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Spinors group field theory and Voros star product: First contact

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In the context of noncommutative geometries, we develop a group Fourier transform for the Lie group SU(2). Our method is based on the Schwinger representation of the Lie algebra $\mathfrak{su}(2)$ in terms of spinors. It allows us to prove that the noncommutative $\mathbb{R}^3$ space dual to the SU(2) group is in fact of the Moyal type and endowed with the Voros star product when expressed in the spinor variables. Finally, from the perspective of quantum gravity, we discuss the application of these new tools to group field theories for spinfoam models and their interpretation as noncommutative field theories with quantum-deformed symmetries.

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I. INTRODUCTION

Spinfoam models provide us with a framework for regularized path integral for quantizing gravity (for a review, see e.g., Ref. [1]). They define transition amplitudes for quantum states of geometry and can be considered as the covariant definition of loop quantum gravity. The first spinfoam model can be seen retrospectively as the Ponzano-Regge model for 3d Euclidian gravity (with no cosmological constant). The path integral is defined on a discretized three-dimensional manifold and the resulting quantum gravity partition function is essentially defined in terms of the $6j$ symbols of the recoupling theory of SU(2) representations. Since then the spinfoam framework has been much developed and generalized to the four-dimensional case and refined in order to account for a Lorentzian signature and a cosmological constant and to incorporate matter fields [2].

In the early 1990s, Boulatov showed [3] that the spinfoam amplitudes of the Ponzano-Regge model could be obtained as Feynman diagram amplitudes of a nonlocal (quantum) field theory defined over a Lie group manifold, in this case $\mathrm{SU}(2)^{\times 3}$. It was later shown that all spinfoam models can be reformulated in such terms and generated from a group field theory (GFT) [4]. The introduction of GFTs to generate spinfoam amplitudes was an important technical development since it allows one to sum in a controlled way over topologies and now the GFTs are considered as the proper nonperturbative definition of spinfoam models. Moreover the GFT framework allows one to discuss the issue of a spinfoam continuum limit and a semiclassical limit in terms of renormalization [5]. A field theory formulation is then a perfect framework to address the typical divergencies one meets in the spinfoam approach, for instance in the infrared regime [6].

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Until recently, the usual point of view on group field theories was to consider the Lie group manifold as the configuration space and perform a Fourier transform using the Peter-Weyl theorem in order to obtain the spinfoam amplitudes in terms of representations and nj symbols for the relevant group. In recent years, noncommutative techniques entered the game thanks to a new type of Fourier transform [7–11]. The general mathematical formalism behind this generalized Fourier transform was mostly developed by Majid [12] and it was rediscovered later on in the context of three-dimensional spinfoam while coupling particles to the Ponzano-Regge model [13,14]. From this perspective, the Lie group manifold is now interpreted as the momentum space and a group Fourier allows us to go from the group to the dual configuration space, which is then a noncommutative space of the Lie algebra type.

In the context of Boulatov's GFT, this generalized Fourier transform formalism has been used to construct the relevant noncommutative $\mathbb{R}^3$ configuration space dual to the group manifold. As a consequence, the GFT can be understood as a noncommutative field theory. However this construction was only done for the group $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$, which is not satisfying. Indeed, to deal with a spinfoam model based on SO(3) instead of SU(2) means that we are missing all the half integer representations. By analogy, this would be similar to working with a field theory based only on the positive energy modes, missing all the negative ones. In some sense we are missing half of the degrees of freedom. More concretely, to consider only the integer representations would prevent introducing fermions in the game. For instance introducing supersymmetry in the game requires one to be able to consider half integer representations in order to define the spinfoam path integral for supergravity [15]. It is also known that a scalar field theory on SO(3) is nonunitary [16]. Dealing instead with SU(2), preliminary results indicate that the scalar field theory is unitary [17]. All these arguments indicate that it is important to have a spinfoam model based on SU(2) instead of SO(3).

The Fourier transform for a momentum space given by SU(2) turns out to be more difficult to construct in order to have a one-to-one map from momentum space to configuration space. In an earlier work [18], the authors analyzed very carefully the details of the group Fourier transform pinpointing that the typical choice of plane waves on SU(2) leads to a two-to-one map. In Ref. [19], the authors nevertheless constructed a one-to-one Fourier transform for SU(2) from a four-dimensional point of view. The first result we will present here is the construction of a well-defined one-to-one Fourier transform for SU(20), using a three-dimensional realization. With respect to the previous proposals, this seems to us the most natural choice of Fourier transform to consider. This choice does not directly impact current spinfoam models that are built usually in momentum space, in terms of group variables. However it affects the understanding we have of the geometric variables. For example, the usual closure relation is now implemented in a fuzzy way as we shall discuss in Secs. III D and V B 1.

This noncommutative perspective for spinfoams allows one to connect the diffeomorphism symmetry (in three dimensions) of spinfoam amplitudes to quantum group symmetries of the field theory [11]. One can expect that these symmetries will be useful in order to discuss the question of renormalization of the field theory by putting constraints on the renormalization scheme and allowed counterterms. Another interesting strength for the spinfoam noncommutative perspective is that it allows one to connect spinfoams to deformed special relativity that is a candidate phenomenological model to encode effective quantum gravity corrections to matter kinematics and dynamics in the semiclassical regime [8]. Unfortunately, quantum field theories based on noncommutative spaces of the Lie algebra type[1] are very poorly understood at this time. For example, there is no integral form for the SO(3) and SU(2) star products; there are no fermionic or Yang-Mills theories defined yet; and the divergence structure of such quantum field theories is barely known, in general.

In fact, the noncommutative field theory that has attracted the most attention is the Moyal-Voros noncommutative field theory, i.e., a noncommutative space of the type $[x_\mu, x_\nu] = \theta_{\mu\nu}$. In this context, Yang-Mills theories have been introduced and very detailed analysis of quantum field theories have been performed [21]. Our second result consists in showing that a GFT based on SU(2) can also be seen as some sort of Moyal field theory, more exactly a noncommutative field using a star product of the Voros type (see Refs. [22,23] for different perspectives on the Voros quantum field theory). We expect that this will open new doors to address the issue of renormalization in quantum gravity. The key idea in deriving this result and obtaining the Voros noncommutative product is to consider the Jordan-Schwinger representation for $\mathfrak{su}(2)$. This representation consists in introducing a pair of harmonic oscillators, or a spinor $|z\rangle \in \mathbb{C}^2$, to describe the Lie algebra $\mathfrak{su}(2)$. This spinor formalism for spin networks and spinfoam models is inspired by the U(N) formalism for intertwiners [24–27] and twisted geometries for loop quantum gravity [28,29]. It has been further developed in Ref. [30].

In Sec. II, we recall the construction of the SO(3) Fourier transform and the issue with generalizing to SU(2). In Sec. III, we recall the spinor construction and introduce the plane waves and star product defined in terms of the spinor variables. We show that the Fourier transform based on this *spinor plane wave* is well defined for SU(2). In Sec. IV, we discuss the implications of the spinor representation. In particular, we show how we can recover the four-dimensional bicovariant differential calculus naturally. We also show that the star product constructed using the spinor plane waves actually coincides with the Voros star product. In Sec. V, we apply the results of the previous sections to the GFT context, focusing in particular on the Boulatov model. Explicitly we present the new shape of the closure constraint using the spinor variables, and make explicit Boulatov action in terms of the spinor variables. We conclude by discussing the quantum group symmetries of the model. We have added two appendices. In the first one, we recall the notion of coherent intertwiners that is relevant to defining the noncommutative delta function in configuration space. In the second one, we discuss the different choices of plane waves one can make using the spinor variables.

In the following we will always work in units $\hbar = c = 1$ and κ is a mass scale, usually taken to be the Planck mass in the context of quantum gravity (phenomenology).

II. STAR PRODUCT FOR SO(3) AND FOURIER TRANSFORM: AN OVERVIEW

In the context of the matter coupling to the Ponzano-Regge spinfoam model for three-dimensional Euclidean quantum gravity, it has been understood that particles and fields behave as in a noncommutative flat geometry [13,14]. Indeed with the particle momenta now living on the Lie group manifold SU(2), which is curved, the natural space-time coordinates defined as dual to the momentum coordinates are naturally noncommutative. This is the same mechanism as happening in deformed or doubly special relativity in four space-time dimensions when deforming the Poincaré symmetry in order to accommodate a universal Planck length (e.g., Ref. [31]).

To make this relation between momentum living in SU(2) and noncommutative three-dimensional coordinate space, a group Fourier transform between SU(2) and $\mathbb{R}^3_\kappa$ was first introduced in Refs. [13,14] and further developed

[1]We have in mind the ones that are not constructed by a simple twist [20], such as $\mathfrak{su}(2)$ or κ Minkowski.

in Refs. [18,19]. This allowed one to describe the propagation of matter coupled to the three-dimensional quantum geometry in terms of actual space-time coordinates. To be more precise, the original group Fourier transform introduced in the context of spinfoam models in Refs. [13,14] maps functions on $SO(3) \sim SU(2)/\mathbb{Z}_2$ to functions on $\mathbb{R}^3$. Later on in Ref. [19], this group Fourier transform was refined to truly go between SU(2) and $\mathbb{R}^3$, but we will here first focus on the original map between SO(3) and $\mathbb{R}^3$, which is currently used to provide spinfoam models and group field theories with a space-time interpretation.

In the framework of the Ponzano-Regge spinfoam model, the natural candidate for a Fourier transform between functions on SU(2) and functions on $\mathbb{R}^3$ is

$$\hat{f}(\vec{X}) = \int_{SU(2)} dg f(g) e^{\frac{\kappa}{2}\mathrm{Tr} gX} \quad \text{with} \quad X = \vec{X}\cdot\vec{\sigma} \in \mathfrak{su}(2), \tag{1}$$

where σ_i are the Pauli matrices (normalized such that $\sigma_i^2 = \mathbb{I}$ for $i = 1, 2, 3$, and $\sigma_i\sigma_j = \delta_{ij}\mathbb{I} + i\epsilon_{ijk}\sigma_k$). Using the standard parametrization of SU(2) group elements as 2×2 matrices,

$$g = \cos\theta\mathbb{I} + i\sin\theta\hat{u}\cdot\vec{\sigma} \quad \text{with} \quad \theta \in [-\pi, \pi], \qquad \hat{u} \in \mathcal{S}^2, \tag{2}$$

we easily evaluate the exponent:

$$\frac{\kappa}{2}\mathrm{Tr} gX = i\vec{p}\cdot\vec{X}, \quad \text{with} \quad \vec{p} = \kappa\frac{1}{2i}\mathrm{Tr} g\vec{\sigma}, \qquad g = \epsilon\sqrt{1-\frac{p^2}{\kappa^2}} + i\frac{\vec{p}}{\kappa}\cdot\vec{\sigma}, \tag{3}$$

where the momentum is bounded in norm, $|p| \leq \kappa$, and $\epsilon = \pm$ registers the sign of $\cos\theta$. In this context, it is natural to introduce a $\star$ product between the plane waves $e_g(X) \equiv e^{\frac{\kappa}{2}\mathrm{Tr} gX}$ that keeps track of the group multiplication on SU(2):

$$(e_{g_1} \star e_{g_2})(X) = e_{g_1 g_2}(X). \tag{4}$$

The problem with this proposal is that the group Fourier transform defined by (1) has a nontrivial kernel:

$$f(-g^{-1}) = -f(g) \Rightarrow \hat{f}(\vec{X}) = 0.$$

This comes because the momentum conjugated to the coordinates $\vec{X}$ is the 3-vector $\vec{p}$ defined as the projection of the group element g onto the Pauli matrices, but that the map $g \rightarrow \vec{p}$ is not a bijection but is two to one. This can be seen directly when trying to recover the δ distribution on SU(2) by the inverse Fourier transform:

$$\int d^3\vec{X} e^{\frac{\kappa}{2}\mathrm{Tr} gX} \propto \delta(g) + \delta(-g) \propto \delta_{SO(3)}(g), \tag{5}$$

as first pointed out in Ref. [32]. Thus it seems more natural to define a group Fourier transform from SO(3) to $\mathbb{R}^3$ if using these plane waves $e_g(X) = e^{\frac{\kappa}{2}\mathrm{Tr} gX}$. Thus, following [13,14], we modify our definition of the group Fourier transform (1) and define instead

$$\hat{f}(\vec{X}) = \int_{SU(2)} dg f(g) e^{\frac{\kappa}{2}\mathrm{Tr}|g|X}, \tag{6}$$

where we define the absolute value of a group element as $|g| = g$ if $\cos\theta \geq 0$ else $|g| = -g$ if $\cos\theta \leq 0$. The plane wave exponent is now

$$\frac{\kappa}{2}\mathrm{Tr}|g|X = i\epsilon\vec{p}\cdot\vec{X}.$$

This absolute value satisfies the obvious identities:

$$|g| = |-g|, \qquad |g_1 g_2| = ||g_1||g_2||.$$

And in terms of p momentum, it reads as

$$g(\vec{p}, \epsilon) = \epsilon\sqrt{1-\frac{p^2}{\kappa^2}} + i\frac{\vec{p}}{\kappa}\cdot\vec{\sigma}, \qquad |g| = \sqrt{1-\frac{p^2}{\kappa^2}} + i\epsilon\frac{\vec{p}}{\kappa}\cdot\vec{\sigma} = g(\epsilon\vec{p}, +). \tag{7}$$

