Hitchin functionals are related to measures of entanglement

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According to the black hole/qubit correspondence (BHQC) certain black hole entropy formulas in supergravity can be related to multipartite entanglement measures of quantum information. Here we show that the origin of this correspondence is a connection between Hitchin functionals used as action functionals for form theories of gravity related to topological strings and entanglement measures for systems with a small number of constituents. The basic idea acting as a unifying agent in these seemingly unrelated fields is stability connected to the mathematical notion of special prehomogeneous vector spaces associated to Freudenthal systems coming from simple Jordan algebras. It is shown that the nonlinear function featuring these functionals and defining Calabi-Yau and generalized Calabi-Yau structures is the Freudenthal dual, a concept introduced recently in connection with the BHQC. We propose to use the Hitchin invariant for three-forms in seven dimensions as an entanglement measure playing a basic role in classifying three-fermion systems with seven modes. The representative of the class of maximal tripartite entanglement is the three-form used as a calibration for compactification on manifolds with G_2 holonomy. The idea that entanglement measures are related to action functionals from which the usual correspondence of the BHQC follows at the tree level suggests that one can use the BHQC in a more general context.

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I. INTRODUCTION

The main motivation of the present paper is to generalize further the recently discovered black hole/qubit correspondence (BHQC) [1,2]. This correspondence is based on striking mathematical connections found recently between two seemingly unrelated research areas: black hole solutions in string theory[3] and the theory of multipartite entanglement measures [4] in quantum information (QI) [5].

The main correspondence is between the structure of the Bekenstein-Hawking entropy formulas of extremal Bogomol'nyi-Prasad-Sommerfeld (BPS) or non-BPS black hole solutions in supergravity and certain multipartite entanglement measures of composite quantum systems with either distinguishable or indistinguishable constituents [6–11]. As another aspect of the correspondence it has also been realized that the classification problem of entanglement types of special entangled systems and special types of black hole solutions can be related [1,7,12]. Using this input coming from the physics of black holes the BHQC motivated the introduction of new entanglement measures [13,14] and helped to classify the entanglement patterns of certain quantum systems with a small number of constituents [12–15].

Apart from structural correspondences between measures and classes of entanglement and entropy formulas and classes of solutions for black holes, the BHQC also addressed issues of dynamics. In particular the dynamics of moduli stabilization related to the attractor mechanism [16] has been shown to correspond to a distillation procedure of entangled states of a very special kind on the black hole event horizon [8,17]. This result has recently been demonstrated in the IIB picture via the use of entangled states associated to wrapping configurations of three-branes [18,19] on T^6. These complex states are depending on the black hole charges and the complex structure moduli [19].

The BHQC has revealed the interesting finite geometric structure of black hole entropy formulas [20,21] and related them to Mermin squares [22], error correcting codes [1], and graph states [17] objects playing an important role in quantum information. It is also important to note that algebraic structures like Freudenthal triple systems [23–25] that have already been well-known to the supergravity community [26] have made their debut to the theory of quantum entanglement via the BHQC [13,14,27]. On the other hand reconsidered in the light of quantum information, applications of these systems to the physics of black holes have also resulted in introducing useful notions such as black holes admitting a Freudenthal and Jordan dual [28].

As the main reason for the BHQC, usually the occurrence of similar *symmetry structures* is emphasized [1,2]. Indeed, on the string theory side there are the U-duality groups [29] leaving invariant the black hole entropy formulas; on the other hand on the quantum information theoretic side there are the groups of admissible transformations [30,31] used to represent local manipulations on the entangled subsystems leaving invariant the corresponding entanglement measures. The U-duality groups in string theory are real but the groups of admissible transformations in QI are complex. However, under special circumstances the real U-duality groups should be embedded into the complex domain using reality conditions [8,12,19,32] rendering the techniques of QI applicable.

In this paper as another reason for the BHQC we would like to propose the notion of *stability*. This notion turns out to be a useful one since via the attractor mechanism it can naturally be related to the dynamic aspects of the BHQC. Interestingly in the separated research areas of string and quantum information theory the idea of stability appeared nearly at the same time. In quantum information Klyachko proposed [33] (semi)stability as a useful idea to capture entanglement for a system characterized by a dynamical symmetry group. In this context the role of special invariants as entanglement measures separating unstable orbits from stable ones has been emphasized. On the string theory side research was initiated by the influential mathematical papers of Hitchin on stable forms [34] and their connection to generalized Calabi-Yau spaces [35,36] and manifolds with special holonomy. In these papers functionals based on nondegenerate stable forms have been constructed. It is then shown that their extremal properties are related to the existence of special geometric structures on six-, seven-, and eight-dimensional manifolds. The possibility for introducing these structures rests on the applicability of an important notion. This idea is well-known to mathematicians, however, not yet fully appreciated by physicists. This is the notion of a *prehomogeneous vector space* (PV) as introduced by Sato and Kimura in their classical work [37,38].

A prehomogeneous vector space is a triple (G, R, V) where V is a finite dimensional vector space over $\mathbb{C}$, G is a linear algebraic group, and R is a rational representation $R\colon G \to GL(V)$ such that for a generic element $v \in V$ G has an *open dense orbit* $\rho(G)v$ in V. An element $v \in V$ is called *stable* if it lies in such an open orbit of G.

In the setting of entanglement V should correspond to the state space of our quantum system consisting of a finite number of subsystems. In the case of distinguishable constituents V has a tensor product structure. For indistinguishable ones we have either the symmetric or the antisymmetric tensor product structure corresponding to bosons or fermions. The group action G and its orbit should represent the admissible local operations on the subsystems and the *generic* entanglement class, respectively. One is then left to define polynomial invariants called entanglement measures, which are usually relative invariants. This means that they are invariants up to a character of G. In this picture the open dense orbit should be characterized by the *nonvanishing* of a particular relative invariant. Stability then would mean that states in a neighborhood of a particular one are equivalent with respect to the group G of local manipulations.

Clearly using stability for a definition of entanglement via PVs is too restrictive. (That was the reason for using the notion of semistability instead.) This is because for prehomogeneous vector spaces one should have [37] $\dim G - \dim G_v = \dim V$ where G_v is the stabilizer of a $v \in V$. Since the dimension of G has a slower growth than the dimension of V the notion of stability as related to PVs works only for characterizing special entangled systems. These are the ones with a small number of constituents. Remarkably such systems are the ones also related to the BHQC.

On the string theory side in form theories of quantum gravity [39] PVs show up via the use of *stable forms* [34]. In six dimensions these form theories are related to topological strings [40]. In seven dimensions they define the low energy limit of topological M theory. Generally these form theories are based on action principles for p-forms on a manifold M. The actions involve a volume element constructed in a nontrivial way from the p-form. At each point of the manifold M the vector space V for the PV is arising as the space $\wedge^p W^*$ where W is the tangent space at a point of M. Now in this context a p-form ϱ is stable at $m \in M$ if it lies in an open orbit of the $GL(W)$ action on $\wedge^p W^*$. ϱ is stable if it is stable at every point of M.

The physical significance of stable forms stems from the Ooguri-Strominger-Vafa (OSV) conjecture [41–43]. OSV suggested a relation between black hole entropy and the partition function of topological strings. Later work has revealed [39,44] that at the classical level black hole entropy and the topological string partition function are also related to Hitchin's functional [35] for real three-forms on a six-dimensional manifold M. The critical points of Hitchin's functional define a Calabi-Yau structure on M. Then the idea was to use Hitchin's functional also to recover the quantum corrections that have already been calculated via topological string techniques [45]. It turned out [46] that in order to achieve agreement at one loop level one has to use the so-called generalized Hitchin functional [36] instead. Now this new generalized functional [36] contains polyforms of either even or odd degree, with its critical points defining generalized Calabi-Yau spaces [36,47]. A further generalization occurs if we are considering form theories of gravity in seven dimensions. Here the PVs in question are based on the vector spaces $\wedge^3 \tilde{W}^*$ and $\wedge^4 \tilde{W}^*$ where now the seven-dimensional vector space $\tilde{W}$ is the tangent space of the 7-manifold $\tilde{M}$ at a point. The group G is $GL(\tilde{W})$ and the stabilizer of a generic form is the exceptional group G_2. Using these stable forms then one can define functionals that via their critical points generate G_2 holonomy on the 7-manifold. There is a natural connection [34] between these functionals and the Hitchin functionals of the 6-manifold M with critical points being manifolds with $SU(3)$ holonomy. This connection gives rise to a relation [39] between topological M theory on the 7-manifold $\tilde{M}$ and topological string theory on the 6-manifold M.

The proposal we would like to put forward in this paper is to regard the invariants underlying these functionals as entanglement measures for special entangled systems with the class of stable forms corresponding to the class of genuine entangled states. This idea makes it possible to generalize the BHQC substantially. First of all entanglement measures are now related to *action functionals* from which one can recover at the *semiclassical level* the usual correspondence of the BHQC found between the Bekenstein-Hawking entropy and some entanglement measure. However, since one loop calculations [46] based on *quantization* of such functionals are also capable of reproducing results obtained by topological string techniques [45], this interpretation also suggests that one can use the entanglement measures of the BHQC in a more general context. Second since possessing a stable form is far less restrictive than the requirement of special holonomy, in this generalized version of the BHQC one does not have to assume the metric to be of the special holonomy (Calabi-Yau, etc.) form. Third, after identifying Hitchin's invariants with entanglement measures a reconsideration of the results of the BHQC on the attractor mechanism [8,17] provides a new way of looking at the dynamic aspects of the BHQC.

The organization of the paper is as follows. In Sec. II. we summarize the basic material concerning the simplest of tripartite entangled systems both for distinguishable and for indistinguishable constituents. In Sec. III. we introduce Hitchin's functional for the real three-form ϱ with the underlying invariant related to the canonical entanglement measure for three fermions with six single particle states. Here as a novelty the usual nonlinear function $\hat{\varrho}(\varrho)$ is expressed in terms of the Freudenthal dual [28] originating from the cubic Jordan algebra of 3×3 complex matrices [23]. In terms of the associated Freudenthal system the symplectic structure and the corresponding Hamiltonian system [35] are expressed in an elegant manner. In special subsections of Sec. III. we also consider truncations giving rise to entangled systems with $SP(6, \mathbb{C})$ and $SP(2, \mathbb{C})^{\times 3}$ as the group of admissible transformations. Stable forms of the corresponding real cases give rise to familiar structures known from type IIB compactifications on T^6 and $T^2 \times T^2 \times T^2$ (STU model). A further truncation with three bosonic qubits corresponds to the t^3 model. In Sec. IV. we consider the generalized Hitchin functional. It is shown that the corresponding invariant gives rise to an entanglement measure for a fermionic system with six single particle states with either an even or an odd number of particles. An alternative interpretation can also be given in terms of the tripartite entanglement of six qubits, a structure living naturally inside the recently discovered tripartite entanglement of seven qubits [9,10,21]. Instead of the usual way of writing this invariant in terms of pure spinors we present a Freudenthal triple based description which is coming from the cubic Jordan algebra of 3×3 matrices with biquaternionic entries. As an illustration on the string theory side we relate this invariant to the work of Pestun for $N = 2$ compactification on T^6 with more general backgrounds. In Sec. V. we propose to use Hitchin's invariant based on three-forms in seven dimensions as an entanglement measure playing a basic role in classifying three-fermion systems with seven single particle states. We reinterpret the classification of three-forms in seven dimensions [48,49] as the classification of entanglement classes under the so-called Stochastic Local Operations and Classical Communication group [30] of quantum information. We emphasize that the representative of the class of maximal tripartite entanglement is related to the usual three-form used as a calibration for compactifications on manifolds with G_2 holonomy and the structure of the octonions. Section VI is left for the comments and the conclusions. Here we also speculate on the meaning of our entangled *states*, and suggest trying to connect them via the OSV conjecture to topological string theory. These attempts might pave the way for finding the physical basis of the BHQC. For the convenience of the reader, in the Appendix we summarized the material needed for a Freudenthal triple based description of the generalized Hitchin functional.

II. ENTANGLEMENT

A. Distinguishable constituents

In QI theory entangled systems with distinguishable constituents are represented by vectors in a tensor product of finite dimensional Hilbert spaces [4,5]. In the special case of pure states of a multiqubit system states are elements of the complex vector space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$ where the number of two-state spaces equals the number of qubits. For example a three-qubit state can be written in the form

$$|\psi\rangle = \sum_{i,j,k=0,1} \psi_{ijk} |i\rangle_1 \otimes |j\rangle_2 \otimes |k\rangle_3 \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2, \quad (1)$$

where the subscripts 1, 2, 3 refer to the distinguishable subsystems. Since entanglement in QI is regarded as a resource for performing different tasks, the central problem is to characterize different types of entanglement. Since entanglement is a global phenomenon, local transformations are supposed to have no effect on the entanglement types. According to this idea entanglement types of, say, three qubits should correspond to different *orbits* under some set of local transformations of the form

$$|\psi\rangle \mapsto (S_1 \otimes S_2 \otimes S_3)|\psi\rangle. \quad (2)$$

Here a specification of the local operators S_1, S_2, S_3 defines a classification scheme of entanglement types. The restriction for these operators to be unitary is an obvious choice; however, for practical reasons other classification schemes proved to be useful. Choosing local equivalence under the action of operators belonging to the group $GL(2, \mathbb{C})$ yields the so-called SLOCC orbits [30,31]. The name comes from the abbreviation of stochastic local operations and classical communication, referring to the particular type of protocols that can mathematically be represented by such invertible complex linear operators.

Entanglement measures are certain polynomials in the amplitudes of $|\psi\rangle$ satisfying a number of physically sensible criteria. For our concern the most important of these criteria is that they should be (relative) invariants under the action of the SLOCC group. In our special case of three qubits the quartic polynomial [50–52]

$$\begin{aligned} D(\psi) = {} & [\psi_0\psi_7 - \psi_1\psi_6 - \psi_2\psi_5 - \psi_3\psi_4]^2 \\ & - 4[(\psi_1\psi_6)(\psi_2\psi_5) + (\psi_2\psi_5)(\psi_3\psi_4) \\ & + (\psi_3\psi_4)(\psi_1\psi_6)] + 4\psi_1\psi_2\psi_4\psi_7 \\ & + 4\psi_0\psi_3\psi_5\psi_6, \end{aligned} \tag{3}$$

where $(\psi_0, \psi_1, \ldots, \psi_7) \equiv (\psi_{000}, \psi_{001}, \ldots, \psi_{111})$, gives rise to a famous entanglement measure called the *three tangle* [51] which for normalized states satisfies

$$0 \leq \tau_{123} = 4|D(\psi)| \leq 1. \tag{4}$$

Under SLOCC transformations $D(\psi)$ transforms as

$$D(\psi) \mapsto (\mathrm{Det}S_1)^2(\mathrm{Det}S_2)^2(\mathrm{Det}S_3)^2 D(\psi), \tag{5}$$

hence this polynomial is a relative invariant.

The classification problem of SLOCC entanglement types has been solved by mathematicians [52], the proof has later been independently rediscovered by physicists [31]. According to this result there are six nontrivial SLOCC entanglement classes. The genuine entanglement class with normalized representative is the so-called Greenberger-Horne-Zeilinger (GHZ) class [53]

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{6}$$

It is characterized by the constraint $D(\psi) \neq 0$. The so-called W class [31] represented by

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \tag{7}$$

has $D(\psi) = 0$; however, states belonging to this class still contain some sort of tripartite entanglement [31]. The remaining four classes are separable. This means that their representatives are either of the form $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |0\rangle$ or two similar states with the qubits cyclically permuted (biseparable states), or represented by $|000\rangle$ (totally separable states).

The corresponding classification of SLOCC entanglement types over the reals (i.e., the classification for three rebits [54]) has also been used by physicists [55]. In this case rebits live in $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ and the SLOCC group is three copies of $GL(2, \mathbb{R})$. The result in this case is that the GHZ class splits into two classes. One of them is the usual one with representative as given by Eq. (6) and $D(\psi) > 0$. However, now we have an extra class with $D(\psi) < 0$ with representative

$$|GHZ\rangle_- = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle). \tag{8}$$

Note that the state

$$|GHZ\rangle_+ = \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle), \tag{9}$$

with $D(\psi) > 0$ is the real SLOCC equivalent to the one of Eq. (6). Indeed

$$|GHZ\rangle = (H \otimes H \otimes H)|GHZ\rangle_+, \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{10}$$

where H is the Hadamard matrix of discrete Fourier transformation. Notice also that the new state $|GHZ\rangle_-$ as a real one can be embedded into $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ as a state

$$\begin{aligned} |GHZ\rangle_- &= \frac{1}{\sqrt{2}}(|\alpha\rangle \otimes |\alpha\rangle \otimes |\alpha\rangle + \overline{|\alpha\rangle \otimes |\alpha\rangle \otimes |\alpha\rangle}), \\ |\alpha\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle). \end{aligned} \tag{11}$$

B. Indistinguishable constituents: Fermions

The notion of entanglement can also be generalized to include systems with indistinguishable parts [56]. In the following we need results merely from the theory of fermionic entanglement. We consider fermions on an M dimensional single particle Hilbert space $V = \mathbb{C}^M$. The observables are generated by the operators $f^\dagger$ and f satisfying the usual canonical anticommutation relations $\{f_k, f_l{}^\dagger\} = \delta_{kl}$, $\{f_k, f_l\} = 0$, $\{f_k{}^\dagger, f_l{}^\dagger\} = 0$. It is clear that $f_j^\dagger$ creates a particle in the mode, or single particle state, corresponding to the basis vector e_j of $\mathbb{C}^M$. The Hilbert space of the fermionic system is spanned by the basis

$$(f_1^\dagger)^{n_1}(f_2^\dagger)^{n_2}\ldots(f_M^\dagger)^{n_M}|0\rangle, \tag{12}$$

where $n_j \in \{0, 1\}$ and the vacuum state $|0\rangle$ satisfies $f_j|0\rangle = 0$, $\forall j$. The N particle subspace of the Fock space is spanned by those vectors that satisfy the constraint $\sum_j n_j = N$.

