\boldsymbol{d} \times \boldsymbol{b}]^i - a^i \boldsymbol{\nabla} \cdot \boldsymbol{d}, \tag{53}$$

$$\Theta^{i0} = [\boldsymbol{e} \times \boldsymbol{h}]^i - a_0 [\partial_0 d^i - (\boldsymbol{\nabla} \times \boldsymbol{h})^i], \tag{54}$$

$$\begin{aligned}
\Theta^{ij} = {} & -d^i e^j - b^i h^j + \frac{1}{2} \delta^{ij} (\boldsymbol{e} \cdot \boldsymbol{d} + \boldsymbol{b} \cdot \boldsymbol{h}) \\
& + a^j [\partial_0 d^i - (\boldsymbol{\nabla} \times \boldsymbol{h})^i],
\end{aligned} \tag{55}$$

where the electric and magnetic induction vectors are defined in Eq. (19). Explicitly,

$$\begin{aligned}
\boldsymbol{d} &= (1 - \mathfrak{L}_{\mathfrak{F}}) \boldsymbol{e} + \mathfrak{L}_{\mathfrak{G}\mathfrak{G}} (\boldsymbol{e} \cdot \boldsymbol{B}) \boldsymbol{B} \quad \text{for } \mathfrak{F} > 0, \\
\boldsymbol{d} &= (1 - \mathfrak{L}_{\mathfrak{F}}) \boldsymbol{e} + \mathfrak{L}_{\mathfrak{F}\mathfrak{F}} (\boldsymbol{e} \cdot \boldsymbol{E}) \boldsymbol{E} \quad \text{for } \mathfrak{F} < 0,
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
\boldsymbol{h} &= (1 - \mathfrak{L}_{\mathfrak{F}}) \boldsymbol{b} - \mathfrak{L}_{\mathfrak{F}\mathfrak{F}} (\boldsymbol{b} \cdot \boldsymbol{B}) \boldsymbol{B} \quad \text{for } \mathfrak{F} > 0, \\
\boldsymbol{h} &= (1 - \mathfrak{L}_{\mathfrak{F}}) \boldsymbol{b} - \mathfrak{L}_{\mathfrak{G}\mathfrak{G}} (\boldsymbol{b} \cdot \boldsymbol{E}) \boldsymbol{E} \quad \text{for } \mathfrak{F} < 0.
\end{aligned} \tag{57}$$

Let us consider the case in which the equations of motion are fulfilled, i.e., when the current $j^{\mu} = 0$ vanishes identically. By integrating the continuity equation [Eq. (51)] with $\nu = 0$ over a final spatial volume V and defining the energy in this volume as

$$\mathcal{P}^0_{\mathrm{V}} = \int_{\mathrm{V}} d^3x \Theta^{00}(x) = \int_{\mathrm{V}} d^3x \Big(\tfrac{1}{2} \boldsymbol{d} \cdot \boldsymbol{e} + \tfrac{1}{2} \boldsymbol{h} \cdot \boldsymbol{b} \Big), \tag{58}$$

we get

$$\frac{\partial \mathcal{P}^0_{\mathrm{V}}}{\partial x^0} = \oint_{\mathrm{S}} \Theta_{i0} d\sigma_i = \oint_{\mathrm{S}} (\boldsymbol{e} \times \boldsymbol{h})_i d\sigma_i, \tag{59}$$

where the integral in the right-hand side is run over the surface S surrounding the volume V. Therefore, the Poynting vector $\Theta^{i0} = (\boldsymbol{e} \times \boldsymbol{h})^i$ accounts for the energy per unit of time, per unit area, transported by the small electromagnetic waves. It is parallel to the group velocity of eigenmodes and to their center-of-mass velocity, as described in Appendix D. In the infinite-volume limit, $\mathrm{V} = \infty$, and under the assumption that the fields $\boldsymbol{e}$, $\boldsymbol{h}$ fall off at spatial infinity, we find that the energy inside the infinite volume $\mathcal{P}^0 = \mathcal{P}^0_{\infty}$ does not depend on time:

$$\frac{\partial \mathcal{P}^0}{\partial x^0} = 0. \tag{60}$$

By integrating the continuity equation [Eq. (51)] with $\nu = j$ over a final spatial volume V, we find that the momentum in this volume,

$$\mathcal{P}^j_{\mathrm{V}} = \int_{\mathrm{V}} d^3x \Theta^{0j}(x) = \int_{\mathrm{V}} d^3x (\boldsymbol{d} \times \boldsymbol{b})^j, \tag{61}$$

satisfies the equation

$$\frac{\partial \mathcal{P}^i_{\mathrm{V}}}{\partial x_0} = \oint_{\mathrm{S}} \Theta^{ij} d\sigma_i, \tag{62}$$

which indicates that the total momentum contained inside the infinite volume $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_\infty$ conserves,

$$\frac{\partial \boldsymbol{\mathcal{P}}}{\partial x^0} = 0, \tag{63}$$

under the same assumption that the fields decrease at spatial infinity. Note that the momentum density $\sim \boldsymbol{d} \times \boldsymbol{b}$ and the Poynting vector $\sim \boldsymbol{e} \times \boldsymbol{h}$ describe different quantities, which does not take place in an empty space-time. Observe that, on the equation of motion, the structure of Eqs. (52)–(55) does not differ from the case of light propagation in an anisotropic material [33].[2]

In the general case in which the current j^μ does not vanish, the spatial integral of Eqs. (52) and (53) defines the translation generators. For further convenience, we express the latter in terms of the momentum $\pi^i = \partial \mathcal{L}/\partial(\partial_0 a_i)$, canonically conjugated to the field a_i taken as a canonical coordinate. It coincides with the electric induction, defined in Eq. (19), $\boldsymbol{d} = -\boldsymbol{\pi}$. We invert Eq. (15) so that the electric field of the wave can be expressed as $e_i = \varepsilon_{ij}^{-1} d_j$, where ε_{ij}^{-1} must be understood as the inverse of the tensors given in Eq. (39):

$$\begin{aligned} \varepsilon_{ij}^{-1} &= \frac{1}{1-\mathfrak{L}_{\mathfrak{F}}}\left[\delta_{ij} - \frac{\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}{1-\mathfrak{L}_{\mathfrak{F}} + 2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}} B_i B_j\right] \mathfrak{F} > 0, \\ \varepsilon_{ij}^{-1} &= \frac{1}{1-\mathfrak{L}_{\mathfrak{F}}}\left[\delta_{ij} - \frac{\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}}{1-\mathfrak{L}_{\mathfrak{F}} - 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}} E_i E_j\right] \mathfrak{F} < 0. \end{aligned} \tag{64}$$

With these details in mind, the translation generators turn out to be

$$\begin{aligned} \boldsymbol{\mathcal{P}} &= -\int d^3x \{\boldsymbol{\pi} \times \boldsymbol{b} - \boldsymbol{a}(\boldsymbol{\nabla} \cdot \boldsymbol{\pi})\}, \\ \mathcal{P}^0 &= \int d^3x \left\{\frac{1}{2}\pi_i \varepsilon_{ij}^{-1} \pi_j + \frac{1}{2} b_i \mu_{ij}^{-1} b_j - a_0(\boldsymbol{\nabla} \cdot \boldsymbol{\pi})\right\}. \end{aligned} \tag{65}$$

Some comments are in order. First of all, we point out that these generators and their respective translational charges, Eqs. (61) and (58) differ from each other in the terms which are proportional to Gauss's law, $\boldsymbol{\nabla} \cdot \boldsymbol{\pi} = 0$. Such terms are intrinsically associated with the constrained Hamiltonian formalism [19] (see the next subsection). Correspondingly, we shall show that the expression involved in Eq. (65) canonically realizes the space-time translations, at least when acting on a phase space defined by constraints associated with the gauge symmetry.

B. Gauge fixing and Dirac brackets

The local approximation [Eq. (43)] of our effective Lagrangian $\mathcal{L}$ does not depend on the velocity $\partial a_0/\partial x^0$, so that the related momentum vanishes identically. Obviously, this leads us to introduce

$$\varphi_1 \equiv \pi^0 \approx 0 \tag{66}$$

as a "primary constraint." Note that the symbol $\approx$ must be understood as "weak equality"; i.e., the constraints cannot be assumed to equal zero until the Poisson bracket between two arbitrary functionals $\hat{\mathcal{Q}}$ and $\mathcal{Q}$ of the field variables (a_μ, π_μ) is calculated:

$$\{\hat{\mathcal{Q}}, \mathcal{Q}\} = \int d^3x \left[\frac{\delta \hat{\mathcal{Q}}}{\delta a_\mu(\boldsymbol{x})} \frac{\delta \mathcal{Q}}{\delta \pi^\mu(\boldsymbol{x})} - \frac{\delta \hat{\mathcal{Q}}}{\delta \pi_\mu(\boldsymbol{x})} \frac{\delta \mathcal{Q}}{\delta a^\mu(\boldsymbol{x})}\right]. \tag{67}$$

According to the Dirac algorithm, φ_1 must be implemented within the canonical Hamiltonian

$$\mathcal{H}_C \equiv \mathcal{P}^0 = \int d^3x \left(\boldsymbol{\pi} \frac{\partial \boldsymbol{a}}{\partial x^0} - \mathcal{L}\right), \tag{68}$$

by means of a Lagrangian multiplier $\mathcal{C}$, so that the "total" Hamiltonian turns out to be

$$\mathcal{H} = \mathcal{P}^0 + \int d^3x \mathcal{C}(x) \pi^0(x). \tag{69}$$

In this context, the equation of motion for $\hat{\mathcal{Q}}$ reads

$$\frac{d\hat{\mathcal{Q}}}{dx^0} = \frac{\partial \hat{\mathcal{Q}}}{\partial x^0} + \{\hat{\mathcal{Q}}, \mathcal{H}\}, \tag{70}$$

and, in particular, for $\hat{\mathcal{Q}} = \pi_0$, one has

$$\frac{d\pi^0}{dx^0} = \{\pi^0, \mathcal{H}\} = \boldsymbol{\nabla} \cdot \boldsymbol{\pi}. \tag{71}$$

The constraint φ_1 should hold at all times. In consequence, Gauss's law

$$\varphi_2 \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \approx 0, \tag{72}$$

which is one of the field equations, $j^0 = \nabla \cdot \boldsymbol{d} = 0$, arises as a "secondary constraint." However, the latter is already present in $\mathcal{H}_C = \mathcal{P}^0$ in the form $a_0 \boldsymbol{\nabla} \cdot \boldsymbol{\pi}$ [see Eq. (65)]. Therefore, a_0 can be considered as a Lagrange multiplier and thus an arbitrary function of x. Its equation of motion implies that

$$\frac{da_0}{dx^0} = \{a_0, \mathcal{H}\} = \mathcal{C}(x). \tag{73}$$

The string of constraints stops here because Gauss's law commutes with the Hamiltonian. Moreover, our primary and secondary constraint are "first class" with a vanishing Poisson bracket:

$$\{\pi_0(x), \boldsymbol{\nabla} \cdot \boldsymbol{\pi}(x')\} = 0. \tag{74}$$

The remaining algebra of the constraints and the Hamiltonian is given by

$$\begin{aligned} \{\pi_0(x), \pi_0(x')\} = 0, &\qquad \{\boldsymbol{\nabla} \cdot \boldsymbol{\pi}(x), \boldsymbol{\nabla} \cdot \boldsymbol{\pi}(x')\} = 0, \\ \{\pi_0, \mathcal{H}_C\} = -\boldsymbol{\nabla} \cdot \boldsymbol{\pi}, &\qquad \{\boldsymbol{\nabla} \cdot \boldsymbol{\pi}, \mathcal{H}_C\} = 0. \end{aligned} \tag{75}$$

[2] Remember that we restrict ourselves to the low-frequency, low-momentum limit in the present subsection and in the rest of the article.

Observe that the Lagrangian multipliers $\mathcal{C}$ and a_0 transfer an arbitrariness to the Hamiltonian [Eq. (69)]. As a consequence, we are forced to deal with a phase space plagued by nonphysical degrees of freedom. This problem is closely associated with the gauge invariance property and is formally removed by imposing two gauge-fixing conditions. Because of this fact, the two existing multipliers are rendered to precise dependences on the physical fields, and can eventually be removed from the theory.

A suitable set of gauge conditions can be found by solving Gauss's law [Eq. (72)] with respect to a_0:

$$a_0 = \frac{1}{\nabla\varepsilon\nabla}\partial_0(\nabla_i\varepsilon_{ij}a_j), \tag{76}$$

where $\nabla\varepsilon\nabla = \nabla_i\varepsilon_{ij}\nabla_j$. Guided by this result, we are led to choose a generalized version of the Coulomb gauge as the third constraint of the theory:

$$\varphi_3 \equiv \nabla_i\varepsilon_{ij}a_j \approx 0. \tag{77}$$

The consistency consequence of this gauge condition can be found by Poisson-commuting the Hamiltonian [Eq. (69)] with φ_3. However, it can be read off directly from Eq. (76) and promotes the last constraint,

$$\varphi_4 \equiv a_0 \approx 0. \tag{78}$$

Since this is found to be stationary as well, Eq. (73) provides a vanishing value of the Lagrangian multiplier $\mathcal{C}$, and there are no further constraints.

The accessibility of these gauge conditions can be checked by using the gauge function

$$\Lambda = -\frac{1}{\nabla\varepsilon\nabla}\nabla_i\varepsilon_{ij}a_j. \tag{79}$$

In fact, for any value of $\boldsymbol{a}$, a^0, the gauge-transformed fields

$$a_i'(x) = a_i(x) + \nabla_i\Lambda, \qquad a_0'(x) = a_0(x) + \partial_0\Lambda, \tag{80}$$

obey the gauge conditions of Eqs. (77) and (78). Explicitly,[3]

$$\begin{aligned}\nabla_i\varepsilon_{ij}a_j' &= \nabla_i\varepsilon_{ij}a_j + \nabla_i\varepsilon_{ij}\nabla_j\Lambda = 0,\\ \boldsymbol{\nabla}\cdot\boldsymbol{\pi}' &= -\nabla_i\varepsilon_{ij}\nabla_j a_0' = \nabla_i\varepsilon_{ij}\nabla_j a_0 + \nabla_i\varepsilon_{ij}\nabla_j\partial_0\Lambda = 0.\end{aligned} \tag{81}$$

Note, in addition, that the constraints $\{\varphi_i\}$ defined above restrict the original phase space of the theory to a four-dimensional hypersurface,

$$\boldsymbol{\Omega} \equiv \{(a_\mu, \pi_\mu)|\varphi_i \approx 0, i = 1\dots 4\}, \tag{82}$$

in which the time evolution of two physical degrees of freedom takes place.

Certainly the set $\{\varphi_i\}$ is second class, with a characteristic matrix $C_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}') \equiv \{\varphi_\alpha(\boldsymbol{x}), \varphi_\beta(\boldsymbol{x}')\}$ given by

$$C_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & \nabla^x\varepsilon\nabla^x & 0\\ 0 & -\nabla^x\varepsilon\nabla^x & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'). \tag{83}$$

Since $C_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}')$ is regular by construction, we can also write down its inverse:

$$C^{-1}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & -\frac{1}{\nabla^x\varepsilon\nabla^x} & 0\\ 0 & \frac{1}{\nabla^x\varepsilon\nabla^x} & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'). \tag{84}$$

With the help of the latter, we introduce the Dirac bracket:

$$\begin{aligned}\{\hat{\mathcal{Q}}(\boldsymbol{x}), \mathcal{Q}(\boldsymbol{x}')\}_* = \{\hat{\mathcal{Q}}(\boldsymbol{x}), \mathcal{Q}(\boldsymbol{x}')\} - \int d^3y\{\hat{\mathcal{Q}}(\boldsymbol{x}), \varphi_\alpha(\boldsymbol{y})\}\\ \times\int d^3z C^{-1}_{\alpha\beta}(\boldsymbol{y}, z)\{\varphi_\beta(z), \mathcal{Q}(\boldsymbol{x}')\}.\end{aligned} \tag{85}$$

In this context, the fundamental bracket of the theory can be calculated straightforwardly and reads

$$\{a_i(\boldsymbol{x}), \pi_j(\boldsymbol{x}')\}_* = t_{ij}(\boldsymbol{x}, \boldsymbol{x}'), \tag{86}$$

where

$$\boldsymbol{t}_{ij}(\boldsymbol{x},\boldsymbol{x}') \equiv \left[\delta_{ij} - \nabla_i^x\frac{1}{\nabla\varepsilon\nabla}\varepsilon_{jk}\nabla_k^x\right]\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') \tag{87}$$

is a projector-valued distribution which fulfills the relation

$$\int d^3y\,\boldsymbol{t}_{il}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{t}_{lj}(\boldsymbol{y},\boldsymbol{x}') = \boldsymbol{t}_{ij}(\boldsymbol{x},\boldsymbol{x}'). \tag{88}$$

This, however, is not symmetric [$\boldsymbol{t}_{ij}(\boldsymbol{x},\boldsymbol{x}') \neq \boldsymbol{t}_{ji}(\boldsymbol{x},\boldsymbol{x}')$][4] and, depending on the field to be projected, one must contract one index or the other. For instance, let us decompose the canonical field into two mutually orthogonal pieces: $\boldsymbol{a}(\boldsymbol{x},t) = \boldsymbol{a}^\tau(\boldsymbol{x},t) + \boldsymbol{a}^\ell(\boldsymbol{x},t)$. Here the transversal component is obtained by contracting the field $\boldsymbol{a}(\boldsymbol{x},\boldsymbol{t})$ with the index of $\boldsymbol{t}_{ij}(\boldsymbol{x},\boldsymbol{x}')$, which is provided by the optical tensor ε_{lj}. Thus,

$$\begin{aligned}a_i^\tau(\boldsymbol{x},t) &= \int d^3x'\boldsymbol{t}_{ij}(\boldsymbol{x},\boldsymbol{x}')a_j(\boldsymbol{x}',t)\\ &= \left[\delta_{ij} - \nabla_i\frac{1}{\nabla\varepsilon\nabla}\nabla_k\varepsilon_{kj}\right]a_j(\boldsymbol{x},t).\end{aligned} \tag{89}$$

[3]It is worth observing at this point that the electric field associated with the small-amplitude waves $\boldsymbol{e} = \boldsymbol{\nabla}a_0 - \partial_0\boldsymbol{a}$ is a gauge-invariant quantity. As a consequence, the canonical momenta $\pi_i = -\varepsilon_{ij}e_j$ are invariant as well. So, under the gauge transformation of Eq. (80), $\boldsymbol{\pi}' = \boldsymbol{\pi}$.