It is then natural to restrict ourselves to functions on SO(3), i.e., even functions on SU(2) satisfying $f(g) = f(-g)$. Thus, defining $f(\vec{p}) = f(\vec{p}, +) = f(-\vec{p}, -)$, the SO(3) group Fourier transform reads

$$\hat{f}(\vec{X}) = \int_{|p|<\kappa} \frac{d^3\vec{p}}{\pi^2\kappa^3\sqrt{1-\frac{p^2}{\kappa^2}}} f(\vec{p}) e^{i\vec{p}\cdot\vec{X}}, \tag{8}$$

where the nontrivial measure in $\vec{p}$ reflects the normalized Haar measure on $SO(3) = SU(2)/\mathbb{Z}_2$. It is then natural to introduce a $\star_s$ product inherited from the group multiplication on SO(3):

$$e^{\frac{\kappa}{2}\mathrm{Tr}|g_1|X} \star_s e^{\frac{\kappa}{2}\mathrm{Tr}|g_2|X} = e^{\frac{\kappa}{2}\mathrm{Tr}||g_1||g_2||X} = e^{\frac{\kappa}{2}\mathrm{Tr}|g_1 g_2|X}. \tag{9}$$

This $\star_s$ product can be translated into a modified addition on momenta in the $\vec{p}$ variables

$$e^{i\vec{p}_1\cdot\vec{X}} \star_s e^{i\vec{p}_2\cdot\vec{X}} = e^{i(\vec{p}_1\oplus\vec{p}_2)\cdot\vec{X}},$$

with the following deformed addition law:

$$\vec{p}_1 \oplus \vec{p}_2 = \epsilon_{12}\left(\sqrt{1-\frac{p_2^2}{\kappa^2}}\vec{p}_1 + \sqrt{1-\frac{p_1^2}{\kappa^2}}\vec{p}_2 - \frac{1}{\kappa}\vec{p}_1\wedge\vec{p}_2\right), \tag{10}$$

where ϵ_{12} is the sign of $\sqrt{1-\frac{p_1^2}{\kappa^2}}\sqrt{1-\frac{p_2^2}{\kappa^2}} - \frac{1}{\kappa^2}\vec{p}_1\cdot\vec{p}_2$. This sign flip is a necessary subtlety of this group Fourier transform for SO(3). Expanding this formula for small momentum, we can compute the commutator between coordinates

$$[X_i, X_j]_{\star_s} = \frac{2}{\kappa} i\epsilon_{ijk}X_k, \tag{11}$$

which shows explicitly the noncommutativity structure of space-time.

Furthermore, using the inverse Fourier transform of the δ distribution,

$$\frac{1}{(2\pi)^3}\int d^3\vec{X}e^{\frac{\kappa}{2}\mathrm{Tr}gX} = \frac{1}{(2\pi)^3}\int d^3\vec{X}e^{\frac{\kappa}{2}\mathrm{Tr}|g|X} = \delta(g)+\delta(-g) = 2\delta_{\mathrm{SO}(3)}(g), \tag{12}$$

we can use the $\star_s$ product to write the inverse Fourier transform for general even functions on SO(3):

$$f(g) = \frac{1}{2(2\pi)^3}\int d^3\vec{X}\,\hat{f}(\vec{X})\star_s e^{\frac{\kappa}{2}\mathrm{Tr}|g^{-1}|X} = \frac{1}{2(2\pi)^3}\int d^3\vec{X}\,\hat{f}(\vec{X})\star_s e^{-\frac{\kappa}{2}\mathrm{Tr}|g|X}. \tag{13}$$

We can also use the explicit parametrization in terms of $\vec{p}$ to give an explicit formula for the inverse:

$$f(\vec{p}) = \sqrt{1-\frac{p^2}{\kappa^2}}\int\frac{\kappa^3}{(2\pi)^3}d^3\vec{X}\,\hat{f}(\vec{X})e^{-i\vec{p}\cdot\vec{X}}, \tag{14}$$

for functions $\hat{f}(\vec{X})$ with a standard Fourier transform with support on momentum bounded by κ in norm.

The Fourier transform of the matrix elements and characters of SO(30) group elements can also be computed. They are expressed in terms of Bessel functions. We refer the interested reader to Refs. [7,14,18,19].

Finally, we would like to remind the reader that there is an ambiguity in the choice of the momentum variable on which is based the whole construction. Instead of choosing plane waves $\exp(i\vec{p}\cdot\vec{X})$ in terms of the momentum $\vec{p}\propto \mathrm{Tr}g\vec{\sigma}$, one could choose plane waves $\exp(i\vec{P}\cdot\vec{X})$ based on different choices of parametrization of the group elements $g\in \mathrm{SO}(3)$. These lead to different Fourier transforms and star products e.g., Refs. [18,33]. For instance, choosing $\vec{P} = \kappa\tan\theta\hat{u} = \kappa\,\mathrm{Tr}g\vec{\sigma}/\mathrm{Tr}g$ avoids the issue of having a bounded momentum and it is still possible to define the star product and deformed addition of momenta. Nevertheless, $\vec{p}$ seems to be the nicest choice with respect to the differential calculus [18,19].

III. SPINOR PLANE WAVES AND $\star$ PRODUCT FOR SU(2)

In this section, we will show how to use the recently developed spinorial tools for SU(2) to define new plane waves and a group Fourier transform on the whole SU(2) group.

A. Spinors and 3-vectors

The spinor formalism for spin networks and spinfoam models [24–27] is based on the simple remark that 3-vectors can be constructed as the projection of spinors on Pauli matrices and that we have the natural action of SU(2) on spinors as 2×2 matrices.

More explicitly, let us start with a spinor $z\in\mathbb{C}^2$. This is a two-dimensional complex vector living in the fundamental representation of SU(2). We will use the ket-bra notations:

$$|z\rangle = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \qquad \langle z| = (\,\bar{z}_0 \quad \bar{z}_1\,).$$

Then we consider the Hermitian matrix $|z\rangle\langle z|$, from which we define the dimensionful vector $\vec{X}\in\mathbb{R}^3$:

$$\vec{X} = \frac{1}{\kappa}\mathrm{Tr}|z\rangle\langle z|\vec{\sigma} = \frac{1}{\kappa}\langle z|\vec{\sigma}|z\rangle = \frac{1}{\kappa}\bar{z}^a\vec{\sigma}_{ab}z^b, \qquad |z\rangle\langle z| = \frac{\kappa}{2}(|\vec{X}|\mathbb{I}+\vec{X}\cdot\vec{\sigma}), \quad \text{with} \quad |\vec{X}| = \frac{\langle z|z\rangle}{\kappa}. \tag{15}$$

The vector $\vec{X}$ entirely determines the original spinor z up to a global phase $z\to e^{i\alpha}z$. Then all U(1)-invariant functions of the spinor z are functions of $\vec{X}$ and vice versa. The change of integration variable from d^4z to a measure $d^4\mu(\vec{X},\phi)$ can be easily computed. In particular, for a U(1)-invariant function f, we can show that

$$\frac{1}{\pi^2}\int d^4z e^{-\langle z|z\rangle}f(\vec{X}(z)) = \frac{1}{4\pi}\int\frac{d^3\vec{X}}{|\vec{X}|}e^{-|\vec{X}|}f(\vec{X}). \tag{16}$$

It is natural to endow the space of spinors $\mathbb{C}^2$ with the canonical Poisson bracket $\{z_a,\bar{z}_b\} = -i\delta_{ab}$. This induces the following brackets on the X_i coordinates:

$$\{X_i, X_j\} = \frac{2}{\kappa}\epsilon_{ijk}X_k, \tag{17}$$

$$\{X_i, |\vec{X}|\} = 0. \tag{18}$$

Thus the X_i's form a $\mathfrak{su}(2)$ algebra and actually generate the fundamental SU(2) action on spinors. $|\vec{X}|$ gives the (square root of the) $\mathfrak{su}(2)$ Casimir. At the quantum level, this simply becomes the Schwinger representation for $\mathfrak{su}(2)$ in terms of a couple of harmonic oscillators.

The present proposal exploits this expression of a 3-vector $\vec{X}$ in terms of a spinor z and uses the fact that the action of SU(2) group elements on $z\in\mathbb{C}^2$ simply induces the corresponding three-dimensional rotation on the vector $\vec{X}$. Using the fundamental two-dimensional representation of SU(2) instead of the three-dimensional action of SU(2) on 3-vectors will avoid the problem of only representing SO(3) and will allow us to define a Fourier transform for all functions on SU(2).

B. Spinor plane waves on SU(2)

Following the previous work on the spinorial formulation of SU(2) and its representation theory [30,34], a natural candidate for the new SU(2) plane wave is

$$E_g(z) \equiv e^{\langle z|g|z\rangle} = e^{\mathrm{Tr} g|z\rangle\langle z|}. \tag{19}$$

This functional of the spinor z is clearly invariant under the multiplication of the spinor by a global phase, so it can be expressed solely in terms of the 3-vector $\vec{X}$:

$$E_g(\vec{X}) = e^{\frac{\kappa}{2}|\vec{X}|\,\mathrm{Tr} g} e^{\frac{\kappa}{2}\mathrm{Tr} gX}. \tag{20}$$

Comparing to the SO(3) plane waves discussed earlier, there are two differences:

(1) There is a new phase factor depending on the norm $|\vec{X}|$ and on the trace of the group element. This trace $\mathrm{Tr} g$ allows one to distinguish g from $-g$ and thus allows us to probe the whole SU(2) group.

(2) We do not need to take the absolute value of the group element and the main factor of the plane wave is $e^{\frac{\kappa}{2}\mathrm{Tr} gX}$ and not $e^{\frac{\kappa}{2}h\mathrm{Tr}|g|X}$ as earlier.

In terms of the $(\vec{p}, \epsilon)$ parametrization, these new spinorial plane waves read

$$E_{g(\vec{p},\epsilon)}(\vec{X}) = e^{\epsilon\kappa|\vec{X}|\sqrt{1-\frac{p^2}{\kappa^2}}} e^{i\vec{p}\cdot\vec{X}}, \tag{21}$$

with the special prefactor depending on the norms of $\vec{X}$ and $\vec{p}$.

As an element of $\mathcal{C}(\mathrm{SU}(2))$, the plane wave $e^{\langle z|g|z\rangle}$ is a square integrable for the Haar measure dg of SU(2). To prove this, we notice that $\overline{\langle z|g|z\rangle} = \langle z|g^{-1}|z\rangle$ and use the SU(2) coherent states technology as well as the Peter-Weyl theorem (cf. Appendix A)

$$\begin{aligned}\int dg|e^{\langle z|g|z\rangle}|^2 &= \int dg e^{\langle z|g|z\rangle} e^{\langle z|g^{-1}|z\rangle} \\ &= \sum_{j,k} \frac{1}{(2j)!(2k)!} \int dg \langle j,z|g|j,z\rangle\langle k,z|g^{-1}|k,z\rangle \\ &= \sum_{j\in\mathbb{N}/2} \frac{(\langle z|z\rangle^2)^{2j}}{(2j)!^2(2j+1)} = \frac{I_1(2\langle z|z\rangle)}{\langle z|z\rangle},\end{aligned} \tag{22}$$

where I_n is the nth modified Bessel function of the first kind. Note that when $\kappa \to \infty$, this becomes divergent, that is we recover in the classical limit plane waves that are not square integrable.

The plane wave $e^{\langle z|g|z\rangle}$ can also be seen as a function of $\vec{X}$. We note $\mathcal{C}_\star(\mathbb{R}^3)$ the set of functions generated by the $\vec{X}$. $\star$ denotes the star product between the elements in $\mathcal{C}_\star(\mathbb{R}^3)$, which generalizes the notion of a point-wise product. We can give a precise definition of the $\star$ product by defining it on the spinorial plane waves; it reflects the group multiplication on SU(2)

$$\begin{aligned}(E_{g_1} \star E_{g_2})(\vec{X}) \equiv E_{g_1g_2}(\vec{X}) &\Longleftrightarrow e^{\langle z|g_1|z\rangle} \star e^{\langle z|g_2|z\rangle} \\ &= e^{\langle z|g_1g_2|z\rangle}.\end{aligned} \tag{23}$$

The identity for this $\star$ product is $\mathbb{1}_\star \equiv E_1(\vec{X}) = e^{\langle z|z\rangle} = e^{\kappa|\vec{X}|}$. This is not the usual identity, given by the constant function equal to 1. Indeed, we check that

$$\begin{aligned}(E_1 \star E_g)(\vec{X}) = (E_g \star E_1)(\vec{X}) \equiv E_g(\vec{X}) &\Longleftrightarrow e^{\langle z|z\rangle} \star e^{\langle z|g|z\rangle} \\ &= e^{\langle z|g|z\rangle} \star e^{\langle z|z\rangle} = e^{\langle z|g|z\rangle}.\end{aligned} \tag{24}$$

Note that the feature of having a nontrivial identity in configuration space was already present in Ref. [19], where another Fourier transform on SU(2) [as opposed to SO(3)] was introduced. In Ref. [19], the authors deal with a four-dimensional Fourier transform with a four-dimensional momentum space defined as $\mathbb{R}^+ \times \mathrm{SU}(2)$. In our scheme, we have not introduced an extra momentum dimension. As a consequence, we shall see in Sec. III E, that this feature of having a nontrivial identity can be easily avoided. Nevertheless, we can also see our construction from a four-dimensional perspective. Indeed the algebra $\mathcal{C}_\star(\mathbb{R}^3)$ can be seen as the subalgebra of $\mathcal{C}_\star(\mathbb{C}^2) = \mathcal{C}_\star(\mathbb{R}^4)$ that is generated by functions of the spinor z invariant under global phase transformations $z \to e^{i\alpha}z$ (or equivalently the functions that $\star$ commute with $|\vec{X}|$ as we shall see in Secs. III F and IV B).