For example an (unnormalized) three-fermion state ($N = 3$) with six single particle states or *modes* ($M = 6$) is represented by the state vector

$$|P\rangle = \sum_{1\leq i_1<i_2<i_3\leq 6} P_{i_1i_2i_3} f_{i_1}^\dagger f_{i_2}^\dagger f_{i_3}^\dagger|0\rangle, \tag{13}$$

where the $P_{i_1i_2i_3}$ are 20 complex amplitudes characterizing the state. It is convenient to use another representation for such fermion states as multilinear forms. Hence the state $|P\rangle$ can also be represented by a *three-form* over the space $V = \mathbb{C}^6$ as

$$P = \sum_{1\le i_1<i_2<i_3\le 6} P_{i_1 i_2 i_3} e^{i_1}\wedge e^{i_2}\wedge e^{i_3} \in \wedge^3 V^*, \tag{14}$$

where $\{e^j\}$, $j = 1, \ldots 6$ are basis vectors of V^* dual to the basis vectors $\{e_j\}$ of V.

Due the indistinguishable nature of the subsystems SLOCC transformations are acting on our fermion states with the *same* $GL(M, \mathbb{C})$ transformations to be applied to each slot. For example for $V = \mathbb{C}^6$ the SLOCC transformation $|P\rangle \mapsto (S \otimes S \otimes S)|P\rangle$ is represented by

$$P_{i_1 i_2 i_3} \mapsto P_{j_1 j_2 j_3} S^{j_1}{}_{i_1} S^{j_2}{}_{i_2} S^{j_3}{}_{i_3}, \qquad S = S^j{}_i e^i \otimes e_j \in GL(V), \tag{15}$$

coming from the transformation rule $P \mapsto S^* P$ for three-forms.

There is a quartic polynomial which is a relative invariant with respect to the SLOCC group [13,37]. In order to define this polynomial we reorganize the 20 independent complex amplitudes $P_{i_1 i_2 i_3}$ into two complex numbers η, ξ and two complex 3×3 matrices X and Y as follows. As a first step we change our labeling convention by using the symbols $\bar{1}, \bar{2}, \bar{3}$ instead of 4, 5, 6, respectively. The meaning of the labels 1, 2, 3 is not changed. Hence for example we can alternatively refer to P_{456} as $P_{\overline{123}}$ or to P_{125} as $P_{12\bar{2}}$. Now we define

$$\eta \equiv P_{123}, \qquad \xi \equiv P_{\overline{123}}, \tag{16}$$

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} \equiv \begin{pmatrix} P_{1\overline{23}} & P_{1\overline{31}} & P_{1\overline{12}} \\ P_{2\overline{23}} & P_{2\overline{31}} & P_{2\overline{12}} \\ P_{3\overline{23}} & P_{3\overline{31}} & P_{3\overline{12}} \end{pmatrix}, \tag{17}$$

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} \equiv \begin{pmatrix} P_{\bar{1}23} & P_{\bar{1}31} & P_{\bar{1}12} \\ P_{\bar{2}23} & P_{\bar{2}31} & P_{\bar{2}12} \\ P_{\bar{3}23} & P_{\bar{3}31} & P_{\bar{3}12} \end{pmatrix}. \tag{18}$$

With this notation the quartic polynomial is

$$\mathcal{D}(P) = [\eta\xi - \mathrm{Tr}(XY)]^2 - 4\,\mathrm{Tr}(X^\sharp Y^\sharp) + 4\eta \mathrm{Det}(X) + 4\xi \mathrm{Det}(Y), \tag{19}$$

where $X^\sharp$ and $Y^\sharp$ correspond to the regular adjoint matrices for X and Y; hence for example $XX^\sharp = X^\sharp X = \mathrm{Det}(X) I$ with I the 3×3 identity matrix [see also Eq. (A23) in the Appendix]. Clearly the structure of our new polynomial $\mathcal{D}(P)$ is very similar to the one of $D(\psi)$ we defined in Eq. (3). Later on this will be important for us. $\mathcal{D}(P)$ defines an entanglement measure similar to the three tangle in the form [13]

$$0 \le \mathcal{T}_{123} = 4|\mathcal{D}(P)|, \tag{20}$$

where $\mathcal{T}_{123} \le 1$ for normalized states.

There is an alternative way of describing this polynomial. Let us define a symplectic form on $\wedge^3 V^*$ as follows:

$$\{\cdot,\cdot\}\colon\ \wedge^3 V^* \times \wedge^3 V^* \to \mathbb{C}, \qquad (P, Q) \mapsto \frac{1}{3!3!}\varepsilon^{abcijk} P_{abc} Q_{ijk}, \tag{21}$$

where the 2×20 amplitudes of P and Q have been extended to totally antisymmetric tensors of rank three and we used the summation convention. Now we define $\tilde{P}$ for the three-form $P \in \wedge^3 V^*$ as

$$\tilde{P} = \frac{1}{3!}\tilde{P}_{abc} e^a \wedge e^b \wedge e^c, \qquad \tilde{P}_{abc} = \frac{1}{2!3!}\varepsilon^{d i_2 i_3 i_4 i_5 i_6} P_{bcd} P_{a i_2 i_3} P_{i_4 i_5 i_6}. \tag{22}$$

The quartic invariant then takes the form

$$\mathcal{D}(P) = \frac{1}{2}\{\tilde{P}, P\}. \tag{23}$$

In the theory of Freudenthal triple systems the quantity $\tilde{P}$ which is cubic in the original amplitudes of P is usually defined via the so-called trilinear form [23]. With the help of $\tilde{P}$ for a state with $\mathcal{D} \neq 0$ one can define the quantity

$$\hat{P} \equiv \frac{-\tilde{P}}{\sqrt{|\mathcal{D}|}}. \tag{24}$$

$\hat{P}$ is the *Freudenthal dual* of P as defined by the paper [28] of Borsten *et al.*

The classification problem for three-forms in $V = \mathbb{C}^6$ was solved long ago by mathematicians [57]; in the context of fermionic entanglement it has recently been rediscovered by physicists [13]. According to this result we have four disjoint SLOCC classes. The representatives of these classes can be brought to the following form:

$$P = \frac{1}{2}(e^1\wedge e^2\wedge e^3 + e^1\wedge e^{\bar{2}}\wedge e^{\bar{3}} + e^2\wedge e^{\bar{3}}\wedge e^{\bar{1}} + e^3\wedge e^{\bar{1}}\wedge e^{\bar{2}}), \qquad \mathcal{D}(P) \neq 0, \tag{25}$$

$$P = \frac{1}{\sqrt{3}}(e^1\wedge e^2\wedge e^3 + e^1\wedge e^{\bar{2}}\wedge e^{\bar{3}} + e^2\wedge e^{\bar{3}}\wedge e^{\bar{1}}), \qquad \mathcal{D}(P) = 0, \qquad \tilde{P} \neq 0, \tag{26}$$

$$P = \frac{1}{\sqrt{2}} e^1\wedge(e^2\wedge e^3 + e^{\bar{2}}\wedge e^{\bar{3}}), \qquad \mathcal{D}(P) = 0, \qquad \tilde{P} = 0, \tag{27}$$

$$P = e^1\wedge e^2\wedge e^3, \qquad \mathcal{D}(P) = 0, \qquad \tilde{P} = 0. \tag{28}$$

In analogy with the three-qubit case we will refer to the first two classes as the GHZ and W class. In order to separate the last two classes (i.e., the biseparable and separable ones) one has to use the Plücker relations [13]. Clearly the GHZ and W classes are the two inequivalent classes for tripartite entangled fermionic systems with six modes. These classes are completely characterized by the relative invariant $\mathcal{D}(P)$ and the dual state $\tilde{P}$ (a covariant). The GHZ class corresponds to a stable orbit. This fact is related to the result that our system corresponds to a PV which is the class no. 5 in the Sato-Kimura classification [37]. As in the previous section, after restricting to the real case, the GHZ class splits into two classes. The canonical states are of the form of Eq. (25) with $D(P) > 0$ and the extra state

$$P = \frac{1}{2}(e^1 \wedge e^2 \wedge e^3 - e^1 \wedge e^{\bar{2}} \wedge e^{\bar{3}} - e^2 \wedge e^{\bar{3}} \wedge e^{\bar{1}} - e^3 \wedge e^{\bar{1}} \wedge e^{\bar{2}}), \tag{29}$$

with $D(P) < 0$.

C. Embedded systems

From the SLOCC classification of fermionic entanglement one can derive other entanglement classes by restricting to a subgroup of the SLOCC group. One way of achieving this is to constrain the set of admissible transformations to ones that are also leaving invariant some extra structure.

In the special case of three fermions with six single particle states we can consider a fixed symplectic form $\omega \in \wedge^2 V^*$ on $V = \mathbb{C}^6$ and constrain the local operations to the set leaving ω invariant. In this way we obtain the subgroup $GL(1, \mathbb{C}) \times SP(6, \mathbb{C}) \subset GL(6, \mathbb{C})$. If we restrict the SLOCC group $GL(6, \mathbb{C})$ to this group the 20-dimensional representation space decomposes to the direct sum of a 14- and a six-dimensional representation irreducible under $Sp(6, \mathbb{C})$.

$$\wedge^3 V^* = \omega \wedge V^* \oplus \wedge^3_0 V^*. \tag{30}$$

Here $\wedge^3_0 V^*$ refers to the space of primitive three-forms P satisfying $\omega \wedge P = 0$. Choosing the fixed symplectic form as the one,

$$\omega = e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6 = e^1 \wedge e^{\bar{1}} + e^2 \wedge e^{\bar{2}} + e^3 \wedge e^{\bar{3}}, \tag{31}$$

one can see that the constraint $\omega \wedge P = 0$ yields the one

$$P_{a14} + P_{a25} + P_{a36} = 0, \qquad 1 \leq a \leq 6. \tag{32}$$

In the language of the 3×3 matrices of Eqs. (17) and (18) this means that

$$X^t = X, \qquad Y = Y^t. \tag{33}$$

Taken together with η and ξ of Eq. (16) we obtain the $1 + 6 + 6 + 1 = 14$ independent components of a three-form in $\wedge^3_0 V^*$.

The entanglement classes under the restricted group of admissible transformations turn out to be of the same structure as the ones under $GL(6, \mathbb{C})$. The orbit corresponding to the GHZ class is again a stable orbit. This property dates back to the fact that the system we have considered is a PV which is class no. 14 in the Sato-Kimura classification [37]. The classification of real entanglement classes is more involved [58]. For an explicit list see the appendix of the paper of Bryant in the first of Ref. [59].

One can even restrict further the SLOCC group $GL(6, \mathbb{C})$ by regarding V as the direct sum of three two-dimensional complex vector spaces. In this case we have $V = \mathbb{C}^6 = V_1 \oplus V_2 \oplus V_3$. Let us furnish V_j, $j = 1, 2, 3$ with the symplectic forms $\omega_j \equiv e^j \wedge e^{\bar{j}}$ and demand that the admissible set of transformations is the one leaving the symplectic forms *one by one* invariant. This means that there is a group action $Sp(2, \mathbb{C})^{\oplus 3} \simeq SL(2, \mathbb{C})^{\oplus 3}$ on $V = V_1 \oplus V_2 \oplus V_3$. Taken together with an overall complex rescaling we obtain the SLOCC group $GL(2, \mathbb{C})^{\oplus 3}$. Now in this way we obtain the constraints $P_{a14} = P_{a25} = P_{a36} = 0$, $1 \leq a \leq 6$. This means that only the 8 amplitudes $P_{123}, P_{12\bar{3}}, P_{1\bar{2}\bar{3}}, \ldots P_{\overline{123}}$ are surviving. In this way the labels of the V_j can be mapped to the labels of three *distinguishable* qubits. Labeling the qubits from the left to the right under the correspondence

$$(P_{123}, P_{12\bar{3}}, P_{1\bar{2}\bar{3}}, \ldots P_{\overline{123}}) \leftrightarrow (\psi_{000}, \psi_{001}, \psi_{010}, \ldots, \psi_{111}), \tag{34}$$

and a similar one for the basis vectors ($e^1 \wedge e^2 \wedge e^3 \mapsto |000\rangle$, etc.), our special three-form can be mapped to a three-qubit state with the usual SLOCC group $GL(2, \mathbb{C})^{\times 3}$ acting on it. Now the labels 1, 2, 3 are referring to the labels of the distinguishable constituents; on the other hand numbers without an overline correspond to "0" and ones with an overline correspond to "1." Notice also that in this case the invariant $\mathcal{D}(\mathcal{P})$ of Eq. (19) restricts to Cayley's hyperdeterminant $D(\psi)$ as given by Eq. (4).

This example can be generalized for different possible splits of $V = \mathbb{C}^6$ with the result of different special entangled systems [14]. All of them and their corresponding restricted sets of SLOCC transformations are embedded into V and the basic $GL(6, \mathbb{C})$ action on it. One particular example that we need later can be obtained as follows. Let us consider the embedding as given by Eq. (34) and as a further restriction demand that the admissible transformation consists of the action of the *same* $S \in GL(2, \mathbb{C})$ on each of the V_js. In the three-qubit picture this means that the restricted SLOCC group now acts as

$$|\psi\rangle \mapsto (S \otimes S \otimes S)|\psi\rangle, \qquad S \in GL(2, \mathbb{C}), \tag{35}$$

with $|\psi\rangle$ of the form

$$|\psi\rangle = \psi_{000}|000\rangle + \ldots + \psi_{111}|111\rangle, \quad \psi_{001} = \psi_{010} = \psi_{110}, \qquad \psi_{110} = \psi_{101} = \psi_{011}. \tag{36}$$

Hence in this case the number of independent complex amplitudes is 4 and the representation space for the $GL(2, \mathbb{C})$ action is the symmetrized tensor product of three $\mathbb{C}^2$s. Clearly this situation is describing three indistinguishable *bosonic* qubits. The relative invariant which is the entanglement measure characterizing this situation is a convenient truncation of Cayley's hyperdeterminant of Eq. (4),

$$d(x) = x_1^2x_4^2 - 6x_1x_2x_3x_4 + 4x_1x_3^3 + 4x_2^3x_4 - 3x_1^2x_4^2, \tag{37}$$

where

$$\begin{aligned} x_1 &= \psi_{000}, & x_2 &= \psi_{001}, \\ x_3 &= \psi_{110}, & x_4 &= \psi_{111}. \end{aligned} \tag{38}$$

This example gives rise to a particular PV called no. 4 in the Sato-Kimura classification scheme of regular PVs [37].

Notice, however, that all of our embedded systems were based on the $V = \mathbb{C}^6$ case which is very special. It is based on the Freudenthal system related to the cubic Jordan algebra of 3×3 complex matrices [23]. This can be regarded as the "complexification" [23] of the Jordan algebra of 3×3 Hermitian matrices. According to Eqs. (16) and (17) this gives $1 + 9 + 9 + 1 = 20$ components for the corresponding Freudenthal triple system. This system also gives rise to a PV. On the other hand the prehomogeneous vector space we encountered in the beginning of this subsection is related to the complexification of the simple Euclidean Jordan algebra of rank three based on the 3×3 symmetric matrices. This gives rise to $1 + 6 + 6 + 1 = 14$ components for the corresponding Freudenthal triple system. There are two more PVs of that type. These are the no. 23 and 29 classes in the Sato-Kimura classification [37]. They are related to Freudenthal systems based on the compexifications of the Jordan algebras of 3×3 quaternion and octonian Hermitian matrices [23]. They give rise to PVs with the corresponding splits and dimensions: $1 + 15 + 15 + 1 = 32$ and $1 + 27 + 27 + 1 = 56$. One expects that the stable orbits of these PVs should give rise to GHZ-like classes and some particular relative invariants that can be used as entanglement measures. Before justifying our expectations we have to turn our attention to string theory, the field where these exotic structures have first been applied.

III. HITCHIN'S FUNCTIONAL

A. Hitchin's invariant as an entanglement measure

Let us consider the *real vector space* $W = \mathbb{R}^6$ and the three-form $\varrho \in \wedge^3 W^*$. Then after introducing the 6×6 matrix

$$(K_\varrho)^a{}_b = \frac{1}{2!3!}\varepsilon^{ai_2i_3i_4i_5i_6}\varrho_{bi_2i_3}\varrho_{i_4i_5i_6}. \tag{39}$$

Hitchin's invariant [35] can be expressed as

$$\lambda(\varrho) = \frac{1}{6}(K_\varrho)^a{}_b(K_\varrho)^b{}_a. \tag{40}$$

Clearly after identifying a *real* three-form $P \in \wedge^3 W^*$ from the previous subsection with ϱ and using Eq. (19), one obtains

$$\lambda(\varrho) = \mathcal{D}(\varrho). \tag{41}$$

Hence Hitchin's invariant $\lambda(\varrho)$ is just our relative invariant used as an entanglement measure in the previous section.