[4]For a vanishing external field, this reduces to the symmetric transversal projector associated with the Coulomb gauge $\boldsymbol{t}^{\mathrm{Coul}}_{ij}(\boldsymbol{x},\boldsymbol{x}') = [\delta_{ij} - \nabla_i\nabla_j/\boldsymbol{\nabla}^2]\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$.

Accordingly, the longitudinal components of $\boldsymbol{a}$ turn out to be $a_i^\ell(\boldsymbol{x}, t) = \int d^3x' l_{ij}(\boldsymbol{x}, \boldsymbol{x}')a_j(\boldsymbol{x}', t)$, where

$$l_{ij}(\boldsymbol{x}, \boldsymbol{x}') \equiv \nabla_i^x \frac{1}{\nabla\varepsilon\nabla} \nabla_k^x \varepsilon_{kj}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \tag{90}$$

is the nonsymmetric longitudinal projector. Likewise, we decompose $\boldsymbol{\pi} = \boldsymbol{\pi}^\tau + \boldsymbol{\pi}^\ell$. However, to extract its transversal and longitudinal elements, the canonical momentum must be contracted with the index provided by the gradient operator:

$$\begin{aligned}\pi_j^\tau(\boldsymbol{x}, t) &= \int d^3x' \boldsymbol{t}_{ij}(\boldsymbol{x}, \boldsymbol{x}')\pi_i(\boldsymbol{x}', t) \\ &= \left[\delta_{ij} - \nabla_i^x \frac{1}{\nabla\varepsilon\nabla}\nabla_k^x \varepsilon_{kj}\right]\pi_i(\boldsymbol{x}, t).\end{aligned} \tag{91}$$

Due to the gauge-fixing condition [Eq. (77)] and Gauss's law [Eq. (72)], the longitudinal components $a_i^\ell = \int d^3x' l_{ij}(\boldsymbol{x}, \boldsymbol{x}')a_j(\boldsymbol{x}', t)$ and $\pi_j^\ell(\boldsymbol{x}, t) = \int d^3x' l_{ij}(\boldsymbol{x}, \boldsymbol{x}') \pi_i(\boldsymbol{x}', t)$ vanish identically. Keeping these details in mind, the fundamental Dirac bracket of our system [Eq. (86)] acquires the following structure:

$$\{a_i^\tau(\boldsymbol{x}), \pi_j^\tau(\boldsymbol{x}')\}_* = \boldsymbol{t}_{ij}(\boldsymbol{x}, \boldsymbol{x}'). \tag{92}$$

On the other hand, the Dirac bracket of a_0 or π_0 with an arbitrary functional $\hat{\mathcal{Q}}$ vanishes identically by construction:

$$\{\hat{\mathcal{Q}}(\boldsymbol{x}), a_0(\boldsymbol{x}')\}_* = 0, \qquad \{\hat{\mathcal{Q}}(\boldsymbol{x}), \pi_0(\boldsymbol{x}')\}_* = 0. \tag{93}$$

Because of this, the sector (a_0, π_0) can be formally eliminated from the phase space, and the theory is fully described in terms of $(\boldsymbol{a}^\tau, \boldsymbol{\pi}^\tau)$. Moreover, with $\hat{\mathcal{Q}}$ being a generic functional, one has

$$\{\hat{\mathcal{Q}}(x), \overbrace{\boldsymbol{\nabla}\cdot\boldsymbol{\pi}(\boldsymbol{x}')}^{\varphi_2}\}_* = \{\hat{\mathcal{Q}}(\boldsymbol{x}), \overbrace{\nabla_i\varepsilon_{ij}a_j(\boldsymbol{x}')}^{\varphi_3}\}_* = 0. \tag{94}$$

Thus, both the second and third constraints [Eqs. (72) and (77)] are no longer "weak" equalities, and instead they can be used as "strong" equations. The latter terminology is conceptually equivalent to replacing the symbol $\approx$ by $=$ in the set of constraints $\{\varphi_i\}$. So, once the Dirac brackets [Eq. (85)] are constructed, they can be set to zero everywhere.

Making use of Eq. (85) and Eqs. (88)–(91), we find

$$\begin{aligned}&\{\mathcal{P}^0, a_i(x)\}_* = \partial^0 a_i(x), \qquad \{\mathcal{P}^0, \pi_i(x)\}_* = \partial^0\pi_i(x), \\ &\{\boldsymbol{\mathcal{P}}, a_i(x)\}_* = \boldsymbol{\nabla}a_i(x), \qquad \{\boldsymbol{\mathcal{P}}, \pi_i(x)\}_* = \boldsymbol{\nabla}\pi_i(x),\end{aligned}$$

which are the well-known time and spatial transformation properties of fields. It follows that for any polynomial functional $\hat{\mathcal{Q}}$ of $\boldsymbol{a}$ and $\boldsymbol{\pi}$ that does not depend explicitly on x, one has

$$\{\mathcal{P}^\mu, \hat{\mathcal{Q}}(x)\}_* = \partial^\mu\hat{\mathcal{Q}}(x). \tag{95}$$

To conclude this subsection, we derive the modified Maxwell equations. Whatever the nature of the electromagnetic background ($\mathfrak{F} > 0$ or $\mathfrak{F} < 0$), the Hamiltonian equation of motion for $\boldsymbol{\pi}$ becomes Ampere's law:

$$\frac{d\boldsymbol{\pi}}{dx^0} = \{\boldsymbol{\pi}, \mathcal{P}^0\}_* = \boldsymbol{\nabla}\times\frac{\delta\mathcal{P}^0}{\delta\boldsymbol{b}} = -\boldsymbol{\nabla}\times\boldsymbol{h}. \tag{96}$$

Together with the constraint $\varphi_2 = 0$ (Gauss's law), they make the second pair of Maxwell equations. The Hamiltonian equation for $\boldsymbol{b}$,

$$\frac{d\boldsymbol{b}}{dx^0} = \{\boldsymbol{b}, \mathcal{P}^0\}_* = \boldsymbol{\nabla}\times\frac{\delta\mathcal{P}^0}{\delta\boldsymbol{\pi}} = -\boldsymbol{\nabla}\times\boldsymbol{e}, \tag{97}$$

where $\boldsymbol{e} = -\partial_0\boldsymbol{a}$, becomes Faraday's equation. It is fulfilled as an identity. Together with another identity, Gauss's law for magnetism, $\boldsymbol{\nabla}\cdot\boldsymbol{b} = 0$, which is not a Hamilton equation of motion, they make the first pair of Maxwell equations.

We stress that the procedure developed in this subsection is also applicable to any other linear approximation of electrodynamics in which the optical tensors depend on neither the space-time coordinates nor the derivates with respect to the latter. Observe that it is even suitable to describe the situation in which there exists a certain biaxiality associated with the crossed fields configuration; i.e., where the external field invariants vanish identically, $\mathfrak{F} = \mathfrak{G} = 0$.

C. Generators of rotations and Lorentz transformations

In this subsection, we first obtain the conserved generators associated with the Amputated Lorentz Group, $SO_A(3, 1)$. The Noether theorem for infinitesimal transformations from the $SO_A(3, 1)$ group over the field $a^\mu(\mathrm{x})$ that leave the action Γ in Eq. (2) invariant reads

$$\partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu a^\nu)}\delta a^\nu + T^\mu{}_\nu\delta x^\nu\right\} = 0, \tag{98}$$

where $\mathcal{L}$ is the quadratic Lagrangian of Eq. (17), $T^\mu{}_\nu$ is the canonical stress-energy tensor of Eq. (44), and the transformation laws are

$$\begin{aligned}\delta a^\rho(x) &= \frac{i}{2}\omega^{\alpha\beta}(\mathfrak{J}_{\alpha\beta})^\rho{}_\sigma a^\sigma(x), \\ \delta x^\rho &= \frac{i}{2}\omega^{\alpha\beta}(\mathfrak{J}_{\alpha\beta})^\rho{}_\sigma x^\sigma,\end{aligned} \tag{99}$$

with $\omega^{\alpha\beta}$ determined by Eq. (27) and with the vectorial representation of the Lie algebra generator of $SO(3, 1)$,

$$(\mathfrak{J}_{\alpha\beta})^\rho{}_\sigma = i(\delta^\rho{}_\beta\eta_{\alpha\sigma} - \delta^\rho{}_\alpha\eta_{\beta\sigma}). \tag{100}$$

With these details in mind, the Noether conservation equation [Eq. (98)] may be written as

$$\partial_\mu\left[\frac{1}{2}\vartheta\mathcal{F}_{\alpha\beta}J^{\mu\alpha\beta}+\frac{1}{2}\xi\tilde{\mathcal{F}}_{\alpha\beta}J^{\mu\alpha\beta}\right]=0. \quad (101)$$

Correspondingly, the conserved currents associated with the $SO_A(3,1)$ symmetry are given by

$$J^\mu=\frac{1}{2}\mathcal{F}_{\alpha\beta}J^{\mu\alpha\beta} \quad \text{and} \quad \tilde{J}^\mu=\frac{1}{2}\tilde{\mathcal{F}}_{\alpha\beta}J^{\mu\alpha\beta}. \quad (102)$$

Their respective continuity equations read

$$\partial_\mu J^\mu=\frac{1}{2}\mathcal{F}_{\alpha\beta}\partial_\mu J^{\mu\alpha\beta}=0, \quad (103)$$

$$\partial_\mu \tilde{J}^\mu=\frac{1}{2}\tilde{F}_{\alpha\beta}\partial_\mu J^{\mu\alpha\beta}=0. \quad (104)$$

To provide the fulfillment of Eqs. (101), (103), and (104), it is sufficient to define the generally nonconserved current $J^{\mu\alpha\beta}$ in the form

$$J^{\mu\alpha\beta}=-\left[T^{\mu\nu}x^\sigma+\frac{\partial\mathcal{L}}{\partial(\partial_\mu a_\nu)}a^\sigma\right]i(\mathfrak{J}^{\alpha\beta})_{\nu\sigma}, \quad (105)$$

which imitates the elements associated with $SO(3,1)$ invariance. For further convenience, we express $J^{\mu\alpha\beta}$ in terms of $\Theta^{\mu\nu}$ [Eq. (50)]. To this end, we substitute Eq. (48) and make use of the identity

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu a^\nu)}a^\sigma=\partial_\lambda\left[x_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu a_\lambda)}a^\sigma\right]-x_\nu\partial_\lambda\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu a_\lambda)}a^\sigma\right].$$

Consequently,

$$J^{\mu\alpha\beta}=-i\Theta^{\mu\nu}x^\sigma(\mathfrak{J}^{\alpha\beta})_{\nu\sigma}=x^\alpha\Theta^{\mu\beta}-x^\beta\Theta^{\mu\alpha}. \quad (106)$$

Observe that the nonconservation of $J^{\mu\alpha\beta}$ is intrinsically related to the nonsymmetric feature of the energy momentum tensor [Eq. (50)]. In fact,

$$\partial_\mu J^{\mu\alpha\beta}=\Theta^{\alpha\beta}-\Theta^{\beta\alpha}. \quad (107)$$

Since the continuity equations [Eqs. (103) and (104)] involve the projection of Eq. (107) onto the external field tensors, we end up with the following identities:

$$\mathcal{F}_{\alpha\beta}\Theta^{\alpha\beta}=0 \quad \text{and} \quad \tilde{\mathcal{F}}_{\alpha\beta}\Theta^{\alpha\beta}=0. \quad (108)$$

Now, the respective spatial integrals of the time components of the currents contained in Eq. (102) provide the conserved charges

$$\mathcal{G}=\frac{1}{2}\mathcal{F}_{\mu\nu}\mathcal{J}^{\mu\nu} \quad \text{and} \quad \tilde{\mathcal{G}}=\frac{1}{2}\tilde{\mathcal{F}}_{\mu\nu}\mathcal{J}^{\mu\nu}, \quad (109)$$

with

$$\frac{\partial\mathcal{G}}{\partial x^0}=0 \quad \text{and} \quad \frac{\partial\tilde{\mathcal{G}}}{\partial x^0}=0. \quad (110)$$

These scalars involve a second-rank tensor whose structure extends the known representation of the Lorentz generator

$$\mathcal{J}^{\mu\nu}\equiv\int d^3xJ^{0\mu\nu}(\boldsymbol{x},t)=\int d^3x(x^\mu\Theta^{0\nu}-x^\nu\Theta^{0\mu}), \quad (111)$$

to the case of the violated Lorentz invariance under consideration. Correspondingly, we can define both the photon angular momentum $\boldsymbol{\mathcal{J}}=(\mathcal{J}^{23},\mathcal{J}^{31},\mathcal{J}^{12})$ and the photon boost generator $\boldsymbol{\mathcal{K}}=(\mathcal{J}^{10},\mathcal{J}^{20},\mathcal{J}^{30})$.

Considering the prescription above, we can express the conserved charges of $SO_A(3,1)$ in the most general case of a constant and homogeneous external field as

$$G=\boldsymbol{B}\cdot\boldsymbol{\mathcal{J}}-\boldsymbol{E}\cdot\boldsymbol{\mathcal{K}}, \qquad \tilde{\mathcal{G}}=\boldsymbol{B}\cdot\boldsymbol{\mathcal{K}}+\boldsymbol{E}\cdot\boldsymbol{\mathcal{J}}. \quad (112)$$

Thus, as soon as the effects of the vacuum polarization tensor are considered, the number of Lorentz generators is reduced from 6 to 2. Therefore, the vacuum symmetry group $ISO_A(3,1)$ has dimension 2.

The explicit structure of $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{K}}$ in terms of the canonical fields $(\boldsymbol{a},\boldsymbol{\pi})$ follows from Eqs. (111) and (47), and reads

$$\boldsymbol{\mathcal{J}}=-\int d^3x\{\boldsymbol{x}\times[\boldsymbol{\pi}\times\boldsymbol{b}]-(\boldsymbol{x}\times\boldsymbol{a})(\boldsymbol{\nabla}\cdot\boldsymbol{\pi})\}, \quad (113)$$

$$\boldsymbol{\mathcal{K}}=-x^0\boldsymbol{\mathcal{P}}+\int d^3x\left\{\boldsymbol{x}\left(\frac{1}{2}\pi_i\varepsilon_{ij}^{-1}\pi_j+\frac{1}{2}b_i\mu_{ij}^{-1}b_ja_0\boldsymbol{\nabla}\cdot\boldsymbol{\pi}\right)\right\}, \quad (114)$$

where $\boldsymbol{b}=\boldsymbol{\nabla}\times\boldsymbol{a}$. Note how the secondary constraint of our problem is implemented in both generators through terms proportional to $\sim\boldsymbol{\nabla}\cdot\boldsymbol{\pi}$. In the phase subspace defined in Eq. (82), such terms vanish identically, and the resulting expression of $\boldsymbol{\mathcal{J}}$ coincides with the angular momentum of light in an optical medium [33].

Now, using the definition of the Dirac bracket [Eq. (85)] and equipped with the "Lorentz-like" generators $\mathcal{J}_i$ and $\mathcal{K}_i$ in Eqs. (113) and (114), we are able to express

$$\{\mathcal{J}_i,a_j(\boldsymbol{x})\}_*=(\boldsymbol{x}\times\boldsymbol{\nabla})_ia_j(\boldsymbol{x})+\epsilon_{ijl}a_l(\boldsymbol{x})-\epsilon_{ikm}\frac{\nabla_j\varepsilon_{lk}\nabla_m}{\nabla\varepsilon\nabla}a_l(\boldsymbol{x}) - \epsilon_{ikl}\frac{\nabla_j\varepsilon_{km}\nabla_m}{\nabla\varepsilon\nabla}a_l(\boldsymbol{x}), \quad (115)$$

$$\{\mathcal{J}_i,\pi_j(\boldsymbol{x})\}_*=(\boldsymbol{x}\times\boldsymbol{\nabla})_i\pi_j(\boldsymbol{x})+\epsilon_{ijl}\pi_l(\boldsymbol{x}), \quad (116)$$

$$\{\mathcal{K}_i,a_j(\boldsymbol{x})\}_*=x^i\partial^0a_j(\boldsymbol{x})-x^0\nabla_ia_j(\boldsymbol{x}) - \frac{\nabla_j\varepsilon_{kl}\nabla_l}{\nabla\varepsilon\nabla}[x_i\partial^0a_k(\boldsymbol{x})], \quad (117)$$

$$\{\mathcal{K}_i,\pi_j(\boldsymbol{x})\}_*=-x^0\nabla_i\pi_j-\epsilon_{ijk}\mu_{km}^{-1}(\boldsymbol{\nabla}\times\boldsymbol{a})_m + x_i\epsilon_{jkl}\mu_{lm}^{-1}\boldsymbol{\nabla}_k(\boldsymbol{\nabla}\times\boldsymbol{x})_m. \quad (118)$$

These expressions deserve some comments. The first pair of brackets realizes the infinitesimal rotation on the canonical variables $\boldsymbol{a}$ and $\boldsymbol{\pi}$, respectively. Note that, in contrast to the momentum $\boldsymbol{\pi}$, the field $\boldsymbol{a}$ transforms as a vector up to terms associated with the gauge-fixing condition [Eq. (77)]. A similar statement applies to the boost transformation properties contained in Eqs. (117) and (118). We find it convenient to emphasize that all these brackets guarantee that the infinitesimal canonical transformations induced by the generating functions $\mathcal{J}_i$, $\mathcal{K}_i$ do not lead out from the constrained phase subspace of Eq. (82). Indeed,

$$\nabla_m \varepsilon_{mj}\{\mathcal{J}_i, a_j(\boldsymbol{x})\}_* = 0, \qquad \nabla_j\{\mathcal{J}_i, \pi_j(\boldsymbol{x})\}_* = 0, \tag{119}$$

$$\nabla_m \varepsilon_{mj}\{\mathcal{K}_i, a_j(\boldsymbol{x})\}_* = 0, \qquad \nabla_j\{\mathcal{K}_i, \pi_j(\boldsymbol{x})\}_* = 0. \tag{120}$$

While the first column realizes $\varphi_3 = 0$ [Eq. (77)], the second one verifies $\varphi_2 = 0$ [Eq. (72)]. It is worth observing at this point that in the limit in which the external field vanishes, $\varepsilon_{ij} = \mu_{ij}^{-1} = \delta_{ij}$, the gauge condition is reduced to the standard Coulomb gauge ($\boldsymbol{\nabla}\cdot\boldsymbol{a} = 0$). Due to this fact, the rotation transformation property of the gauge field [Eq. (115)] becomes similar to the one associated with the canonical momentum [Eq. (116)].