C. Fourier transform on SU(2) and its inverse

We use the plane wave $E_g(z) = e^{\langle z|g|z\rangle}$ based on the spinor variable z to define a new Fourier transform $\mathcal{F}$ between $\mathcal{C}(\mathrm{SU}(2))$ and $\mathcal{C}_\star(\mathbb{R}^3)$

$$\begin{aligned}&\mathcal{F}\colon \mathcal{C}(\mathrm{SU}(2)) \to \mathcal{C}_\star(\mathbb{R}^3), \\ &\quad f \mapsto \hat{f}(z) = \int_{\mathrm{SU}(2)} dg f(g) E_g(z) \quad \text{or equivalently} \\ &\qquad \hat{f}(\vec{X}) = \int dg f(g) e^{\frac{\kappa}{2}|\vec{X}|\,\mathrm{Tr} g} e^{\frac{\kappa}{2}\mathrm{Tr} gX}.\end{aligned} \tag{25}$$

Since the plane waves $E_g(z)$ are square integrable with respect to the Haar measure dg on SU(2), this Fourier transform is a well-defined map[2] in the sense that $\hat{f}(z)$ is finite for all $z \in \mathbb{C}^2$ provided that f is in $L^2(\mathrm{SU}(2))$.

The $\star$ product between $\hat{\phi}$, $\hat{\psi} \in \mathcal{C}_\star(\mathbb{R}^3)$ is as usual the Fourier transform of the convolution product between the functions ϕ $\psi \in \mathcal{C}_\star(\mathrm{SU}(2))$:

[2]This can be checked directly using the Cauchy-Schwarz inequality in order to derive an explicit bound on the norm of $\hat{f}(z)$:

$$\begin{aligned}|\hat{f}(z)|^2 &= \left| \int_{\mathrm{SU}(2)} dg f(g) e^{\langle z|g|z\rangle} \right|^2 \\ &\le \left(\int dg|f(g)|^2\right)\left(\int dg|e^{\langle z|g|z\rangle}|^2\right) \\ &\le \frac{I_1(2\langle z|z\rangle)}{\langle z|z\rangle} \int |f|^2 < +\infty.\end{aligned}$$

$$
\begin{aligned}
(\widehat{\phi \circ \psi})(z) &= \int [dg]^2 dh \phi(g_1)\psi(g_2)\delta(g_1 g_2 h^{-1}) E_h(z) \\
&= \int [dg]^2 \phi(g_1)\psi(g_2) E_{g_1 g_2}(z) \\
&= \int [dg]^2 \phi(g_1)\psi(g_2)(E_{g_1} \star E_{g_2})(z) \\
&= (\hat{\phi} \star \hat{\psi})(z). \qquad (26)
\end{aligned}
$$

To prove that we are really dealing with SU(2) and not SO(3), we can compute the Fourier transform of the matrix elements of the SU(2) group elements, which form a basis of L^2 functions over SU(2) by the Peter-Weyl theorem. Following the approach of [27,30,34,35], we use the overcomplete basis of SU(2) coherent states labeled by a spin $j \in \mathbb{N}/2$, indicating the SU(2) irreducible representation, and by a spinor $z \in \mathbb{C}^2$ defining the state. The reader can find more details on these coherent states and the corresponding decomposition of the identity in Appendix A. Here, we will simply use the fact that the matrix elements[3] of $g^{-1} \in \mathrm{SU}(2)$ on these coherent states have a simple expression:

$$
\langle j, w | g^{-1} | j, \tilde{w} \rangle = \langle w | g^{-1} | \tilde{w} \rangle^{2j}. \qquad (27)
$$

Their Fourier transform is straightforward to compute:

$$
\begin{aligned}
\hat{f}^{(j)}_{w,\tilde{w}}(z) &= \int dg e^{\langle z|g|z\rangle} \langle j, w | g^{-1} | j, \tilde{w} \rangle \\
&= \sum_k \frac{1}{(2k)!} \int dg \langle k, z | g | k, z \rangle \langle j, w | g^{-1} | j, \tilde{w} \rangle \\
&= \frac{1}{(2j+1)!} \langle w | z \rangle^{2j} \langle z | \tilde{w} \rangle^{2j} \\
&= \frac{1}{(2j+1)!} \langle w | z \rangle^{2j} \langle z | \tilde{w} \rangle^{2j} e^{-\langle z|z\rangle} \mathbb{1}_\star. \qquad (28)
\end{aligned}
$$

Note that we have made apparent the nontrivial identity, which brings the extra factor $e^{-\langle z|z\rangle}$. The matrix elements are therefore maps to linear combinations of polynomials of the type $\langle w|z\rangle^{2j}\langle z|\tilde{w}\rangle^{2j}\langle z|z\rangle^{2k}$ that are homogenous of identical degree in $|z\rangle$ and $\langle z|$. They can be expressed in terms of the 3-vector, $\forall j, k \in \mathbb{N}/2$

$$
\begin{aligned}
&\langle w|z\rangle^{2j}\langle z|\tilde{w}\rangle^{2j}\langle z|z\rangle^{2k} \\
&\quad = \left(\frac{\kappa}{2}\right)^{2j} (|\vec{X}|\langle w|\tilde{w}\rangle + \vec{X} \cdot \langle w|\vec{\sigma}|\tilde{w}\rangle)^{2j} (\kappa|\vec{X}|)^{2k}. \qquad (29)
\end{aligned}
$$

Polynomials with terms of identical degree in $|z\rangle$ and $\langle z|$ are clearly a basis of all U(1)-invariant polynomials of z that generate $\mathcal{C}_\star(\mathbb{R}^3)$. This direct calculation of the Fourier transform of the matrix element functionals on SU(2) ensures that our Fourier transform does not have any nontrivial kernel as the SO(3) group Fourier transform reviewed in Sec. II. Moreover, it shows that every function in $L^2(\mathrm{SU}(2))$ has a finite well-defined Fourier transform since they can be decomposed onto the matrix elements.

We can now recover the δ distribution on SU(2) as a superposition of our new spinorial plane waves. The fastest way to proceed is to use the SU(2) coherent state technology as reviewed in Appendix A. Then, as was previously shown in Refs. [30,34], we obtain

$$
\begin{aligned}
\delta(g) &= \frac{1}{\pi^2} \int d^4 z (\langle z|z\rangle - 1) e^{-\langle z|z\rangle} e^{\langle z|g|z\rangle} \\
&= \frac{1}{\pi^2} \int d^4 z (\langle z|z\rangle - 1) e^{-\langle z|z\rangle} E_g(z). \qquad (30)
\end{aligned}
$$

The Haar measure dg on $\mathcal{C}(\mathrm{SU}(2))$ allows one to determine the standard scalar product $\langle , \rangle_{\mathrm{SU}(2)}$. The Fourier transform should define by construction an isometry between $(\mathcal{C}(\mathrm{SU}(2)), \langle , \rangle_{\mathrm{SU}(2)})$ and $\mathcal{C}_\star(\mathbb{R}^3)$ equipped with a scalar product $\langle \hat{\phi}, \hat{\psi} \rangle = \int d\mu(z)[(\bar{\hat{\phi}} \star \hat{\psi})(z)]$ built from a measure $d\mu(z)$, which we determine as

$$
\begin{aligned}
\int dg \bar{\phi}(g)\psi(g) &= \int dg_1 dg_2 \bar{\phi}(g_1)\psi(g_2)\delta(g_1^{-1} g_2) \\
&= \frac{1}{\pi^2} \int dg_1 dg_2 \bar{\phi}(g_1)\psi(g_2) \\
&\quad \times \int d^4 z (\langle z|z\rangle - 1) e^{-\langle z|z\rangle} E_{g_1^{-1} g_2}(z) \\
&= \frac{1}{\pi^2} \int d\mu(z)[(\bar{\hat{\phi}} \star \hat{\psi})(z)]. \qquad (31)
\end{aligned}
$$

The measure is therefore[4]

$$
\begin{aligned}
d\mu(z) &\equiv \frac{1}{\pi^2} d^4 z (\langle z|z\rangle - 1) e^{-\langle z|z\rangle} \Leftrightarrow [dX] \\
&= \frac{1}{\pi^2} d^3 X \frac{\kappa|\vec{X}| - 1}{\kappa|\vec{X}|} e^{-\kappa|\vec{X}|}. \qquad (32)
\end{aligned}
$$

With all this in hand, we infer that the Fourier transform is well defined in the sense that it takes a function in $L^2(\mathrm{SU}(2), dg)$ to a function in $L^2_\star(\mathbb{R}^3, d\mu(z))$.

Since the Fourier transform is an isometry, we can define the inverse Fourier transform:

[3]A similar calculation can be done for $\langle j, w|g|j, \tilde{w}\rangle$ by using the fact that g is unitary and taking it complex conjugate:

$$
\begin{aligned}
\langle j, w|g|j, \tilde{w}\rangle &= \overline{\langle j, \tilde{w}|g^{-1}|j, w\rangle} = \langle j, \bar{\tilde{w}}|\bar{g}^{-1}|j, \bar{w}\rangle \\
&= \langle j, \bar{\tilde{w}}|\epsilon^{-1} g^{-1} \epsilon|j, \bar{w}\rangle = \langle j, \epsilon\bar{\tilde{w}}|g^{-1}|j, \epsilon\bar{w}\rangle,
\end{aligned}
$$

$$
\text{with} \quad \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

[4]The curious feature of this scalar product is that the measure factor $(\langle z|z\rangle - 1)e^{-\langle z|z\rangle}$ on the space of spinors is not positive. However, this deviation from the Gaussian measure does not mean that the norm of functions of z will be possibly negative. Indeed, since the scalar product defined with this measure factor and the $\star$ product is strictly equal to the standard scalar product between functions on SU(2), the norm will always be strictly positive unless the function vanishes.

$$\mathcal{F}^{-1}:\mathcal{C}_\star(\mathbb{R}^3)\rightarrow\mathcal{C}(\mathrm{SU}(2))$$
$$\hat{f}(z)\mapsto f(g)=\int d\mu(z)[e^{\langle z|g^{-1}|z\rangle}\star\hat{f}(z)]$$
$$=\frac{\kappa^3}{4\pi}\int[dX]e^{-\kappa|\vec{X}|}[(E_{g^{-1}}\star\hat{f})(\vec{X})]. \quad (33)$$

A last remark on the definition of this SU(2) group Fourier transform is on taking its complex conjugate:

$$\overline{\hat{f}(z)}=\int dg\overline{f(g)}e^{\langle z|g^{-1}|z\rangle}=\int dg\overline{f(g^{-1})}e^{\langle z|g|z\rangle}, \quad (34)$$

so that a real Fourier transform $\hat{f}(z)\in\mathbb{R}$ is equivalent to $\bar{f}(g)=f(g^{-1})$.