Regarding as an endomorphism of V one can write $K_\varrho = (K_\varrho)^a{}_b e^b \otimes e_a$. It is known [35] that $\mathrm{Tr}K_\varrho = 0$, hence $K_\varrho \in sl(6, \mathbb{R}) \subset gl(6, \mathbb{R})$. As a Lie-algebra element with trace zero, K_ϱ acts on a three-form as Ref. [47] (see also the Appendix in this respect)

$$K_\varrho \cdot \varphi = -\frac{1}{2}\mathrm{Tr}(K_\varrho)\varphi + (K_\varrho)^a{}_b e^b \wedge i_{e_a}\varphi = (K_\varrho)^*\varphi. \tag{42}$$

Then the correspondence between the two alternative ways of describing $\mathcal{D}(\varrho)$, namely, the one of Eqs. (23) and (40), is given by

$$\tilde{\varrho} = \frac{1}{3}(K_\varrho)^*\varrho, \tag{43}$$

giving rise to the formula $\tilde{\varrho}_{abc} = \varrho_{dbc}(K_\varrho)^d{}_a$.

Notice also that by virtue of Eq. (23) for real three-forms ϱ after employing the Freudenthal dual of Eq. (24) one can write

$$2\mathrm{sgn}(\mathcal{D})\sqrt{|\mathcal{D}(\varrho)|}\epsilon = \varrho \wedge \hat{\varrho}(\varrho), \tag{44}$$

where $\epsilon = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6$. We recall that

$$\mathcal{D}(\hat{\varrho}) = \mathcal{D}(\varrho), \tag{45}$$

and

$$\hat{\hat{\varrho}} = -\varrho. \tag{46}$$

It is important to realize that the latter three identities are satisfied for *all* Freudenthal triple systems [28], not merely the ones related to three-forms.

Our result is that Hitchin's nonlinear function $\hat{\varrho}(\varrho)$ is the Freudenthal dual of ϱ. Apart from giving an explicit formula for $\hat{\varrho}$ this result also elucidates many of the important formulas obtained in Ref. [35]. Notice for example that Eq. (11) of that paper is just a special case of our Eq. (44) when $\lambda(\varrho) = \mathcal{D}(\varrho) < 0$. Moreover, for *both* of the two real GHZ-like entanglement classes the important identity Eq. (46) holds. For these two classes the forms

$$\alpha = \varrho + \hat{\varrho}(\varrho), \qquad \beta = \varrho - \hat{\varrho}(\varrho), \qquad \mathcal{D}(\varrho) > 0, \tag{47}$$

and

$$\Omega = \varrho + i\hat{\varrho}(\varrho), \quad \bar{\Omega} = \varrho - i\hat{\varrho}(\varrho), \quad \mathcal{D}(\varrho) < 0 \tag{48}$$

belong to the fully separable entanglement class [35]. In either case we have a two term decomposition for ϱ, namely, $\varrho = (\alpha + \beta)/2$ and $\varrho = (\Omega + \bar{\Omega})/2$ which is up to normalization of the canonical GHZ form. [In the case of the three-qubit embedding just have a look at Eqs. (6) and (11)] This trick clearly also works in the complex case; hence we have an explicit method for calculating the canonical form of entangled states belonging to the stable orbit. For $\mathcal{D} < 0$ via the property [35] $K_\varrho^2 = \mathcal{D}(\varrho)1$ the 6×6 matrix $I_\varrho = K_\varrho/\sqrt{-\mathcal{D}(\varrho)}$ defines a complex structure on W. With respect to this complex structure Ω is of type (3, 0).

Another important property that the Freudenthal formalism automatically takes care of is the nice symplectic geometry on the space of real three-forms [35]. The phase space is $\wedge^3 W^*$ with the symplectic form as defined by Eq. (21). According to Eq. (44) one can see that $H(\varrho) = \sqrt{|\mathcal{D}(\varrho)|}$ can be regarded as the Hamiltonian and the Hamiltonian vector field $X_H = \mp\hat{\varrho}(\varrho)$ for $\mathrm{sgn}(\mathcal{D}) = \mp 1$; i.e., up to sign it is just the Freudenthal dual of ϱ. Moreover one can see that K_ϱ is related to the moment map. Moreover, for the special case of real three-forms with $\mathcal{D}(\varrho) < 0$ according to proposition 5 of Ref. [35], the derivative of $-\hat{\varrho}$ at ϱ defines an integrable complex structure J_ϱ on the corresponding open orbit of stable forms. Note that we already have a complex structure on W defined by I_ϱ according to which we have the decomposition

$$\wedge^3 W^* \otimes \mathbb{C} = \wedge^{3,0} \oplus \wedge^{2,1} \oplus \wedge^{1,2} \oplus \wedge^{0,3}. \tag{49}$$

One can clarify the relationship between J_ϱ and I_ϱ by checking the action of J_ϱ on the type decomposition above. The result is that J_ϱ acts as i on $\wedge^{3,0} \oplus \wedge^{2,1}$ and as $-i$ on $\wedge^{1,2} \oplus \wedge^{0,3}$.

Generally since the symplectic properties rest on the ones of Freudenthal triple systems we can regard these as nice examples of classical mechanical systems. Then the symplectic form is the usual one defined for such systems and the square root of the magnitude of the quartic invariant [23] is playing the role of the Hamiltonian. The Freudenthal dual in all cases can then be regarded as the Hamiltonian vector field. This observation will be playing a role later.

B. Hitchin's functional and semiclassical black hole entropy

Let us now consider a closed oriented 6-manifold M and a real three-form ϱ with local coordinates x^a in a coordinate patch expressed as

$$\varrho = \frac{1}{3!}\varrho_{abc}(x)dx^a \wedge dx^b \wedge dx^c \in \wedge^3 T^* M. \tag{50}$$

Then Hitchin's functional is defined as

$$V_H(\varrho) = \int_M \sqrt{|\mathcal{D}(\varrho)|}d^6x, \tag{51}$$

where $\mathcal{D}(\varrho)$ related to our entanglement measure of Eq. (20) is defined by either Eq. (19) with P replaced by ϱ or Eqs. (40) and (41). Using the observation that $\hat{\varrho}$ is the Freudenthal dual of ϱ by virtue of Eq. (44), an alternative formula for this functional is

$$V_H(\varrho) = \frac{1}{2}\,\mathrm{sgn}(\mathcal{D}(\varrho)) \int_M \varrho \wedge \hat{\varrho}(\varrho). \tag{52}$$

In the special case when $\mathcal{D}(\varrho) < 0$ everywhere on M, each differential three-form ϱ defines an almost complex structure I_ϱ on M. If ϱ is a critical point of $V_H(\varrho)$ on a cohomology class of $H^3(M, \mathbb{R})$ ($d\varrho = 0$) then it follows [35] that we also have $d\hat{\varrho} = 0$. Hence the separable three-form $\Omega = \varrho + i\hat{\varrho}(\varrho)$ of type (3, 0) introduced in the previous subsection is closed and the almost complex structure I_ϱ defined by ϱ is integrable. As a result of these considerations a critical point or a classical solution of $V_H(\varrho)$ defines a complex structure on M with a nonvanishing holomorphic three-form Ω. Note that in terms of $\Omega = \varrho + i\hat{\varrho}(\varrho)$ Hitchin's functional is just the holomorphic volume of M,

$$V_H(\varrho) = -\frac{i}{4}\int_M \Omega \wedge \bar{\Omega}. \tag{53}$$

In particular Calabi-Yau threefolds used by string theorists in models of string compactification are Kähler manifolds with a nonvanishing holomorphic three-form Ω. Hence the *complex structure* of such manifolds can be derived from the critical points of $V_H(\varrho)$. The phenomenon of obtaining a particular complex structure from a fixed three-form ϱ also occurs in the case of $4D$ BPS black holes in type IIB string theory compactified on Calabi-Yau threefolds via the attractor mechanism [3,16]. In this case fixing the BPS charge configuration of the black hole solution amounts to fixing a homology class $\gamma \in H_3(M, \mathbb{Z})$ corresponding to a wrapping configuration of 3-branes. The cohomology class of ϱ then equals the Poincaré dual $\Gamma \in H^3(M, \mathbb{Z})$ of γ. The attractor mechanism provides a particular holomorphic three-form Ω at the black hole horizon in terms of the charges. Identifying the *real part* of Ω with ϱ the attractor mechanism gives the imaginary part $\hat{\varrho}$ in terms of $[\varrho] = \Gamma$.

This argument has been suggested in Ref. [39] to relate the value of $V_H(\varrho)$ at the critical point to the semiclassical Bekenstein-Hawking entropy. Since the main correspondence of the BHQC is the one existing between the semiclassical black hole entropy and certain entanglement measures, it is instructive to revisit this argument in the context of the BHQC using type IIB string theory. In type IIB compactification on a Calabi-Yau threefold M with holomorphic three-form Ω the resulting low energy theory is four-dimensional $N = 2$ supergravity. In this theory we have $h^{2,1}(M)$ vector multiplets. Let us denote by X^I, $I = 1, 2, \ldots h^{2,1}$ the scalar components of these multiplets describing the *complex structure* moduli of M. The vector multiplet part of the effective action is fully specified by the holomorphic prepotential $\mathcal{F}(X)$ defining a special Kähler geometry (with Kähler potential K) of the moduli space of M. Denoting $F_I = \partial_I \mathcal{F}$ we have

$$X^I = \int_{A^I} \Omega, \qquad F_I = \int_{B^I} \Omega, \tag{54}$$

$$\Omega = X^I \alpha_I - F_I(X)\beta^I, \tag{55}$$

and

$$K = -\log i(\bar{X}^I F_I - X^I \bar{F}_I). \tag{56}$$

Here $\{A^I, B_I\}$ form a basis for the three-cycles in $H_3(M, \mathbb{Z})$ and $\{\alpha_I, \beta^I\}$ are the dual basis three-forms of $H^3(M, \mathbb{Z})$. In this setting the holomorphic volume is

$$V_H(\varrho) = \frac{1}{4i}\int_M \Omega \wedge \bar{\Omega} = \frac{1}{2}\operatorname{Im}(X^I \bar{F}_I) = \frac{1}{4}e^{-K}. \tag{57}$$

Let us introduce for a $\gamma \in H_3(M, \mathbb{Z})$ its Poincaré dual Γ as

$$\Gamma = p^I \alpha_I - q_I \beta^I. \tag{58}$$

Then the central charge field is

$$\begin{aligned} Z(\gamma) &= e^{K/2}\int_\gamma \Omega = e^{K/2}\int_M \Omega \wedge \Gamma \\ &= e^{K/2}(p^I F_I - q_I X^I). \end{aligned} \tag{59}$$

One can show [3] that for static spherically symmetric extremal BPS black hole solutions the semiclassical black hole entropy is

$$S = \pi |Z|^2 = \pi \frac{|p^I F_I - q_I X^I|^2}{2\operatorname{Im} X^I \bar{F}_I}. \tag{60}$$

Here it is understood that S is depending on the moduli X^I and the charges p^I, q_I; moreover the values of the moduli fields should be taken at the black hole horizon. According to the attractor mechanism [3,16] these values for the moduli can be expressed in terms of the charges via the attractor equations

$$\operatorname{Re}(CX^I) = p^I, \quad \operatorname{Re}(CF_I) = q_I, \quad C = -2i\bar{Z}e^{K/2}. \tag{61}$$

Since the formula for the entropy is invariant under dilatations one can set $C = 1$ in Eq. (61). As a consequence of this the charges are just the *real parts* of the quantities of X^I and F_I; hence after putting this into Eq. (60) we get

$$S = \frac{\pi}{2}\operatorname{Im}(X^I \bar{F}_I). \tag{62}$$

Here the charges via Eq. (61) also determine the *imaginary* parts of X^I and F_I; hence S can be expressed entirely in terms of the charges p^I and q_I.

Let us now compare this implicit expression for the entropy as given by Eq. (62) and using Eq. (57), the similar expression for Hitchin's functional of Eq. (53). At the critical point of $V_H(\varrho)$ where ϱ determines the imaginary part $\hat{\varrho}$ of Ω we clearly have

$$S_{\mathrm{BH}} = \pi V_H(\varrho_{\mathrm{crit}}), \qquad [\varrho] = \Gamma. \tag{63}$$

This establishes a link between the value of the extremized action $V_H(\varrho)$ based on an entanglement measure $\mathcal{D}(\varrho)$ and the semiclassical black hole entropy.

C. An example: T^6

In order to elucidate the meaning of Eq. (63) we consider as our oriented closed 6-manifold the torus T^6. We choose real coordinates u^i, v^i, $i = 1, 2, 3$ and the orientation $\int_{T^6} du^1 \wedge dv^1 \wedge du^2 \wedge dv^2 \wedge du^3 \wedge dv^3 = 1$. Next we consider a wrapping configuration $\gamma \in H_3(T^6, \mathbb{Z})$ and we expand its Poincaré dual $\Gamma \in H^3(T^6, \mathbb{Z})$ in the basis satisfying $\int_{T^6} \alpha^I \wedge \beta_J = \delta^I{}_J$, $\quad I, J = 1, 2, \ldots 10$,

$$\begin{aligned} \alpha_0 &= du^1 \wedge du^2 \wedge du^3, \\ \alpha_{ij} &= \frac{1}{2}\varepsilon_{ii'j'} du^{i'} \wedge du^{j'} \wedge dv^j, \end{aligned} \tag{64}$$

$$\begin{aligned} \beta^0 &= -dv^1 \wedge dv^2 \wedge dv^3, \\ \beta^{ij} &= \frac{1}{2}\varepsilon_{ji'j'} du^i \wedge dv^{i'} \wedge dv^{j'}, \end{aligned} \tag{65}$$

as

$$\Gamma = p^0 \alpha_0 + P^{ij}\alpha_{ij} - Q_{ij}\beta^{ij} - q_0 \beta^0. \tag{66}$$

We write the nondegenerate real three-form ϱ belonging to the class with $\mathcal{D}(\varrho) < 0$ featuring Hitchin's functional $V_H(\varrho)$ as

$$\varrho = \sum_{1 \le a < b < c \le 6} \varrho_{abc} f^a \wedge f^b \wedge f^c, \tag{67}$$

where

$$\begin{aligned} (f^1, f^2, f^3, f^4, f^5, f^6) &\equiv (f^1, f^2, f^3, f^{\bar{1}}, f^{\bar{2}}, f^{\bar{3}}) \\ &= (du^1, du^2, du^3, dv^1, dv^2, dv^3). \end{aligned} \tag{68}$$

According to Eq. (63) up to an exact form we should identify ϱ with Γ. Explicitly this identification is given by the expressions

$$p^0 = \varrho_{123}, \quad \begin{pmatrix} P^{11} & P^{12} & P^{13} \\ P^{21} & P^{22} & P^{23} \\ P^{31} & P^{32} & P^{33} \end{pmatrix} = \begin{pmatrix} \varrho_{23\bar{1}} & \varrho_{23\bar{2}} & \varrho_{23\bar{3}} \\ \varrho_{31\bar{1}} & \varrho_{31\bar{2}} & \varrho_{31\bar{3}} \\ \varrho_{12\bar{1}} & \varrho_{12\bar{2}} & \varrho_{12\bar{3}} \end{pmatrix}, \tag{69}$$

$$q^0 = \varrho_{\overline{123}}, \quad \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} = \begin{pmatrix} \varrho_{1\overline{23}} & \varrho_{1\overline{31}} & \varrho_{1\overline{12}} \\ \varrho_{2\overline{23}} & \varrho_{2\overline{31}} & \varrho_{2\overline{12}} \\ \varrho_{3\overline{23}} & \varrho_{3\overline{31}} & \varrho_{3\overline{12}} \end{pmatrix}. \tag{70}$$

Now a critical point of $V_H(\varrho)$ gives rise to a fully separable state of the form $\Omega = \varrho + i\hat{\varrho}(\varrho)$ where $\hat{\varrho}$ is the Freudenthal dual of ϱ expressed in terms of the charges. For $\hat{\varrho}$ one can use Eq. (24) or the formulas

$$\hat{p}^{\,0} = \frac{-\tilde{p}^0}{\sqrt{-\mathcal{D}}}, \qquad \hat{P} = \frac{-\tilde{P}}{\sqrt{-\mathcal{D}}}, \tag{71}$$

$$\hat{q}^{\,0} = \frac{-\tilde{q}^0}{\sqrt{-\mathcal{D}}}, \qquad \hat{Q} = \frac{-\tilde{Q}}{\sqrt{-\mathcal{D}}}, \tag{72}$$

valid for *all* Freudenthal triple systems. Here

$$\mathcal{D} = [p^0q_0 - (P,Q)]^2 - 4(P^\sharp, Q^\sharp) + 4p^0N(Q) + 4q_0N(P), \tag{73}$$

$$\begin{aligned} \tilde{p}^0 &= -2N(P) - p^0(p^0q_0 - (P,Q)),\\ \tilde{P} &= 2(p^0Q^\sharp - Q\times P^\sharp) - (p^0q_0 - (P,Q))P, \end{aligned} \tag{74}$$

$$\begin{aligned} \tilde{q}^0 &= 2N(Q) + q^0(p^0q_0 - (P,Q)),\\ \tilde{Q} &= -2(q^0P^\sharp - P\times Q^\sharp) + (p^0q_0 - (P,Q))Q. \end{aligned} \tag{75}$$

Here in our special case $(A,B) = \mathrm{Tr}(AB)$ and $N(A) = \mathrm{Det}(A)$; for the remaining definitions see the Appendix.