The transformation properties of the gauge field $\boldsymbol{a}(x)$ in Eqs. (115) and (117) are not very helpful as they stand. For computing more cumbersome brackets involving gauge-invariant quantities, it is more convenient to represent the $\boldsymbol{a}$-containing part of Eqs. (115)–(118) in terms of the magnetic field $\boldsymbol{h}$ and magnetic induction $\boldsymbol{b}$, respectively. To this end, we apply $\boldsymbol{\nabla}\times$ to Eqs. (115) and (117). Then

$$\{\mathcal{J}_i, b_j(\boldsymbol{x})\}_* = (\boldsymbol{x}\times\boldsymbol{\nabla})_i b_j(\boldsymbol{x}) + \epsilon_{ijk} b_k(\boldsymbol{x}), \tag{121}$$

$$\{\mathcal{J}_i, \pi_j(\boldsymbol{x})\}_* = (\boldsymbol{x}\times\boldsymbol{\nabla})_i \pi_j(\boldsymbol{x}) + \epsilon_{ijk} \pi_k(\boldsymbol{x}), \tag{122}$$

$$\{\mathcal{K}_i, b_j(\boldsymbol{x})\}_* = -\epsilon_{ijk} e_k + x_i(\boldsymbol{\nabla}\times\boldsymbol{e})_j - x^0\nabla_i b_j, \tag{123}$$

$$\{\mathcal{K}_i, \pi_j(\boldsymbol{x})\}_* = -\epsilon_{ijk} h_k + x_i(\boldsymbol{\nabla}\times\boldsymbol{h})_j - x^0\nabla_i \pi_j. \tag{124}$$

These brackets constitute the starting point for determining the effects induced by the vacuum polarization within Lorentz algebra. Note the remarkable symmetry under the interchange $\boldsymbol{\pi}\leftrightarrows\boldsymbol{b}$ of the first pair of these equations. The second pair, however, is invariant under the simultaneous replacement $\boldsymbol{\pi}\leftrightarrows\boldsymbol{b}$ and $\boldsymbol{e}\leftrightarrows\boldsymbol{h}$.

IV. DIRAC COMMUTATORS AND EQUATIONS OF MOTION FOR GENERATORS

The goal of this section is to determine the equations of motion of the generators of the Lorentz rotations and to establish the Dirac commutation relations distorting the Lie algebra of the Poincaré group. By contracting these relations with the field tensor and with its dual, the closed algebra of the vacuum symmetry subgroup is obtained. As with other algebraic relations, the vacuum invariance holds in the physical subspace of the phase space specialized by constraints, where the evolution of the physical degrees of freedom takes place.

A. Equations of motion for the angular momentum and for the boost generator

The vacuum polarization tensor provides an effective coupling between photons and the external field $\mathcal{F}$. In order to explore the role of this quantity within LSB, we first consider the equation of motion associated with the total angular momentum of the electromagnetic waves:

$$\frac{d\mathcal{J}}{dx^0} = \frac{\partial\mathcal{J}}{\partial x^0} + \{\mathcal{J}, \mathcal{P}^0\}_*. \tag{125}$$

The first term on the right-hand side vanishes identically, since $\mathcal{J}$ does not depend explicitly on time ($x^0 = t$), whereas the Dirac bracket provides the total torque exerted over the photon field. In our context (see details in Appendix A, Sec. A), this can be expressed as

$$\{\mathcal{J}, \mathcal{P}^0\}_* = \int d^3x[\boldsymbol{\pi}\times\boldsymbol{e} + \boldsymbol{h}\times\boldsymbol{b}]. \tag{126}$$

In the rotation-invariant case—say, a vacuum or isotropic material—when the dielectric permeability is a unit tensor, Eq. (126) disappears in correspondence with the momentum conservation. In this case, $\boldsymbol{\pi}\parallel\boldsymbol{e}$, $\boldsymbol{h}\parallel\boldsymbol{b}$, and their vector products are zero. As a consequence, the equation of motion for the photon angular momentum can be written in terms of the spatial part of the energy momentum tensor [Eq. (55)]:

$$\frac{d\mathcal{J}^i}{dx^0} = \frac{1}{2}\epsilon^{ijk}\int d^3x(\Theta^{jk} - \Theta^{kj}). \tag{127}$$

On the contrary, the equation of motion associated with the photon boost generator is given by

$$\frac{d\mathcal{K}}{dx^0} = \frac{\partial\mathcal{K}}{\partial x^0} + \{\mathcal{K}, \mathcal{P}^0\}_*. \tag{128}$$

Because of the explicit dependence of $\mathcal{K}$ on time [see Eq. (114)], the first term on the right-hand side contributes to the equation of motion $\partial\boldsymbol{\mathcal{K}}/\partial x^0 = -\boldsymbol{\mathcal{P}}$. The Dirac bracket involved in this expression is calculated in Appendix A, Sec. B and reads

$$\{\mathcal{K}, \mathcal{P}^0\}_* = \int d^3x\boldsymbol{e}\times\boldsymbol{h}. \tag{129}$$

Combining these details, Eq. (128) acquires the following structure:

$$\begin{aligned}\frac{d\mathcal{K}^i}{dx^0} &= \int d^3x[(\boldsymbol{e}\times\boldsymbol{h})^i + (\boldsymbol{\pi}\times\boldsymbol{b})^i]\\ &= \int d^3x[\Theta^{i0} - \Theta^{0i}],\end{aligned} \tag{130}$$

where Θ^{i0} and Θ^{0i} are the Poynting vector and the density of momentum, respectively.

Both Eqs. (127) and (129) can be embedded in a compact four-dimensional expression

$$\frac{d\mathcal{J}^{\mu\nu}}{dx^0} = \mathcal{T}^{\mu\nu}, \tag{131}$$

in which the antisymmetric tensor $\mathcal{T}^{\mu\nu}$ is connected to the stress-energy tensor by means of the following expression:

$$\mathcal{T}^{\mu\nu} \equiv \int d^3x \tau^{\mu\nu}(x), \qquad \tau^{\mu\nu} \equiv \Theta^{\mu\nu} - \Theta^{\nu\mu}. \tag{132}$$

The structure of Eq. (131) with Eq. (132) is somewhat expected: the spatial integration of Eq. (107) reproduces the same equation for the Lorentz-like generators up to a surface integral $\sim \oint_S d\sigma_i J^{i\alpha\beta}$, which vanishes identically when the rapid falloff of the canonical fields at spatial infinity ($S \to \infty$) is provided.

Observe that the projections of Eq. (131) onto the external field tensors are consistent with the conservation law of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ [Eq. (110)]. In fact, due to Eq. (108), we obtain that

$$\frac{1}{2}\mathcal{F}_{\mu\nu}\mathcal{T}^{\mu\nu} = 0 \quad \text{and} \quad \frac{1}{2}\tilde{F}_{\mu\nu}\mathcal{T}^{\mu\nu} = 0. \tag{133}$$

We shall see that these identities, together with the equation of motion of $\mathcal{J}^{\mu\nu}$, turn out to be very convenient to evaluate the vacuum polarization effects on "Poincaré-like" algebra.

B. Distorted Poincaré algebraic relations

Let us consider Eq. (95) with $\hat{\mathcal{Q}}$ replaced by $\Theta^{0\nu}$. After an integration over $\boldsymbol{x}$, we obtain

$$\{\mathcal{P}^\mu, \mathcal{P}^\nu\}_* = \int d^3x \partial^\mu \Theta^{0\nu}(\boldsymbol{x}, t). \tag{134}$$

Using the divergence theorem and assuming that the canonical fields $\boldsymbol{a}$ and $\boldsymbol{\pi}$ vanish sufficiently rapidly at infinity, we find

$$\{\mathcal{P}^i, \mathcal{P}^\nu\}_* = \int d^3x \partial^i \Theta^{0\nu}(\boldsymbol{x}, t) = 0. \tag{135}$$

Therefore,

$$\{\mathcal{P}^i, \mathcal{P}^j\}_* = 0 \quad \text{and} \quad \{\mathcal{P}^i, \mathcal{P}^0\}_* = 0. \tag{136}$$

Because of the antisymmetry of the Dirac bracket, we have that $\{\mathcal{P}^0, \mathcal{P}^0\}_*$ vanishes identically as well. Having these aspects in mind, we can write Eq. (134) as

$$\{\mathcal{P}^\mu, \mathcal{P}^\nu\}_* = 0. \tag{137}$$

We also apply Eq. (95) to the following case:

$$\begin{aligned}\{\mathcal{P}^\mu, J^{0\lambda\sigma}\}_* &= x^\lambda\{\mathcal{P}^\mu, \Theta^{0\sigma}\}_* - x^\sigma\{\mathcal{P}^\mu, \Theta^{0\lambda}\}_* \\ &= x^\lambda \partial^\mu \Theta^{0\sigma} - x^\sigma \partial^\mu \Theta^{0\lambda},\end{aligned} \tag{138}$$

where the integrand in Eq. (111) has been considered. Thanks to the identity $x^\alpha \partial^\beta \Theta^{0\tau} = \partial^\beta(x^\alpha \Theta^{0\tau}) - \eta^{\beta\alpha}\Theta^{0\tau}$, we are able to express Eq. (138) as

$$\{\mathcal{P}^\mu, J^{0\lambda\sigma}\}_* = \eta^{\mu\sigma}\Theta^{0\lambda} - \eta^{\mu\lambda}\Theta^{0\sigma} - \partial^\mu J^{0\lambda\sigma}. \tag{139}$$

Integrating the expressions above over $\boldsymbol{x}$, we find

$$\{\mathcal{P}^\mu, \mathcal{J}^{\lambda\alpha}\}_* = \eta^{\mu\sigma}\mathcal{P}^\lambda - \eta^{\mu\lambda}\mathcal{P}^\sigma - \int d^3x \partial^\mu J^{0\lambda\sigma}. \tag{140}$$

Because of the rapid vanishing of the field at infinity, we obtain that, for $\mu = i$, $\int d^3x \partial^i J^{0\lambda\sigma}$ vanishes identically. However, if $\mu = 0$, the last integral can be written as $\partial^0 \int d^3x J^{0\lambda\sigma} = \partial^0 \mathcal{J}^{\lambda\sigma}$. With these details in mind, and using Eq. (131), we end up with

$$\{\mathcal{P}^\mu, \mathcal{J}^{\lambda\sigma}\}_* = \eta^{\mu\sigma}\mathcal{P}^\lambda - \eta^{\mu\lambda}\mathcal{P}^\sigma - \eta^{\mu 0}\mathcal{T}^{\lambda\sigma}, \tag{141}$$

where $\mathcal{T}^{\lambda\sigma}$ is specified in Eq. (132). The bracket above reproduces the commutators associated with $ISO(3,1)$ Lie algebra up to a term manifesting LSB.

Our analysis on the Poincaré-like algebra is to be completed by deriving the Lorentz-like algebra, which includes $\{\mathcal{K}^i, \mathcal{J}^j\}_*$, $\{\mathcal{J}^i, \mathcal{J}^j\}_*$, and $\{\mathcal{K}^i, \mathcal{K}^j\}_*$. A detailed derivation of these Dirac brackets can be found in Appendix B. In particular, we have obtained that

$$\begin{aligned}&\{\mathcal{J}^i, \mathcal{J}^j\}_* = \epsilon^{ijk}\mathcal{J}^k, \\ &\{\mathcal{J}^i, \mathcal{K}^j\}_* = \epsilon^{ijk}\mathcal{K}^k + \frac{1}{2}\epsilon^{ilm}\int d^3x x^j \tau^{lm}, \\ &\{\mathcal{K}^i, \mathcal{K}^j\}_* = -\epsilon^{ijk}\mathcal{J}^k - \int d^3x(x^i\tau^{j0} - x^j\tau^{i0}),\end{aligned} \tag{142}$$

with τ^{ij} being the spatial part of the tensorial density $\tau^{\mu\nu}$ defined in Eq. (132). Naturally, the brackets above can be combined in a four-dimensional expression,

$$\begin{aligned}\{\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}\}_* &= \eta^{\mu\rho}\mathcal{J}^{\nu\sigma} - \eta^{\nu\rho}\mathcal{J}^{\mu\sigma} + \eta^{\sigma\mu}\mathcal{J}^{\rho\nu} - \eta^{\sigma\nu}\mathcal{J}^{\rho\mu} \\ &\quad - \int d^3x(x^\nu\eta^{\mu 0} - x^\mu\eta^{\nu 0})\tau^{\rho\sigma} \\ &\quad - \int d^3x(x^\rho\eta^{\sigma 0} - x^\sigma\eta^{\rho 0})\tau^{\mu\nu},\end{aligned} \tag{143}$$

where the right-hand side reproduces the standard well-known result of the Lorentz algebra up to terms that contain the antisymmetric part of the energy-momentum tensor. Note that in spite of its nonconservation, the spatial rotation generators retain the standard angular momentum algebra according to the first line in Eq. (142). This is a usual situation in classical and quantum mechanics with broken symmetries under transformations that do not touch the time variable. It cannot be the same with the whole of the Poincaré algebra, because the Hamiltonian itself is one of its generators, whereas its commutators with other members of the algebra that do not conserve cannot help being affected. That is why the violation of the algebra in the second and third lines of Eq. (142) is not unexpected. This result indicates, in addition, that the Lorentz invariance in pure Coulomb-gauge $\boldsymbol{\nabla}\cdot\boldsymbol{a} = 0$ Maxwell theory is established only when the Lorentz generators are conserved. We conclude our analysis by pointing out that the brackets in Eqs. (141) and (143) coincide with algebra obtained in Ref. [21] within the context of the free field quantization in a noncovariant gauge.

C. Algebra of the symmetry subgroup

In order to pursue our research, we proceed to contract Eq. (141) with $\frac{1}{2}\mathcal{F}_{\lambda\sigma}$ and $\frac{1}{2}\tilde{\mathcal{F}}_{\lambda\sigma}$. As a consequence,

$$\{\mathcal{G}, \mathcal{P}^{\mu}\}_{*} = \mathcal{F}^{\mu\nu}\mathcal{P}_{\nu} - \eta^{\mu 0}\frac{1}{2}\mathcal{F}_{\lambda\sigma}\mathcal{T}^{\lambda\sigma}, \qquad (144)$$

$$\{\tilde{\mathcal{G}}, \mathcal{P}^{\mu}\}_{*} = \tilde{\mathcal{F}}^{\mu\nu}\mathcal{P}_{\nu} - \eta^{\mu 0}\frac{1}{2}\tilde{\mathcal{F}}_{\lambda\sigma}\mathcal{T}^{\lambda\sigma}, \qquad (145)$$

where the definitions of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ have been used. According to Eq. (133), the last terms on the right-hand sides of these brackets vanish identically. Therefore,

$$\{\mathcal{G}, \mathcal{P}^{\mu}\}_{*} = \mathcal{F}^{\mu\nu}\mathcal{P}_{\nu}, \qquad \{\tilde{\mathcal{G}}, \mathcal{P}^{\mu}\}_{*} = \tilde{\mathcal{F}}^{\mu\nu}\mathcal{P}_{\nu}. \quad (146)$$

The remaining Dirac bracket involving $\mathcal{G}$ and $\tilde{\mathcal{G}}$ can be determined by projecting Eq. (143) twice onto the external field tensors. This procedure generates the following brackets:

$$\{\mathcal{G}, \mathcal{G}\}_{*} = \mathcal{F}_{\mu\lambda}\mathcal{F}_{\nu}{}^{\lambda}\mathcal{J}^{\mu\nu}, \qquad \{\tilde{\mathcal{G}}, \tilde{\mathcal{G}}\}_{*} = \tilde{\mathcal{F}}_{\mu\lambda}\tilde{\mathcal{F}}_{\nu}{}^{\lambda}\mathcal{J}^{\mu\nu},$$
$$\{\mathcal{G}, \tilde{\mathcal{G}}\}_{*} = \frac{1}{2}\int d^3x[\mathcal{F}_{\mu}{}^{0}x^{\mu}\tilde{\mathcal{F}}_{\rho\sigma}\tau^{\rho\sigma} + \tilde{\mathcal{F}}_{\mu}{}^{0}x^{\mu}\mathcal{F}_{\rho\sigma}\tau^{\rho\sigma}]. \qquad (147)$$

Both $\mathcal{F}_{\mu\lambda}\mathcal{F}_{\nu}{}^{\lambda}$ and $\tilde{\mathcal{F}}_{\mu\lambda}\tilde{\mathcal{F}}_{\nu}{}^{\lambda} = 2\mathfrak{F}\eta_{\mu\nu} + \mathcal{F}_{\mu\lambda}\mathcal{F}_{\nu}{}^{\lambda}$ are symmetric tensors. Hence, their contractions with the antisymmetric tensor $\mathcal{J}^{\mu\nu}$ vanish identically, and $\{\mathcal{G}, \mathcal{G}\}_{*} = \{\tilde{\mathcal{G}}, \tilde{\mathcal{G}}\}_{*} = 0$. The latter result is expected because it comes out from Eq. (67) that $\{\hat{\mathcal{Q}}, \hat{\mathcal{Q}}\} = 0$, with $\hat{\mathcal{Q}}$ being a generic function of the canonical variables. Furthermore, by considering Eq. (133), we can claim that the right-hand side of the last Dirac bracket vanishes identically as well.