D. Defining the δ distribution on the noncommutative space

An important missing ingredient is the noncommutative delta function $\delta^\star(\vec{X})$ over $\mathbb{R}^3$. As usual we can determine it as a superposition of the plane wave.

$$\delta^\star(\vec{X})=\int dgE_g(\vec{X})=e^{-\langle z|z\rangle}\mathbb{1}_\star=e^{-\kappa|\vec{X}|}\mathbb{1}_\star. \quad (35)$$

This shows that with this choice of parametrization the δ distribution is actually regularized as a Gaussian in the spinor variables. This definition extends to the case where the delta function projects over an arbitrary point $\vec{X}_2$:

$$\delta^\star_{\vec{X}_2}(\vec{X}_1)\equiv\int dgE_g(\vec{X}_1)E_{g^{-1}}(\vec{X}_2)=\int dge^{\langle z_1|g|z_1\rangle}e^{\langle z_2|g^{-1}|z_2\rangle}$$
$$=\frac{I_1(2|\langle z_2|z_1\rangle|)}{|\langle z_2|z_1\rangle|}=\frac{I_1(2|\langle z_2|z_1\rangle|)}{|\langle z_2|z_1\rangle|}e^{-\langle z_1|z_1\rangle}\mathbb{1}_\star, \quad (36)$$

where z_1 is the spinor for $\vec{X}_1$ and z_2 the spinor for $\vec{X}_2$. Also we have made apparent the identity $\mathbb{1}_\star$ in the $\vec{X}$ variable in the last equality. We can actually relate the scalar product $\langle z_2|z_1\rangle$ between the spinors to the one between the vectors:

$$|\langle z_2|z_1\rangle|^2=\mathrm{Tr}|z_2\rangle\langle z_2||z_1\rangle\langle z_1|=\frac{\kappa^2}{2}(|\vec{X}_1||\vec{X}_2|+\vec{X}_1\cdot\vec{X}_2), \quad (37)$$

which simply vanishes if $\vec{X}_2=0$. In particular, when $\vec{X}_2=\vec{0}$, i.e., $z_2=0$, the previous definition of $\delta^\star_{\vec{X}_2}$ gives back (35) as expected. However, it is important to notice that $\delta^\star_{\vec{X}_2}(\vec{X}_1)$ is different from the more naïve definition $\delta^\star(\vec{X}_1-\vec{X}_2)=\int dgE_g(\vec{X}_1-\vec{X}_2)$, since the plane wave is not linear in X_1 and X_2. Nevertheless, this delta function $\delta^\star_{\vec{X}_2}(\vec{X}_1)$ satisfies the usual properties of the delta function since $\forall\hat{f}\in\mathcal{C}_\star(\mathbb{R}^3)$,

$$\int[dX]\delta^\star(\vec{X})=\frac{1}{\pi^2}\int d^4zdg(\langle z|z\rangle-1)e^{-\langle z|z\rangle}e^{\langle z|g|z\rangle}$$
$$=\int dg\delta(g)=1, \quad (38)$$

$$\int[dX_1](\delta^\star_{\vec{X}_2}\star\hat{f})(\vec{X}_1)=\int[dX_1](\hat{f}\star\delta^\star_{\vec{X}_2})(\vec{X}_1)=\hat{f}(\vec{X}_2). \quad (39)$$

It is furthermore very interesting that the expression of the delta function $\delta^\star_{\vec{X}_2}(\vec{X}_1)$ defined in terms of the z_i variables can be related to the notion of coherent intertwiners as introduced in Ref. [26]. Indeed, as we recall in Appendix A, an n-valent coherent intertwiner $|\{z_i\}\rangle$ is given by

$$|\{z_i\}\rangle\equiv\sum_{\{j_i\}}\frac{1}{\prod_i\sqrt{(2j_i)!}}\int dg\bigotimes_i g|j,z_i\rangle, \qquad i=1,\ldots,n, \quad (40)$$

where the $|j,z_i\rangle$ are the SU(2) coherent states following the conventions of [26,34,35]. From this definition, we see that the norm of this coherent intertwiner gives the integral over SU(2) of products of n plane waves $E_g(z_i)$:

$$\langle\{z_i\}|\{z_i\}\rangle=\int dg\prod_i^n E_g(z_i). \quad (41)$$

This norm was fortunately already computed explicitly in Refs. [34,35]:

$$\langle\{z_i\}|\{z_i\}\rangle=\sum_{J\in\mathbb{N}}\frac{(\det\Omega)^J}{J!(J+1)!}=\frac{I_1(2\sqrt{\det\Omega})}{\sqrt{\det\Omega}}, \text{ with}$$
$$\Omega=\sum_i|z_i\rangle\langle z_i|,$$
$$\det\Omega=\frac{\kappa^2}{2^2}\Big[\Big(\sum_i|\vec{X}_i|\Big)^2-\Big|\sum_i\vec{X}_i\Big|^2\Big]\geq0, \quad i=1,\ldots,n. \quad (42)$$

We notice that the norm is maximal when $\sum_i\vec{X}_i=\vec{0}$, i.e., the closure constraint is satisfied or equivalently $\sum_{i=1,\ldots,n}|z_i\rangle\langle z_i|\propto\mathbb{I}_2$. Moreover, the norm becomes more and more peaked around this maximal value in the classical limit as κ grows to ∞. In that sense, the integral $\int dg\prod_i^n E_g(z_i)$ can be interpreted as defining a smooth delta function δ_κ peaked around the closure $\sum_i\vec{X}_i=0$ (Fig. 1):

$$\delta_\kappa(\vec{X}_1,\ldots,\vec{X}_n)\equiv\int dg\prod_i^n E_g(z_i)=\langle\{z_i\}|\{z_i\}\rangle. \quad (43)$$

In particular, in the case of the bivalent intertwiner, when $n=2$, this reduces to our previous definition of $\delta^\star_{\vec{X}}$. Indeed, we have

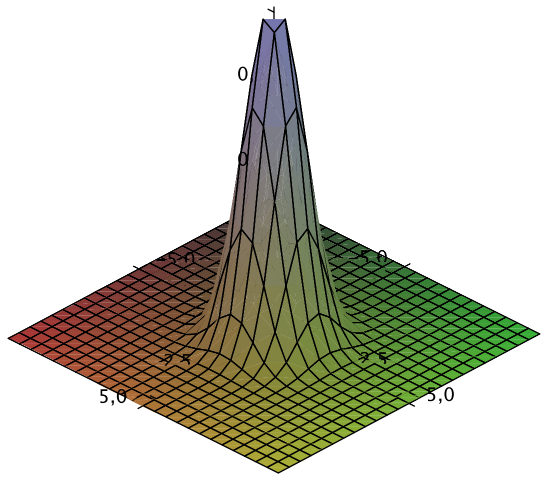

FIG. 1 (color online). The (two-dimensional realization of the) delta function $\tilde{\delta}_\kappa(\vec{X}_1, -\vec{X}_2)$ peaked on $\vec{X}_2 = (1, 1)$.

$$\delta^\star_{\vec{X}_2}(\vec{X}_1) = \int dg E_g(\vec{X}_1) E_{g^{-1}}(\vec{X}_2) = \int dg E_g(\vec{X}_1) E_g(-\vec{X}_2) = \delta_\kappa(\vec{X}_1, -\vec{X}_2).$$

We find it very interesting that the noncommutative delta function we constructed can be defined in terms of loop quantum gravity tools. This is an another example of the interplay between structures of noncommutative geometry and of loop quantum gravity [36].

Notice nevertheless that the delta function $\delta_\kappa(\vec{X}_1, \ldots, \vec{X}_n)$ is not in general a straightforward function of $\sum_i \vec{X}_i$ due to the nonlinearity of the plane wave. We can see from the explicit expression that it also depends on the total norm $\sum_i |\vec{X}_i|$, which cannot be simply factored out of the formula.

Using this delta function $\delta^\star_{\vec{X}_2}(\vec{X}_1)$ as well as the delta function on the group $\delta(g)$, it is then straightforward to check explicitly that $\mathcal{F} \circ \mathcal{F}^{-1} = \mathbb{1}_{\mathcal{C}_\star(\mathbb{R}^3)}$ and $\mathcal{F}^{-1} \circ \mathcal{F} = \mathbb{1}_{\mathcal{C}(\mathrm{SU}(2))}$, where $\mathcal{F}$ is the Fourier transform.

E. On the choice of plane wave: Using normalized plane waves

The $\star$ product representation of a noncommutative algebra is a highly nonunique representation. There exists actually many different star products that can be introduced through different choices of momentum variables or more generally different choices of plane waves. We can thus change our plane waves $E_g(z) = e^{\langle z|g|z\rangle}$ for other [U(1)-invariant] functions of the spinor z. The general construction is described in Appendix B. Here we would like to focus on a particular choice of normalized plane waves so that the identity for the star product remains the trivial constant function on $\mathbb{R}^3$. To this purpose, we rescale the plane waves $E_g(z)$ by an appropriate factor:

$$\tilde{E}_g(z) = e^{-\langle z|z\rangle} e^{\langle z|g|z\rangle} = e^{\langle z|g-1|z\rangle} \Leftrightarrow \tilde{E}_g(\vec{X}) = e^{\frac{\kappa}{2}|\vec{X}|(-1+\mathrm{Tr}g)+\frac{\kappa}{2}\mathrm{Tr}gX}. \tag{44}$$

We note in fact that the normalizing Gaussian factor is already present in the integral so that the delta function over the group is

$$\delta(g) = \frac{1}{\pi^2}\int d^4z(\langle z|z\rangle - 1)e^{-\langle z|z\rangle}e^{\langle z|g|z\rangle} = \frac{1}{\pi^2}\int d^4z(\langle z|z\rangle - 1)\tilde{E}_g(z). \tag{45}$$

In this normalized case, the $\star$ product becomes

$$(\tilde{E}_{g_1} \star \tilde{E}_{g_2})(z) \equiv \tilde{E}_{g_1g_2}(z) \Leftrightarrow (e^{\langle z|g_1-1|z\rangle}) \star e^{\langle z|g_2-1|z\rangle}) \equiv e^{\langle z|g_1g_2-1|z\rangle} \Leftrightarrow (\tilde{E}_{g_1} \star \tilde{E}_{g_2})(\vec{X}) \equiv \tilde{E}_{g_1g_2}(\vec{X}). \tag{46}$$

The previous construction of the Fourier transform goes along the same way as in the previous section, but the identity is now trivial, $\mathbb{1}_\star = \mathbb{1}$. The delta function over configuration space is given by the Gaussian in the spinor variables, or in the $\vec{X}$ variables as

$$\tilde{\delta}(\vec{X}) = e^{-\kappa|\vec{X}|}. \tag{47}$$

The generalized delta functions $\tilde{\delta}^\star(\vec{X})$, and more generally the distributions $\tilde{\delta}_\kappa(\vec{X}_1, \ldots, \vec{X}_n)$, are now given as a function of z_i as

$$\tilde{\delta}_\kappa(\vec{X}_1, \ldots, \vec{X}_n) = \frac{\langle\{z_i\}|\{z_i\}\rangle}{\prod_i e^{\langle z_i|z_i\rangle}}. \tag{48}$$

Their expressions in terms of X_i can be easily read from (42). In Fig. 2 'e' illustrates the shape of $\tilde{\delta}_\kappa(\vec{X}_1, -\vec{X}_2)$.

F. Computing the $\star$ product

It is now natural to ask what is the structure of the $\star$ product we have introduced. Although we study here the star products induced by the choices of plane waves E_g and $\tilde{E}_g$, the discussion below applies to all plane waves of the type $\mathcal{K}^*_g$ and their induced $\star$ products as introduced in Appendix B. We first analyze the $\star$ product between coordinates to check that we recover the $\mathfrak{su}(2)$ noncommutative structure.

Calling now p^i and q^i the coordinates for, respectively, the group elements g_1 and g_2 such that

$$g_1g_2 \rightsquigarrow \vec{p} \oplus \vec{q} = \sqrt{1 - \frac{\vec{q}^2}{\kappa^2}}\vec{p} + \sqrt{1 - \frac{\vec{p}^2}{\kappa^2}}\vec{q} - \frac{1}{\kappa}\vec{p} \wedge \vec{q}, \tag{49}$$

the $\star$ product between coordinates for the plane waves E_g is then[5]

$$X_i \mathbb{1}_\star \star X_j \mathbb{1}_\star = -\int dg_1 dg_2 \delta(g_1)\delta(g_2)\partial_{p^i}\partial_{q^j} E_{g_1 g_2}(z) = \left(X_i X_j + \frac{1}{\kappa}(\delta_{ij}|\vec{X}| + i\epsilon_{ij}^k X_k)\right)\mathbb{1}_\star, \quad (50)$$

where $\mathbb{1}_\star = e^{\kappa|\vec{X}|}$. It is interesting to notice that the previous formula (50) is actually very similar to the one derived by the authors of Ref. [19] and obtained when considering the four-dimensional Fourier transform for SU(2), with $T = |\vec{X}|$ in their notations.

In the case of the plane waves $\tilde{E}_g(z)$, the nontrivial identity $\mathbb{1}_\star$ drops out and we have more simply

$$X_i \star X_j = X_i X_j + \frac{1}{\kappa}(\delta_{ij}|\vec{X}| + i\epsilon_{ij}^k X_k). \quad (51)$$

In both cases, it is direct to see that the $\star$ product we introduced is in fact a realization of the $\mathfrak{su}(2)$ noncommutative structure since

$$[X_i\mathbb{1}_\star, X_j\mathbb{1}_\star]_\star = X_i\mathbb{1}_\star \star X_j\mathbb{1}_\star - X_j\mathbb{1}_\star \star X_i\mathbb{1}_\star = \frac{2}{\kappa} i\epsilon_{ij}^k X_k \mathbb{1}_\star. \quad (52)$$

Earlier we defined $\mathcal{C}_\star(\mathbb{R}^3)$ to be the functions that are generated by the $X_i(z)$ and as such invariant under the phase rescaling $z \to e^{i\alpha}z$. This algebra can also be characterized as the algebra generated by the functions that $\star$ commute with $|\vec{X}|$. Indeed, we have

$$|\vec{X}|\mathbb{1}_\star \star X_i\mathbb{1}_\star = \left(|\vec{X}|X_i + \frac{1}{\kappa}X_i\right)\mathbb{1}_\star = X_i\mathbb{1}_\star \star |\vec{X}|\mathbb{1}\ \star, \qquad [|\vec{X}|\mathbb{1}_\star, X_i\mathbb{1}_\star]_\star = 0. \quad (53)$$

From this we deduce that any function built out from the X_i will $\star$ commute with $|\vec{X}|$.