Now this particular Ω arising from the critical point of $V_H(\varrho)$ can be expanded as

$$\Omega = C\Omega_0 = C(\alpha_0 + \tau^{jk}\alpha_{jk} + \tau^\sharp{}_{jk}\beta^{kj} - (\mathrm{Det}\tau)\beta^0), \tag{76}$$

where we put back the factor C of Eq. (61). One can then introduce complex coordinates

$$z^i = u^i + \tau^{ij}v^j, \tag{77}$$

such that the separable form is manifest,

$$\Omega = C\Omega_0 = Cdz^1\wedge dz^2\wedge dz^3 = \varrho + i\hat{\varrho}(\varrho), \tag{78}$$

i.e., Ω_0 is a holomorphic three-form for the torus. Here for the expansion coefficients τ^{ij} fixing the complex structure of T^6 we choose the convention

$$\tau^{ij} = x^{ij} - iy^{ij}, \qquad y^{ij} > 0. \tag{79}$$

One can also check that by virtue of Eq. (57)

$$e^{-K} = 8\mathrm{Det}y. \tag{80}$$

From Eqs. (76) and (78) one can see that the complex structure obtained from the extremization of Hitchin's functional is

$$\tau = \frac{P + i\hat{P}}{p^0 + i\hat{p}^0}, \tag{81}$$

or after performing standard manipulations [19,60] using identities for Freudenthal systems

$$\tau = \frac{1}{2}[-(2PQ + [p^0q_0 - (P,Q)]) + i\sqrt{-\mathcal{D}}](P^\sharp - p^0Q)^{-1}. \tag{82}$$

According to Eq. (57) the value of V_H at the critical point is $\frac{1}{4}|C|^2e^{-K}$. Using this we obtain the final result

$$S_{\mathrm{BH}} = \pi V_H(\varrho_{\mathrm{crit}}) = \pi\sqrt{-\mathcal{D}}, \tag{83}$$

where $\mathcal{D}$ is given by Eq. (73). This result shows that the semiclassical black hole entropy is given by the entanglement measure $\mathcal{D}$ for the three-fermion state as given by Eqs. (67)–(70).

It is instructive to express $[\varrho] = \Gamma$ from Eq. (78) in the form

$$\begin{aligned} \Gamma &= \frac{1}{2}(C\Omega_0 + \overline{C\Omega_0})\\ &= \bar{Z}(-ie^{K/2}\Omega_0) + (-Z)(-ie^{K/2}\bar{\Omega}_0). \end{aligned} \tag{84}$$

Let us introduce the Hermitian inner product for three-forms as

$$\langle\varphi|\psi\rangle = \int_{T^6}\varphi\wedge *\bar{\psi}, \tag{85}$$

where $*$ is the Hodge star. One can then regard $H^3(T^6,\mathbb{C})$ equipped with $\langle\cdot|\cdot\rangle$ as a 20-dimensional Hilbert space. One can then see [19] that Eq. (84) can be written in the form

$$|\Gamma\rangle = \Gamma_{123}|123\rangle + \Gamma_{\overline{123}}|\overline{123}\rangle, \tag{86}$$

where $|123\rangle$ and $|\overline{123}\rangle$ are orthonormal basis vectors. Since $|Z|^2 = \sqrt{-\mathcal{D}}$ one can show that

$$|\Gamma\rangle = (-\mathcal{D})^{1/4}(e^{i\alpha}|123\rangle - e^{-i\alpha}|\overline{123}\rangle), \quad \tan\alpha = \frac{p^0}{\hat{p}^0}. \tag{87}$$

Notice that this *state* is of the GHZ-form, with the phase factors coming from the phase of the central charge expressed in terms of the charges p^0 and its Freudenthal dual $\hat{p}^0$. The quantity [17,19] $\frac{1}{2}\pi||\Gamma||^2$ is just the semiclassical entropy S_{BH}.

Now we employ an extra constraint and consider our torus equipped with a symplectic form ω giving rise to the volume form compatible with the orientation, and we also restrict ϱ featuring Hitchin's functional by the constraint $\omega\wedge\varrho = 0$. In this case we have to find the constrained critical points [34] of V_H. In the language of entanglement the new ϱ arising from Eq. (67) can now be regarded as an embedded real three-fermion state, with the extra constraint restricting the SLOCC group from $GL(6,\mathbb{R})$ to $GL(1,\mathbb{R})\times SP(6,\mathbb{R})$. According to Eq. (32) these considerations give the restriction on the charge configuration

$$P^t = P, \qquad Q^t = Q, \tag{88}$$

yielding 14 independent charges. Moreover, due to our restrictions corresponding to Eq. (88) T^6 with the arising complex structure will be a principally polarized Abelian variety with $\tau^{ij} = \tau^{ji}$. The example we obtain in this way is just the one discussed by Moore [19,60] when studying BPS attractor varieties in *IIB* string theory compactified on T^6. Black hole entropy is again given by Eq. (83) with the corresponding formula the one depending on 14 charges. This is the square root of the quartic invariant of the Freudenthal triple system based on the cubic Jordan algebra of real 3×3 symmetric matrices.

D. STU truncation

Let us now choose an M having the product form $M = M_1\times M_2\times M_3$ where $M_{1,2,3}$ are two-dimensional tori T^2 with coordinates u^i, v^i. Here $i = 1,2,3$ labels the different tori. Now using the notation $e^i = du^i$, $e^{\bar{i}} = dv^i$, an element of $H^3(M, \mathbb{R})$ can be written as

$$\varrho = \varrho_{123} e^1 \wedge e^2 \wedge e^3 + \varrho_{12\bar{3}} e^1 \wedge e^2 \wedge e^{\bar{3}} + \dots + \varrho_{\overline{123}} e^{\bar{1}} \wedge e^{\bar{2}} \wedge e^{\bar{3}}, \tag{89}$$

i.e., ϱ has merely 8 nonzero amplitudes. It is convenient to relabel them in a notation reminiscent of the amplitudes of three qubits,

$$(\varrho_{123}, \varrho_{12\bar{3}}, \varrho_{1\bar{2}3}, \dots, \varrho_{\overline{123}}) \leftrightarrow (\varrho_{000}, \varrho_{001}, \varrho_{010}, \dots, \varrho_{111}). \tag{90}$$

In the language of embedded systems (see Sec. II C) one can obtain this case from the results of the previous subsection by employing the constraint $\omega_i \wedge \varrho = 0$, where ω_i are the symplectic forms of the tori.

Now a calculation shows that K_ϱ has the form

$$(K_\varrho)^a{}_b = \begin{pmatrix} U_\varrho & 0 & 0 \\ 0 & T_\varrho & 0 \\ 0 & 0 & S_\varrho \end{pmatrix}, \quad a, b = 1, \bar{1}, 2, \bar{2}, 3, \bar{3}, \tag{91}$$

where

$$S_\varrho = \begin{pmatrix} (\varrho_0 \cdot \varrho_1)_1 & (\varrho_1 \cdot \varrho_1)_1 \\ -(\varrho_0 \cdot \varrho_0)_1 & -(\varrho_0 \cdot \varrho_1)_1 \end{pmatrix}, \quad T_\varrho = \begin{pmatrix} (\varrho_0 \cdot \varrho_1)_2 & (\varrho_1 \cdot \varrho_1)_2 \\ -(\varrho_0 \cdot \varrho_0)_2 & -(\varrho_0 \cdot \varrho_1)_2 \end{pmatrix}, \tag{92}$$

$$U_\varrho = \begin{pmatrix} (\varrho_0 \cdot \varrho_1)_3 & (\varrho_1 \cdot \varrho_1)_3 \\ -(\varrho_0 \cdot \varrho_0)_3 & -(\varrho_0 \cdot \varrho_1)_3 \end{pmatrix}. \tag{93}$$

Here for $A, B \in \mathbb{R}^4$ an $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ invariant inner product is defined as

$$A \cdot B = A_1 B_4 - A_2 B_3 - A_3 B_2 + A_4 B_1, \tag{94}$$

and the subscripts 1, 2, 3 of the inner products mean that the splitting of the 8 amplitudes into two four component vectors is effected by assigning to qubit 1, 2, 3 a special role. Hence for example for calculating $(\varrho_0 \cdot \varrho_1)_1$ the four component vectors to be used in Eq. (94) are

$$\varrho_0 = \begin{pmatrix} \varrho_{000} \\ \varrho_{001} \\ \varrho_{010} \\ \varrho_{011} \end{pmatrix}, \qquad \varrho_1 = \begin{pmatrix} \varrho_{100} \\ \varrho_{101} \\ \varrho_{110} \\ \varrho_{111} \end{pmatrix}, \tag{95}$$

where now the first qubit (labeling the qubits from the left to the right) plays a special role.

In this notation Hitchin's invariant is

$$D(\varrho) = (\varrho_0 \cdot \varrho_1)_n^2 - (\varrho_0 \cdot \varrho_0)_n (\varrho_1 \cdot \varrho_1)_n, \quad \forall n = 1, 2, 3, \tag{96}$$

which is just another form for Cayley's hyperdeterminant of Eq. (4) related to the entanglement measure the three tangle. Notice that the independence of $D(\varrho)$ on the particular split expresses the permutation invariance of D. After these considerations one can immediately check that [35]

$$\mathrm{Tr}\, K_\varrho = 0, \qquad K_\varrho^2 = D(\varrho)\mathbf{1}, \tag{97}$$

where $\mathbf{1}$ is the 6×6 identity matrix.

For $\mathcal{D}(\varrho) < 0$ the GHZ components Ω and $\bar{\Omega}$ of such a ϱ are given by Eq. (48) where now the Freudenthal dual $\hat{\varrho}$ is arising from the Freudenthal system based on the Jordan algebra of 3×3 *diagonal* matrices. There is a geometric description of three-qubit entanglement in terms of twistors [8]. In this picture finding the canonical GHZ components of ϱ amounts to finding the principal null directions [with respect to the symmetric bilinear form of Eq. (94)] of bivectors like $\varrho_{0a}\varrho_{1b} - \varrho_{0b}\varrho_{1a}$ formed from the four-vectors of Eq. (95). For a real bivector with $\mathcal{D} \neq 0$ we have two principal null directions. We have either two real directions $\mathcal{D} > 0$ or they are coming in complex conjugate pairs if $\mathcal{D} < 0$. These cases correspond to the two inequivalent real GHZ SLOCC classes.

Now for a three-form representing the cohomology class of a wrapped $D3$-brane configuration we take

$$\Gamma = p^I \alpha_I - q_I \beta^I \in H^3(T^6, \mathbb{Z}), \tag{98}$$

with summation on $I = 0, 1, 2, 3$ and

$$\alpha_0 = du^1 \wedge du^2 \wedge du^3, \quad \beta^0 = -dv^1 \wedge dv^2 \wedge dv^3, \tag{99}$$

$$\alpha_1 = dv^1 \wedge du^2 \wedge du^3, \qquad \beta^1 = du^1 \wedge dv^2 \wedge dv^3, \tag{100}$$

with the remaining ones obtained via cyclic permutation.

Let us now pretend that we have an *arbitrary* complex structure on $M = T^2 \times T^2 \times T^2$. We introduce the coordinates

$$z^j = u^j + \tau^j v^j, \qquad \tau^j = x^j - i y^j \qquad y^j > 0, \qquad j = 1, 2, 3, \tag{101}$$

and the holomorphic three-form

$$\Omega_0 = dz^1 \wedge dz^2 \wedge dz^3, \tag{102}$$

with τ^j labeling the complex structure. It is well-known [61] that we can express Γ in a basis where the Hodge star is diagonal as

$$\begin{aligned} \Gamma &= e^{K/2}(iZ(\Gamma)\bar{\Omega}_0 - i g^{j\bar{k}} D_j Z(\Gamma) \bar{D}_{\bar{k}} \bar{\Omega}_0 + \text{c.c.}) \\ &= e^{K/2}(iZ(\Gamma)\bar{\Omega}_0 - i\delta^{\hat{j}\hat{\bar{k}}} D_{\hat{j}} Z(\Gamma) \bar{D}_{\hat{\bar{k}}} \bar{\Omega} + \text{c.c.}). \end{aligned} \tag{103}$$

Here $Z(\Gamma) = e^{K/2} \int_{T^6} \Gamma \wedge \Omega_0$ is the central charge, and the flat covariant derivative in our special case is defined as

$$D_{\hat{\tau}} \Omega_0 \equiv (\bar{\tau} - \tau) D_\tau \Omega_0 \equiv (\bar{\tau} - \tau)(\partial_\tau + \partial_\tau K)\Omega_0, \tag{104}$$

where for simplicity we have omitted the labels of τ. For the STU truncation the explicit form of $Z(\Gamma)$ is

$$Z(\Gamma) = e^{K/2} W(\tau^3, \tau^2, \tau^1), \tag{105}$$

where

$$W(\tau^3, \tau^2, \tau^1) = q_0 + q_1\tau^1 + q_2\tau^2 + q_3\tau^3 + p^1\tau^2\tau^3 + p^2\tau^1\tau^3 + p^3\tau^1\tau^2 - p^0\tau^1\tau^2\tau^3. \tag{106}$$

Let us now restrict the Hermitian inner product of Eq. (85) to the eight-dimensional untwisted primitive part of $H^3(T^6, \mathbb{C})$. One can then introduce a Hodge diagonal basis in this space as follows [19]:

$$\begin{aligned} -ie^{K/2}\Omega_0 &\leftrightarrow |000\rangle, & -ie^{K/2}D_{\hat{1}}\Omega_0 &\leftrightarrow |001\rangle, \\ -ie^{K/2}D_{\hat{2}}\Omega_0 &\leftrightarrow |010\rangle, & -ie^{K/2}D_{\hat{3}}\Omega_0 &\leftrightarrow |100\rangle, \\ -ie^{K/2}\bar{\Omega}_0 &\leftrightarrow |111\rangle, & -ie^{K/2}\bar{D}_{\hat{1}}\Omega_0 &\leftrightarrow |110\rangle, \\ -ie^{K/2}\bar{D}_{\hat{2}}\Omega_0 &\leftrightarrow |101\rangle, & -ie^{K/2}\bar{D}_{\hat{3}}\Omega_0 &\leftrightarrow |011\rangle. \end{aligned}$$

Now with these definitions our eight-dimensional space is isomorphic to $(\mathbb{C}^2)^{\times 3}$ equipped with a Hermitian inner product; i.e., it is the space of states for three qubits. We must note, however, two important peculiarities. First of all the state $\Gamma \leftrightarrow |\Gamma\rangle$ defined as the qubit version of Eq. (103) is complex in appearance. However, its explicit form

$$|\Gamma\rangle = \Gamma_{000}|000\rangle + \Gamma_{001}|001\rangle + \ldots + \Gamma_{110}|110\rangle + \Gamma_{111}|111\rangle, \tag{107}$$

where

$$\Gamma_{111} = -e^{K/2}W(\tau^3, \tau^2, \tau^1) = -\bar{\Gamma}_{000}, \tag{108}$$

$$\Gamma_{001} = -e^{K/2}W(\bar{\tau}^3, \bar{\tau}^2, \tau^1) = -\bar{\Gamma}_{110}, \qquad \text{etc.}, \tag{109}$$

shows that it satisfies an extra reality condition coming from Eq. (98). Second for BPS states with $D < 0$ the amplitudes Γ_{000} and Γ_{111} are obviously playing a special role since they are connected to the holomorphic structure via the appearance of Ω_0 in Eq. (103).