We can then summarize the Lie algebra of $ISO_A(3,1)$ as follows:

$$\{\mathcal{P}^{\mu}, \mathcal{P}^{\nu}\}_{*} = 0, \qquad \{\mathcal{G}, \tilde{\mathcal{G}}\}_{*} = 0,$$
$$\{\tilde{\mathcal{G}}, \mathcal{P}^{\mu}\}_{*} = \tilde{\mathcal{F}}^{\mu\nu}\mathcal{P}_{\nu}, \qquad \{\mathcal{G}, \mathcal{P}^{\mu}\}_{*} = \mathcal{F}^{\mu\nu}\mathcal{P}_{\nu}, \qquad (148)$$

with the external field tensors $\mathcal{F}$ and $\tilde{\mathcal{F}}$ playing the roles of group structure constants. Certainly the translation generators induce an Abelian-invariant subalgebra which defines the nonsemisimple structure of $SO_A(3,1)$. Moreover, the Casimir invariants of our problem are given as

$$\mathcal{P}^2 = \mathcal{Z}_1 + \mathcal{Z}_2, \qquad \mathcal{Z}_1 = \frac{\mathcal{P}\tilde{\mathcal{F}}^2\mathcal{P}}{2\mathfrak{F}},$$
$$\text{and} \quad \mathcal{Z}_2 = -\frac{\mathcal{P}\mathcal{F}^2\mathcal{P}}{2\mathfrak{F}}. \qquad (149)$$

It is worth mentioning at this point that the scalars involved in Eq. (9) are, therefore, maps of the invariants above: $\mathcal{P}^2 \mapsto k^2$, $\mathcal{Z}_1 \mapsto z_1$, $\mathcal{Z} \mapsto z_2$. Moreover, in the special frame where the field is purely magnetic ($\mathfrak{F} > 0$) or purely electric ($\mathfrak{F} < 0$), Eq. (143) expands into

$$\{\mathcal{P}_x, \mathcal{P}_y\}_{*} = 0, \qquad \{\mathcal{P}_z, \mathcal{P}_0\}_{*} = 0,$$
$$\{\mathcal{J}_z, \mathcal{P}_x\}_{*} = \mathcal{P}_y, \qquad \{\mathcal{K}_z, \mathcal{P}_0\}_{*} = -\mathcal{P}_z,$$
$$\{\mathcal{J}_z, \mathcal{P}_y\}_{*} = -\mathcal{P}_x, \qquad \{\mathcal{K}_z, \mathcal{P}_z\}_{*} = -\mathcal{P}_0,$$

where Eq. (112) has been used. Each column in this set of commutators manifests a subalgebra: the first one corresponds to the two-dimensional Euclidean group $ISO(2)$, whereas the second one corresponds to the $(1+1)$-dimensional pseudo-Euclidean group $ISO(1,1)$. The latter groups are associated with the transverse and pseudoparallel planes with respect to the $\boldsymbol{B}(\boldsymbol{E})$ direction. Therefore, the symmetry subgroup $SO_A(3,1)$, down to which the Poincaré group is broken due to the presence of an external field, reduces in the reference frame—where that field is purely magnetic or electric—to the direct product of $ISO(2)$ and $ISO(1,1)$. Besides, we want to remark that as long as a photon propagates transverse to the external field, the square of the Pauli-Lubanski operator $w^{\mu} = 1/2\epsilon^{\mu\lambda\sigma\varrho}\mathcal{J}_{\lambda\sigma}\mathcal{P}_{\varrho}$ is no longer a Casimir invariant. This fact reflects the underlying difference between the vacuum in an external field $\mathcal{F}$ and the case of an empty space-time in which the $SO(3,1)$ symmetry is preserved and all particles are classified according to the spin and helicity representations encoded in the w^2 eigenvalues.

V. MAGNETIC MOMENT OF SMALL ELECTROMAGNETIC PERTURBATIONS OF THE VACUUM

In this section, we analyze some consequences associated with the equation of motion for the photon angular momentum [Eqs. (125) and (126)]:

$$\frac{d\mathcal{J}}{dx^0} = \mathcal{T}, \quad \text{with} \quad \mathcal{T} = \int d^3x(\boldsymbol{\pi}\times\boldsymbol{e} + \boldsymbol{h}\times\boldsymbol{b}). \qquad (150)$$

In the special case where the external field is magneticlike ($\mathfrak{F} > 0$), the explicit substitution of $\boldsymbol{\pi}$ and $\boldsymbol{h}$ [Eqs. (56) and (57)] into $\mathcal{T}$ [Eq. (150)] allows us to express the latter in a rather meaningful form:

$$\mathcal{T} = 2\mathcal{M}\times\boldsymbol{B},$$
$$\mathcal{M} = \frac{1}{2}\int d^3x\left\{\frac{\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}{\varepsilon_{\perp}\varepsilon_{\parallel}}(\boldsymbol{\pi}\cdot\boldsymbol{B})\boldsymbol{\pi} + \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}(\boldsymbol{b}\cdot\boldsymbol{B})\boldsymbol{b}\right\}, \qquad (151)$$

where Eq. (40) has been used. The expression above mimics the torque exerted by the external field on the magnetic dipole $\mathcal{M}$. The torque $\mathcal{T}$ [Eq. (151)] vanishes when projected onto $\boldsymbol{B}$, so Eq. (150) implies that the parallel component of the photon angular momentum, $\mathcal{J}_{\parallel}$, is a constant of motion. On the contrary, the projection (helicity) $\mathfrak{h} \sim \mathcal{J}\cdot\mathcal{P}$ of the angular momentum of a photon onto its canonical momentum [Eq. (65)][5] is not a conserved quantity unless $\mathcal{P}$ turns out to be parallel to the external field.

The magnetic moment $\mathcal{M}$ is a feature of the small electromagnetic perturbations of the vacuum (photons in the first place), when the former interact with it through virtual electron-positron pairs. This interaction makes the photon behave like a magnetic dipole. Note that $\mathcal{M}$ [Eq. (151)] is a gauge-invariant quantity, also orthogonal to the photon canonical momentum, $\mathcal{P}\cdot\mathcal{M}=0$. Moreover, it has two components in correspondence with the cylindrical symmetry imposed by the external magnetic field. One of them,

$$\mathcal{M}_\perp=\frac{1}{2}\int d^3x\left\{\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}\frac{\boldsymbol{\pi}\cdot\boldsymbol{B}}{\varepsilon_\perp\varepsilon_\parallel}\boldsymbol{\pi}_\perp+\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}(\boldsymbol{b}\cdot\boldsymbol{B})\boldsymbol{b}_\perp\right\}, \quad (152)$$

is perpendicular to $\boldsymbol{B}$; whereas the remaining one,

$$\mathcal{M}_\parallel=\frac{1}{2}\int d^3x\left\{\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}\frac{\boldsymbol{\pi}\cdot\boldsymbol{B}}{\varepsilon_\perp\varepsilon_\parallel}\boldsymbol{\pi}_\parallel+\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}(\boldsymbol{b}\cdot\boldsymbol{B})\boldsymbol{b}_\parallel\right\} \quad (153)$$

is parallel to the axis in which the external field lies, and therefore invariant under rotation about the magnetic field direction. This, however, does not contribute to $\mathcal{T}$ [Eq. (151)]. Hence, it does not play any role within the equation of motion of the photon angular momentum [Eq. (150)]. Nevertheless, $\mathcal{M}_\parallel$ contributes to the effective Hamiltonian [Eq. (65), $H\equiv\mathcal{P}^0$], and thus to the photon energy. Note that with the use of Eq. (40), the latter can be conveniently written as

$$H=\int d^3x\left(\frac{1}{2\varepsilon_\perp}\boldsymbol{\pi}^2+\frac{1}{2}\mu_\perp^{-1}\boldsymbol{b}^2\right)-\mathcal{M}\cdot\boldsymbol{B}. \quad (154)$$

In contrast to $\mathcal{M}_\parallel$, the perpendicular component of $\mathcal{M}$ neither remains invariant under a rotation around the $\boldsymbol{B}$ direction nor contributes to the photon energy, since it is projected out from the scalar product involved in Eq. (154). However, it turns out to be a clear manifestation of LSB, since it specifies, by means of Eq. (150), that not all components of the photon angular momentum are conserved quantities. Furthermore, it follows from Eqs. (40) and (153) that $\mathcal{M}_\parallel\geq 0$ behaves paramagnetically. By contrast, it is not possible to establish a definite magnetic behavior in $\mathcal{M}_\perp$, because it contains terms which mix not only $\pi_\parallel$ and $\pi_\perp$, but also $b_\perp$ and $b_\parallel$.

Let us consider the case in which the external magnetic field is asymptotically large, $\mathfrak{b}=|\boldsymbol{B}|/\mathrm{B_c}\to\infty$. In this limit, the basic entities contained in Eq. (154) [see Eq. (41)] are written in Ref. [28], referring to $\mathfrak{L}$ as the one-loop term of the Euler-Heisenberg Lagrangian [34–37]:

$$\mathfrak{L}_{\mathfrak{F}}\approx\frac{\alpha}{3\pi}\ln\mathfrak{b},\qquad 2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}\approx\frac{\alpha}{3\pi}\mathfrak{b},\qquad 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}\approx\frac{\alpha}{3\pi}. \quad (155)$$

Observe that for a magnetic field with $10<\mathfrak{b}\ll 3\pi/\alpha$, one can treat $\varepsilon_\perp=\mu_\perp^{-1}\sim 1$ and therefore $\boldsymbol{\pi}\sim-\boldsymbol{e}-\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}(\boldsymbol{e}\cdot\boldsymbol{B})\boldsymbol{B}$. The resulting effective Hamiltonian H reads

$$H\approx\int d^3x\left(\frac{1}{2}\boldsymbol{\pi}^2+\frac{1}{2}\boldsymbol{b}^2\right)-\mathcal{M}\cdot\boldsymbol{B}. \quad (156)$$

In this approximation, the second term in Eq. (151) decreases as $1/\mathfrak{b}$, and thus contributes in H as a small constant. Hence, it can be disregarded in comparison with the term provided by the first term of Eq. (151), which turns out to be a linear function on the external field strength. As a consequence, the magnetic dipole acquires the following structure:

$$\mathcal{M}\approx g\frac{\mathrm{e}}{2\mathrm{m}}\mathcal{S}, \quad (157)$$

with $g=\alpha/3\pi$ being a sort of Landé factor, whereas

$$\mathcal{S}=\frac{1}{\varepsilon_\parallel\mathrm{m}}\int_{\mathrm{V}}d^3x\,\pi_\parallel(\boldsymbol{x},x^0)\boldsymbol{\pi}(\boldsymbol{x},x^0)$$
$$\approx\frac{1}{\mathrm{m}}\int_{\mathrm{V}}d^3x\,e_\parallel(\boldsymbol{x},x^0)\boldsymbol{e}(\boldsymbol{x},x^0). \quad (158)$$

It is worth mentioning that $\mathcal{S}$ is only determined by the electric induction vector associated with the small electromagnetic waves. Thereby $\mathcal{M}$ can be interpreted as a magnetic moment, with $\mathcal{S}$ playing the role of the "spin" of the small electromagnetic waves. This terminology, however, is used just to establish an analogy with the case of the electron magnetic moment. In contrast to any massive particle, a photon lacks a rest frame, and thus one cannot define a spin for it. Besides, $\mathcal{S}$ does not fulfill the standard Dirac bracket of the angular momentum, i.e., the first bracket in Eq. (142).

We continue our research by considering the electric field of each eigenmode as a monochromatic plane wave, so that

$$\boldsymbol{e}^{(\lambda)}(\boldsymbol{x},x^0)=\mathcal{E}_0^{(\lambda)}\frac{\boldsymbol{e}^{(\lambda)}(\boldsymbol{k})}{|\boldsymbol{e}^{(\lambda)}(\boldsymbol{k})|}\cos[\omega_\lambda x^0-\boldsymbol{k}\cdot\boldsymbol{x}]. \quad (159)$$

Here $\mathcal{E}_0^{(\lambda)}$ and $\omega_\lambda(\boldsymbol{k})$ are the amplitude and frequency of mode λ, respectively. Note that the shape of $e^{(\lambda)}(\boldsymbol{k})$ can be found below Eq. (13). Observe, in addition, that only mode 2 has an electric field parallel to $\boldsymbol{B}$. As a consequence, $\mathcal{S}$ becomes physically relevant for the second polarization mode. Inserting the expression above into Eq. (158), we find

$$\mathcal{S}\simeq\int_{\mathrm{V}}d^3x\frac{k_\perp\mathcal{E}_0^{(2)2}}{\mathrm{m}|\boldsymbol{k}|}\boldsymbol{s}\cos^2[\omega_2x^0-\boldsymbol{k}\cdot\boldsymbol{x}], \quad (160)$$

where the leading term has been withheld so that $\omega^{(2)}\approx|\boldsymbol{k}|$ and $\boldsymbol{s}\equiv\boldsymbol{e}^{(2)}/|\boldsymbol{e}^{(2)}|\approx\boldsymbol{n}\times(\boldsymbol{n}_\parallel\times\boldsymbol{n}_\perp)$. Equipped with these approximations, the time average of $\mathcal{M}$ reads

[5] Recall that this is the direction of the momentum flux and the wave vector (see Appendix C), not of the energy flux, whose direction coincides with the Poynting vector and with that of the group velocity.

$$\langle \mathcal{M} \rangle = \wp \frac{\mathrm{e}}{2\mathrm{m}} \langle \mathcal{S} \rangle, \quad \text{with} \quad \langle \mathcal{S} \rangle \approx \frac{1}{2} \frac{\mathcal{E}_0^{(2)2} \mathrm{V} k_\perp}{|\boldsymbol{k}|\mathrm{m}} \boldsymbol{s}. \tag{161}$$

Here, V denotes the volume over which the integral contained in $\mathcal{S}$ is considered. It is worth observing at this point that $\mathcal{M}$ is proportional to the average energy associated with the second propagating mode: $\sim \frac{1}{2}\mathcal{E}_0^{(2)2}$.

We should also mention at this point that the notion of the photon magnetic moment was introduced in Ref. [23] and discussed in Refs. [24,25]. In those works, it was defined as contributing to the photon energy as a function of both its momentum and the external field strength. Moreover, it was conceptually analyzed in different energy regimes and magnetic field contexts of the photon dispersion curve. Only in Ref. [25] were the asymptotic conditions investigated in the present work considered. The connection between the respective photon magnetic moment and $\mathcal{M}$ in Eq. (161) can only be established under the second quantization of our problem, which requires us to substitute the Dirac brackets with standard commutator relations.[6] In such a case, the frequencies $\omega_2(\boldsymbol{k})$ and $\omega_3(\boldsymbol{k})$ are related to the respective photon energies, and in correspondence, the mode-2 photon density turns out to be $\mathcal{N} = \frac{1}{2}\mathcal{E}_0^{(2)2}/\omega_2$. Dividing $\langle \mathcal{M} \rangle$ by the total number of mode-2 photons contained in the volume V, we obtain

$$\boldsymbol{m} = \frac{\langle \mathcal{M} \rangle}{\mathcal{N}\mathrm{V}} = \wp \frac{\mathrm{e}}{2\mathrm{m}} f(k_\perp) \boldsymbol{s}, \tag{162}$$

where m is the electron mass and $f(k_\perp) = k_\perp/\mathrm{m}$ is a dimensionless form factor which guarantees the gauge invariance of the theory; i.e., $\boldsymbol{m}$ does not provide a photon rest mass $\boldsymbol{m}(k_\perp \to 0) \to 0$. Equation (162) coincides with the photon anomalous magnetic moment previously obtained by one of the authors in Ref. [25]. However, in that work, $\boldsymbol{m}$ was defined as the coefficient of $\boldsymbol{B}$ when the dispersion curve $\omega_2(\boldsymbol{k})$ is linearly approximated in terms of the external magnetic field.

VI. CORRESPONDENCES WITH THE GENERAL LORENTZ-VIOLATING ELECTRODYNAMICS

The present work is not concerned with all thinkable Lorentz violations associated with extensions beyond the standard model like Refs. [9–11]. Just the opposite; it is developed entirely within standard model and deals specifically with such Lorentz symmetry violations as are stimulated by a background electromagnetic field, mostly by a time- and space-independent magneticlike field ($\mathfrak{G} = 0$, $\mathfrak{F} > 0$) within quantum electrodynamics (QED).

[6]The second quantization of the small electromagnetic field in an external field becomes a necessary issue as far as one wishes to go beyond the purely electromagnetic sector by including not only virtual, as here, but also free charged particles. This task, however, is not the issue of the present work, although some needed building blocks are prepared in it.