We shall show in Sec. IV B that these two characterizations of $\mathcal{C}_\star(\mathbb{R}^3)$ are indeed equivalent.

IV. $\mathfrak{su}(2)$NONCOMMUTATIVITY FROM THE QUANTUM OSCILLATOR PERSPECTIVE

In the previous section we showed that the spinorial approach allows one to introduce a well-defined Fourier transform for SU(2). In this section, we want to show that this approach allows one to shed a new light on the $\mathfrak{su}(2)$ noncommutative structure. First, we show that the four-dimensional (bicovariant) differential calculus on $\mathfrak{su}(2)$ can be naturally recovered from this approach. Second, we prove that the Voros star product between the spinor variables gives exactly the SU(2) $\star$ product of the previous section based on the normalized plane waves $\tilde{E}_g(z)$. This provides a simple representation of our $\star$ product as a differential operator.

A. $\mathfrak{su}(2)$ bicovariant differential calculus

As we recalled in Sec. III A, the spinorial approach developed in Refs. [24–27] relies on the Schwinger representation of $\mathfrak{su}(2)$. More explicitly the spinor variables $z \in \mathbb{C}^2$ are quantized, i.e., $z_a \to a_a$, $\bar{z}_a \to a_a^\dagger$

$$\{z_a, \bar{z}_b\} = -i\delta_{ab} \to [a_a, a_b^\dagger] = \delta_{ab}, \quad (54)$$

and the dimensionful $\mathfrak{su}(2)$ generators are simply

$$\vec{\mathbf{X}} = \frac{1}{\kappa} a_a^\dagger \vec{\sigma}_{ab} a_b, \qquad \kappa|\vec{\mathbf{X}}| = \sum_a a_a^\dagger a_a = \sum_a N_a, \quad (55)$$

where N_a is the number operator. Using the commutation relations of the creation/annihilation operators, we recover

$$[\mathbf{X}_i, \mathbf{X}_j] = \frac{2}{\kappa} i\epsilon_{ij}^k \mathbf{X}_k, \qquad \left[\sum_a N_a, X_j\right] = 0. \quad (56)$$

The noncommutative space $\hat{\mathbb{R}}^4 \sim \hat{\mathbb{R}}^2 \times \hat{\mathbb{R}}^2$ generated by the operators $\alpha_\mu = (a_0, a_0^\dagger, a_1, a_1^\dagger)$ is equipped with a differential structure that satisfies the Leibniz law,

$$d(\alpha_\mu\alpha_\nu) = (d\alpha_\mu)\alpha_\nu + \alpha_\mu(d\alpha_\nu), \qquad \forall \mu, \nu, \quad (57)$$

and such that the oneforms commute with α_μ

$$[d\alpha_\mu, \alpha_\nu] = 0. \quad (58)$$

This last property can be also understood as the fact that the translation symmetry is not deformed in this case.[6]

Using this, we can now calculate in a direct manner, the commutators of $\mathbf{X}_i$ and $d\mathbf{X}^j = da_b^\dagger \sigma_{bc}^j a_c + a_b^\dagger \sigma_{bc}^j da_c$,

$$[\mathbf{X}_i, d\mathbf{X}_j] = \frac{1}{\kappa^2}\left(-i\epsilon_{ilj}(da_a^\dagger \sigma_{ab}^l a_b + a_a^\dagger \sigma_{ab}^l da_b) + \delta_i^j \sum_c (a_c^\dagger da_c - a_c da_c^\dagger)\right) = \frac{1}{\kappa}(i\epsilon_{ij}^k d\mathbf{X}_k + \delta_{ij}\Theta). \quad (59)$$

We see therefore that there is an extra contribution $\Theta = \frac{1}{\kappa}\sum_c (a_c^\dagger da_c - a_c da_c^\dagger)$ that appears. This is the nontrivial fourth component of the $\mathfrak{su}(2)$ bicovariant

[5] To prove this we have used the following identities:

$$\partial_{q^j}\langle z|g_1 g_2|z\rangle_{|_{\vec{q}=\vec{p}=\vec{0}}} = \partial_{p^j}\langle z|g_1 g_2|z\rangle_{|_{\vec{q}=\vec{p}=\vec{0}}} = iX_j$$
$$\partial^2_{p^i q^j}\langle z|g_1 g_2|z\rangle_{|_{\vec{q}=\vec{p}=\vec{0}}} = -\frac{1}{\kappa}(\delta_{ij}|\vec{X}| + i\epsilon_{ij}^k X_k).$$

[6] It is only when a Poincaré transformation is performed—that is in the relevant 2d Euclidian case, both a rotation and a translation—that the differential structure is nontrivial. In fact there is no pure rotation transformation. For further details see Ref. [37].

differential calculus [38]. Using again the quantum harmonic oscillator expressions, we get

$$[\Theta, \mathbf{X}_i] = -\frac{1}{\kappa} d\mathbf{X}_i, \tag{60}$$

which is consistent with Ref. [38].

B. Voros $\star$ product versus SU(2) $\star$ product

The quantum harmonic oscillators can naturally be defined in the Moyal representation of quantum mechanics [39]. More precisely, we start with c numbers, here the spinorial variables z, and we introduce a $\star$ product encoding the quantum structure and realizing an exact quantization through the Weyl map:

$$[a_i, a_i^\dagger] = \delta_{ij} \to [z_i, \bar{z}_j]_* = z_i * \bar{z}_j - \bar{z}_j * z_i = \delta_{ij}. \tag{61}$$

There exist many realizations of such $*$ product, the most well known being the Moyal product and the Voros product defined, respectively, as

$$(f_1 \star_m f_2)(z) = f_1(z) e^{\frac{1}{2}(\overleftarrow{\partial}_z \vec{\partial}_{\bar{z}} - \overleftarrow{\partial}_{\bar{z}} \vec{\partial}_z)} f_2(z), \tag{62}$$

$$(f_1 \star_v f_2)(z) = f_1(z) e^{\overleftarrow{\partial}_z \vec{\partial}_{\bar{z}}} f_2(z). \tag{63}$$

Both of these $*$ products allow us to recover the commutation relation in (61). Let us point out that these two $*$ products define unitary-equivalent quantization maps—the Moyal product corresponding to the Weyl symmetric ordering while the Voros product corresponds to the normal ordering.

Following this logic, we would like to reexpress the Schwinger representation as $\mathbf{X}_i \to X_i = \frac{1}{\kappa} \bar{z}_a \vec{\sigma} z_b$ using a Weyl map and define the relevant $*$ product between the spinors z to calculate the commutators of the X_i, seen as a function of the spinor variables z, such that we get

$$[X_i, X_j]_* = \frac{2}{\kappa} \epsilon_{ij}^k X_k, \qquad [|\vec{X}|, X_i]_\star = 0. \tag{64}$$

A natural question is now to wonder if one such $*$ product between the spinorial variables is actually equivalent to the SU(2) $\star$ product that we have defined in the previous Sec. III F. The Moyal and Voros products give, respectively,

$$X_i *_m X_j = X_i X_j + \frac{1}{\kappa} \epsilon_{ij}^k X_k, \tag{65}$$

$$X_i *_v X_j = X_i X_j + \frac{1}{\kappa} (\delta_{ij} |\vec{X}| + i\epsilon_{ij}^k X_k). \tag{66}$$

We see therefore that the Voros $*$ product gives the same star product between coordinates as our one normalized $\star$ product considered in (51):

$$X_i \star X_j = X_i *_v X_j. \tag{67}$$

Let us extend this analysis to the plane waves $\tilde{E}_g(z)$ and calculate the Voros $*$ product between them in order to fully check that we recover the $\star$ product. Writing the normalized plane waves as $\tilde{E}_g(z) = e^{-\langle z|z\rangle} e^{\langle z|g|z\rangle} = e^{\langle z|g-1|z\rangle}$ in terms of the spinors z proves efficient and we compute

$$\begin{aligned}(\tilde{E}_{g_1} *_v \tilde{E}_{g_2})(z) &= (e^{\langle z|g_1-1|z\rangle}) e^{\overleftarrow{\partial}_z \vec{\partial}_{\bar{z}}} (e^{\langle z|g_2-1|z\rangle}) \\ &= (e^{\langle z|g_1-1|z\rangle} e^{(\langle z|(g_1-1))\cdot \vec{\partial}_{\bar{z}}}) (e^{\langle z|g_2-1|z\rangle}) \\ &= e^{\langle z|g_1-1|z\rangle} e^{\langle z|g_2-1|z\rangle} e^{\langle z|(g_1-1)(g_2-1)|z\rangle} \\ &= (e^{\langle z|g_1 g_2-1|z\rangle}) = \tilde{E}_{g_1 g_2}(z) = (\tilde{E}_{g_1} \star \tilde{E}_{g_2})(z).\end{aligned} \tag{68}$$

This shows explicitly that the Voros $*$ product reproduces exactly our SU(2) $\star$ product defined in (46) and provides a proper representation of the $\mathfrak{su}(2)$ noncommutative structure:

$$(\hat{\phi} \star \hat{\psi})(z) = (\hat{\phi} *_v \hat{\psi})(z), \qquad \forall \hat{\psi}, \hat{\phi} \in \mathcal{C}_\star(\mathbb{R}^3). \tag{69}$$

Furthermore, using the definition for the Voros $*$ product and

$$\partial_{z_a} = \frac{1}{\kappa} \bar{z}_m \sigma_{ma}^i \frac{\partial}{\partial X_i}, \qquad \partial_{\bar{z}_a} = \frac{1}{\kappa} \sigma_{an}^i z_n \frac{\partial}{\partial X_i},$$

we can use this equality to give a nice and simple expression for the SU(2) $\star$ product [40],

$$\begin{aligned}(\hat{\phi} \star \hat{\psi})(\vec{X}) = \hat{\phi}(\vec{Y}) e^{\frac{1}{\kappa}(|\vec{X}|\delta_{ij} + i\epsilon_{ij}^k X_k) \overleftarrow{\partial}_{y_i} \vec{\partial}_{z_j}} \hat{\psi}(\vec{Z})_{|_{Y=Z=X}}, \\ \forall \hat{\phi}, \hat{\psi} \in \mathcal{C}_\star(\mathbb{R}^3).\end{aligned} \tag{70}$$

This provides us with an expression as a differential operator for the SU(2) $\star$ product based on the normalized plane waves $\tilde{E}_g$. By reinserting the Gaussian normalization, we can easily deduce from it a differential operator representation for the SU(2) $\star$ product based on the original spinorial plane waves E_g, which will be nevertheless less elegant.

Thanks to the Voros realization of the SU(2) $\star$ product, we can reexamine the equivalent definitions of the $\mathcal{C}_\star(\mathbb{R}^3)$. We have seen it is given by the subalgebra of functions of $\mathcal{C}(\mathbb{C}^2)$ that $\star$ commute with $|\vec{X}|$. Using the Voros representation, this becomes then

$$[|\vec{X}|, f(z, \bar{z})]_{\star_v} = 0 \Leftrightarrow (\bar{z}\partial_{\bar{z}} - z\partial_z) f(z, \bar{z}) = 0. \tag{71}$$

On the other hand, $\mathcal{C}_\star(\mathbb{R}^3)$ is generated by the functions that are invariant under the rescaling $z \to e^{i\alpha} z$, that is

$$f(z, \bar{z}) = f(e^{i\alpha} z, e^{-i\alpha} \bar{z}), \qquad \forall f \in \mathcal{C}_\star(\mathbb{R}^3). \tag{72}$$

Considering a small α and expanding (72), we get

$$-i\alpha(\bar{z}\partial_{\bar{z}} - z\partial_z) f(z, \bar{z}) = 0, \tag{73}$$

which is equivalent to (71).

We can recap the present situation: there are different ways to construct a star product representation of a noncommutative structure. One consists in identifying the momentum addition structure and from this we infer the star product between coordinates using a Fourier transform. This is the approach we followed in Sec. III C. Another one consists in defining a Weyl map by brute force. In this section we have constructed such a Weyl map, starting from the Voros representation of the quantum oscillators. We have shown that the two representations are the same.

As a final comment, we note that considering the momentum structure, and, in particular, the delta function over momentum space, allows us to recover the right measure in the configuration space. We have seen that in the case of $\tilde{E}_g(z)$, the relevant measure is given by $d\mu(z) = d^4z(\langle z|z\rangle - 1)$. In Ref. [40], the authors did not consider the momentum structure and therefore only took the standard measure d^4z, which is not the correct measure, as we have shown.