These considerations show that in the case of the STU truncation by putting an *arbitrary* complex structure on $M = T^2 \times T^2 \times T^2$ one can define a charge and moduli dependent three-qubit state of the (107) form. The critical point of Hitchin's functional on the other hand defines a *special* complex structure on M. It is given by Eq. (82) keeping only the diagonal entries of the 3×3 matrices showing up in this formula. In this case our three-qubit state has the special form

$$|\Gamma_{\mathrm{crit}}\rangle = (-D)^{1/4}(e^{i\alpha}|000\rangle - e^{-i\alpha}|\overline{111}\rangle), \qquad \tan\alpha = \frac{p^0}{\hat{p}^0}, \tag{110}$$

where now D is Cayley's hyperdeterminant related to the three tangle of Eq. (4) as the canonical entanglement measure for three-qubit systems. The Freudenthal dual component $\hat{p}^0$ is also modified accordingly. The semiclassical black hole entropy is given by

$$S_{\mathrm{BH}} = \pi V_H(\varrho_{\mathrm{crit}}) = \pi\sqrt{-D}, \tag{111}$$

where D is given by Eq. (4). It is important to realize that again $S_{\mathrm{BH}} = \frac{1}{2}\pi||\Gamma_{\mathrm{crit}}||^2$. On the other hand the norm squared of the state $|\Gamma\rangle$ featuring an arbitrary complex structure is the black hole potential [8,42,62]

$$V_{\mathrm{BH}} = \frac{1}{2}||\Gamma||^2. \tag{112}$$

For BPS black holes we have $D < 0$. In this case studying the explicit form of the full radial flow from the asymptotically Minkowski region to the event horizon, one can follow the transition from a three-qubit state of the form Eq. (107) to the one of the Eq. (110) form. In the literature of the BHQC this process is called a distillation procedure of a GHZ state. This process is terminated at the horizon where $\Gamma_{001} = \Gamma_{010} = \Gamma_{100} = 0$. This is just the usual process well-known in the string theory literature for which Γ has only $H^{3,0}$ and $H^{0,3}$ components. This observation is originally due to Moore [60]. We must stress that the formalism based on V_H is more general.

We would like to close this subsection with an important comment on *non-BPS* black holes [63]. In the most general setting it is tempting to relate the critical points in the $\mathcal{D} > 0$ branch of real states of the Hitchin functional of the form as given by Eq. (52) with the critical points of the general expression for the black hole potential [42]. It is easy to see that using the Hermitian inner product of Eq. (85) the "norm squared" interpretation of Eq. (112) survives even in this case. In the special case of the STU truncation studied here the explicit form of these solutions is known [62]. In the framework of the BHQC the non-BPS analogues of the state of Eq. (110) are again of special form. They belong to the GHZ class with $D > 0$ and are called graph states in the QI literature [17]. The interpretation for the attractor mechanism as being some sort of distillation procedure also works in this case [17]. An especially interesting feature of these solutions is that for the non-BPS branch the special role of the holomorphic three-form in the expansion of Eq. (107) is lost. This can be seen most clearly for non-BPS solutions giving rise to attractors with vanishing central charge [64]. In this case Γ_{000} and Γ_{111}, i.e., precisely the canonical GHZ amplitudes, are vanishing. Hence also including the non-BPS branch into the picture renders the interpretation of Γ as a quantity relating to some sort of "state" more natural. We will have something more to say about these interesting issues in Sec. VI.

E. t^3 truncation

The t^3 truncation is the diagonal torus example where $M = T^2 \times T^2 \times T^2$ with the tori regarded indistinguishable. In the entanglement picture this case amounts to considering three indistinguishable bosonic qubits. Now $\Gamma \in H^3(M, \mathbb{Z})$ is expanded as

$$\Gamma = p^0\alpha_0 + p(\alpha_1 + \alpha_2 + \alpha_3) - q(\beta^1 + \beta^2 + \beta^3) - q_0\beta^0. \qquad (113)$$

In the Hodge diagonal basis we have $\Gamma_{001} = \Gamma_{010} = \Gamma_{100}$, etc.; hence our charge and moduli dependent state will be of the form

$$|\Gamma\rangle = \Gamma_{000}|000\rangle + \Gamma_{001}(|001\rangle + |010\rangle + |100\rangle) + \Gamma_{110}(|110\rangle + |101\rangle + |011\rangle) + \Gamma_{111}|111\rangle. \qquad (114)$$

The critical point of Hitchin's functional gives the particular complex structure labeled by a single τ with its expression obtained in a straightforward manner from the general formula of Eq. (82) or directly from the corresponding formula of the STU truncation. For this truncation and special complex structure the "bosonic" three-qubit state is of the usual form

$$|\Gamma\rangle = (-d)^{1/4}(e^{i\alpha}|000\rangle - e^{-i\alpha}|\overline{111}\rangle), \quad \tan\alpha = \frac{p^0}{\hat{p}^0}, \qquad (115)$$

where now d is given by the expression of Eq. (37). As usual the charge states supporting BPS black holes (among other conditions [62]) have $d < 0$. The semiclassical black hole entropy is given by

$$S_{\mathrm{BH}} = \pi V_H(\varrho_{\mathrm{crit}}) = \pi\sqrt{-d}. \qquad (116)$$

IV. GENERALIZED HITCHIN FUNCTIONAL

A. Quantum corrections

We have seen in the previous section that Hitchin's functional $V_H(\varrho)$ at its critical point has a very important physical interpretation. According to Eq. (63) it is just proportional to the semiclassical black hole entropy S_{BH} which can also be related to an entanglement measure. Now we will regard S_{BH} merely as the leading-order contribution to the black hole entropy. In order to do this one can consider the quantum theory with action V_H. Let us formally define the partition function [39]

$$Z_H(\gamma) = \int_{[\varrho]=\Gamma} e^{V_H(\varrho + d\sigma)}\mathcal{D}\sigma, \qquad (117)$$

where as usual Γ is the Poincaré dual to γ. In Ref. [39] it was conjectured that the partition function $Z_H(\gamma)$ on a manifold M is the Wigner transform of the partition functions of B and $\bar{B}$ topological strings on M. Based on our considerations of Sec. III B using the method of steepest descent it is easy to demonstrate that this conjecture is correct at the classical level. Then the idea was to use Hitchin's functional also to recover the quantum corrections that have already been calculated via topological string techniques [45]. However, it turned out [46] that after appropriate gauge fixing at the one loop level there is a discrepancy between the result based on Hitchin's functional and the result of topological string theory. In order to resolve this discrepancy Pestun and Witten suggested using a partition function based on the *generalized Hitchin functional* instead. Hitchin's functional is connected to Calabi-Yau structures; on the other hand the generalized Hitchin functional (GHF) is connected to generalized Calabi-Yau structures [36,47]. For the resolution they chose manifolds with $b_1(M) = 0$ where the critical points and classical values of both functionals coincide; however, the quantum fluctuating degrees of the two functionals are different. The upshot of these considerations was that after a convenient interpretation [46] the conjecture of Ref. [39] remains true even at the one loop level.

Hence we have an interesting possibility of using form theories of gravity in a much wider context which is also capable of describing quantum fluctuations provided we are willing to use a generalization of Hitchin's functional. Within the framework of the BHQC we have seen that Hitchin's functional is related to an entanglement measure and its descendants that can describe entangled systems with a small number of constituents. Now the question is whether we can also find a quantum information theoretic interpretation of the GHF. The aim of this section is to show that a variant of the entangled system which is directly connected to the invariant underlying the GHF has already been discussed in the BHQC [9,10]. This entangled system lends itself to precisely such an interpretation. Hence as an extra bonus its intimate connection to the GHF enables a further generalization of the BHQC.

The GHF for a six-dimensional manifold M is defined by replacing the three-form ϱ_3 in the usual formulation of the Hitchin functional by a *polyform* $\varphi = \varphi_1 + \varphi_3 + \varphi_5$ of odd degree. It was shown [36] that if this polyform is *nondegenerate* in a suitable sense then it defines a generalized almost complex structure [36] on M. The nondegeneracy is defined via a quartic invariant, our main concern here, which is invariant under Spin(6,6). Then the generalized complex structure is given by a pure spinor [36,47,65] of the form $\varphi + i\hat{\varphi}(\varphi)$ with respect to Spin(TM, TM^*) where TM and TM^* are the tangent and cotangent bundles of M. It is then shown that if in addition $d\varphi = 0$ and $d\hat{\varphi} = 0$ then the generalized almost complex structure is integrable giving rise to a *generalized complex manifold.*

As far as physics is concerned the interest in such manifolds stems from the fact that a special case of such manifolds is the class of *generalized Calabi-Yau* manifolds. Such manifolds are showing up in strings propagating in general backgrounds. A generalized Calabi-Yau structure can be regarded as one which is interpolating in a suitable sense between the symplectic structure and the Calabi-Yau one of a given manifold M. Similarly to the Hitchin functional case the condition which is crucial for the generalized Calabi-Yau structure, namely, $d\hat{\varphi} = 0$, is arising from the extremization of the GHF which is constructed from the quartic invariant. There is an alternative formulation of the GHF based on polyforms of even rank.

These polyforms are of the form $\varphi = \varphi_0 + \varphi_2 + \varphi_2 + \varphi_6$. Note that the dimension of the space of such polyforms is 32 in both cases. This is connected to the fact that polyforms with odd (even) degree form an irrep of negative (positive) chirality with respect to Spin(6,6). Our aim is to present a form for the quartic invariant related to the GHF based on the Freudenthal system corresponding to the Jordan algebra of quaternion Hermitian 3×3 matrices. Note that the usual form of the quartic invariant underlying the GHF is based on the moment map and the properties of pure spinors [36]. This alternative form given below is convenient for our entanglement based considerations. In the Appendix for the convenience of the reader we have collected the mathematical results on polyforms and the Freudenthal system we need for our presentation.

Recall first that after complexification $\mathcal{D}$ of Eq. (19) underlying the construction of Hitchin's functional was a relative invariant under $G \equiv GL(6, \mathbb{C}) = GL(1, \mathbb{C}) \times SL(6, \mathbb{C})$. Moreover, this relative invariant was related to the PV of Sato and Kimura of class 5. We also used a suitable restriction of this relative invariant for the description of embedded entangled systems. The restricted Hitchin functional in this case was based on the group $G \equiv GL(1, \mathbb{C}) \times Sp(6, \mathbb{C})$. This relative invariant was related to the PV of class 14. We also observed that our relative invariants could elegantly be described by Freudenthal systems based on the *complexifications* of the Jordan algebras of 3×3 complex Hermitian or real symmetric matrices. The crucial identity in this respect was the complex analogue of Eq. (44),

$$2\sqrt{|\mathcal{D}(P)|}e^{i \arg \mathcal{D}(P)} = \{P, \hat{P}\}, \tag{118}$$

where $P \in \wedge^3 \mathbb{C}^6$ or $P \in \wedge_0^3 \mathbb{C}^6$, respectively, and $\hat{P}$ was the corresponding complex extension of the Freudenthal dual. Nondegenerate three-forms then corresponded to three-fermion states belonging to the *complex* GHZ class characterized by the property $\mathcal{D} \neq 0$. This class was then a *stable orbit* under G. Moreover, the relative invariant $\mathcal{D}$ and the product $\{\cdot, \cdot\}$ was just the *negative* of the quartic invariant and the symplectic form of the corresponding Freudenthal system.

We have seen that over the reals the complex GHZ class splits into two real GHZ orbits with the corresponding states having $\mathcal{D} < 0$ and $\mathcal{D} > 0$, respectively. For the construction of Calabi-Yau structures the first of the *real orbits* was needed. As we already know a real state belonging to this orbit can be exressed as a real part of a complex separable state. On this open orbit the real SLOCC group $GL(1, \mathbb{R}) \times SL(6, \mathbb{R})$ acts transitively. For the generalized Calabi-Yau structures it is known [36] that there is a real spinor which is the real part of a complex *pure* spinor. This spinor belongs to the 32-dimensional spinor representation with the corresponding quartic invariant being negative. Moreover, spinors with this property form an open set and the real group $GL(1, \mathbb{R}) \times \mathrm{spin}(6, 6)$ acts on it transitively. More importantly this open orbit can be regarded as one of the two real orbits arising from the stable complex orbit of a PV with group $G = GL(1, \mathbb{C}) \times \mathrm{spin}(12, \mathbb{C})$ which is class 23 in the Sato-Kimura scheme [37]. This is as we expected the next item in the line of Freudenthal systems, namely, the one which is based on the complexification of the Jordan algebra of 3×3 quaternion Hermitian matrices.

The upshot of these investigations is that a nice alternative way for expressing the GHF is simply using the real version of Eq. (118) with $\mathcal{D} < 0$ where now $\mathcal{D}$ is replaced by a $\mathbb{D}$ which is the negative of the quartic invariant of the corresponding Freudenthal system. What is left to be established is the precise dictionary between the components of a polyform φ and the components of the Freudenthal system. A detailed derivation of this is given in the Appendix. Choosing a polyform of even degree $\varphi = \varphi_0 + \varphi_2 + \varphi_4 + \varphi_6$ the GHF is

$$V_{GH}(\varphi) = \int_M \sqrt{-\mathbb{D}(\varphi)} d^6x = -\frac{1}{2} \int_M \varphi \wedge \hat{\varphi}(\varphi), \tag{119}$$

where

$$\mathbb{D}(\varphi) = -q(p) \tag{120}$$

is the quartic invariant of p which is the element of the Freudenthal system. For the explicit form of $q(p)$ and the correspondence $\varphi \leftrightarrow p$ see Eqs. (A36), (A44), and (A45) of the Appendix. Notice that for the nonlinear expression $\hat{\varphi}(\varphi)$ now we have an explicit form in terms of the Freudenthal dual of the Freudenthal system. For explicit formulas one just has to use the quaternionic analogues of Eqs. (74) and (75). The Freudenthal formalism again automatically takes care of the nice symplectic interpretation. Namely, the space of polyforms can be regarded as a phase space of a classical mechanical system. The symplectic form is given by the pairing $\{\cdot, \cdot\}$ and $\sqrt{|\mathbb{D}(\varphi)|}$ is the Hamiltonian. The Freudenthal dual $\hat{\varphi}$ is up to sign just the Hamiltonian vector field. The generalized almost complex structure is integrable for $d\hat{\varphi} = 0$.

B. The generalized Hitchin invariant as an entanglement measure

Interestingly an entangled system that we can relate to the invariant underlying the GHF has already appeared in the literature of the BHQC [9,10]. In order to see this, notice that there is yet another PV that we have not discussed yet. It is class 29 in the Sato-Kimura list. The group of the PV in this case is $GL(1, \mathbb{C}) \times E_7(\mathbb{C})$. The corresponding Freudenthal system is the one based on the complexification of the Jordan algebra of 3×3 octonion Hermitian matrices. The quartic invariant of the associated Freudenthal system is well-known in the string theory literature. Indeed the most general class of black holes in $\mathcal{N} = 8$ supergravity/M theory is defined by 56 charges and the entropy formula is given by the square root of the quartic Cartan-Cremmer-Julia $E_{7(7)}$ invariant [66–68] which is a real version of our quartic invariant.

It can be shown that the 56-dimensional fundamental representation of $E_7(\mathbb{C})$ can be decomposed with respect to the $SL(2,\mathbb{C})^{\times 7}$ subgroup as follows [9,10,21]:

$$\begin{aligned}\mathbf{56} \to &(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})\\ &+ (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2})\\ &+ (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})\\ &+ (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}). \end{aligned} \tag{121}$$

Let us now replace formally the 2's with 1's, and the 1's with 0's, and form a 7×7 matrix by regarding the seven vectors obtained in this way as its rows. Let the rows correspond to lines and the columns to points, and the location of a "1" in the corresponding slot correspond to incidence. Then this correspondence results in the incidence matrix of the Fano plane Let us reproduce here this incidence matrix with the following labeling for the rows (r) and columns (c):

$$\begin{pmatrix} r/c & A & B & C & D & E & F & G \\ a & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ c & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ d & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ f & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ g & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a_{ABD} \\ b_{BCE} \\ c_{CDF} \\ d_{DEG} \\ e_{EFA} \\ f_{FGB} \\ g_{GAC} \end{pmatrix}, \tag{122}$$

where we also displayed the important fact that this labeling automatically defines the index structure for the amplitudes of seven three-qubit states formed out of seven distinguishable qubits A, B, C, D, E, F, G. If we introduce the notation $V_{ijk} \equiv V_i \otimes V_j \otimes V_k$ where $i, j, k \in \{A, B, C, D, E, F, G\}$ then the 56 of E_7 denoted by $\mathcal{H}$ decomposes as

$$\begin{aligned}\mathcal{H} = &V_{ABD} \oplus V_{BCE} \oplus V_{CDF} \oplus V_{DEG} \oplus V_{EFA}\\ &\oplus V_{FGB} \oplus V_{GAC}. \end{aligned} \tag{123}$$

Clearly this structure encompasses an unusual type of entanglement, for entanglement is usually associated with tensor products, but here we also encounter *direct sums*. One can regard the seven tripartite sectors as seven superselection sectors which in the black hole context correspond to seven different STU truncations [9,10]. This structure is usually referred to in the literature as the tripartite entanglement of seven qubits [9].

Now one can express the quartic invariant in terms of the 56 amplitudes of the seven copies of three-qubit systems [2,9,10,21]. Especially, one can establish a precise dictionary between this qubit based description and the Freudenthal one [2]. Then one can consider the decomposition of the **56** of $E_7(\mathbb{C})$

$$\mathbf{56} \to (\mathbf{2},\mathbf{12}) \oplus (\mathbf{1},\mathbf{32}), \tag{124}$$

with respect to the subgroup $SL(2,\mathbb{C}) \times \mathrm{spin}(12,\mathbb{C})$. It can then be shown that the $(\mathbf{1},\mathbf{32})$ part consists of those amplitudes that are excluding one particular qubit. Hence for example the space

$$V_{BCE} \oplus V_{CDF} \oplus V_{DEG} \oplus V_{FGB}, \tag{125}$$

excluding qubit A forms a representation space for $\mathrm{spin}(12,\mathbb{C})$. This representation space comprises the tripartite entanglement of *six* qubits. Now after using the correspondence between the 32 amplitudes b_{BCE}, c_{CDF}, d_{DEG}, and f_{FGB} and the $1 \oplus 15 \oplus 15 \oplus 1$ structure of the relevant Freudenthal system, one can see that the quartic invariant is just the same as the one underlying the GHF.