Nevertheless, it makes sense to try to reduce the approaches of Refs. [9–11] and of related studies [henceforth referred to as General Lorentz-Violating Electrodynamics (GLVE)] to a common denominator with the results of many works dealing in a relativistic, covariant way with external fields, nontrivial metrics, and/or a medium, as was proposed by a referee of PRD. When treated in the framework of conventional physics, these are acting as Lorentz and $SO(3)$ invariance-violating agents, and therefore may supply special examples to serve as models for verifying general constructions in GLVE. A full analysis in this field would require a quite separate study. So as not to deviate too far from our principal theme, we are now only listing—for an external magneticlike field—the CPT-even $(\kappa_F)^{\kappa\mu\lambda\nu}$ coefficients, whose combinations are subject to measurements in various experiments intended for detecting Lorentz violations, as they follow from the general covariant decomposition of the polarization tensor in a nonlinear electrodynamics. We shall see that, contrary to the postulate accepted in GLVE, in our context this tensor is not double traceless, but quite the opposite; its trace is physically meaningful and associated with the diffractive properties of the anisotropic medium formed by the background field. Even if admitted, the case where the double trace would vanish identically could not introduce any modification to the distorted Poincaré algebra, i.e., Eqs. (141) and (143). The double tracelessness condition modifies the optical tensors of the theory, but the Poincaré-like generators keep their structure as long as they are expressed in terms of the canonical variables. Besides, the energy-momentum tensor remains nonsymmetric, also as in Ref. [9], a fact needed to save the nontrivial equation of motion of the angular momentum of light.

We also comment on how the sensitivity achieved in experiments aimed at detecting possible Lorentz violations in the vacuum might be confronted with measuring equivalent magnetic fields. We made sure that the accuracy available would not be sufficient to detect the Lorentz violation produced even by the hitherto strongest laboratory magnetic field in the vacuum. To detect an effect of Lorentz violation presumably inherent in the vacuum, which might be equivalent to the one stemming from QED with an external magnetic field on the order of the cosmic background (10^{-6} G), the experimentalist would need to achieve a sensitivity of 10^{-44} in experiments that might, besides, exclude the influence of any dielectric material involved in the experimental device. This surpasses the boldest prospects of sensitivity under present-time considerations by at least 30 orders of magnitude.

A. The components of the CPT-even tensor

The quadratic part of the effective Lagrangian $\mathfrak{L}$ [Eqs. (17), (18), and (43)] in an external magnetic or electric field can be expressed as

$$\mathfrak{L} = -\frac{1}{4}(\kappa_F)^{\kappa\lambda\mu\nu} f_{\kappa\lambda} f_{\mu\nu}. \quad (163)$$

In accordance with Eq. (2), this is the same as

$$\mathfrak{L} = \frac{1}{2} a^\mu \Pi_{\mu\nu} a^\nu.$$

Thanks to the antisymmetry of the field tensor $f_{\mu\nu}$, the coefficient tensor $(\kappa_F)^{\kappa\lambda\mu\nu}$ is not defined by Eq. (163) in a unique way: its parts, symmetric under the permutations within the first and second pair of indices, are left undetermined. For this reason, this tensor is to be understood as antisymmetric under these permutations:

$$(\kappa_F)^{\kappa\lambda\mu\nu} = -(\kappa_F)^{\lambda\kappa\mu\nu} = -(\kappa_F)^{\kappa\lambda\nu\mu}. \quad (164)$$

Analogously, only the part of the tensor $(\kappa_F)^{\kappa\lambda\mu\nu}$, which is symmetric under the permutation of the first and second pairs of indices, contributes to Eq. (163). For this reason, this tensor is also to be understood as symmetric under these permutations:

$$(\kappa_F)^{\kappa\lambda\mu\nu} = (\kappa_F)^{\mu\nu\kappa\lambda}. \quad (165)$$

By using the definition $f_{\mu\nu} = i(k_\mu a_\nu - k_\nu a_\mu)$ in Eq. (163), we see that the polarization tensor that has direct physical meaning is connected with combinations of tensor $(\kappa_F)^{\kappa\lambda\mu\nu}$ components in momentum space as

$$\Pi_{\mu\nu}(k) = \frac{1}{2} k_\kappa k_\lambda [(\kappa_F)^{\kappa\nu\lambda\mu} - (\kappa_F)^{\kappa\mu\nu\lambda} - (\kappa_F)^{\mu\kappa\lambda\nu} + (\kappa_F)^{\nu\kappa\mu\lambda}] = 2k_\kappa k_\lambda (\kappa_F)^{\lambda\mu\kappa\nu}. \quad (166)$$

The properties of Eqs. (164) and (165) provide that this polarization tensor will be symmetric, $\Pi_{\mu\nu} = \Pi_{\nu\mu}$, as it should be in the vacuum with a background field [27]; and transverse, $\Pi_{\mu\nu} k_\nu = 0$, as is prescribed by the gauge invariance. Using the relation

$$\tilde{\mathcal{F}}^{\alpha\beta} \tilde{\mathcal{F}}_\xi{}^\sigma + \mathcal{F}^{\alpha\beta}\mathcal{F}_\xi{}^\sigma = 2\mathfrak{F}(\eta^{\alpha\sigma}\delta^\beta{}_\xi - \eta^{\sigma\beta}\delta^\alpha{}_\xi) + \eta^{\alpha\sigma}\mathcal{F}^\beta{}_\lambda \mathcal{F}^\lambda{}_\xi - \delta^\alpha{}_\xi \mathcal{F}^{\beta\lambda}\mathcal{F}_\lambda{}^\sigma + \delta^\beta{}_\xi \mathcal{F}^{\alpha\lambda}\mathcal{F}_\lambda{}^\sigma - \eta^{\beta\sigma}\mathcal{F}^{\alpha\lambda}\mathcal{F}_{\lambda\xi},$$

in Eq. (43), the coefficients $(\kappa_F)^{\kappa\lambda\mu\nu}$ for the infrared limit in QED with a constant magnetic field may be chosen as

$$(\kappa_F)^{\kappa\lambda\mu\nu} = \frac{1}{2}(-\mathfrak{L}_\mathfrak{F} + 2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}})(\eta^{\kappa\mu}\eta^{\lambda\nu} - \eta^{\kappa\nu}\eta^{\lambda\mu}) - \frac{1}{2}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}(\eta^{\lambda\mu}\mathcal{F}^{\kappa\sigma}\mathcal{F}_\sigma{}^\nu - \eta^{\lambda\nu}\mathcal{F}^{\kappa\sigma}\mathcal{F}_\sigma{}^\mu + \eta^{\kappa\nu}\mathcal{F}^{\lambda\sigma}\mathcal{F}_\sigma{}^\mu - \eta^{\kappa\mu}\mathcal{F}^{\lambda\sigma}\mathcal{F}_\sigma{}^\nu) + \frac{1}{2}(\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} - \mathfrak{L}_{\mathfrak{F}\mathfrak{F}})\mathcal{F}^{\kappa\lambda}\mathcal{F}^{\mu\nu}. \quad (167)$$

The (anti)symmetry properties of Eqs. (164) and (165) are obeyed by Eq. (167). The Lorentz violation induced by the magnetic field is characterized by $6 - 1 = 5$ components of the antisymmetric external field tensor $\mathcal{F}_{\mu\nu}$ subjected to one condition $\mathfrak{G} = 0$, and by the three scalars $\mathfrak{L}_\mathfrak{F}$, $\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}$, $\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}$ determined by the dynamics of the interaction.

Beyond the infrared limit, Eqs. (17) and (18) imply that the extension of the tensor $(\kappa_F)^{\kappa\mu\lambda\nu}$ to include the infinite series of space-time derivatives $(\hat{\kappa}_F)^{\kappa\mu\lambda\nu} = \sum_n (\kappa_F)^{\kappa\mu\lambda\nu\alpha_1\ldots\alpha_n}\partial_1 \ldots \partial_n$ reduces to multiplications—in the momentum space—of its separate parts by the scalar functions $\frac{\varkappa_1}{k^2}$ $\frac{\varkappa_1-\varkappa_2}{k\mathcal{F}^2k}$, and $\frac{\varkappa_1-\varkappa_3}{k\mathcal{F}^2k}$ built of the polarization operator eigenvalues, or to equivalent action of the corresponding (nonlocal) integral operators in the coordinate space. This factorization feature is not a consequence of Eqs. (14) and (20), since these equations hold true already in a more general case of nonlocality [38]. On the contrary, it follows from the structure of the polarization operator in a magnetic field [Eq. (10)].

B. The double trace

The double trace of Eq. (167) turns out to be

$$(\kappa_F)^{\mu\nu}{}_{\mu\nu} = -6\mathfrak{L}_\mathfrak{F} + 2\mathfrak{F}(\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} - \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}). \quad (168)$$

This is, in general, a nonvanishing quantity. This statement can be verified, for instance, by considering the weak- field approximation of the Euler-Heisenberg Lagrangian. In QED, "weak" means small as compared to the Schwinger characteristic value, $B_c = \mathrm{m}^2/\mathrm{e} = 4.42 \times 10^{13}$ G provided $\mathfrak{F} > 0$. In this asymptotic regime, the field derivatives read [39]

$$\mathfrak{L}_\mathfrak{F} = \frac{2\alpha}{45\pi}\frac{B^2}{B_c^2}, \quad \mathfrak{L}_{\mathfrak{F}\mathfrak{F}} = \frac{4\alpha}{45\pi}\frac{1}{B_c^2}, \quad \mathfrak{L}_{\mathfrak{G}\mathfrak{G}} = \frac{7\alpha}{45\pi}\frac{1}{B_c^2}. \quad (169)$$

Then the double trace of Eq. (168) becomes

$$(\kappa_F)^{\mu\nu}{}_{\mu\nu} = -\frac{\alpha}{3\pi}\frac{B^2}{B_c^2}.$$

It is worth mentioning at this point that the double tracelessness condition is customarily taken in GLVE on the grounds that the trace may be absorbed into the field renormalization [9]. In QED, the (infinite) renormalization has been fulfilled at the stage of the one-loop calculations that underlie the QED expressions for $\mathfrak{L}$ and for its field derivatives, the polarization tensors. The standard renormalization procedure of QED relates only to the zero-background field limit, while the background field-dependent part is fixed and obeys the condition reflecting the correspondence principle: $\mathfrak{L}_\mathfrak{F} \to 0$ as $B \to 0$. This condition is respected by Eq. (169) and establishes the absence of radiative corrections to the Maxwell Lagrangian for small and steady fields. So, the Maxwell Lagrangian remains untouched in this limit, which is the physically necessary requirement. Therefore, no renormalization of the electromagnetic field additional to the one performed in the course of the infinite renormalization procedure is admitted. All the terms in

Eq. (168) are physically important, as they serve various components responsible for the vacuum refraction processes [Eq. (40)]. We shall see in Sec. VI D that this situation is retained even in the nonbirefringent case.

C. Magnetoelectric coefficients

Let us introduce, as is customary, the matrix combinations $(\kappa_{DE})^{ij} = -2(\kappa_F)^{0i0j}$, $(\kappa_{HB})^{ij} = \frac{1}{2}\epsilon^{ikl}\epsilon^{jpq}(\kappa_F)^{klpq}$, and $(\kappa_{DB})^{ij} = -(\kappa_{HE})^{ji} = \epsilon^{kpq}(\kappa_F)^{0jpq}$. In terms of these quantities, the most general form of the quadratic Lagrangian is [10]

$$\mathcal{L} = \frac{1}{2}e_i[\delta^{ij} + (\kappa_{DE})^{ij}]e_j - \frac{1}{2}b_i[\delta^{ij} + (\kappa_{HB})^{ij}]b_j + e_i(\kappa_{DB})^{ij}b_j. \tag{170}$$

We wish to specialize this expression to the case where a magneticlike background ($\mathfrak{F} > 0$, $\mathfrak{G} = 0$) induces LSB. To this end, we compare Eq. (170) with Eq. (14). As a consequence, the following relations are established:

$$\begin{aligned} \varepsilon_{ij} &= \delta_{ij} + (\kappa_{DE})_{ij}, \quad \mu_{ij}^{-1} = \delta_{ij} + (\kappa_{HB})_{ij}, \\ (\kappa_{DE})_{ij} &= -\mathfrak{L}_{\mathfrak{F}}\delta_{ij} + \mathfrak{L}_{\mathfrak{G}\mathfrak{G}}B^iB^j, \\ (\kappa_{HB})_{ij} &= -\mathfrak{L}_{\mathfrak{F}}\delta_{ij} - \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}B^iB^j. \end{aligned} \tag{171}$$

The derivation of these relations requires the use of the corresponding optical tensors given in Eq. (39) and is in agreement with Eq. (167). The remaining matrices, i.e., $(\kappa_{DB})^{ij}$ and $(\kappa_{HE})^{ji}$, are responsible for the magnetoelectric effect, which is the magnetic linear response to an applied electric field and, reciprocally, the electric linear response to an applied magnetic field. These matrices vanish identically in our framework. For the magnetoelectric effect, and hence for the matrices $(\kappa_{DB})^{ij}$ and $(\kappa_{HE})^{ji}$, to exist, it is necessary to admit [5] the nonvanishing of the pseudoscalar invariant of the external field, $\mathfrak{G} \neq 0$, i.e., to take a general combination of an electric and a magnetic field as the external field.

D. The anisotropic nonbirefringent case

The special option of anisotropy without birefringence is often paid attention in GLVE; for example, in Ref. [40]. To establish the conditions for the absence of birefringence in our context, we must equalize the dispersion laws for the two different eigenmodes of Eq. (12) $f_2(k_\perp^2) = f_3(k_\perp^2)$. It follows from Eq. (38) that in the infrared limit, one has [28]

$$f_2(k_\perp^2) = k_\perp^2\left(\frac{1-\mathfrak{L}_{\mathfrak{F}}}{1-\mathfrak{L}_{\mathfrak{F}} + 2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}\right), \tag{172}$$

$$f_3(k_\perp^2) = k_\perp^2\left(1 - \frac{2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}}{1-\mathfrak{L}_{\mathfrak{F}}}\right). \tag{173}$$

Hence, there is no birefringence, provided that the effective Lagrangian is subject to the condition

$$2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} = (1-\mathfrak{L}_{\mathfrak{F}})(\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} - \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}). \tag{174}$$

This condition is Lorentz invariant: once there is no birefringence in a special frame, there is none in any inertial frame. Note that the nonbirefringence condition is not the condition of coincidence of the eigenvalues $\varkappa_2$ and $\varkappa_3$ [Eq. (38)] (which does not take place even on the common mass shell of the two eigenmodes [28]), nor the coincidence of the two dielectric permeability values [Eq. (40)], contrary to what one might think. The condition of Eq. (174) is not fulfilled in QED, where the Heisenberg-Euler Lagrangian [34–37] is taken for $\mathfrak{L}$. The only Lagrangian where the background field tensor makes up its single argument free of birefringence is (as inferred in Ref. [41]) the Born-Infeld Lagrangian [42],[7] wherein

$$\mathfrak{L}_{\mathfrak{F}}{}^{\rm BI} = 1 - \left(1 + \frac{2\mathfrak{F}}{\mathfrak{a}^2}\right)^{-1/2}, \quad \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}{}^{\rm BI} = \frac{1}{\mathfrak{a}^2}\left(1 + \frac{2\mathfrak{F}}{\mathfrak{a}^2}\right)^{-3/2}, \quad \mathfrak{L}_{\mathfrak{G}\mathfrak{G}}^{\rm BI} = \frac{1}{\mathfrak{a}^2}\left(1 + \frac{2\mathfrak{F}}{\mathfrak{a}^2}\right)^{-1/2}.$$

Here, $\mathfrak{a}$ is the dimensional parameter inherent in that model, and the requirement of the correspondence principle $\mathfrak{L}_{\mathfrak{F}}|_{\mathfrak{F}=0} = 0$ is obeyed. In the small external field domain, the two dispersion curves [Eqs. (172) and (173)] become

$$f_2(k_\perp^2)|_{\mathfrak{F}\to 0} = k_\perp^2(1 - 2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}), \tag{175}$$

$$f_3(k_\perp^2)|_{\mathfrak{F}\to 0} = k_\perp^2(1 - 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}). \tag{176}$$

Therefore, in this domain, the nonbirefringence condition [Eq. (174)] reduces to $\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} = \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}|_{\mathfrak{F}=0}$ and to the disappearance of the last (Weyl-like) term in Eq. (167), the same as in Ref. [40]. This fact allows us to represent Eq. (167) in the same form as in GLVE [40]:

$$(\kappa_F)^{\kappa\lambda\mu\nu} = \frac{1}{2}(\eta^{\kappa\mu}\kappa^{\lambda\nu} - \eta^{\kappa\nu}\kappa^{\lambda\mu} - \eta^{\lambda\mu}\kappa^{\kappa\nu} + \eta^{\lambda\nu}\kappa^{\kappa\mu}). \tag{177}$$

However, the symmetric tensor $\kappa^{\mu\nu}$ contained in this expression,

$$\kappa^{\mu\nu} \equiv (\kappa_F)_\lambda{}^{\mu\lambda\nu} + \mathfrak{L}_{\mathfrak{F}}\eta^{\mu\nu} = \frac{1}{2}(-\mathfrak{L}_{\mathfrak{F}} + 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}})\eta^{\mu\nu} + \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}F^{\mu\lambda}F_\lambda{}^\nu, \tag{178}$$

where $(\kappa_F)_\lambda{}^{\mu\lambda\nu}$ is the trace of Eq. (167), is not traceless. Its trace, $\kappa_\mu^\mu = -2\mathfrak{L}_{\mathfrak{F}}$, disappears only in the zero external field limit, but otherwise, in the small-field regime, it depends quadratically on the external field strength—the same as other terms in Eq. (167) or in Eq. (178)—and

[7] Another example of the absence of birefringence is supplied by noncommutative electrodynamics in an external field [43].

forms the whole of the isotropic part of the dielectric and magnetic permeability tensors in Eq. (39). In other words, it determines the isotropic part of the vacuum polarization in the external magnetic field. This is a physically meaningful quantity that cannot be expelled from the Born-Infeld model, the same as the double trace [Eq. (168)] from QED.