V. NCQFT REPRESENTATION OF GROUP FIELD THEORY

Spinfoam models (for quantum gravity and topological field theories) can be defined at the nonperturbative level through group field theories, which are nonlocal field theories defined on Lie group manifolds (for a review, see e.g., Ref. [5]). For instance, the most studied case is the Boulatov group field theory for three-dimensional Euclidean quantum gravity [3]. It is indeed the standard model to discuss the issue of renormalization in the context of GFT before addressing the more complicated GFT's describing spinfoam models for four-dimensional gravity. The model is formulated on the manifold $\mathrm{SU}(2)^3$, satisfies a SU(2) gauge invariance, and has a nonlocal interaction term.

In the previous section, we defined a SU(2) $\star$ product and showed its equivalence to the Voros $*$ product in the spinor variables. This opens new possibilities. Indeed, we can use our new SU(2) Fourier transform and write the GFT in terms of the spinor variables. As we have seen, the noncommutativity is then a *standard* one, of the Moyal type, and we hope to be able to use standard renormalization techniques already developed for noncommutative quantum field theories based on the Moyal and Voros $*$ products.

In this section we will describe explicitly Boulatov's model in terms of the spinor variables and discuss the realization of the quantum symmetries given by the quantum double $DSU(2)$. As a warmup, we first consider the two-dimensional GFT on SU(2) [7,41].

A. Two-dimensional GFT on SU(2)

We consider a (real) scalar field theory on SU(2) with a field $\phi \in \mathcal{C}(\mathrm{SU}(2))$ and the action is given by

$$S_{2d}[\phi] = \frac{1}{2}\int dg\phi(g_1)\phi(g_2)\delta(g_1g_2) + \sum_{n\geq 3}\frac{\alpha_n}{n!}\int [dg]^n\phi(g_1)\dots\phi(g_n)\delta(g_1\dots g_n). \tag{74}$$

From the perspective of noncommutative field theories, SU(2) is the momentum space and the term $\delta(g_1\dots g_n)$ corresponds to the conservation law of momenta. We can now use the normalized plane wave $\tilde{E}_g(z)$, to implement our Fourier transform and express the action $S_{2d}[\phi]$ in configuration space. We consider, therefore, $\hat{\phi} \in \mathcal{C}_\star(\mathbb{R}^3)$ with

$$\hat{\phi}(z) = \int dg\phi(g)\tilde{E}_g(z) \qquad \tilde{E}_g(z) = \frac{e^{\langle z|g|z\rangle}}{e^{\langle z|z\rangle}},$$
$$\tilde{E}_{g_1}\star\tilde{E}_{g_2} = \tilde{E}_{g_1g_2}.$$

A straightforward implementation of the Fourier transform gives

$$S[\phi] = \frac{1}{2}\int d\mu(z)(\hat{\phi}\star\hat{\phi})(z) + \sum_{n\geq 3}\frac{\alpha_n}{n!}\int d\mu(z)\hat{\phi}^{\star n}(z), \tag{75}$$

with the measure $d\mu(z) = (\langle z|z\rangle - 1)d^4z$. As we have shown the equivalence of the $\star$ product and the Voros $*$ product, this action can also be written as

$$S[\phi] = \frac{1}{2}\int d\mu(z)(\hat{\phi}*_v\hat{\phi})(z) + \sum_{n\geq 3}\frac{\alpha_n}{n!}\int d\mu(z)\hat{\phi}^{*_v n}(z). \tag{76}$$

We recall that unlike the Moyal star product, we have

$$\int dz(\hat{\phi}\star_v\hat{\psi})(z) \neq \int dz\hat{\phi}(z)\hat{\psi}(z). \tag{77}$$

The field theory defined by $S_{2d}[\phi]$ can also be written as a (sum over) matrix model [7]. We have shown here that we can write this GFT as a Voros noncommutative field theory, with a scalar field $\hat{\phi}(z) \in \mathcal{C}(\mathbb{R}^3)$ invariant under $z \to e^{i\alpha}z$. It would be interesting to explore further the properties of this new formulation, especially with respect of the renormalization of the theory. In any case, the immediate advantage of our formalism over the previous works [7,9–11] is that we are truly dealing with a field living in SU(2), and we do not restrict ourselves to even fields living only in SO(3). At the spinfoam level, this means that we will get all SU(2) representations of arbitrary spin $j \in \mathbb{N}/2$ and not only even representations with integer spins.

As soon as we have a law of conservation of momenta, one expects some translational symmetry to be involved. Indeed there exists an action on our field $\phi \in \mathcal{C}(\mathrm{SU}(2))$ of the quantum double $D\mathrm{SU}(2)$, which is a deformation of the Euclidian group $I\mathrm{SO}(3)$ [18]. We emphasize again that our spinor formalism allows for the full action of $D\mathrm{SU}(2)$ and not only DSO(3). The quantum double is given as an algebra in terms of the cross product between the algebra of the functions of $\mathcal{C}(\mathrm{SU}(2))$ and the group algebra $\mathbb{C}\mathrm{SU}(2)$, as $D\mathrm{SU}(2)=\mathcal{C}(\mathrm{SU}(2))\rtimes\mathbb{C}\mathrm{SU}(2)$ [12]. The action of DSU(2) on $\phi\in\mathcal{C}(\mathrm{SU}(2))$ is given by the action of translations parametrized by an arbitrary spinor $a\in\mathbb{C}^2$ and the action of rotations parametrized by a group element $u\in\mathrm{SU}(2)$ that are, respectively,

$$\begin{aligned}\phi(g)&\to\tilde{E}_g(a)\phi(g),\\ \phi(g_1)\otimes\phi(g_2)&\to(\tilde{E}_{g_1}(a)\star\tilde{E}_{g_2}(a))\phi(g_1)\otimes\phi(g_2)\\ &=\tilde{E}_{g_1g_2}(a)\phi(g_1)\otimes\phi(g_2),\qquad a\in\mathbb{C}^2,\end{aligned}\tag{78}$$

$$\begin{aligned}\phi(g)&\to\phi(ugu^{-1}),\\ \phi(g_1)\otimes\phi(g_2)&\to\phi(ug_1u^{-1})\otimes\phi(ug_2u^{-1})\quad u\in\mathrm{SU}(2).\end{aligned}\tag{79}$$

It is not difficult to check that due to the conservation of momenta, the action $S_{2d}(\phi)$ is invariant under such translations, and that thanks to the Haar measure and the invariance of δ under rotations, $S_{2d}(\phi)$ is also invariant under the rotations.

Now that we have defined the realization of the DSU(2) symmetries in momentum space, we can determine their realization in configuration space. We are looking for the analogue of (78) and (79) for functions in the variables $\vec{X}$ or z. To this purpose, we perform the Fourier transform of (79) and (78). The rotations simply read

$$\begin{aligned}&\mathcal{F}(\phi(ugu^{-1}))(z)=\hat{\phi}(u\vec{X}u^{-1}),\quad\text{with}\\ &\vec{X}==\frac{1}{\kappa}\langle z|\vec{\sigma}|z\rangle=\frac{1}{\kappa}\bar{z}^a\vec{\sigma}_{ab}z^b,\end{aligned}\tag{80}$$

and we recover the standard adjoint action of SU(2) on the coordinates $\vec{X}$, as expected. On the other hand, the realization of the translations in the X space or z space is trickier. In that case, the Fourier transform reads

$$\begin{aligned}&\mathcal{F}(\tilde{E}_g(a)\phi(g))(z_1)\\ &\quad=\int dg\phi(g)\tilde{E}_g(z_1)\tilde{E}_g(a)\\ &\quad=\int dgd\mu(z_2)\hat{\phi}(z_2)\star\tilde{E}_{g^{-1}}(z_2)\tilde{E}_g(z_1)\tilde{E}_g(a)\\ &\quad=\int d\mu(z_2)\hat{\phi}(z_2)\star\tilde{\delta}_\kappa(\vec{X}_1,-\vec{X}_2,\vec{A})\\ &\quad=\int[dX_2]\hat{\phi}(\vec{X}_2)\star\tilde{\delta}_\kappa(\vec{X}_1,-\vec{X}_2,\vec{A}).\end{aligned}\tag{81}$$

We have an *explicit* realization of the translation symmetry in configuration space given by (81). However, unlike the commutative or SO(3) cases, this formula does not simplify to something like

$$\begin{aligned}&\mathcal{F}(\tilde{E}_g(a)\phi(g))(z)=\hat{\phi}(z+a)\quad\text{or}\\ &\mathcal{F}(\tilde{E}_g(a)\phi(g))(X)=\hat{\phi}(X+a),\end{aligned}\tag{82}$$

due to the nonlinearity of the delta function δ_κ. Hence the translations cannot be realized exactly as

$$z\to z+a\quad\text{or}\quad X\to X+a.\tag{83}$$

We have therefore a *fuzzy* implementation of the translations in configuration space. *A priori,* there could exist a different choice of configuration variables $\vec{Y}$ for which the translations would be realized in a linear way. We have not been able to identify such variables.

B. Boulatov model

We can easily adapt the procedure described above to the three-dimensional case. Indeed, let us consider the colored Boulatov GFT [9–11,42] with complex fields $\phi_{c=1\ldots4}\in\mathcal{C}(\mathrm{SU}(2)^3)$ that are right translational invariant

$$\phi_c(g_1,g_2,g_3)\equiv\int dh\phi_a(g_1h,g_2h,g_3h),\quad\forall c=1\ldots4.\tag{84}$$

The colored Boulatov action is given by

$$\begin{aligned}S_b[\phi_c]=&\frac{1}{2}\int[dg]^3\sum_c^4\phi_c(g_1,g_2,g_3)\bar{\phi}_c(g_1,g_2,g_3)\\ &+\frac{\alpha}{4!}\int[dg]^6\phi_1(g_1,g_2,g_3)\phi_2(g_3,g_4,g_5)\\ &\times\phi_3(g_5,g_2,g_6)\phi_4(g_6,g_4,g_1)+\text{c.c.}\end{aligned}\tag{85}$$

This GFT generates the Ponzano-Regge amplitudes for Euclidian gravity, as it is easy to see by looking at the Feynman amplitudes and using the Peter-Weyl theorem. This same field theory defined over $\mathrm{SO}(3)^3$ has been recently studied using noncommutative techniques based on the plane waves $e^{\mathrm{Tr}|g|X}$ discussed in Sec. II. This approach allowed us, on one hand, to connect GFT with simplicial geometry, since the noncommutative variable X can be interpreted as a discretized B field [10] (also see Ref. [33] for a discussion on the extent of the validity of the identification of X as the discretization of the B field) and on the other hand to connect the quantum group symmetries of the GFT to the diffeomorphism symmetry of the BF action [11]. Since the spinfoam Ponzano-Regge model is defined for SU(2), we intend now to discuss Boulatov's model on SU(2) in the light of the new spinorial Fourier transform.

1. Noncommutative variables and discretization of the BF action

We first perform the Fourier transform on the fields ϕ_c and consider[7] $\hat{\phi}_c(z_1, z_2, z_3) \in \mathcal{C}_\star(\mathbb{R}^{3\times3})$, $\forall c = 1, \ldots, 4$,

$$\begin{aligned}\hat{\phi}_c(z_1, z_2, z_3) &\equiv \int [dg]^3 dh \phi_c(g_1 h, g_2 h, g_3 h) \\ &\quad \times \tilde{E}_{g_1}(z_1)\tilde{E}_{g_2}(z_2)\tilde{E}_{g_3}(z_3) \\ &= \hat{\phi}_c(z_1, z_2, z_3)\star_{1,2,3} \\ &\quad \times \int dh \tilde{E}_h(z_1)\tilde{E}_h(z_2)\tilde{E}_h(z_3) \end{aligned} \tag{86}$$

$$= \hat{\phi}_c(z_1, z_2, z_3) \star_{1,2,3} \hat{C}(z_1, z_2, z_3), \tag{87}$$

where $\hat{C}(z_1, z_2, z_3) = \frac{\langle\{z_i\}|\{z_i\}\rangle}{\prod_{i=1}^3 e^{\langle z_i|z_i\rangle}} = \tilde{\delta}_\kappa(\vec{X}_1, \vec{X}_2, \vec{X}_3)$. As we discussed earlier, $\tilde{\delta}_\kappa(\vec{X}_1, \vec{X}_2, \vec{X}_3)$ is the norm of the trivalent coherent intertwiner and it defines a smooth delta function peaked around the closure $\sum_i \vec{X}_i = 0$. Geometrically, as in the SO(3) case, we still interpret $\hat{\phi}_c(z_1, z_2, z_3) = \hat{\phi}_c(X_1, X_2, X_3)$ as representing a quantized triangle where the vectors $\vec{X}_i$ are considered as the normals to the edges and the closure of the triangle $\sum_i \vec{X}_i = 0$ is implemented in a *fuzzy* way.