As far as string theory is concerned this entanglement based interpretation is useful because it reveals four STU subsectors hidden in the structure of the GHF. However, there is an even more useful interpretation. It is just the one of directly regarding polyforms as representatives of fermionic entangled systems and the relative invariant like the one underlying the GHF as an entanglement measure. Of course since the structure of polyforms is also intimately connected to the structure of the underlying manifold M and its moduli space, this interpretation is again an unusual one. Recall also that the number of modes or single particle states just equals the dimension of M.

Notice moreover that the fermion number is not conserved. For our case of the GHF we have either $\varphi = \varphi_1 + \varphi_3 + \varphi_5$ or $\varphi = \varphi_0 + \varphi_2 + \varphi_4 + \varphi_6$. The analogue of the SLOCC group is now $GL(1,\mathbb{C}) \times \mathrm{spin}(12,\mathbb{C})$ which is mixing the forms of different degree but respecting the parity. One can write these polyforms as a sum of objects like in Eq. (13) expressed in terms of different numbers of fermionic creation operators. Alternatively one can regard the polyforms as spinors [36,47]. It is easy to see that *pure spinors* should correspond in this picture to separable states. Classifying the entanglement types of spinors then should correspond to finding the SLOCC orbits and the stabilizers of the representatives. This problem has been solved up to dimension twelve in the classical paper of Igusa [69]. Many results can also be found in the book of Chevalley [65].

Notice that the B transform of the pure spinor 1 (i.e., a φ_0), $e^{-B} \cdot 1$ (another pure spinor), can be expressed in terms of the Pfaffian combinations of the 6×6 matrix $\mathcal{B}$ underlying the two-form B. [See Eq. (A36).] There is a similar phenomenon occurring in the theory of fermionic Gaussian states where the higher order correlations can be obtained from the quadratic ones via Wick's theorem. [See for example Eqs. (4) and (8) of the paper of Kraus *et al.* [70].] This can yield a standard form for Gaussian states that look like "paired states" known from the BCS theory of superconductivity [70,71]. These states look similar to GHZ states coming from combinations of two pure spinors. There are many more mathematical correspondences with fermionic systems in many body physics which deserve some attention. However, as far as the authors are aware this perspective has not made its debut to the quantum information community. The elaboration of these ideas within the field of quantum entanglement could be another useful input string theory can provide.

C. An example: T^6 revisited

A simple example illustrating the difference between the Hitchin and generalized Hitchin functionals was given by Pestun [72]. Here we briefly revisit this example putting the emphasis on the entanglement interpretation.

We already know that for Calabi-Yau compactification in the supergravity approximation black hole entropy is equal to the Hitchin functional taken at its critical point. We have also seen that at the critical point the resulting expression can be interpreted as an entanglement measure. Moreover, according to the OSV relation [41] black hole entropy can be related to the topological string partition function. Hence in terms of partition functions at the classical level we have checked the chain of relations symbolically written as $Z_H = Z_{BH} = |Z_{\rm TOP}|^2$. For Calabi-Yau manifolds M with $b_1(M) = 0$ in order to have these relations even at one loop level we have learned that we have to replace Z_H with $Z_{\rm GH}$ where the latter is the partition function based on the GHF. Can we relate the GHF taken at its critical point to black hole entropy as an entanglement measure already at the tree level? Clearly to have this situation we need a manifold where $b_1(M) \neq 0$. The simplest example of that kind [72] is T^6. This is of course our example already used in connection with the Hitchin functional. However, now we will suppose that the extra fields featuring the GHF have nonzero expectation values even at tree level. Note that although this illustrative case has more than $N = 2$ supersymmetry (the setting needed for topologiclal strings), hence we do not expect agreement at the one loop level, at the tree level it still has a consistent $N = 2$ truncation.

In this T^6 example the $N = 8$ supergravity multiplet is truncated by disregarding the gravitini multiplet with a result of having instead of the $1 + 12 + 15$ gauge fields the $1 + 15$ ones of the $N = 2$ sector. As a result of this we are merely having those vector multiplets at our disposal whose corresponding scalars are giving rise to the generalized complex moduli of T^6. In the case of the IIA picture related to the topological A model the 32 charges correspond to the wrapping configurations of the D0-, D2-, D4-D6-branes. The periods X^I, $I = 0, 1, \ldots 15$ are arising from integrals of the complexified Kähler class Ω,

$$\Omega = e^{b+i\omega} = \varphi + i\hat{\varphi}(\varphi), \qquad \varphi = \varphi_0 + \varphi_2 + \varphi_4 + \varphi_6, \tag{126}$$

where now

$$\varphi = [\Gamma], \qquad \Gamma = p^0\alpha_0 + P^{\mu\nu}\alpha_{\mu\nu} - Q_{\mu\nu}\beta^{\mu\nu} - q_0\beta^0. \tag{127}$$

Here α_0, $\alpha_{\mu\nu}$, $\beta^{\mu\nu}$, $\mu, \nu = 1, 2, \ldots 6$, and β^0 are a basis of 0-, 2-, 4-, and 6-forms. Now an expansion similar to the one as given by Eq. (76) gives for the complexified Kahler class $\tau^{\mu\nu} = X^{\mu\nu}/X^0$ corresponding to a critical point of the GHF with an expression for the generalized complex structure as given by Eq. (82). Note that this formula is the same in appearance; however, now the matrices are 6×6 antisymmetric ones or 3×3 ones with biquaternionic entries. This is, according to the Appendix, just the complexification of the Freudenthal system based on the Jordan algebra of 3×3 quaternion Hermitian matrices. For an explicit mapping between these matrices see Eq. (A21). Note that in this formalism $\hat{\varphi}$ is again the Freudenthal dual as determined by the cohomology class $[\Gamma]$ of the charge polyform. Now after performing manipulations and using identities for the corresponding Freudenthal system the GHF at the critical point can be evaluated. The result for the semiclassical black hole entropy is as expected [72,73]

$$S_{\rm BH} = \pi V_{\rm GH}(\varphi_{\rm crit}) = \pi\sqrt{-\mathbb{D}}, \tag{128}$$

where $\mathbb{D}$ is related to the quartic invariant of the quaternionic Freudenthal system of Eq. (A44) as

$$\mathbb{D}(p^0, q_0, P, Q) = -q(\xi, \eta, X, Y). \tag{129}$$

According to the results of the previous subsection we can reinterpret this formula as an entanglement measure describing the tripartite entanglement of six qubits, or of a fermionic system with six modes and even parity. Notice also that the distillation interpretation of the "attractor states" of Eq. (87) also holds in this case—the basis "states" like $|123\rangle$ and $|\overline{123}\rangle$, $\mathcal{D}$ and $\hat{p}^0$ should be replaced with the corresponding pure spinors, $\mathbb{D}$ and the relevant component of the Freudenthal dual.

V. ENTANGLEMENT OF THREE FERMIONS WITH SEVEN SINGLE PARTICLE STATES

In this section as a further step we would like to propose a reinterpretation of the invariant underlying Hitchin's functional for three-forms in seven dimensions as an entanglement measure for three fermions with seven single particle states. As a byproduct of this we can simply use the classification theorem [48,49] of $GL(7, \mathbb{C})$ orbits of three-forms in seven dimensions to give a full list of SLOCC entanglement classes. Note that this invariant integrated on a real 7-manifold is well-known in string theory. The critical points of the associated functional give rise to manifolds with G_2 holonomy. At the critical point the nondegenerate three-forms not only determine a metric of G_2 holonomy but also special three-dimensional submanifolds with minimal volume. The interpretation of this new functional as one related to another entanglement measure provides a further support for the BHQC.

Let us denote the octonionic units as $e_1, e_2, \ldots e_7$, and for their multiplication table use the conventions of Günaydin and Gürsey [74]. With this notation an octonion $x \in \mathbb{O}$ and its conjugate $\bar{x}$ can be written as $x = x_0 + x_A e_A$ and $\bar{x} = x_0 - x_A e_A$ where summation for $A = 1, 2, \ldots 7$ is implied. An imaginary octonion $x = x_A e_A$ in the basis of e_A has the usual norm $Q(x) = x\bar{x} = x_1^2 + \ldots x_7^2$.

Let us now consider the seven-dimensional complex vector space U. By an abuse of notation we also denote its canonical basis vectors by e_A. Let us denote the six-dimensional subspace V of U spanned by e_a, $a = 1, \ldots 6$. The basis vectors for the dual U^* will be denoted by e^A. As a complex basis of U^* we define

$$\begin{aligned} E^{1,2,3} &= e^{1,2,3} + ie^{4,5,6}, \\ E^{\bar{1},\bar{2},\bar{3}} &= e^{1,2,3} - ie^{4,5,6}, \\ E^7 &= ie^7. \end{aligned} \tag{130}$$

Let us use in the following the shorthand notation $e^{ABC} \equiv e^A \wedge e^B \wedge e^C$. For $1 \leq A < B < C \leq 7$ the e^{ABC} form a basis for $\wedge^3 U^*$. Then a GHZ-like state in the subspace $\wedge^3 V^*$ can be written as

$$E^{123} + E^{\overline{123}} = 2(e^{123} - e^{156} + e^{246} - e^{345}). \tag{131}$$

With the usual relabeling 4, 5, $6 \mapsto \bar{1}, \bar{2}, \bar{3}$ and up to normalization the state on the right-hand side is just the one of Eq. (29) with its Hitchin invariant of Eq. (19) being negative. Let us add to this state the one $(E^{1\bar{1}} + E^{2\bar{2}} + E^{3\bar{3}}) \wedge E^7$. Then we obtain the three-fermion state with seven single particle states

$$\begin{aligned} \phi \equiv& \frac{1}{2}(E^{123} + E^{\overline{123}} + (E^{1\bar{1}} + E^{2\bar{2}} + E^{3\bar{3}}) \wedge E^7) \\ =& e^{123} - e^{156} + e^{246} - e^{345} + e^{147} + e^{257} + e^{367}. \end{aligned} \tag{132}$$

Notice that the structure of our tripartite state ϕ is encoded into the incidence structure of the *lines* of the *oriented* Fano plane which is also encoding the multiplication table of the octonions [74]. As a *complex* three-form it can be shown [48,59] that the subgroup of the SLOCC group $GL(7, \mathbb{C})$ that fixes ϕ is $G_2^{\mathbb{C}} \times \{\omega\mathbf{1} | \omega^3 = 1\}$ where $\mathbf{1}$ is the 7×7 identity matrix.

Rather than using ϕ as an entangled state, in string theory it is used as a *real* differential form on a seven-dimensional real manifold. In this context instead of the complex SLOCC group the real one, i.e., $GL(7, \mathbb{R})$, is used. The stabilizer of ϕ as a real three-form is the compact real form G_2 which is the automorphism group of the octonions. In the theory of special holonomy manifolds invariant forms like ϕ are called calibrations. Note that after the permutation $e^5 \leftrightarrow e^7$ we obtain the form for ϕ usually used in the literature [3,59].

Let us now take an *arbitrary* element $\Phi \in \wedge^3 U^*$ with $U = \mathbb{C}^7$. Such an element can be written in the form

$$\Phi = \frac{1}{3!}\Phi_{ABC} e^A \wedge e^B \wedge e^C. \tag{133}$$

Define [35,39] the matrix of a symmetric bilinear form as

$$\mathcal{B}_{AB} = -\frac{1}{144}\Phi_{AC_1C_2}\Phi_{BC_3C_4}\Phi_{C_5C_6C_7}\varepsilon^{C_1C_2C_3C_4C_5C_6C_7}. \tag{134}$$

Then it can be shown [35,37] that

$$I_7(\Phi) \equiv \mathrm{Det}\mathcal{B} \tag{135}$$

is a relative invariant under the action of the SLOCC group $GL(7, \mathbb{C})$. This means that under the action of a $g \in GL(7, \mathbb{C})$ the invariant transforms as $g^* I_7 = (\mathrm{Det} g)^9 I_7$. Especially choosing ϕ shows that $\mathcal{B}_{AB} = \delta_{AB}$, hence $I_7(\phi) = 1$.

We propose the three fermionic state ϕ of Eq. (132) as a generalization of the tripartite GHZ state. It has a nonvanishing relative invariant just like the GHZ state for three qubits, and the GHZ-like states for three fermions with six single particle states. The invariant I_7 for three fermionic states plays a similar role to Cayley's hyperdeterminant Eq. (4) for three qubits. There is another similarity with the canonical GHZ state and ϕ. If we suitably normalize ϕ, hence producing a $|\phi\rangle$ with unit norm, and calculate the reduced density matrix (since the constituents are identical any of such reduced density matrices will do), we get

$$\rho_1 \equiv \mathrm{Tr}_{23}|\phi\rangle\langle\phi| = \frac{1}{7}\mathbf{1}. \tag{136}$$

This reduced density matrix is the one representing the *totally mixed state* for any of the subsystems. This relation is coming from the identity $f_{ACD}f_{BCD} = 6\delta_{AB}$ for the octonionic structure constants, i.e., $e_A e_B = f_{ABC}e_C$. The two-partite reduced density matrix of $|\phi\rangle$,

$$\rho_{23} = \mathrm{Tr}_1|\phi\rangle\langle\phi|, \tag{137}$$

will be a 21×21 matrix. The structure of this matrix can be worked out using the identity

$$f_{ABC}f_{ADE} = f_{BCDE} + \delta_{BD}\delta_{CE} - \delta_{BE}\delta_{CD}. \tag{138}$$

This formula shows that the structure of bipartite density matrices is controlled by the octonionic structure constants f_{BCDE} connected to the incidence structure of the complement of the lines of the Fano plane. Hence regarded as an entangled state, $|\phi\rangle$ is connected in many ways to the structure of the octonions. It would be an interesting possibility to use the properties of $|\phi\rangle$ as a manifestation of the algebra of octonions in quantum information.

In string theory instead of the complex vector space U the real tangent space of a 7-manifold M_7 is used. This can be regarded as the real version of the state space for our

TABLE I. Entanglement classes of three fermions with seven single particle states.

Type	Canonical form	Instructive $SL(7)$ equivalent	Name
f_1	E^{123}	E^{123}	Separable
f_2	$E^{123}+E^{145}$	$E^1\wedge(E^{23}+E^{\bar{2}\bar{3}})$	Biseparable
f_3	$E^{123}+E^{456}$	$E^{123}+E^{\bar{1}\bar{2}\bar{3}}$	GHZ
f_4	$E^{162}+E^{243}+E^{135}$	$E^{12\bar{3}}+E^{1\bar{2}3}+E^{\bar{1}23}$	W
f_5	$E^{123}+E^{456}+E^{147}$	$E^{1\bar{1}}\wedge E^7+E^{123}+E^{\bar{1}\bar{2}\bar{3}}$	Sympl_1/GHZ
f_6	$E^{152}+E^{174}+E^{163}+E^{243}$	$(E^{1\bar{1}}+E^{2\bar{2}}+E^{3\bar{3}})\wedge E^7+E^{\bar{1}\bar{2}\bar{3}}$	Sympl_3/Sep
f_7	$E^{146}+E^{157}+E^{245}+E^{367}$	$(E^{2\bar{2}}+E^{3\bar{3}})\wedge E^7+E^{123}+E^{\bar{1}\bar{2}\bar{3}}$	Sympl_2/GHZ
f_8	$E^{123}+E^{145}+E^{167}$	$(E^{1\bar{1}}+E^{2\bar{2}}+E^{3\bar{3}})\wedge E^7$	Sympl_3
f_9	$E^{123}+E^{456}+(E^{14}+E^{25}+E^{36})\wedge E^7$	$(E^{1\bar{1}}+E^{2\bar{2}}+E^{3\bar{3}})\wedge E^7+E^{123}+E^{\bar{1}\bar{2}\bar{3}}$	Sympl_3/GHZ

tripartite states with the amplitudes now depending on the coordinates of the manifold. Hence the state in this case is a *real* differential three-form. For nondegenerate three-forms Φ taken from the stable orbit of $GL(7,\mathbb{R})$ represented by ϕ, one can define a metric [35,39]

$$g_{AB}=\mathrm{Det}(\mathcal{B})^{-1/9}\mathcal{B}_{AB}. \tag{139}$$

Since $\mathrm{Det}g=(\mathrm{Det}\mathcal{B})^{2/9}$ one can define Hitchin's functional

$$V_7(\Phi)\equiv\int_{M_7}I_7^{1/9}(\Phi)d^7x=\int_{M_7}\sqrt{g_\Phi}d^7x. \tag{140}$$

This formula shows that Hitchin's functional is simply the volume of M_7 with respect to a metric determined by the nondegenerate three-form Φ according to the formulas of Eqs. (134) and (139). The relative invariant I_7 is just the entanglement measure of Eq. (135). The important property of $V_7(\Phi)$ is that its critical points in a fixed cohomology class give [35,39]

$$d\Phi=0,\qquad d*\Phi=0, \tag{141}$$

where the Hodge star is the one defined with respect to the metric determined by Φ. These are the conditions for our three-form Φ defining a metric of G_2 holonomy [59].