The case under consideration is parameterized by two quantities $\mathfrak{L}_{\mathfrak{F}}$ and $\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}$, with the external magnetic field $\boldsymbol{B}$ driving LSB. There exists, however, an additional parameter $\tilde{\kappa}_{\rm tr} = \frac{2}{3}\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} - \mathfrak{L}_{\mathfrak{F}}$, whose definition is intrinsically associated with the matrices that characterize the theory [see Eq. (180) in the next subsection]. We combine the latter expression with the convexity properties of the effective Lagrangian [Eq. (41)] to establish the condition[8]

$$\tilde{\kappa}_{\rm tr} \geq -1. \tag{179}$$

Let us finally remark that the relation above must be understood as a direct consequence of fulfilling the fundamental unitarity and causality principles.

E. Numerical estimates

We now estimate to what precision the coefficients $(\kappa_F)^{\kappa\mu\lambda\nu}$ should be measured in order that the Lorentz violation caused by the magnetic field of a given magnitude might be detected in the vacuum, and confront it with sensitivities attained in existing experiments aimed at detecting the intrinsic Lorentz violations within GLVE, as these are listed in Ref. [44]. To this end, we begin with the matrix combinations $(\tilde{\kappa}_{e^+})^{ij} = \frac{1}{2}[(\kappa_{DE})^{ij} + (\kappa_{HB})^{ij}]$, $(\tilde{\kappa}_{e^-})^{ij} = \frac{1}{2}[(\kappa_{DE})^{ij} - (\kappa_{HB})^{ij}] - \frac{1}{3}\delta^{ij}(\kappa_{DE})^{ll}$, and $(\tilde{\kappa}_{o^\pm})^{ij} = \frac{1}{2}[(\kappa_{DB})^{ij} \pm (\kappa_{HE})^{ij}]$, which are frequently used in determining the parameter space to which the experiments on birefringence are sensitive. We find it convenient to remark that these refer to the context of GLVE. The substitution of Eq. (171) into the latter set of matrices allows us to express $(\tilde{\kappa}_{o^\pm})^{ij} = 0$:

$$\begin{aligned} \tilde{\kappa}_{\rm tr} &\equiv \frac{1}{3}(\kappa_{DE})^{ll} = \frac{2}{3}\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} - \mathfrak{L}_{\mathfrak{F}} < 10^{-14}, \\ (\tilde{\kappa}_{e^+})^{ij} &= -\mathfrak{L}_{\mathfrak{F}}\delta^{ij} + \frac{1}{2}[\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} - \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}]B^iB^j < 10^{-32}, \\ (\tilde{\kappa}_{e^-})^{ij} &= \tilde{\kappa}_{\rm tr}\delta^{ij} + \frac{1}{2}[\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} + \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}]B^iB^j < 10^{-17}. \end{aligned} \tag{180}$$

Here the inequalities indicate the experimental sensitivity related to the measurement of the corresponding coefficient combinations.

In considering a possible magnetic field-like Lorentz symmetry violation, we must restrict ourselves to very small magnetic fields. With QED in mind, we should then refer to Eq. (169) in this case. Note that $\frac{2\alpha}{45\pi} = 1.03 \times 10^{-4}$. The accuracy of 10^{-32} would be enough to detect the Lorentz violation produced by the magnetic field of the Earth, $B_{\rm earth} = 0.3$–0.6 G; the accuracy of 10^{-14} would allow one to fix a Lorentz violation equivalent to the presence of a magnetic field on the order of 10^9 G. This is an unearthly large value, on the pulsar scale. However, an increase in experimental accuracy—say, to 10^{-21}—would give us the possibility of detecting the electromagnetic wave refraction due to an external magnetic field already at the laboratory value of 10^5 G. That would be great, because up to now no effect of birefringence of QED has been seen in the vacuum in a direct experiment [45,46]. Unfortunately, as long as any material strongly enhances the effect of the magnetic field as compared to the vacuum, the above considerations may only relate to experimental devices that do not exploit matter as a medium for electromagnetic wave propagation. Apart from such devices, electron-positron pair creation by a single photon, and the photon splitting and merging are well-recognized, efficient processes in pulsar magnetospheres at magnetic field strengths above 10^{12} G. The exclusion of one-photon pair creation at the accuracy level of 10^{-20} from an ultrahigh-energy cosmic ray event in Ref. [40], if viewed within QED, only implies that there is no magnetic field larger than 10^6 G in the space region where that event occurred anyway, which is not unexpected.

The general conclusion of this subsection is that the prospects of detecting Lorentz violations in the vacuum by perfecting the existing experimental means might be based only on the belief that these violations are for some reason much larger than the ones induced via QED by magnetic fields (presumably present in the Galactic background).

VII. SUMMARY AND OUTLOOK

The main effort in this work has been to perform an analysis of Lorentz symmetry breaking by an external field in nonlinear electrodynamics, the gauge sector of QED included. We defined how transformations from the residual symmetry space-time subgroup, left after the external time- and space-independent magnetic field had been imposed, act on coordinates and other vector entities via the external field tensor. For small and steady electromagnetic excitations over the magnetic field background, we have developed the Hamiltonian formalism to serve the linear electrodynamics of the equivalent anisotropic medium, for which purpose a quadratic Lagrangian of these excitations is written with the help of the polarization operator in the external magnetic field. The electric and magnetic permeability tensors in this linear electrodynamics are shown to be those of an equivalent uniaxial medium in any special Lorentz frame, where the external field is purely magnetic or purely electric. Their principal values are expressed in the paper in terms of the field derivatives of the effective Lagrangian. The fields and inductions are given the sense of canonical variables, the necessary

[8]Inequalities [Eq. (41)] are fulfilled in the Born-Infeld model.

primary and secondary constraints are determined, and the Dirac brackets are defined on constrained physical phase space in accordance with the $U(1)$ gauge invariance of the theory. The conserved Noether currents, corresponding to the residual symmetry transformations, as well as the nonconserved Noether currents, corresponding to the Lorentz transformations, the symmetry under which is violated by the external field, are defined on the physical phase subspace of the problem. Among the former, and of some importance, is a nonsymmetric but gauge-invariant energy-momentum tensor, used to form the Poynting vector and the momentum density, which are not the same quantities due to the antisymmetric part of the tensor. We have calculated the Dirac bracket commutation relations between all the generators of infinitesimal space-time rotations and translations to see that the $SO(3)$ algebra of the photon angular momentum remains intact, despite the violation of the rotation symmetry by the external field, whereas the Poincaré algebra is distorted. We derived the evolution equation for the photon angular momentum, which is governed by the photon magnetic moment depending on the antisymmetric part of the energy-momentum tensor. The polarization effects entering the equation of motion of the photon angular momentum are closely associated with the existence of an optical torque. This is a phenomenon inherent to conventional electrodynamics in anisotropic media, which manifests the breakdown of the rotational invariance. We argued that a small-amplitude electromagnetic wave propagating in a strong magnetic field behaves as a quasiparticle carrying a gauge-invariant magnetic moment orthogonal to the wave-vector. The corresponding analysis of the equation of motion for the angular momentum of light in a weak magnetic field was not developed here. This limiting case seems to be very convenient for probing the nonlinear behavior of the quantum vacuum. The latter could be achieved by transferring the angular momentum from the small waves to a microscopic absorptive object (e.g., tweezers). However, a detailed analysis of this issue will be given in a forthcoming publication.

It would be also a challenge to find a closure to the distorted Poincaré algebra for the present case—or, perhaps, other, simpler, cases—of the Lorentz-symmetry violations.

In Sec. VI, we confronted the structure of the polarization operator in the magnetic field with prescriptions of a general Lorentz-violating electrodynamics, and discussed the common features and differences.

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APPENDIX A: TWO IMPORTANT BRACKETS

1. The case of $\{\mathcal{J}, \mathcal{P}^0\}_*$

The aim of this appendix is to compute the Dirac bracket $\{\mathcal{J}, \mathcal{P}^0\}_*$. In order to obtain the latter, we divide it into three terms:

$$\{\mathcal{J}^i, \mathcal{P}^0\}_* = I_1 + I_2 + I_3, \quad I_1^i = \int d^3x \frac{\delta \mathcal{J}^i}{\delta a_l}\frac{\delta \mathcal{P}^0}{\delta \pi^l},$$
$$I_2^i = \int d^3x \frac{\delta \mathcal{J}^i}{\delta \pi_l}\frac{\delta \mathcal{P}^0}{\delta a^l}, \tag{A1}$$
$$I_3 = -\int d^3y d^3z \{\mathcal{J}^i, \varphi_\alpha(\boldsymbol{y})\} C^{-1}_{\alpha\beta}(\boldsymbol{y},\boldsymbol{z})\{\varphi_\beta(\boldsymbol{z}), \mathcal{P}^0\}.$$

An explicit derivation of the expressions above requires us to know the functional derivatives associated with the photon angular momentum [Eq. (113)]. For further convenience, we write the latter as

$$\mathcal{J} = \int d^3x[-(\boldsymbol{x}\cdot\boldsymbol{b})\boldsymbol{\pi} + (\boldsymbol{x}\cdot\boldsymbol{\pi})\boldsymbol{b} + (\boldsymbol{x}\times\boldsymbol{a})\boldsymbol{\nabla}\cdot\boldsymbol{\pi}]. \tag{A2}$$

In correspondence, we obtain

$$\frac{\delta \mathcal{J}^j}{\delta \pi^l} = -(\boldsymbol{x}\cdot\boldsymbol{b})\delta^{lj} + b^j x^l - \nabla^l(\boldsymbol{x}\times\boldsymbol{a})^j,$$
$$\frac{\delta \mathcal{J}^i}{\delta a^l} = \epsilon^{jlk}\nabla^k(\boldsymbol{x}\cdot\boldsymbol{\pi}) + (\boldsymbol{x}\times\boldsymbol{\nabla})^l\pi^j - \epsilon^{jlk}x^k\boldsymbol{\nabla}\cdot\boldsymbol{\pi}. \tag{A3}$$

The explicit structure of $\mathcal{P}^0$ can be found in Eq. (65). Because of Eq. (93), we will ignore the contribution proportional to a_0. Having this in mind, the respective derivatives turn out to be

$$\frac{\delta \mathcal{P}^0}{\delta \pi^l} = -e^l, \qquad \frac{\delta \mathcal{P}^0}{\delta a^l} = (\boldsymbol{\nabla}\times\boldsymbol{h})^l. \tag{A4}$$

Substituting Eqs. (A3) and (A4) into $I^i_{1,2}$, one finds

$$I_1^i = \int d^3x[-\epsilon^{ijk}e^j\nabla^k(\boldsymbol{x}\cdot\boldsymbol{\pi}) - e^j(\boldsymbol{x}\times\boldsymbol{\nabla})^j\pi^i + \epsilon^{ijk}x^k e^j\boldsymbol{\nabla}\cdot\boldsymbol{\pi}], \tag{A5}$$

$$I_2^i = \int d^3x[b^i x^j(\boldsymbol{\nabla}\times\boldsymbol{h})^j - (\boldsymbol{\nabla}\times\boldsymbol{h})^i(\boldsymbol{x}\cdot\boldsymbol{b}) - (\boldsymbol{\nabla}\times\boldsymbol{h})^l\nabla^l(\boldsymbol{x}\times\boldsymbol{a})^i]. \tag{A6}$$

The last term in Eq. (A6) vanishes identically, provided an integration by parts.

We expand the derivative in the first integrand of I_1^i and use the vectorial identity

$$\begin{aligned} \boldsymbol{u} \times [(\boldsymbol{x}\times\boldsymbol{\nabla})\times\boldsymbol{o}] &= u^l(\boldsymbol{x}\times\boldsymbol{\nabla})o^l - \boldsymbol{u}\cdot(\boldsymbol{x}\times\boldsymbol{\nabla})\boldsymbol{o} \\ &= -\epsilon^{ijk}u^jx^k(\boldsymbol{\nabla}\cdot\boldsymbol{o}) + \epsilon^{ijk}u^jx^m\nabla^k o^m. \end{aligned} \tag{A7}$$

As a consequence,

$$\mathcal{I}_1^i = \int d^3x\{-(\boldsymbol{e}\times\boldsymbol{\pi})^i - e^l(\boldsymbol{x}\times\boldsymbol{\nabla})^i\pi^l\}. \tag{A8}$$

Let us turn our attention to $\mathcal{I}_2^i$. Note that an integration by parts in the second term of the latter leads to

$$\mathcal{I}_2^i = \int d^3x[b^ix^j(\boldsymbol{\nabla}\times\boldsymbol{h})^j + (\boldsymbol{h}\times\boldsymbol{b})^i + \epsilon^{ijk}h^kx^l\nabla^jb^l]. \tag{A9}$$

Adding to this expression a vanishing term, $-\int d^3x\epsilon^{ijk}h^jx^k\boldsymbol{\nabla}\cdot\boldsymbol{b}$, and using Eq. (A7), we find

$$\mathcal{I}_2^i = \int d^3x[(\boldsymbol{h}\times\boldsymbol{b})^i - h^l(\boldsymbol{x}\times\boldsymbol{\nabla})^ib^l]. \tag{A10}$$

Now, we focus ourselves on $\mathcal{I}_3$. Expanding the sum over α and β and taking into account Eq. (84), it reduces to

$$\begin{aligned} \mathcal{I}_3 = &\int d^3yd^3z\nabla_k^y\{\mathcal{J}^i,\pi_k(\boldsymbol{y})\}\frac{1}{\nabla^y\varepsilon\nabla^y}\delta^{(3)} \\ &\times(\boldsymbol{y}-\boldsymbol{z})\nabla_m^z\varepsilon_{mn}\{a_n(\boldsymbol{z}),\mathcal{P}^0\} \\ &-\int d^3yd^3z\nabla_m^y\varepsilon_{mn}\{\mathcal{J}^i,a_n(\boldsymbol{y})\}\frac{1}{\nabla^y\varepsilon\nabla^y}\delta^{(3)} \\ &\times(\boldsymbol{y}-\boldsymbol{z})\nabla_k^z\{\pi_k(\boldsymbol{z}),\mathcal{P}^0\}. \end{aligned} \tag{A11}$$

With the help of Eqs. (119) and (96), it is simple to obtain the following identities:

$$\nabla_k^y\{\mathcal{J}^i,\pi_k(\boldsymbol{y})\} = 0, \qquad \nabla_k^z\{\pi_k(\boldsymbol{z}),\mathcal{P}^0\} = 0. \tag{A12}$$

In correspondence, $\mathcal{I}_3$ vanishes identically as well. We then substitute Eqs. (A8) and (A10) into Eq. (A1). As a matter of fact,

$$\{\mathcal{J}^i,\mathcal{P}^0\}_* = \int d^3x\{(\boldsymbol{x}\times\boldsymbol{\nabla})^i\Theta^{00} + \boldsymbol{\pi}\times\boldsymbol{e} + \boldsymbol{h}\times\boldsymbol{b}\}, \tag{A13}$$

where the identity

$$(\boldsymbol{x}\times\boldsymbol{\nabla})\Theta^{00} = -e^l(\boldsymbol{x}\times\boldsymbol{\nabla})\pi^l + h^l(\boldsymbol{x}\times\boldsymbol{\nabla})b^l \tag{A14}$$

has been taken into account. Note that an integration by parts carries out a change of sign in the integral $\int d^3x(\boldsymbol{x}\times\boldsymbol{\nabla})\Theta^{00} = -\int d^3x(\boldsymbol{x}\times\boldsymbol{\nabla})\Theta^{00}$. Therefore, the latter vanishes identically and can be ignored in Eq. (A13). This operation leaves us with the terms which appear in Eq. (126).