The Fourier transform can be performed on the Boulatov action $S_b[\phi_c]$. Since the $\star$ product is the dual of the convolution product, the combinatorial structure of the action is preserved. Using the equivalence between the SU(2) $\star$ product and Voros $*$ product, we write it as a Voros noncommutative field theory. To keep the notations simple, we define $\hat{\phi}_1(X_1, X_2, X_3) \equiv \hat{\phi}_{(123)}$, $\hat{\phi}_2(X_3, X_4, X_5) \equiv \hat{\phi}_{(345)}$ and so on and so forth. The action becomes then

$$\begin{aligned} S_b[\phi_c] = \frac{1}{2}\int [dX]^3 \sum_c^4 \hat{\phi}_{(123)} * \bar{\hat{\phi}} \\ + \frac{\alpha}{4!}\int [dX]^6 \hat{\phi}_{(123)} * \hat{\phi}_{(345)} * \hat{\phi}_{(526)} * \hat{\phi}_{(641)} + \text{c.c.}, \end{aligned} \tag{88}$$

where $\hat{\phi}_{(i)} \star \hat{\phi}_{(i)} \equiv (\hat{\phi} \star \hat{\psi})(X_i)$, with $\hat{\psi}(X_i) = \hat{\phi}(-X_i)$ and where $[dX]$ is the nontrivial measure $[dX] = d^3X \frac{|\vec{X}|-1}{|\vec{X}|}$. It would be interesting to understand if this new formulation can provide new angles of attack for the renormalization analysis.

In the SO(3) case, the interpretation of the X variables came when looking at the Feynman amplitudes of the GFT written in terms of the configuration variables X. Indeed these Feynman amplitudes give the spinfoam amplitudes for BF theory that are understood as discretization of the path integral for three-dimensional quantum gravity. This provides the Feynman amplitudes and the variables X with a clear geometrical interpretation. In the present case in the colored GFT defined on SU(2), the kinetic and interaction terms provide the following propagator $\mathcal{P}(\vec{X}, \vec{Y})$ and vertex contribution $\mathcal{V}(\vec{X}, \vec{Y})$:

$$\begin{aligned} \mathcal{P}(\vec{X}, \vec{Y}) &= \int dh_t \prod_{i=1}^3 (\tilde{\delta}_{-X_i} \star \tilde{E}_{h_{tt'}})(Y_i), \\ \mathcal{V}(\vec{X}, \vec{Y}) &= \int \prod_t dh_{\tau t} \prod_{i=1}^6 (\delta_{-X_i} \star \tilde{E}_{h_{t\tau}h_{\tau t'}})(Y_i). \end{aligned} \tag{89}$$

Following [10], t denotes the relevant triangle and τ the tetrahedron. The group variables $h_{tt'}$ and $h_{t\tau}$ come from the right invariance of the fields and are interpreted as parallel transport, respectively, through the triangle t identified with t' and from the tetrahedron τ to the triangle t, cf. Fig. 2. This is exactly the same combinatorial and algebraic structure as the one derived for the SO(3) case, expect for the fact that the plane waves are different in the two cases. As a result, we can follow the steps of Baratin and Oriti [10] and we obtain the Feynman amplitude for a graph Γ:

$$\begin{aligned} Z(\Gamma) &= \int \prod_t dh_t \prod_e [dX_e]\tilde{E}_{H_e}(X_e) \\ &= \int \prod_t dh_t \prod_e [dX_e] e^{\frac{\kappa}{2}|X_e|(-1+\text{Tr}H_e)+\frac{\kappa}{2}\text{Tr}X_e H_e} \\ &= \int \prod_t dh_t \prod_e d\mu(z_e) e^{\langle z_e|H_e - 1|z_e\rangle}, \end{aligned} \tag{90}$$

where $H_e = \prod_i h^{\tau_i}_{t_i t'_i} h_{t'_i t_{i+1}}$ is the holonomy around the face dual to the edge e, with $h^{\tau_i}_{t_i t'_i} = h_{t_i \tau_i} h_{\tau_i t'_i}$ and τ_i, $i \in \{1, \cdots, N\}$ the ith tetrahedron of the link around the edge e (cf. Fig. 2). We identify $t_{n+1} = t_1$. This expression (90) provides an expression in terms of the spinor variables z_e of the path integral for the BF theory over the triangulation dual to the Feynman diagram Γ. It was already derived

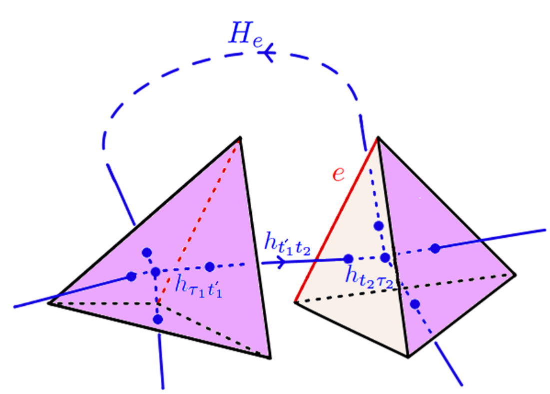

FIG. 2 (color online). Holonomy H_e around the face dual to the edge e. The face t'_1 is identified with the face t_2.

[7] The algebra $\mathcal{C}_\star(\mathbb{R}^{3\times3})$ is seen as a subalgebra of $\mathcal{C}(\mathbb{C}^{2\times3})$. The fields ϕ_c are therefore invariant under the rescaling by independent phases of the spinors: $z_i \to e^{i\alpha_i} z_i$, $\bar{z}_i \to e^{-i\alpha_i}\bar{z}_i$.

using coherent intertwiner techniques in Ref. [34]. We compare it to the expression for the discretized path integral for BF theory in the SO(3) case [13,32]:

$$Z_{SO(3)}(\Gamma) = \int \prod_t dh_t \prod_e dX_e e_{H_e}(X_e) = \int \prod_t dh_t \prod_e dX_e e^{\frac{\kappa}{2}\mathrm{Tr} X_e |H_e|}. \quad (91)$$

The key differences between (91) and (90) lie in the choices of measure dX_e [which is trivial for the SO(3) case] and the plane waves. The term $\frac{\kappa}{2}\mathrm{Tr} X_e H_e$ in (91) corresponds to the *natural* discretization of the BF action. However *rigorously*, from the noncommutative point of view, this discretization loses track of the full SU(2) structure and keeps only SO(3) as explained in Sec. II. Our spinorial approach suggests here that a *good* discretization, which would keep track of the full SU(2), should involve a nontrivial discretization of the BF action as follows:

$$\mathrm{Tr}\, B \wedge F \rightarrow \frac{1}{2}|X_e|(-1 + \mathrm{Tr} H_e) + \frac{1}{2}\mathrm{Tr} X_e H_e, \quad (92)$$

and the associated nontrivial measure $[dX_e] = d^3 X_e \frac{|\vec{X}_e| - 1}{|\vec{X}_e|}$. As we have shown, this choice allows us to recover rigourously the discretized path integral for BF theory with gauge group SU(2):

$$Z(\Gamma) = \int \prod_t dh_t \prod_e [dX_e] \tilde{E}_{H_e}(X_e) = \int \prod_t dh_t \prod_e \delta(H_e), \quad \text{with} \quad H_e \in \mathrm{SU}(2). \quad (93)$$

2. Boulatov symmetries

The invariance of the colored Boulatov action defined over $\mathrm{SO}(3)^{\times 3}$ under the action of four copies of DSO(3) has been explained explicitly in Ref. [11]. We are now considering the Boulatov model defined over $\mathrm{SU}(2)^{\times 3}$, and we can perform a similar analysis by generalizing (78) and (79) to $\mathrm{DSU}(2)^{\times 3}$. The symmetry analysis of Boulatov action can be performed in a momentum space given by either $\mathrm{SO}(3)^{\times 3}$ or $\mathrm{SU}(2)^{\times 3}$ in an analogous manner. As was shown in Ref. [11], the quantum group symmetry of the interaction term corresponds to the invariance of the spinfoam amplitudes under moving the four summits of a tetrahedron.

The key difference between the SU(2) and the SO(3) cases comes when implementing the symmetries at the configuration space level. Indeed the translational symmetry of the Boulatov model is related to the translational symmetry of the BF action thanks to the Bianchi identity [32,43],

$$B \rightarrow B + d_A \phi, \quad (94)$$

where d_A is the covariant derivative with respect to the connection A, whose curvature is F. When discretizing the BF action into $\frac{1}{2}\mathrm{Tr} X_e H_e$ for SO(3), the discretized Bianchi identity still implies invariance under the transformation $X_e \rightarrow X_e + A_e$, which is the noncommutative realization of the translation. In the spinorial approach for SU(2), the discretized BF action is $\frac{1}{2}|X_e|(-1 + \mathrm{Tr} H_e) + \frac{1}{2}\mathrm{Tr} X_e H_e$ and is therefore nonlinear in X. From this perspective it is clear that the translational symmetry should be realized in a nonstandard way. This is precisely what we have obtained in (81), where we have argued that the translations are implemented in a fuzzy way.

To summarize, even though the BF action is discretized in a nonstandard way with a nonlinear term in X_e [necessary to account for the full SU(2) structure], a translational symmetry still exists and is implemented in a nontrivial way.

VI. CONCLUSION AND OUTLOOK

Let us summarize what we have done before presenting the new directions to which our approach leads. Essentially we have applied the spinor representation to the GFT context. This has a number of nice implications. First, this means that we can use the noncommutative tools for a GFT defined on SU(2) and not only on SO(3). Thanks to the link between GFT and simplicial geometry, the spinor representation points towards a different discretization of the BF action (as was already shown in Ref. [34]), such that the full SU(2) structure is taken into account. Second, we have pinpointed that the use of the spinor representation allows for a natural derivation of the four-dimensional structure of the bicovariant calculus on $\mathfrak{su}(2)$. Third, we have shown that our $\star$ product for SU(2) is given by the Voros $*$ product between the spinors, unlike the SO(3) star product that still remains rather mysterious despite several studies. Finally, we have discussed the implementation of the quantum group symmetries given by DSU(2). If in the momentum representation, there is not much difference between the action of DSU(2) and DSO(3), in configuration space the difference is important since in the SU(2) case, the translation symmetry is implemented in a nonlinear manner in configuration space.

These different results points toward new interesting ideas to develop.

(i) GFT model for a four-dimensional Euclidean quantum gravity: In Ref. [34], a new spinfoam model for Euclidean quantum gravity was introduced using the spinor representation. It has the nice feature that the simplicity constraints are implemented through a Gupta-Bleuer procedure at the level of the spinors in a strong way [35]. The construction we have presented here extends in a natural way to Ooguri's GFT on Spin(4) $\sim$ SU(2) $\times$ SU(2) that describes BF theory in four dimensions. The next step would be to derive the GFT for the spinfoam model presented in Ref. [34] and understand how these spinfoam amplitudes can be written as the Feynman diagrams of a noncommutative field theory in the spinor variables. This is currently under development.

(ii) Loop quantum gravity as Voros noncommutative geometry: Recently, noncommutative techniques were applied to loop quantum gravity to provide a noncommutative representation of the flux algebra [44]. The key idea was to consider the plane wave used for SO(3). We can now reproduce this analysis using the spinor representation together with the $\star$ product we have defined. As a consequence, the flux algebra would be written as a Voros noncommutative algebra. The implications of this new representation should be explored.

(iii) Renormalization of GFT: The renormalization of the GFT's is a necessary step towards understanding the semiclassical regime and continuum limit of spinfoam models. The usual formulation of spinfoam models in terms of representations or group variables makes it difficult to renormalize the infinities that arise. We have a new formulation of the spinfoam models based on a new set of variables. This new formulation falls now in a well-known framework (Moyal-Voros noncommutative field theory) where many rigorous results on renormalization have been obtained. A fundamental ingredient for this success is, for example, the notion *Moyality* [45] that replaces the usual notion of locality, and which allowed us to tame infinities. In the Voros formulation, the spinfoam GFT action is different than the well-understood standard Voros noncommutative field action, but the noncommutative features are well under control, unlike other noncommutative spinfoam formulations based on a Lie algebra noncommutativity.[8] Since we are using different variables than the momentum variables (group variables or representations), we can identify the source of the divergences in a different way than the usual spinfoam formalism. Hopefully, they will be easier to cope with. Hence another outcome of our approach is that we have potentially a new tool to identify and control infinities. Exploring how the renormalization techniques of the standard Voros scalar field action can be extended to the GFT action is left for future investigations.

(iv) Four-dimensional bicovariant differential calculus: It is striking that the four-dimensional bicovariant differential calculus naturally emerges from the spinor representation of SU(2). It would be interesting to see if there exists (already?) a deeper mathematical structure that would explain this.

(v) Generalization of the $\star$ product to arbitrary Lie groups: In the present paper, we have presented a group Fourier transform for the Lie group SU(2) on its Schwinger representation in terms of spinors (at the classical level). Introducing the spinorial plane waves allowed us to define a $\star$ product dual to the convolution on SU(2), which actually matches the Voros product on the spinor variables. This procedure seems to be easily generalizable to more complicated semisimple Lie groups that admit such a spinorial representation. We would then be able to define the $\star$ products dual to the convolution on these groups and relate them to the much simpler Voros product defined from the spinorial phase space structure.