Apart from the nondegenerate class (i.e., the one with $I_7\neq 0$) in quantum information one is also interested in the full structure of $GL(7,\mathbb{C})$ orbits and their stabilizers. These classes are precisely the SLOCC entanglement classes. The orbit structure over the complex field has been given by Schouten [48]; over finite fields it has been obtained by Cohen and Helminck [49]. Here we need the result over $\mathbb{C}$. In the notation of Ref. [49] these are the classes of type f_1-f_9; see the first column of Table I. Here, in accordance with the notation of Ref. [49], in the second column we expressed the representatives of these classes in the basis $\{E^A\}$. Again it is instructive to relabel the basis vectors using the mapping $\{1,2,3,4,5,6,7\}\mapsto\{1,2,3,\bar{1},\bar{2},\bar{3},7\}$. In this new notation the representatives of the SLOCC classes are given in the third column of Table I. Note that arriving at these forms for the classes f_2, f_6, f_7, f_8 we have chosen different representatives by applying suitable permutations that are still elements of the SLOCC group. The corresponding permutations are (456), (17346)(25), (1765342), (176342), respectively.

Notice that in the third column of Table I the canonical forms are written in the form of $\varrho+\omega\wedge E^7$ where ϱ is a three-form based on the six-dimensional subspace V^* spanned by the basis vectors E^a, $a=1,\dots 6$ and ω is either zero or a two-form on V of Slater [56] rank 1,2, or 3. For $\omega\equiv 0$ the three-form ϱ can belong to the four classes well-known from Sec. II B. They are the separable, biseparable, W, and GHZ states. There is a class with $\varrho\equiv 0$; i.e., our state is of the form $\omega\wedge E_7$ with ω a nondegenerate symplectic form. We have three classes with ϱ being a GHZ state combined with the term $\omega\wedge E_7$ with the two-form being Slater rank 1, 2, 3. Notice that the case of maximal Slater rank plus a GHZ state is just the nondegenerate state ϕ belonging to the complex stable orbit of $GL(7,\mathbb{C})$. There is still one class we have not mentioned; it is the one with a representative consisting of a fully separable ϱ plus $\omega\wedge E_7$ with ω full rank.

It is important to note that over the reals we have *two* stable $GL(7,\mathbb{R})$ orbits. One of them is just the one with the usual representative ϕ of Eq. (132) expressed in the real basis e_A. Its stabilizer is the compact real from G_2 of the complex group $G_2^{\mathbb{C}}$. The other orbit has the representative

$$\tilde{\phi}=e^{123}+e^{345}+e^{156}-e^{246}-e^{147}-e^{257}-e^{367}, \tag{142}$$

with its stabilizer being $\tilde{G}_2$ the noncompact real form of $G_2^{\mathbb{C}}$, i.e., the automorphism group of the split octonions. Using the new basis

$$\begin{aligned} F^{1,2,3}&=e^{1,2,3}+e^{4,5,6},\\ F^{\bar{1},\bar{2},\bar{3}}&=e^{1,2,3}-e^{4,5,6},\\ F_7&=e^7, \end{aligned} \tag{143}$$

$\tilde{\phi}$ can be written as

$$\tilde{\phi}=\frac{1}{2}(F^{123}+F^{\overline{123}}+(F^{1\bar{1}}+F^{2\bar{2}}+F^{3\bar{3}})\wedge F^7). \tag{144}$$

Comparing Eqs. (132) and (144) we see that ϕ and $\tilde{\phi}$ are of the same form in the basis $\{E^A\}$ and $\{F^A\}$, respectively.

From the definitions of Eqs. (130) and (143) it is clear that although ϕ and $\tilde{\phi}$ are $GL(7,\mathbb{R})$ inequivalent, they are $GL(7,\mathbb{C})$ equivalent. Observe that ϕ and $\tilde{\phi}$ can be written in the canonical form $\varrho_{\mp} \pm \omega \wedge e^7$ where $\varrho_{\mp}$ are three-forms with Hitchin's invariant of Eq. (19) negative or positive.

The two real SLOCC classes can alternatively be characterized by the property that $\mathcal{B}_{AB} = \delta_{AB}$ or of the form diag$\{1, 1, 1, -1, -1, -1, -1\}$. In the first case one can calculate the Hodge dual $*\phi$ of $\phi = \varrho + \omega \wedge e^7$ with respect to the metric of Eq. (139):

$$* \phi = \hat{\varrho} \wedge e^7 - \sigma, \qquad \sigma = \frac{1}{2}\omega \wedge \omega, \tag{145}$$

$$\begin{aligned} \hat{\varrho} &= e^{456} - e^{234} + e^{135} - e^{126}, \\ \omega &= e^{14} + e^{25} + e^{36}. \end{aligned} \tag{146}$$

Notice that a calculation shows that $\hat{\varrho}$ is the Freudenthal dual of $\varrho = e^{123} - e^{156} + e^{246} - e^{345}$. According to the formula

$$\frac{1}{4}\varrho \wedge \hat{\varrho} = \frac{1}{6}\omega \wedge \omega \wedge \omega, \tag{147}$$

hence using $\omega \wedge \varrho = 0$ and the invariance properties of $V(\Phi)$, an alternative formula for Hitchin's functional on M_7 is [35,39]

$$V_7(\Phi) = \int_{M_7} \Phi \wedge *_{\Phi}\Phi. \tag{148}$$

Recall that the SLOCC classes are all of the canonical form $\omega_i \wedge e^7 + \varrho_a$ where $i = 0, 1, 2, 3$ refers to the Slater rank [56] of the ω (for $i = 3$ we have a full rank symplectic form) and $a = 0, 1, 2, 3, 4$ labels the five entanglement classes for six fermions with six modes. For the six mode case and its STU truncation the degenerate cases have the interpretation as small black holes [7]. What is the physical interpretation of the degenerate cases of the seven mode case, i.e., the classes $f_1, \ldots f_8$ of Table I?

VI. CONCLUSIONS

In this paper we put forward the proposal to regard the invariants underlying the Hitchin functionals as entanglement measures for special entangled systems. In this picture the nondegenerate class of stable forms corresponds to the class of genuine entangled (GHZ-like) states. This idea makes it possible to generalize the BHQC substantially. Unlike in conventional treatments of the subject where entanglement measures were directly related to the Bekenstein-Hawking entropy formulas, we showed that it is more natural to connect them to *action functionals*. From such functionals one can recover the usual correspondence with the Bekenstein-Hawking entropy merely at the *semi-classical level*. Furthermore since one loop calculations based on *quantization* of such functionals are also capable of reproducing results obtained by topological string techniques, via the OSV conjecture this interpretation also hints that one can use the BHQC beyond the semi-classical level.

We must stress, however, that the aim of the present paper was merely to *relate* Hitchin functionals to entanglement measures via the use of the original OSV conjecture [41]; hence the fine details of this "correspondence" are yet to be worked out. For example it is known [43] that the OSV conjecture has to be refined in many respects. An important departure from the original formulation is the inclusion of an additional measure factor in the usual integral expression of the index of BPS states of a given charge expressed in terms of the topological string partition function. However, the presence of this extra measure does not affect the leading saddle point calculation of the entropy, revealing the fact that the entropy is related to measures of entanglement. A similar trivial calculation demonstrates the same link with a measure of entanglement and the quantization of Hitchin's functional in the leading order. In order to develop further the BHQC by taking into account the refinement of the OSV conjecture, one should have to study the structure of split attractor flows related to multicenter black hole solutions in this entanglement based picture. One solid piece of evidence that entanglement measures might be important even in this multicenter case comes from a recent paper of one of us on two-center stu black holes [75] building on the idea of a "horizontal" symmetry group of Ferrara *et al.* [76] which classifies invariants for p-center black holes. Hopefully further investigations in this spirit will help to fill in the gaps and to connect our entanglement based considerations to the refined version of the OSV conjecture.

The approach based on form theories also has the advantage that it suggests that one does not have to assume the underlying manifold to be furnished with a special holonomy (Calabi-Yau, G_2, etc.) structure from the start. On the contrary these structures are arising as critical points of functionals coming from measures of entanglement. Identifying Hitchin's invariants with measures of entanglement also makes it possible to reconsider previous results of the BHQC on the attractor mechanism as a distillation procedure within a nice and unified framework. As a side result we connected the notion of the Freudenthal dual to the one of almost complex and generalized almost complex structures on M. These structures are integrable precisely when the Freudenthal dual form (or state) is closed. Finally as an application to quantum information we saw that Hitchin's functional for seven-dimensional manifolds gives rise to a natural measure of entanglement playing a basic role in understanding the SLOCC classes of three fermionic states with seven modes. We observed that the analogue of the GHZ class provides a representative state (the calibration form) which via the correlations in its reduced density matrices might serve as a candidate for "seeing the octonions in the lab."

Notice that for six-dimensional manifolds all of our functionals were based on special PVs coming from Freudenthal systems of simple cubic Jordan algebras. These Jordan algebras are the complexifications of the cubic ones of Hermitian matrices with real, complex, and quaternionic entries. In Table II we briefly summarized the properties of the relevant PVs as related to Freudenthal systems. In this paper we have not yet mentioned the string theoretical background of the octonionic case. This case with the corresponding functional based on the quartic invariant of E_7 should be connected to the important new development of generalized exceptional geometry [77,78]. In this field there are reformulations of the $N = 2$ supergravity backgrounds arising in Type II string theory in terms of quantities transforming under the $E_{7(7)}$ U-duality group. This formalism combines the pure spinors of the Neveu-Schwartz sector connected to the degrees of freedom of generalized complex geometry with the Ramond-Ramond sector giving rise to an extended version of generalized geometry. It would be instructive to connect our approach based on Freudenthal systems to these results.

Let us finally discuss some of the important conceptual issues we have not yet investigated. Throughout this paper we called entangled "states" objects like $\Gamma \in H^3(M, \mathbb{Z})$ and $\varrho \in H^3(M, \mathbb{R})$ or $\varphi \in H^\bullet(M, \mathbb{R})$ (where the latter is a polyform of either even or odd degree). In particular we called the representatives of cohomology classes of the $M = T^2 \times T^2 \times T^2$ STU case "3-qubit states." Is there a physical basis for calling such constructs "entangled states" of some kind?

First of all let us notice that the spaces of *real* cohomology classes that show up in the Hitchin and generalized Hitchin functionals are all *phase spaces* in the conventional sense. The symplectic form is the usual one defined for Freudenthal systems which is just the Mukai pairing for polyforms. The Hamiltonians on these phase spaces are the functionals themselves; the Freudenthal duals are the corresponding Hamiltonian vector fields. Thanks to these properties in all cases of the PVs of Table II we can regard the elements of such Freudenthal systems as "classical states."

On the other hand our classical phase spaces are locally the moduli spaces of complex [35], generalized complex

TABLE II. Freudenthal triple systems ($\mathfrak{M}(\mathfrak{J})$) over cubic Jordan algebras ($\mathfrak{J}$), their automorphism groups ($\mathrm{Inv}\mathfrak{M}(\mathfrak{J})$), and the corresponding Hitchin functional.

$\mathfrak{J}$	$\mathrm{Inv}(\mathfrak{M})$	$\dim\mathfrak{M}$	Hitchin functional
$\mathcal{H}_3(\mathbb{R})$	$Sp(6, \mathbb{C})$	14	Constrained Hitchin
$\mathcal{H}_3(\mathbb{C})$	$SL(6, \mathbb{C})$	20	Hitchin
$\mathcal{H}_3(\mathbb{H})$	$\mathrm{Spin}(12, \mathbb{C})$	32	Generalized Hitchin
$\mathcal{H}_3(\mathbb{O})$	$E_7(\mathbb{C})$	56	Generalized exceptional

[36], and probably generalized exceptional structures. However, these spaces are in turn also complex ones so we should see a complex structure on them. As a byproduct of this observation beyond the classical one there should be extra structures playing an important role. In the case of the Hitchin functional we can illustrate this as follows.

(1) One can embed the real cohomology classes into $H^3(M, \mathbb{C})$. This corresponds to the fact that the stable open orbit can be given a structure of a pseudo-Kähler manifold [35] with signature $(1, h^{2,1})$ with the complex structure defined by the derivative of the map that associates to a state its Freudenthal dual. This complex structure is acting on $H^3(M, \mathbb{C})$ as $+i$ on $H^{3,0} \oplus H^{2,1}$ and as $-i$ on $H^{0,3} \oplus H^{1,2}$

(2) One can also embed the real cohomology classes into the space of complex ones furnished with a Hermitian inner product of Eq. (85). The rationale for doing this is encoded into the expansion of $[\Gamma] = \varrho$ in the Hodge diagonal basis [e.g., like the expansion of Eq. (103)]. Notice that the Hodge star is acting on $H^3(M, \mathbb{R})$ as $+i$ on $H^{3,0} \oplus H^{1,2}$ and as $-i$ on $H^{0,3} \oplus H^{2,1}$ (in the STU case $*$ is just i times the parity check operator, i.e., $i\sigma_3 \otimes \sigma_3 \otimes \sigma_3$). This defines an alternative complex structure and embedding for $H^3(M, \mathbb{R})$. Notice also that in this case $||\Gamma||^2$ is positive and related to the black hole potential. Since the Hodge diagonal basis is depending on the coordinates $\tau, \bar{\tau}$ of the moduli space $\mathcal{M}$ of M, we obtain states with complex amplitudes depending on the charges and the moduli. This is the setting which made it possible to regard our real states as also elements of a complex finite dimensional Hilbert space making the entanglement interpretation useful.

Do not confuse our entangled states with the ones discussed in topological string theory. The two different states are related by geometric quantization.

First recall the physical meaning of case 1. According to the OSV conjecture the partition function for BPS black holes in Calabi-Yau compactifications of type II string theory is equal to the product of partition sums of topological strings. The topological string partition function can also be interpreted [79] as a wave function obtained by quantizing our classical phase space $H^3(M, \mathbb{R})$. The idea is that there should be a state $|\Psi\rangle$ which contains the background independent information of topological string theory. In order to carry out this (geometric) quantization a polarization is needed. The polarization which is used for this quantization is the one of case 1, and again depending on the coordinates $\tau, \bar{\tau}$ of the moduli space $\mathcal{M}$ of M. The dependence on these coordinates is expressed in the holomorphic anomaly equation [45,80]. As we know [81] the Hermitian metric is constructed from the canonical symplectic structure on $H^3(M, \mathcal{R})$, and this complex structure is not positive definite, but rather of signature $(1, h^{2,1})$. The quantization is carried out by elevating the expansion coefficients of Eq. (103) (i.e., the amplitudes of our entangled states) to moduli-dependent annihilation operators [81] and then constructing coherent states. This results in non-normalizable states. However, in this approach the holomorphic dependence of the complex structure on τ is manifest.

On the other hand using case 2, the Weyl polarization [81] provided by the Hodge star, we have a positive definite metric; however, the holomorphic dependence of the complex structure is lost. This polarization is not suitable for studying the holomorphic anomaly equations, however directly connected to our entanglement interpretation. Moreover, it is probably more natural for finding its role in the non-BPS version of the OSV conjecture [42] where the holomorphic structure is lost. Can we relate somehow this non-BPS branch to the real orbit with $\mathcal{D} > 0$ of Hitchin's functional?

APPENDIX

In this Appendix we would like to establish a dictionary for the generalized Hitchin functional between the languages based on polyforms and the Freudenthal systems based on the Jordan algebra of quaternion Hermitian 3×3 matrices. Let W be a six-dimensional *real* vector space and W^* its dual. The basis vectors for these spaces will be denoted by $\{e_i\}$ and $\{e^i\}$ $i = 1, 2 \ldots 6$, respectively. There is a natural symmetric bilinear form on the space $W \oplus W^*$ given by

$$(v + \omega, u + \sigma) = \frac{1}{2}(\omega(u) + \sigma(v)), \quad v, u \in W, \quad \omega, \sigma \in W^*. \tag{A1}$$

This symmetric form has signature (6,6) and defines the noncompact orthogonal group $O(W \oplus W^*) \simeq O(6, 6)$. By noticing that

$$\wedge^{12}(W \oplus W^*) = \wedge^6 W \otimes \wedge^6 W^*, \tag{A2}$$

and using the natural pairing between the latter two terms on the right hand side of Eq. (A2) one can define a canonical orientation. The group preserving the symmetric form taken together with this orientation is $SO(W \oplus W^*) \simeq SO(6, 6)$. The Lie algebra of this group is defined as usual by

$$so(W \oplus W^*) = \{T | (Tu, v) + (u, Tv) = 0, u, v \in W \oplus W^*\}, \tag{A3}$$

and can be parametrized as

$$T = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}. \tag{A4}$$

Here

$$A \in \mathrm{End}(W), \qquad A = A^i{}_j e^j \otimes e_i, \tag{A5}$$

$$B \in \Lambda^2 W^*: W \to W^*, \qquad B = \frac{1}{2} B_{ij} e^i \wedge e^j, \tag{A6}$$

$$\beta \in \Lambda^2 W: W^* \to W, \qquad \beta = \frac{1}{2} \beta^{ij} e_i \wedge e_j. \tag{A7}$$

This shows that $so(W \oplus W^*) = \Lambda^2(W \oplus W^*) = \mathrm{End}(W) \oplus \Lambda^2 W^* \oplus \Lambda^2 W$.