2. The case of $\{\mathcal{K},\mathcal{P}^0\}_*$

Let us consider the Dirac bracket involved in Eq. (128). To determine the latter, we express the photon boost in Eq. (114) in the following form:

$$\begin{aligned} &\mathcal{K} = \mathcal{K}_{\mathcal{P}} + \mathcal{K}_{\Theta^{00}}, \\ &\mathcal{K}_{\mathcal{P}} = -x^0\boldsymbol{\mathcal{P}}, \quad \mathcal{K}_{\Theta^{00}} = \int d^3x(\boldsymbol{x}\Theta^{00}), \end{aligned} \tag{A15}$$

where $\boldsymbol{\mathcal{P}}$ is the spatial translation generator and Θ^{00} is the energy density [see Eq. (65)]. As before, we neglect the contributions involving the Lagrangian multiplier a_0. Considering Eq. (A15), one finds

$$\{\mathcal{K}^i,\mathcal{P}^0\}_* = \{\mathcal{K}_{\mathcal{P}}^i,\mathcal{P}^0\}_* + \{\mathcal{K}_{\Theta^{00}}^i,\mathcal{P}^0\}_*. \tag{A16}$$

The first bracket on the right-hand side can be computed by applying Eq. (135). According to this equation, $\{\mathcal{P}^i,\mathcal{P}^0\}_*$ vanishes identically. Thus

$$\{\mathcal{K}_{\mathcal{P}}^i,\mathcal{P}^0\}_* = -x^0\{\mathcal{P}^i,\mathcal{P}^0\}_* = 0. \tag{A17}$$

In order to analyze the remaining terms on the right-hand side of Eq. (A16), it is rather convenient to have at our disposal the following set of derivatives:

$$\frac{\mathcal{K}_{\Theta^{00}}^i}{\delta\pi^l} = -x^ie^l, \qquad \frac{\delta\mathcal{K}_{\Theta^{00}}^i}{\delta a^l} = -\epsilon^{ilm}h^m + x^i(\nabla\times\boldsymbol{h})^l. \tag{A18}$$

With the above expressions in mind, one can undertake the calculation of the second bracket in Eq. (A16). We write the latter as

$$\{\mathcal{K}_{\Theta^{00}}^i,\mathcal{P}^0\}_* = \{\mathcal{K}_{\Theta^{00}}^i,\mathcal{P}^0\} + \mathcal{W}, \tag{A19}$$

with

$$\mathcal{W} \equiv -\int d^3yd^3z\{\mathcal{K}_{\Theta^{00}}^i,\varphi_\alpha(\boldsymbol{y})\}C_{\alpha\beta}^{-1}(\boldsymbol{y},\boldsymbol{z})\{\varphi_\beta(\boldsymbol{z}),\mathcal{P}^0\}. \tag{A20}$$

From Eqs. (A18) and (A4), it is straightforward to get

$$\{\mathcal{K}^i,\mathcal{P}^0\} = \int d^3x(\boldsymbol{e}\times\boldsymbol{h})^i. \tag{A21}$$

With the help of Eq. (84), we express $\mathcal{W}$ as

$$\begin{aligned} \mathcal{W} = &\int d^3yd^3z\nabla_k^y\{\mathcal{K}_{\Theta^{00}}^i,\pi_k(\boldsymbol{y})\}\frac{1}{\nabla^y\varepsilon\nabla^y}\delta^{(3)} \\ &\times(\boldsymbol{y}-\boldsymbol{z})\nabla_m^z\varepsilon_{mn}\{a_n(\boldsymbol{z}),\mathcal{P}^0\} \\ &-\int d^3yd^3z\nabla_m^y\varepsilon_{mn}\{\mathcal{K}_{\Theta^{00}}^i,a_n(\boldsymbol{y})\}\frac{1}{\nabla^y\varepsilon\nabla^y}\delta^{(3)} \\ &\times(\boldsymbol{y}-\boldsymbol{z})\nabla_k^z\{\pi_k(\boldsymbol{z}),\mathcal{P}^0\}. \end{aligned} \tag{A22}$$

Observe that the last integral vanishes identically because $\nabla_k^z\{\pi_k(\boldsymbol{z}),\mathcal{P}^0\} = 0$ [see Eq. (A12)]. The first integral vanishes identically as well because

$$\nabla_k^y\{\mathcal{K}_{\Theta^{00}}^i,\pi_k(\boldsymbol{y})\} = \nabla_k^y\frac{\delta\mathcal{K}_{\Theta^{00}}^i}{\delta a_k(\boldsymbol{y})} = x^i\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{h}) = 0. \tag{A23}$$

We then conclude that $\mathcal{W} = 0$, and the Dirac bracket $\{\mathcal{K}^i,\mathcal{P}^0\}_*$ is just as it appears in Eq. (129).

APPENDIX B: DERIVING THE ALTERATIONS TO THE LORENTZ ALGEBRAIC RELATIONS

1. Dirac bracket between $\mathcal{K}$ and $\mathcal{J}$

In order to determine the Dirac bracket $\{\mathcal{K}, \mathcal{J}\}_*$ we consider Eq. (A15). This allows us to express

$$\{\mathcal{K}^i, \mathcal{J}^j\}_* = \{\mathcal{K}^i_P, \mathcal{J}^j\}_* + \{\mathcal{K}^i_{\Theta^{00}}, \mathcal{J}^j\}_*. \tag{B1}$$

The first bracket on the right-hand side can be computed by applying Eq. (141). Indeed, according to this equation, $\{\mathcal{P}^i, \mathcal{J}^j\}_* = \epsilon^{ijk}\mathcal{P}^k$ provided that the fields vanish at infinity. Therefore,

$$\{\mathcal{K}^i_P, \mathcal{J}^j\}_* = \epsilon^{ijk}[-x^0\mathcal{P}^k]. \tag{B2}$$

Analogously to Eq. (A1), we split the second Dirac bracket in Eq. (B1) into three different contributions:

$$\{\mathcal{K}^i_{\Theta^{00}}, \mathcal{J}^j\}_* = I_1 - I_2 + I_3, \tag{B3}$$

with

$$\begin{aligned} I_1 &= \int d^3x \frac{\delta\mathcal{K}^i_{\Theta^{00}}}{\delta a_l}\frac{\delta\mathcal{J}^j}{\delta\pi^l}, \\ I_2 &= \int d^3x \frac{\delta\mathcal{K}^i_{\Theta^{00}}}{\delta \pi_l}\frac{\delta\mathcal{J}^j}{\delta a^l}, \\ I_3 &= -\int d^3y d^3z \{\mathcal{K}, \varphi_\alpha(\boldsymbol{y})\}C^{-1}_{\alpha\beta}(\boldsymbol{y},z)\{\varphi_\beta(z), \mathcal{J}\}. \end{aligned} \tag{B4}$$

The explicit insertion of the derivatives involved in I_1 allows us to write

$$\begin{aligned} I_1 = \int d^3x[&\epsilon^{ijm}h^m(\boldsymbol{x}\cdot\boldsymbol{b}) - \epsilon^{ilm}h^m b^j x^l - x^i(\boldsymbol{\nabla}\times\boldsymbol{h})^j \\ &\times(\boldsymbol{x}\cdot\boldsymbol{b}) + x^i(\boldsymbol{\nabla}\times\boldsymbol{h})^l b^j x^l - x^i(\boldsymbol{\nabla}\times\boldsymbol{h})^l\nabla^l \\ &\times(\boldsymbol{x}\times\boldsymbol{a})^j + \epsilon^{ilm}h_m\nabla^l(\boldsymbol{x}\times\boldsymbol{a})^j]. \end{aligned} \tag{B5}$$

Thanks to the vectorial identity given in Eq. (A7), we are able to express the integral of the first four terms in this equation as

$$-\int d^3x[x^i(h^l(\boldsymbol{x}\times\boldsymbol{\nabla})^j b^l + (\boldsymbol{h}\times\boldsymbol{b})^j)]. \tag{B6}$$

Note that the fifth and sixth terms in Eq. (B5) cancel each other, since an integration by parts leads to

$$\int d^3x x^i(\boldsymbol{\nabla}\times\boldsymbol{h})^l\nabla^l(\boldsymbol{x}\times\boldsymbol{a})^j = \int d^3x\epsilon^{ilm}h^m\nabla_l(\boldsymbol{x}\times\boldsymbol{a})^j.$$

Once this fact is taken into account and Eq. (B6) is inserted into Eq. (B5), its right-hand side acquires the following structure:

$$I_1 = \int d^3x\{-x^i[h^l(\boldsymbol{x}\times\boldsymbol{\nabla})^j b^l + (\boldsymbol{h}\times\boldsymbol{b})^j]\}. \tag{B7}$$

Furthermore, the substitution of Eqs. (A3) and (A18) into I_2 yields

$$\begin{aligned} I_2 = \int d^3x\{&-x^i\epsilon^{jlk}e^l\nabla^k(\boldsymbol{x}\cdot\boldsymbol{\pi}) - x^ie^l(\boldsymbol{x}\times\boldsymbol{\nabla})^l\pi^j \\ &+ x^i\epsilon^{jlk}e^lx^k(\boldsymbol{\nabla}\cdot\boldsymbol{\pi})\}. \end{aligned} \tag{B8}$$

Nevertheless, the desirable expression of I_2 is obtained by expanding the derivative present in the first term and using the identity given in Eq. (A7). With these details in mind, we find

$$I_2 = \int d^3x\{-x^i[e^l(\boldsymbol{x}\times\boldsymbol{\nabla})^j\pi^l + (\boldsymbol{e}\times\boldsymbol{\pi})^j]\}. \tag{B9}$$

On the other hand, once I_3 is expanded over α and β, one obtains

$$\begin{aligned} I_3 = &\int d^3yd^3z\nabla^y_k\{\mathcal{K}^i_{\Theta^{00}}, \pi_k(\boldsymbol{y})\}\frac{1}{\nabla^y\varepsilon\nabla^y}\delta^{(3)} \\ &\times(\boldsymbol{y}-z)\nabla^z_m\varepsilon_{mn}\{a_n(z), \mathcal{J}^j\} \\ &-\int d^3yd^3z\nabla^y_m\varepsilon_{mn}\{\mathcal{K}^i_{\Theta^{00}}, a_n(\boldsymbol{y})\}\frac{1}{\nabla^y\varepsilon\nabla^y}\delta^{(3)} \\ &\times(\boldsymbol{y}-z)\nabla^z_k\{\pi_k(z), \mathcal{J}^j\}, \end{aligned} \tag{B10}$$

where the relevant elements of $C^{-1}_{\alpha\beta}$ [Eq. (84)] were inserted. Thanks to Eq. (119), both integrals in I_3 vanish identically, and one ends up with $I_3 = 0$. Hence, by substituting Eqs. (B7) and (B9) into Eq. (B3), we get

$$\{\mathcal{K}^i, \mathcal{J}^j\}_* = \epsilon^{ijk}\mathcal{K}^k - \int d^3x\{x^i[(\boldsymbol{\pi}\times\boldsymbol{e})^j + (\boldsymbol{h}\times\boldsymbol{b})^j]\}, \tag{B11}$$

where Eq. (A15) is taken into account. We then use the definition of $\tau^{\mu\nu}$ [Eq. (132)] to write the second line of this equation as it stands in Eq. (132).

2. Dirac bracket between $\mathcal{J}^i$ and $\mathcal{J}^j$

We start off by expressing the Dirac bracket between $\mathcal{J}^i$ and $\mathcal{J}^j$ in terms of two elements:

$$\begin{aligned} &\{\mathcal{J}^i, \mathcal{J}^j\}_* = I_1 + I_2, \\ I_1 &= \int d^3x\left\{\frac{\delta\mathcal{J}^i}{\delta a^l}\frac{\delta\mathcal{J}^j}{\delta\pi^l} - \frac{\delta\mathcal{J}^i}{\delta\pi^l}\frac{\delta\mathcal{J}^j}{\delta a^l}\right\}, \\ I_2 &= -\int d^3yd^3z\{\mathcal{J}, \varphi_\alpha(\boldsymbol{y})\}C^{-1}_{\alpha\beta}(\boldsymbol{y},z)\{\varphi_\beta(z), \mathcal{J}\}. \end{aligned} \tag{B12}$$

An explict substitution of Eq. (A3) into I_1 allows us to write

$$I_1 = \int d^3x\{2\epsilon^{ijk}[-(\boldsymbol{x}\cdot\boldsymbol{b})\nabla^k(\boldsymbol{x}\cdot\boldsymbol{\pi}) + (\boldsymbol{\nabla}\cdot\boldsymbol{\pi})x^k(\boldsymbol{x}\cdot\boldsymbol{b})] + \epsilon^{ilm}b^j x^l\nabla^m(\boldsymbol{x}\cdot\boldsymbol{\pi}) - \epsilon^{jlm}b^i x^l\nabla^m(\boldsymbol{x}\cdot\boldsymbol{\pi})$$
$$+ \epsilon^{ilm}x^m(\boldsymbol{\nabla}\cdot\boldsymbol{\pi})\nabla^l(\boldsymbol{x}\times\boldsymbol{a})^j - \epsilon^{jlm}x^m(\boldsymbol{\nabla}\cdot\boldsymbol{\pi})\nabla^l(\boldsymbol{x}\times\boldsymbol{a})^i - (\boldsymbol{x}\cdot\boldsymbol{b})(\boldsymbol{x}\times\boldsymbol{\nabla})^j\pi^i$$
$$- (\boldsymbol{x}\cdot\boldsymbol{b})(\boldsymbol{x}\times\boldsymbol{\nabla})^i\pi^j - \epsilon^{ilm}\nabla^m(\boldsymbol{x}\cdot\boldsymbol{\nabla})\nabla^l(\boldsymbol{x}\times\boldsymbol{a})^j + \epsilon^{jlm}\nabla^m(\boldsymbol{x}\cdot\boldsymbol{\pi})\nabla^l(\boldsymbol{x}\times\boldsymbol{a})^i$$
$$- \nabla^l(\boldsymbol{x}\times\boldsymbol{a})^j(\boldsymbol{x}\times\boldsymbol{\nabla})^l\pi^i + \nabla^l(\boldsymbol{x}\times\boldsymbol{a})^i(\boldsymbol{a}\times\boldsymbol{\nabla})^l\pi^j\}, \tag{B13}$$

where the terms that vanish due to the antisymmetric property of ϵ^{ijk} have been omitted. Note that the last four terms of this expression vanish as well, provided an integration by parts. The remaining integrant can be written as

$$I_1 = \epsilon^{ijk}\int d^3x\{2[-(\boldsymbol{x}\cdot\boldsymbol{b})\nabla^k(\boldsymbol{x}\cdot\boldsymbol{\pi}) + (\boldsymbol{\nabla}\cdot\boldsymbol{\pi})x^k(\boldsymbol{x}\cdot\boldsymbol{b})]$$
$$- (\boldsymbol{x}\cdot\boldsymbol{\pi})[(\boldsymbol{x}\times\boldsymbol{\nabla})\times\boldsymbol{b}]^k - (\boldsymbol{\nabla}\cdot\boldsymbol{\pi})[(\boldsymbol{x}\times\boldsymbol{\nabla})$$
$$\times(\boldsymbol{x}\times\boldsymbol{a})]^k + (\boldsymbol{x}\cdot\boldsymbol{b})[(\boldsymbol{x}\times\boldsymbol{\nabla})\times\boldsymbol{\pi}]^k\}. \tag{B14}$$

We then use the identity $(\boldsymbol{x}\times\boldsymbol{\nabla})\times\boldsymbol{A} = -\boldsymbol{x}\boldsymbol{\nabla}\cdot\boldsymbol{A} + x^l\boldsymbol{\nabla}A^l$. As a consequence,

$$I_1 = \epsilon^{ijk}\int d^3x\{-2(\boldsymbol{x}\cdot\boldsymbol{b})\nabla^k(\boldsymbol{x}\cdot\boldsymbol{\pi}) + (\boldsymbol{\nabla}\cdot\boldsymbol{\pi})x^k(\boldsymbol{x}\cdot\boldsymbol{b})$$
$$- (\boldsymbol{x}\cdot\boldsymbol{\pi})x^l\nabla^k b^l - (\boldsymbol{\nabla}\cdot\boldsymbol{\pi})x^k(\boldsymbol{x}\cdot\boldsymbol{b})$$
$$- (\boldsymbol{\nabla}\cdot\boldsymbol{\pi})(\boldsymbol{x}\cdot\boldsymbol{b})x^l\nabla^k(\boldsymbol{x}\times\boldsymbol{a}) + (\boldsymbol{x}\cdot\boldsymbol{b})x^l\nabla^k\pi^l\}. \tag{B15}$$

Expanding the derivative of the first integrand and integrating by parts the third and fifth terms, we end up with

$$I_1 = \epsilon^{ijk}\int d^3x\{-(\boldsymbol{x}\cdot\boldsymbol{b})\pi^k + (\boldsymbol{x}\cdot\boldsymbol{\pi})b^k$$
$$+ (\boldsymbol{\nabla}\cdot\boldsymbol{\pi})(\boldsymbol{x}\times\boldsymbol{a})^k\}, \tag{B16}$$

where Eq. (A2) was inserted in order to obtain the second line.

Now, it is rather clear that I_2 reduces to

$$I_2 = \int d^3y d^3z\nabla_k^y\{\mathcal{J}^i, \pi_k(\boldsymbol{y})\}\frac{1}{\nabla^y\varepsilon\nabla^y}\delta^{(3)}$$
$$\times(\boldsymbol{y}-\boldsymbol{z})\nabla_m^z\varepsilon_{mn}\{a_n(\boldsymbol{z}), \mathcal{J}^j\}$$
$$- \int d^3y d^3z\nabla_m^y\varepsilon_{mn}\{\mathcal{J}^i, a_n(\boldsymbol{y})\}\frac{1}{\nabla^y\varepsilon\nabla^y}\delta^{(3)}$$
$$\times(\boldsymbol{y}-\boldsymbol{z})\nabla_k^z\{\pi_k(\boldsymbol{z}), \mathcal{J}^j\}, \tag{B17}$$

where Eq. (84) is used. Since $\nabla_k^y\{\mathcal{J}^i, \pi_k(\boldsymbol{y})\}$ vanishes identically [see Eq. (119)], I_2 does not contribute to the Dirac bracket between $\mathcal{J}^i$ and $\mathcal{J}^j$. Equipped with this result and substituting Eq. (B16) into Eq. (B12), we end up with the bracket written in Eq. (142).