APPENDIX A: SU(2) COHERENT STATES IN TERMS OF SPINORS

Starting with a spinor $z \in \mathbb{C}^2$, for which we use a bra-ket notation,

$$|z\rangle = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \qquad \langle z| = (\,\bar{z}_0 \quad \bar{z}_1\,),$$

with the canonical Poisson bracket $\{z_a, \bar{z}_b\} = -i\delta_{ab}$, we quantize the components of $|z\rangle$ and $\langle z|$, respectively, as annihilation and creation operators $a_{0,1}, a_{0,1}^\dagger$ acting on the Hilbert space $\mathcal{H}_{HO} \otimes \mathcal{H}_{HO}$ where $\mathcal{H}_{HO}$ is the standard Hilbert space for a harmonic oscillator with basis $|n\rangle$:

$$a_0|n_0, n_1\rangle_{HO} = \sqrt{n_0}|n_0 - 1, n_1\rangle_{HO},$$
$$a_0^\dagger|n_0, n_1\rangle_{HO} = \sqrt{n_0 + 1}|n_0 + 1, n_1\rangle_{HO}.$$

Then quantizing the components of the 3-vectors $\vec{X} = \langle z|\vec{\sigma}|z\rangle$, we get the generators of the $\mathfrak{su}(2)$ Lie algebra and its Casimir:

$$J_3 = \frac{1}{2}(a_0^\dagger a_0 - a_1^\dagger a_1), \quad J_+ = a_0^\dagger a_1, \quad J_- = a_0 a_1^\dagger = J_+^\dagger,$$
$$\hat{j} = \frac{1}{2}(a_0^\dagger a_0 + a_1^\dagger a_1), \quad [J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3,$$
$$[\hat{j}, \vec{J}] = 0. \tag{A1}$$

Diagonalizing the operators $\hat{j}$ and J_3, we recover the standard basis $|j, m\rangle$ of $\mathfrak{su}(2)$ irreducible representations and show that $\mathcal{H}_{HO} \otimes \mathcal{H}_{HO} = \bigoplus_{j\in\mathbb{N}/2} V^j$:

$$|j, m\rangle = |n_0, n_1\rangle_{HO}, \quad \text{with} \quad \left| \begin{array}{l} n_0 = j + m \\ n_1 = j - m \end{array} \right. . \tag{A2}$$

Next, we introduce the coherent states for the harmonic oscillators:

$$|z_0, z_1\rangle_{HO} \equiv \sum_{n_0,n_1} \frac{z_0^{n_0} z_1^{n_1}}{\sqrt{(n_0)!(n_1)!}} |n_0, n_1\rangle,$$

[8] For example, there is no closed realization of the star product for $\mathfrak{so}(3)$ in terms of differential operators.

from which we define the SU(2) coherent states by projecting them onto fixed values of the spin $j \in \mathbb{N}/2$:

$$|j,z\rangle \equiv \frac{(z_0 a_0^\dagger + z_1 a_1^\dagger)^{2j}}{\sqrt{(2j)!}}|0\rangle = \sum_{m=-j}^{+j} \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} z_0^{j+m} z_1^{j-m} |j,m\rangle,$$
$$|z_0,z_1\rangle_{HO} = \sum_j \frac{1}{\sqrt{(2j)!}}|j,z\rangle. \tag{A3}$$

These coherent states transform covariantly under the SU(2) action generated by the operators $\vec{J}$ (for more details, see e.g., Refs. [26,35]):

$$e^{i\vec{u}\cdot\vec{J}}|j,z\rangle = |j, e^{\frac{i}{2}\vec{u}\cdot\vec{\sigma}}z\rangle, \tag{A4}$$

where $e^{\frac{i}{2}\vec{u}\cdot\vec{\sigma}}$ is the representation for the group element $e^{i\vec{u}\cdot\vec{J}}$ in the fundamental two-dimensional representation of SU(2). From this fundamental property of the SU(2) coherent states, it is straightforward to deduce that they are all obtained through the action of SU(2) group elements on the highest weight vector $|j,j\rangle$ and that they are simply the tensorial powers of the coherent states in the fundamental $j=\frac{1}{2}$ representation:

$$|j,z\rangle = (\sqrt{\langle z|z\rangle})^{2j} g(z)|j,j\rangle, \quad g(z) = \frac{1}{\sqrt{\langle z|z\rangle}}\begin{pmatrix} z_0 & -\bar{z}_1 \\ z_1 & \bar{z}_0 \end{pmatrix}$$
$$g(z)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{\langle z|z\rangle}}\begin{pmatrix}z_0\\z_1\end{pmatrix}, \quad |j,j\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle^{\otimes 2j},$$
$$|j,z\rangle = |\tfrac{1}{2},z\rangle^{\otimes 2j} = |z\rangle^{\otimes 2j}.$$

In particular, this allows us to compute the matrix elements of SU(2) group elements on the coherent states:

$$\langle j,w|g|j,z\rangle = \langle w|g|z\rangle^{2j}, \qquad \langle j,w|j,z\rangle = \langle w|z\rangle^{2j}. \tag{A5}$$

Moreover, we can write a decomposition of the identity on the Hilbert space V^j of the irreducible representation of spin j (for more details, see e.g., Refs. [26,34,35]):

$$\mathbb{1}_j = \frac{1}{(2j)!}\int_{\mathbb{C}^2} \frac{d^4 z}{\pi^2} e^{-\langle z|z\rangle}|j,z\rangle\langle j,z|. \tag{A6}$$

In particular, this allows us to write the decomposition of the δ distribution on SU(2) onto characters as a Gaussian integral over the spinor variables:

$$\delta(g) = \sum_{j\in\mathbb{N}/2}(2j+1)\chi_j(g) = \frac{1}{\pi^2}\int d^4 z e^{-\langle z|z\rangle}\sum_j \frac{(2j+1)}{(2j)!}\langle j,z|g|j,z\rangle = \frac{1}{\pi^2}\int d^4 z e^{-\langle z|z\rangle}(\langle z|g|z\rangle+1)e^{\langle z|g|z\rangle}. \tag{A7}$$

Working from there, we get by integration by parts

$$\delta(g) = \frac{1}{\pi^2}\int d^4 z e^{-\langle z|z\rangle}(z\partial_z + 1)e^{\langle z|g|z\rangle} = \frac{1}{\pi^2}\int d^4 z(-\partial_z z + 1)e^{-\langle z|z\rangle}e^{\langle z|g|z\rangle} = \frac{1}{\pi^2}\int d^4 z(\langle z|z\rangle - 1)e^{-\langle z|z\rangle}e^{\langle z|g|z\rangle}.$$

Following [26], we can then define by group averaging the Livine-Speziale coherent intertwiners, which are SU(2)-invariant states in the tensor product of N irreducible representations:

$$||\{j_i,z_i\}\rangle \equiv \int_{\mathrm{SU}(2)} dg \otimes_i^N g|j_i,z_i\rangle \ \in \mathrm{Inv}[\bigotimes_i^N V^{j_i}]. \tag{A8}$$

Finally, by summing over the spin labels, we define the coherent intertwiner states, which diagonalize the intertwiner annihilation operators and which are labeled only by spinor variables as introduced in Refs. [34,35]:

$$|\{z_i\}\rangle \equiv \sum_{\{j_i\}}\prod_i \frac{1}{\sqrt{(2j_i)!}}||\{j_i,z_i\}\rangle = \int_{\mathrm{SU}(2)} dg \otimes_i^N g|z_i\rangle_{OH}. \tag{A9}$$

These coherent intertwiners are covariant under the action of U(N) as shown in Refs. [26,34,35]. We can compute their norms and scalar products either by using the U(N) structure or by computing directly the integrals over SU(2).

APPENDIX B: ON THE CHOICE OF PLANE WAVE

An interesting choice for the plane wave, motivated from Ref. [34], is to consider the plane wave and measure

$$\mathcal{E}_g(z) = e^{-\langle z|z\rangle}\frac{(\langle z|g|z\rangle + 1)}{\langle z|z\rangle + 1}e^{\langle z|g|z\rangle}, \quad d\mu(z) = d^4 z(\langle z|z\rangle + 1), \tag{B1}$$

with the $\star$ product, as usual reflecting the SU(2) group structure and relevant measure

$$(\mathcal{E}_{g_1}\star\mathcal{E}_{g21})(z) = \mathcal{E}_{g_1 g_2}(z) \equiv e^{-\langle z|z\rangle}\frac{(\langle z|g_1 g_2|z\rangle + 1)}{\langle z|z\rangle + 1}e^{\langle z|g_1 g_2|z\rangle}.$$

It is normalized since $\mathcal{E}_1(z) = \mathbb{1}$. This plane wave and $\star$ product were suggested in Ref. [34] to construct the partition function for the BF theory. In this case, the delta function in the configuration space is given by

$$\delta_{\star_{bf}}(\vec{X}) = 2\frac{e^{-\kappa|\vec{X}|}}{1+\kappa|\vec{X}|}. \tag{B2}$$

Different plane waves lead therefore to different realizations of the delta function on configuration space.

In a general manner, we can construct a plane wave $\mathcal{K}_g(z)$ and introduce a measure $d\mu(z)$ such that

$$\left.\begin{aligned}\mathcal{K}_g(z) &= \sum_j \frac{\alpha_j}{(2j)!}\langle z|g|z\rangle^{2j}\\ d\mu(z) &= \sum_j \frac{\beta_j}{(2j)!}\langle z|z\rangle^{2j}\end{aligned}\right\} \Rightarrow \delta(g) = \int d\mu(z)\mathcal{K}_g(z), \tag{B3}$$

which puts constraints[9] on the coefficients α_i and β_i. The $\star$ product is defined as usual to reflect the group product.

$$\mathcal{K}_{g_1} \star \mathcal{K}_{g_2}(z) = \mathcal{K}_{g_1g_2}(z). \tag{B4}$$

The plane wave $\mathcal{K}_g(z)$ is in general not normalized, in the sense that $\mathcal{K}_1(z) \neq \mathbb{1}$. We can demand to normalize it, in which case we consider the renormalized plane wave, new $\star$ product, and new measure

$$\begin{aligned}\tilde{\mathcal{K}}_g(z) &= \mathcal{K}_1^{-1}(z)\mathcal{K}_g(z),\\ (\tilde{\mathcal{K}}_{g_1} \star \tilde{\mathcal{K}}_{g_2})(z) &= \tilde{\mathcal{K}}_{g_1g_2}(z),\\ d\mu(z) \to d\mu(z)\mathcal{K}_1(z) &= d\tilde{\mu}(z).\end{aligned} \tag{B5}$$

[9]We can further demand it to be an element of $L^2(\mathrm{SU}(2))$, which translates into

$$\sum_j \frac{|\alpha_j|^2}{(2j)!^2}(\langle z|z\rangle^2)^{2j} < +\infty, \qquad \forall z.$$

Finally, we can demand as well that the coordinates $X_i\mathbb{1}_\star$ are given in terms of the derivative of the plane wave evaluated at the identity,

$$\begin{aligned}&-i\int dg\delta(g)\frac{\partial}{\partial p^i}\mathcal{K}_g(z)\\ &= X_i\mathbb{1}_\star \Leftrightarrow X_i\sum_{k\in\mathbb{N}^\star/2}\frac{\alpha_k}{(2k-1)!}(\kappa|\vec{X}|)^{2k-1} = X_i\mathcal{K}_1(\vec{X}).\end{aligned} \tag{B6}$$

This leads to a plane wave $\mathcal{K}_g^*(z)$, $\star$ product, and measure

$$\begin{aligned}\mathcal{K}_g^*(z) &= \mathcal{X}(g,z)e^{\langle z|g|z\rangle},\\ (\mathcal{K}_{g_1}^* \star \mathcal{K}_{g_2}^*)(z) &= \mathcal{K}_{g_1g_2}^*(z),\\ d\mu(z) &= \frac{\langle z|z\rangle - 1}{\mathcal{X}(z)}e^{-\langle z|z\rangle},\end{aligned} \tag{B7}$$

where $\mathcal{X}(g,z)$ is function invariant under $z \to e^{i\alpha}z$ [that is in $\mathcal{C}_\star(\mathbb{R}^3)$] that is zero nowhere and that satisfies $\frac{\partial\mathcal{X}(g,z)}{\partial p}) = |_{p=0} = 0$. As we mentioned before, we can then normalize $\mathcal{K}_g^\star(z)$ so that $\mathbb{1}_\star = \mathbb{1}$, in which case, $\mathcal{K}_g^\star(z) = e^{-\langle z|z\rangle}e^{\langle z|g|z\rangle} = \tilde{E}_g(z)$ if $\mathcal{X}(g,z) = 1$ and $\mathcal{K}_g^*(z) = e^{i\vec{X}\cdot\vec{p}}$ if $\mathcal{X}(g,z) = e^{-|\vec{X}|\mathrm{Tr}g}$. Note, however, that the latter case does not describe the full structure of SU(2) as recalled in Sec. II.

As a conclusion of this discussion, we see that the nicest plane waves relevant for SU(2) are with no surprise the exponential type $\mathcal{K}_g^*(z) = e^{\langle z|g|z\rangle} = E_g(z)$ or $\mathcal{K}_g^*(z) = \tilde{E}_g(z) = e^{-\langle z|z\rangle}e^{\langle z|g|z\rangle}$ that we have considered earlier.

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