Let us now define the Clifford algebra $\mathrm{Cliff}(W \oplus W^*)$ by the relation

$$w^2 = (w, w)\mathbf{1}, \qquad \forall w \in W \oplus W^*. \tag{A8}$$

The Clifford algebra can be represented on the space $\wedge^\bullet W^*$ of polyforms by

$$(v + \omega) \cdot \varphi = i_v \varphi + \omega \wedge \varphi, \qquad \varphi \in \wedge^\bullet W^*. \tag{A9}$$

Indeed,

$$\begin{aligned}(v + \omega)^2 \cdot \varphi &= i_v(\omega \wedge \varphi) + \omega \wedge (i_v \varphi) = (i_v \omega)\varphi \\ &= \langle v + \omega, v + \omega \rangle,\end{aligned} \tag{A10}$$

hence we have an algebra representation. This formula also gives rise to the standard spin representation; hence the exterior algebra provides a natural description of spinors provided [65] we tensor with the one-dimensional space $(\wedge^6 W)^{1/2}$. Hence the representation space is

$$S = \wedge^\bullet W^* \otimes (\wedge^6 W)^{1/2}. \tag{A11}$$

We can decompose the space of spinors to positive and negative chirality elements $S = S^+ \oplus S^-$ under the ± 1 eigenspaces of the volume element of the Clifford algebra. These are simply exterior forms of even and odd degree:

$$\begin{aligned} S^+ &= \wedge^{\mathrm{ev}} W^* \otimes (\wedge^6 W)^{1/2}, \\ S^- &= \wedge^{\mathrm{odd}} W^* \otimes (\wedge^6 W)^{1/2}. \end{aligned} \tag{A12}$$

They are irreducible under the double cover of $SO(W \oplus W^*)$, the spin group Spin $(W \oplus W^*)$ consisting of products with an *even* number of elements $w_1 w_2 \ldots w_{2r}$, where $(w_i, w_i) = \pm 1$.

Since $so(W \oplus W^*)$ can also be embedded in the Clifford algebra one can calculate the spinorial action of A, B, and β of Eqs. (A5)–(A7) on $\wedge^\bullet W^*$. One can then show that [47] the spinorial versions of A, B, and β are, respectively, $\frac{1}{2} A^i{}_j (e_i e^j - e^j e_i)$, $\frac{1}{2} B_{ij} e^j e^i$, and $\frac{1}{2}\beta^{ij} e_j e_i$. As a result the spinorial actions take the form

$$A \cdot \varphi = \frac{1}{2}\mathrm{Tr}A - A^* \varphi = \frac{1}{2}\mathrm{Tr}A - A^i{}_j e^j \wedge i_{e_i}\varphi, \tag{A13}$$

$$B \cdot \varphi = -B \wedge \varphi = \frac{1}{2} B_{ij} e^j \wedge (e^i \wedge \varphi), \tag{A14}$$

$$\beta \cdot \varphi = i_\beta \varphi = \frac{1}{2}\beta^{ij} i_{e_j}(i_{e_i})\varphi. \tag{A15}$$

An important corollary of Eq. (A13) is that after exponentiation the spinorial action of an element $L \in GL^+(W)$ can be expressed as

$$L \cdot \varphi = \sqrt{\mathrm{Det} L}(L^*)^{-1}\varphi, \tag{A16}$$

giving a rationale for the appearance of the factor $(\wedge^6 W)^{1/2}$ in Eq. (A11).

Let us now complexify our W to $V = W \otimes \mathbb{C} = \mathbb{C}^6$ and let $\varphi \in \wedge^{ev} V^*$ be a polyform of even degree. Then we have

$$\varphi = \varphi_0 + \varphi_2 + \varphi_4 + \varphi_6, \qquad \varphi_p \in \wedge^p V^*. \tag{A17}$$

Since $B \cdot \varphi = -B \wedge \varphi$, for a $B = \frac{1}{2}B_{ij}e^i \wedge e^j \in \wedge^2 V^*$ we have the spinorial action of e^{-B} on the special form $\varphi_0 \equiv 1$ as

$$\begin{aligned} e^{-B} \cdot 1 &= \left(1 + B + \frac{1}{2}B \wedge B + \frac{1}{6}B \wedge B \wedge B\right) \cdot 1 \\ &= 1 + \sum_{i<j} B_{ij}e^i \wedge e^j + \sum_{i<j} \mathrm{Pf}(B_{(ij)}) * (e^i \wedge e^j) \\ &\quad + \mathrm{Pf}(B)\epsilon. \end{aligned} \tag{A18}$$

Here $\epsilon = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6$ and the Pfaffian of the 6×6 *complex* matrix B_{ij} is

$$\mathrm{Pf}\,(B) = \frac{1}{3!2^3}\varepsilon^{ijklmn}B_{ij}B_{kl}B_{mn}. \tag{A19}$$

On the other hand $\mathrm{Pf}(B_{(ij)})$ is the Pfaffian of the 4×4 matrix obtained from the original 6×6 one after omitting the (i, j)th rows and columns. Hence for example

$$\begin{aligned} &\mathrm{Pf}(B_{(56)}) = B_{12}B_{34} - B_{13}B_{24} + B_{14}B_{23}, \\ &*(e^5 \wedge e^6) = e^1 \wedge e^2 \wedge e^3 \wedge e^4. \end{aligned} \tag{A20}$$

Let us now recall that Skew $(6, \mathbb{C}) \simeq \mathrm{Herm}(3, \mathbb{H}) \otimes \mathbb{C}$; i.e., the space of 6×6 skew-symmetric matrices with complex entries can be identified with the cubic Jordan algebra of quaternion Hermitian matrices when the quaternions are replaced by biquaternions. An identification of these objects is given as follows:

$$\mathcal{B} = \begin{pmatrix} \alpha & c & \bar{b} \\ \bar{c} & \beta & a \\ b & \bar{a} & \gamma \end{pmatrix} \leftrightarrow B = \begin{pmatrix} \alpha\epsilon & c\epsilon & \tilde{b}\epsilon \\ \tilde{c}\epsilon & \beta\epsilon & a\epsilon \\ b\epsilon & \tilde{a}\epsilon & \gamma\epsilon \end{pmatrix}. \tag{A21}$$

Here on the left-hand side $\alpha, \beta, \gamma \in \mathbb{C}$, $a, b, c \in \mathbb{H} \otimes \mathbb{C}$, and overline refers to quaternionic conjugation. On the right-hand side we have $\alpha, \beta, \gamma \in \mathbb{C}$, $a, b, c \in \mathrm{Matr}(2, \mathbb{C})$, i.e., 2×2 complex matrices, $\tilde{a} \equiv -\epsilon a^T \epsilon$ with ϵ the standard $SL(2, \mathbb{C})$ invariant antisymmetric 2×2 matrix with $\epsilon_{12} = 1$. One can check that $B^T = -B$.

Now in the language of cubic Jordan algebras the cubic norm $N(\mathcal{B}) = \mathrm{Det}(\mathcal{B})$ is the determinant of the 3×3 matrix with biquaternionic entries. It can be checked that it corresponds to the Pfaffian of the 6×6 antisymmetric matrix with complex entries, i.e.,

$$\mathrm{Det}\,(\mathcal{B}) \leftrightarrow \mathrm{Pf}(B). \tag{A22}$$

Moreover, for elements of $\mathrm{Herm}(3, \mathbb{H}) \otimes \mathbb{C}$ one can define the quadratic *sharp* map by

$$\mathcal{B} \mapsto \mathcal{B}^\sharp = \mathcal{B}^2 - \mathrm{Tr}(\mathcal{B})\mathcal{B} + \frac{1}{2}((\mathrm{Tr}(\mathcal{B}))^2 - \mathrm{Tr}(\mathcal{B}^2))I, \tag{A23}$$

satisfying $\mathcal{B}\mathcal{B}^\sharp = \mathrm{Det}(\mathcal{B})I$ with I the 3×3 unit matrix. The polarization of the sharp map is

$$\mathcal{B} \times \mathcal{C} = (\mathcal{B} + \mathcal{C})^\sharp - \mathcal{B}^\sharp - \mathcal{C}^\sharp. \tag{A24}$$

Now one can check that

$$\mathcal{B}^\sharp \leftrightarrow \mathrm{Pf}(B_{(\cdot\cdot)}). \tag{A25}$$

As a result of these considerations one can have the correspondence

$$e^{-B} \cdot 1 \leftrightarrow (1, \mathcal{B}, \mathcal{B}^\sharp, \mathrm{Det}(\mathcal{B})) \in \mathbb{C} \oplus \mathcal{J} \oplus \mathcal{J} \oplus \mathbb{C}, \tag{A26}$$

where we denoted the cubic Jordan algebra $\mathrm{Herm}(3, \mathbb{H}) \otimes \mathbb{C}$ by $\mathcal{J}$. The algebraic object $\mathbb{C} \oplus \mathcal{J} \oplus \mathcal{J} \oplus \mathbb{C}$ is called the Freudenthal triple system $\mathcal{F}(\mathcal{J})$ associated to the cubic Jordan algebra $\mathcal{J}$. In particular one can see that $e^{-B} \cdot 1$ can be mapped to a special element of $\mathcal{F}(\mathcal{J})$. Now it is straightforward to elaborate the whole correspondence between the action of $\mathrm{Spin}(12, \mathbb{C})$ on the space of spinors $S = S^+ = \Lambda^{\mathrm{ev}} V^* \otimes (\Lambda^6 V)^{1/2}$ and the *conformal group* of $\mathcal{J}$, $\mathrm{Conf}(\mathcal{J})$, acting on $\mathcal{F}$.

The conformal group of $\mathcal{J}$ is the group of rational transformations of $\mathcal{J}$ generated by the translations ($\mathcal{T}$), inversions ($\mathcal{I}$), and transformations ($\mathcal{L}$) belonging to the structure group of $\mathcal{J}$ (linear bijections of $\mathcal{J}$ leaving invariant the norm N up to a character χ). The translations and inversions are of the following form:

$$\mathcal{T}_{\mathcal{B}}\colon \mathcal{Z} \mapsto \mathcal{Z} + \mathcal{B}, \tag{A27}$$

$$\mathcal{I}\colon \mathcal{Z} \mapsto -\mathcal{Z}^{-1}. \tag{A28}$$

It is known [82] that there is a projective irreducible representation of $\mathrm{Conf}(\mathcal{J})$ on $\mathcal{F}(\mathcal{J})$ which is of the form

$$\pi(g)(\eta, y, x, \xi) = (\eta', y', x', \xi') \in \mathcal{F}, \quad g \in \mathrm{Conf}(\mathcal{J}), \tag{A29}$$

with the translations $\pi(\mathcal{T}_{-\mathcal{B}})$ acting as

$$\eta' = \eta, \tag{A30}$$

$$y' = y + \eta\mathcal{B}, \tag{A31}$$

$$x' = x + \mathcal{B} \times y + \eta\mathcal{B}^\sharp, \tag{A32}$$

$$\xi' = \xi + \mathrm{Tr}(\mathcal{B}x) + \mathrm{Tr}(\mathcal{B}^\sharp y) + \eta\mathrm{Det}\mathcal{B}, \tag{A33}$$

[for the definition of $\mathcal{B} \times y$ see Eq. (A24)]. For the inversions $\pi(\mathcal{I})$ we have

$$\eta' = \xi, \quad y' = -x, \quad x' = y, \quad \xi' = -\eta, \tag{A34}$$

and finally for $\pi(\mathcal{L})$ one gets

$$\eta' = \chi(\mathcal{L})^{-1/2}\eta, \qquad y' = \chi(\mathcal{L})^{-1/2}\mathcal{L}(y),$$
$$x' = \chi(\mathcal{L})^{1/2}\mathcal{L}^{*-1}(x), \qquad \xi' = \chi(\mathcal{L})^{1/2}\xi. \tag{A35}$$

By virtue of Eqs. (A26) and (A33) we have

$$e^{-B}\cdot 1 \leftrightarrow \pi(\mathcal{T}_{-\mathcal{B}})(1,0,0,0) = (1,\mathcal{B},\mathcal{B}^{\#},\mathrm{Det}\mathcal{B}). \tag{A36}$$

Now by associating a polyform to an element of $\mathcal{F}$ as

$$(\varphi_0, \varphi_2, \varphi_4, \varphi_6) \leftrightarrow (\eta, y, x, \xi), \tag{A37}$$

one can check that

$$e^{-B}\cdot\varphi \leftrightarrow \pi(\mathcal{T}_{-\mathcal{B}})(\eta, y, x, \xi). \tag{A38}$$

Similarly recalling Eqs. (A16) and (A35) for the $L \in GL(6,\mathbb{C})$ action we get the correspondence

$$L\cdot\varphi \leftrightarrow \pi(\mathcal{L})(\eta, y, x, \xi), \tag{A39}$$

with the character $\chi(\mathcal{L}) \leftrightarrow (\mathrm{Det}L)^{-1}$. Finally the correspondence for the β transform takes the form

$$e^{\beta}\cdot\varphi \leftrightarrow \pi(\mathcal{I}^{-1}\circ\mathcal{T}_{\mathcal{B}}\circ\mathcal{I})(\eta, y, x, \xi). \tag{A40}$$

The upshot of these considerations is that we managed to represent the $\mathrm{Spin}(12,\mathbb{C})$ action on polyforms of even degree as the action of $\mathrm{Conf}(\mathcal{J})$ on the Freudenthal triple system $\mathcal{F}(\mathcal{J})$. This construction enables the identification of the generalized Hitchin functional with the quartic invariant for $\mathcal{F}(\mathcal{J})$.

In order to do this, recall that for $\mathcal{F}(\mathcal{J})$ we can define a symplectic form and a quartic polynomial, both invariant under $\mathrm{Conf}(\mathcal{J})$. The symplectic form can easily be related to the symplectic form of Hitchin [36]. The latter is defined as

$$\langle\varphi, \psi\rangle = \varphi_0\psi_6 - \varphi_2\psi_4 + \varphi_4\psi_2 - \varphi_6\psi_0$$
$$\in \wedge^6 V^* \otimes ((\wedge^6 V)^{1/2})^2 = \mathbb{C}. \tag{A41}$$

On the other hand the symplectic form on $\mathcal{F}(\mathcal{J})$ takes the form

$$\{p, p'\} = \eta\xi' - \mathrm{Tr}(y\bullet x') + \mathrm{Tr}(x\bullet y') - \xi\eta',$$
$$p = (\eta, y, x, \xi), \qquad p' = (\eta', y', x', \xi'). \tag{A42}$$

Here $\eta, \xi \in \mathbb{C}$ and $x, y \in \mathrm{Herm}(3,\mathbb{H})\otimes\mathbb{C}$, and $x\bullet y = \frac{1}{2}(xy+yx)$ is the Jordan product. Clearly, by virtue of the correspondence Eq. (A37) and the identity $\varphi_2\psi_4 \leftrightarrow \mathrm{Tr}(y\bullet x')$ these structures are mapped to each other.

The quartic invariant for $\mathcal{F}(\mathcal{J})$ takes the following form [23]:

$$q(p) = -[\eta\xi - \mathrm{Tr}(x\bullet y)]^2 + 4\mathrm{Tr}(x^{\#}\bullet y^{\#})$$
$$- 4\eta\mathrm{Det}(x) - 4\xi\mathrm{Det}(y),$$
$$p = (\eta, y, x, \xi). \tag{A43}$$

By virtue of the identification in Eq. (A21) an alternative formula can also be given:

$$q(p) = -[\eta\xi - \sum_{i<j} x_{ij}y_{ij}]^2 + 4\sum_{i<j}\mathrm{Pf}(x_{(ij)})\mathrm{Pf}(y_{(ij)})$$
$$- 4\eta\mathrm{Pf}(x) - 4\xi\mathrm{Pf}(y), \tag{A44}$$

where $\eta, \xi \in \mathbb{C}$ and x, y are 6×6 skew-symmetric matrices with complex entries. The last version of the quartic invariant can easily be related to the coefficients of the polyforms $(\varphi_0, \varphi_2, \varphi_4, \varphi_6)$ needed for the explicit expression of the generalized Hitchin functional. For this we just have to parametrize these component forms as

$$\varphi_0 = \eta\otimes(\epsilon^*)^{1/2}, \qquad \varphi_2 = \tfrac{1}{2!}y_{ij}e^i\wedge e^j\otimes(\epsilon^*)^{1/2},$$
$$\varphi_4 = \frac{1}{4!}\frac{1}{2!}x_{ij}\varepsilon^{ij}{}_{klmn}e^k\wedge e^l\wedge e^m\wedge e^n\otimes(\epsilon^*)^{1/2},$$
$$\varphi_6 = \xi\epsilon\otimes(\epsilon^*)^{1/2}, \tag{A45}$$

where $\epsilon = e^1\wedge e^2\wedge\ldots\wedge e^6$ and $\epsilon^* \equiv e_1\wedge e_2\wedge\ldots\wedge e_6$.

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