3. Dirac bracket between $\mathcal{K}^i$ and $\mathcal{K}^j$

Let us conclude the derivation of the modified Lorentz algebra by obtaining the Dirac bracket $\{\mathcal{K}^i, \mathcal{K}^j\}_*$. A straightforward substitution of Eq. (A15) leads us to write this as

$$\{\mathcal{K}^i, \mathcal{K}^j\}_* = \{\mathcal{K}^i_{\mathcal{P}}\mathcal{K}^j_{\mathcal{P}}\}_* + \{\mathcal{K}^i_{\mathcal{P}}, \mathcal{K}^j_{\Theta^{00}}\}_*$$
$$+ \{\mathcal{K}^i_{\Theta^{00}}, \mathcal{K}^j_{\mathcal{P}}\}_* + \{\mathcal{K}^i_{\Theta^{00}}, \mathcal{K}^j_{\Theta^{00}}\}_*. \tag{B18}$$

The first three brackets are easy to compute. For instance,

$$\{\mathcal{K}^i_{\mathcal{P}}, \mathcal{K}^i_{\mathcal{P}}\}_* = x^{02}\{\mathcal{P}^i, \mathcal{P}^j\}_* = 0, \tag{B19}$$

where Eq. (137) is used. Also, by considering Eq. (95) and using integration by parts, we find that

$$\{\mathcal{K}^i_{\mathcal{P}}, \mathcal{K}^j_{\Theta^{00}}\}_* = x^0\mathcal{P}^0\eta^{ij}. \tag{B20}$$

As a consequence,

$$\{\mathcal{K}^i_{\mathcal{P}}, \mathcal{K}^j_{\Theta^{00}}\}_* + \{\mathcal{K}^i_{\Theta^{00}}, \mathcal{K}^j_{\mathcal{P}}\}_* = 0. \tag{B21}$$

All that remains is to compute the last bracket on the right-hand side of Eq. (B18); i.e., the Dirac bracket between $\mathcal{K}^i_{\Theta^{00}}$ and $\mathcal{K}^j_{\Theta^{00}}$. To compute this, we write

$$\{\mathcal{K}^i_{\Theta^{00}}, \mathcal{K}^j_{\Theta^{00}}\}_* = \Sigma_1 + \Sigma_2,$$
$$\Sigma_1 = \int d^3x\left\{\frac{\delta\mathcal{K}^i_{\Theta^{00}}}{\delta a^l}\frac{\delta\mathcal{K}^j_{\Theta^{00}}}{\delta\pi^l} - \frac{\delta\mathcal{K}^i_{\Theta^{00}}}{\delta\pi^l}\frac{\delta\mathcal{K}^j_{\Theta^{00}}}{\delta a^l}\right\},$$
$$\Sigma_2 = -\int d^3y d^3z\{\boldsymbol{\mathcal{K}}_{\Theta^{00}}, \varphi_\alpha(\boldsymbol{y})\}C^{-1}_{\alpha\beta}(\boldsymbol{y},\boldsymbol{z})\{\varphi_\beta(\boldsymbol{z}), \boldsymbol{\mathcal{K}}_{\Theta^{00}}\}.$$

Inserting Eq. (A18) into Σ_1, one finds

$$\Sigma_1 = \epsilon^{ijk}\int d^3x\{-[\boldsymbol{x}\times(\boldsymbol{e}\times\boldsymbol{h})]^k\}. \tag{B22}$$

This bracket can be written in a more appropriate form by using the relation

$$(\boldsymbol{e}\times\boldsymbol{h})^i + (\boldsymbol{\pi}\times\boldsymbol{b})^i - a^i\boldsymbol{\nabla}\cdot\boldsymbol{\pi} = \tau^{i0}, \tag{B23}$$

where a term proportional to a_0 has been ignored. Indeed, the substitution of the expression above into Eq. (B22) allows us to get

$$\Sigma_1 = -\epsilon^{ijk}\mathcal{J}^k - \int d^3x(x^i\tau^{j0} - x^j\tau^{i0}). \tag{B24}$$

Thanks to Eq. (84), Σ_2 reduces to

$$\Sigma_2 = \int d^3y d^3z \nabla_k^y \{\mathcal{K}^i, \pi_k(\boldsymbol{y})\} \frac{1}{\nabla^y \varepsilon \nabla^y} \delta^{(3)} \times (\boldsymbol{y} - \boldsymbol{z}) \nabla_m^z \varepsilon_{mn} \{a_n(\boldsymbol{z}), \mathcal{K}^j\} - \int d^3y d^3z \nabla_m^y \varepsilon_{mn} \{\mathcal{K}^i, a_n(\boldsymbol{y})\} \frac{1}{\nabla^y \varepsilon \nabla^y} \delta^{(3)} \times (\boldsymbol{y} - \boldsymbol{z}) \nabla_k^z \{\pi_k(\boldsymbol{z}), \mathcal{K}^j\}. \quad \text{(B25)}$$

But, since $\nabla_k^y\{\mathcal{K}^i, \pi_k(\boldsymbol{y})\}$ vanishes identically [see Eq. (120)], we can assert that Σ_2 does not contribute to the Dirac bracket of $\mathcal{K}^i$ and $\mathcal{K}^j$ Due to this fact,

$$\{\mathcal{K}^i, \mathcal{K}^j\}_* = -\epsilon^{ijk}\mathcal{J}^k - \int d^3x(x^i \tau^{j0} - x^j \tau^{i0}). \quad \text{(B26)}$$

APPENDIX C: THE MOMENTUM AND PHASE VELOCITY OF THE EIGENWAVES

We find it convenient to determine a connection between the translational generator associated with each degree of freedom and its respective phase velocity. In order to do this, we express the electric field of each eigenmode as it is given in Eq. (159). Likewise, the magnetic field of each eigenwave turns out to be

$$\boldsymbol{b}^{(\lambda)}(\boldsymbol{x}, x^0) = \mathcal{E}_0^{(\lambda)} \frac{\boldsymbol{b}^{(\lambda)}(\boldsymbol{k})}{|\boldsymbol{b}^{(\lambda)}(\boldsymbol{k})|} \cos[\omega_\lambda x^0 - \boldsymbol{k} \cdot \boldsymbol{x}]. \quad \text{(C1)}$$

As in Sec. V, $\mathcal{E}_0^{(\lambda)}$ and $\omega_\lambda(\boldsymbol{k})$ are the amplitude and frequency of mode λ, respectively. Besides, whatever the nature of the external field, the unit vectors $\sim\boldsymbol{e}^{(\lambda)}(\boldsymbol{k})/|\boldsymbol{e}^{(\lambda)}(\boldsymbol{k})|$ in Eq. (159) and $\sim\boldsymbol{b}^{(\lambda)}(\boldsymbol{k})/|\boldsymbol{b}^{(\lambda)}(\boldsymbol{k})|$ in Eq. (C1) must be understood as the respective electric and magnetic polarizations.[9] We remark that the plane wave decomposition for the induction vectors $\boldsymbol{d}$ and $\boldsymbol{h}$ follows from Eqs. (159), (C1), and (15). With these details in mind, the Maxwell equations (72), (96), and (97) read

$$\boldsymbol{k} \cdot \boldsymbol{d}^{(\lambda)}(\boldsymbol{x}, x^0) = 0, \qquad \boldsymbol{k} \times \boldsymbol{e}^{(\lambda)}(\boldsymbol{x}, x^0) = \omega_\lambda \boldsymbol{b}^{(\lambda)}(\boldsymbol{x}, x^0),$$
$$\boldsymbol{k} \cdot \boldsymbol{b}^{(\lambda)}(\boldsymbol{x}, x^0) = 0, \qquad \boldsymbol{k} \times \boldsymbol{h}^{(\lambda)}(\boldsymbol{x}, x^0) = -\omega_\lambda \boldsymbol{d}^{(\lambda)}(\boldsymbol{x}, x^0). \quad \text{(C2)}$$

Thanks to the Faraday equation, the momentum associated with each propagation mode

$$\boldsymbol{\mathcal{P}}^{(\lambda)} = \int d^3x (\boldsymbol{d}^{(\lambda)} \times \boldsymbol{b}^{(\lambda)}) \quad \text{(C3)}$$

can be written as

$$\boldsymbol{\mathcal{P}}^{(\lambda)} = \int d^3x \frac{1}{\omega_\lambda(\boldsymbol{k})} [\boldsymbol{d}^{(\lambda)} \times (\boldsymbol{k} \times \boldsymbol{e}^{(\lambda)})] = \int d^3x \frac{\boldsymbol{d}^{(\lambda)} \cdot \boldsymbol{e}^{(\lambda)}}{u_\lambda(\boldsymbol{k})} \boldsymbol{n}, \quad \text{(C4)}$$

where $u_\lambda = \omega_\lambda/|\boldsymbol{k}|$ is the phase velocity and $\boldsymbol{n} = \boldsymbol{k}/|\boldsymbol{k}|$ denotes the wave vector.

Now, the Ampere law allows us to obtain the following relation: $\boldsymbol{h}^{(\lambda)} = u_\lambda(\boldsymbol{n} \times \boldsymbol{d}^{(\lambda)}) + (\boldsymbol{n} \cdot \boldsymbol{h}^{(\lambda)})\boldsymbol{n}$. Multiplying the latter by $\boldsymbol{b}$, we end up with

$$\boldsymbol{h}^{(\lambda)} \cdot \boldsymbol{b}^{(\lambda)} = u_\lambda(\boldsymbol{n} \times \boldsymbol{d}^{(\lambda)}) \cdot \boldsymbol{b}^{(\lambda)} = u_\lambda \boldsymbol{n} \cdot (\boldsymbol{d}^{(\lambda)} \times \boldsymbol{b}^{(\lambda)}) = \boldsymbol{d}^{(\lambda)} \cdot \boldsymbol{e}^{(\lambda)}. \quad \text{(C5)}$$

Its substitution into the energy [Eq. (58)] yields $\mathcal{P}^{0(\lambda)} = \int d^3x[\boldsymbol{d}^{(\lambda)} \cdot \boldsymbol{e}^{(\lambda)}]$. We use this identity to express Eq. (C4) in the following form:

$$\boldsymbol{\mathcal{P}}^{(\lambda)} = \frac{\mathcal{P}^{0(\lambda)}}{u_\lambda(\boldsymbol{k})} \boldsymbol{n}. \quad \text{(C6)}$$

Thus, the translation generator associated with each $\Pi_{\mu\nu}$ eigenmode turns out to be parallel to the wave vector. Observe, in addition, that Eq. (C6) allows to write the phase velocity as $\boldsymbol{u}_\lambda = \mathcal{P}^{0(\lambda)}/|\boldsymbol{\mathcal{P}}^{(\lambda)}|\boldsymbol{n}$.

APPENDIX D: THE POYNTING VECTOR AND GROUP VELOCITY OF THE EIGENWAVES

Let us turn our attention to the Poynting vector given in Eq. (54). In order to simplify our exposition, we will confine ourselves to the case in which the external background is a magneticlike field ($\mathfrak{F} > 0, \mathfrak{G} = 0$). The results, however, are easily extensible to the case of an electriclike vector ($\mathfrak{F} < 0, \mathfrak{G} = 0$). To establish a comparison with the previously discussed translation generator, it is rather convenient to work with the spatial integral of Eq. (54):

$$\hat{\boldsymbol{\mathcal{P}}} = \int d^3x \Theta^{i0}(\boldsymbol{x}, x^0). \quad \text{(D1)}$$

It is also advantageous to express the Poynting vector in terms of $\boldsymbol{\pi}$ and $\boldsymbol{b}$. This is carried out by inserting the relations $\boldsymbol{e} = -\boldsymbol{\pi}/\varepsilon_\perp + \frac{\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}{\varepsilon_\perp \varepsilon_\parallel}(\boldsymbol{\pi} \cdot \boldsymbol{B})\boldsymbol{B}$ and $\boldsymbol{h} = \varepsilon_\perp \boldsymbol{b} - \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}(\boldsymbol{b} \cdot \boldsymbol{B})\boldsymbol{B}$ into Eq. (54). Considering these details, we obtain

$$\hat{\boldsymbol{\mathcal{P}}} = \boldsymbol{\mathcal{P}} + \boldsymbol{\mathcal{N}} \times \boldsymbol{B}, \quad \text{(D2)}$$

where Eq. (61) has been used, and

$$\boldsymbol{\mathcal{N}} = \int d^3x \left\{ \frac{\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}}{\varepsilon_\perp} (\boldsymbol{b} \cdot \boldsymbol{B})\boldsymbol{\pi} - \frac{\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}{\varepsilon_\parallel} (\boldsymbol{\pi} \cdot \boldsymbol{B})\boldsymbol{b} \right\}. \quad \text{(D3)}$$

We remark that the last term in Eq. (D2) does not point in the same direction of $\boldsymbol{k}$. Of course, each mode has a spatial integral of the Poynting vector given by $\hat{\boldsymbol{\mathcal{P}}}^{(\lambda)} = \boldsymbol{\mathcal{P}}^{(\lambda)} + \boldsymbol{\mathcal{N}}^{(\lambda)} \times \boldsymbol{B}$, with

$$\boldsymbol{\mathcal{N}}^{(1)} = 0, \quad \text{(D4)}$$

$$\boldsymbol{\mathcal{N}}^{(2)} = -\frac{\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}{\varepsilon_\parallel} \int d^3x (\boldsymbol{\pi}^{(2)} \cdot \boldsymbol{B}) \boldsymbol{b}^{(2)}, \quad \text{(D5)}$$

[9] When the external background is a magneticlike field tensor, i.e., $\mathfrak{F} > 0$ and $\mathfrak{G} = 0$, the behavior of $e^{(\lambda)}(\boldsymbol{k})$ and $b^{(\lambda)}(\boldsymbol{k})$ can be found below Eq. (13).

$$\mathcal{N}^{(3)} = \frac{\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}}{\varepsilon_\perp} \int d^3x (\boldsymbol{b}^{(3)} \cdot \boldsymbol{B}) \boldsymbol{\pi}^{(3)}. \tag{D6}$$

To derive these expressions, we have considered Eq. (D3) and the equations below Eq. (13). One must note that, for the second and third propagation modes, the following relations hold: $(\boldsymbol{\pi}^{(2)} \cdot \boldsymbol{B})\boldsymbol{b}^{(2)} = -\boldsymbol{B} \times (\boldsymbol{\pi}^{(2)} \times \boldsymbol{b}^{(2)})$ and $(\boldsymbol{b}^{(3)} \cdot \boldsymbol{B})\boldsymbol{\pi}^{(3)} = \boldsymbol{B} \times (\boldsymbol{\pi}^{(3)} \times \boldsymbol{b}^{(3)})$, respectively. This allows us to express Eqs. (D4)–(D6) as

$$\mathcal{N}^{(1)} = 0, \qquad \mathcal{N}^{(2)} = -\frac{\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}{\varepsilon_\parallel} \boldsymbol{B} \times \boldsymbol{\mathcal{P}}^{(2)}, \qquad \mathcal{N}^{(3)} = -\frac{\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}}{\varepsilon_\perp} \boldsymbol{B} \times \boldsymbol{\mathcal{P}}^{(3)}, \tag{D7}$$

with $\boldsymbol{\mathcal{P}}^{(\lambda)}$ given in Eq. (C6). As a consequence, the spatial integral of the Poynting vector associated with each eigenmode reads

$$\hat{\boldsymbol{\mathcal{P}}}^{(1)} = \boldsymbol{\mathcal{P}}^{(1)}, \qquad \hat{\boldsymbol{\mathcal{P}}}^{(2)} = \boldsymbol{\mathcal{P}}^{(2)} - \frac{2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}{\varepsilon_\parallel} \boldsymbol{\mathcal{P}}_\perp^{(2)}, \qquad \hat{\boldsymbol{\mathcal{P}}}^{(3)} = \boldsymbol{\mathcal{P}}^{(3)} - \frac{2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}}{\varepsilon_\perp} \boldsymbol{\mathcal{P}}_\perp^{(3)}. \tag{D8}$$

In accordance with the expression above, we can conclude that as long as $k_\perp \neq 0$, the direction of the energy propagation in each physical mode differs from its respective momentum.

To proceed in our analysis, we consider the center-of-mass energy associated with the electromagnetic wave. The latter can be defined by

$$\boldsymbol{x}_{\rm cm} = \frac{1}{\mathcal{P}^0} \int d^3x \boldsymbol{x} \Theta^{00}, \tag{D9}$$

where Θ^{00} and $\mathcal{P}^0$ are given in Eqs. (52) and (58), respectively. Note that the derivative with respect to time allows us to define the velocity of energy transport

$$\boldsymbol{v}_{\rm cm} = \frac{d\boldsymbol{x}_{\rm cm}}{dx^0} = \frac{1}{\mathcal{P}^0} \int d^3x \boldsymbol{x} \frac{d\Theta^{00}}{dx^0}, \tag{D10}$$

where the energy conservation ($d\mathcal{P}^0/dx^0 = 0$) is taken into account. Making use of the continuity [Eq. (51)] and integrating by parts, one obtains

$$\boldsymbol{v}_{\rm cm} = \frac{\hat{\boldsymbol{\mathcal{P}}}}{\mathcal{P}^0} = \frac{1}{u}\boldsymbol{n} + \frac{\mathcal{N} \times \boldsymbol{B}}{\mathcal{P}^0}, \tag{D11}$$

where Eq. (D2) has been considered as well, and $u = \omega(\boldsymbol{k})/|\boldsymbol{k}|$ denotes the phase velocity of the small electromagnetic wave. Obviously, the velocity of energy transport associated with each eigenwave follows from this expression and Eqs. (D7) and (D8). In this context,

$$\boldsymbol{v}_{\rm cm1} = \frac{1}{u_1}\boldsymbol{n}, \qquad \boldsymbol{v}_{\rm cm2} = \frac{1}{u_2}\boldsymbol{n} - \frac{2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}{\varepsilon_\parallel u_{2\perp}}\boldsymbol{n}_\perp, \qquad \boldsymbol{v}_{\rm cm3} = \frac{1}{u_3}\boldsymbol{n} - \frac{2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}}{\varepsilon_\perp u_{3\perp}}\boldsymbol{n}_\perp, \tag{D12}$$

with $u_{\lambda\perp} = \mathcal{P}^{0(\lambda)}/|\boldsymbol{\mathcal{P}}_\perp^{(\lambda)}|$.

Now, we consider the dispersion equation [Eq. (11)] with the infrared approximation of the vacuum polarization tensor given in Eq. (38). The corresponding solutions are given by

$$\omega_1 = |\boldsymbol{k}|, \qquad \omega_2 = \sqrt{\boldsymbol{k}_\parallel^2 + \boldsymbol{k}_\perp^2 \frac{\mu_\perp^{-1}}{\varepsilon_\parallel}}, \qquad \omega_3 = \sqrt{\boldsymbol{k}_\parallel^2 + \boldsymbol{k}_\perp^2 \frac{\mu_\perp^{-1}}{\varepsilon_\parallel}}. \tag{D13}$$

With these expressions in mind, it is a straightforward calculation to show that the group velocity $\boldsymbol{v}_\lambda = \partial\omega_\lambda/\partial\boldsymbol{k}$ of each eigenwave coincides with the respective velocity of energy transport [Eq. (D12)].

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