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CP($N-1$) model on a finite interval in the large N limit

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The CP($N-1$) σ model on finite interval of length R with Dirichlet boundary conditions is analyzed in the $1/N$ expansion. The theory has two phases, separated by a phase transition at $R \sim 1/\Lambda$, and Λ is a dynamical scale of the CP($N-1$) model. The vacuum energy dependence of R, and especially Casimir-type scaling $1/R$, is discussed.

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I. INTRODUCTION

The large N expansion is suitable to study nonperturbative behavior of a variety of models in different physical situations (see Ref. [1] for a review). Within this technique many important features such as dynamical mass generation, asymptotic freedom, and an absence of spontaneous continuous symmetry breaking in two dimensions could be seen.

In what follows, we will consider two-dimensional nonlinear CP($N-1$) σ model on a finite interval of length R with Dirichlet boundary conditions, that is, on a ribbon. In infinite space it was solved by D'Adda, Di Vecchia, and Lüscher [2] (and later independently by Witten [3]) by means of the large N expansion. The theory is asymptotically free and possesses dynamical mass generation via dimensional transmutation:

$$\Lambda^2 = \Lambda_{uv}^2 \exp\left(\frac{-4\pi}{g^2}\right), \tag{1}$$

where Λ is a dynamical scale, Λ_{uv} is an ultraviolet cutoff, and g is a bare coupling constant. It is well known that the CP($N-1$) model is the effective low-energy theory on a non-Abelian string worldsheet [4]. Therefore, such a geometry with two Dirichlet boundary conditions can be thought of as a non-Abelian string between two branes.

In this article, we will obtain the following results. The theory has nontrivial R dependence: at $R \gg 1/\Lambda$ it is in the "confining phase" and the mass gap is present; at $R \ll 1/\Lambda$ it is in the "Higgs phase" and there is no mass gap. Very similar behavior occurs in the "twisted mass" deformed CP($N-1$) model, where the twisted mass parameter plays the role of R (see Ref. [5] where the names of the phases were taken). Despite the existence of the mass gap, the vacuum energy has Casimir-type behavior $1/R$. We will discuss it in the light of the works [6,7].

II. GAP EQUATION

The considerations below are very similar to those in Ref. [5]. We start with the action

$$\mathcal{L} = \frac{N}{g^2}(\partial_\mu - iA_\mu)n_i(\partial^\mu + iA^\mu)n^{*i} - \lambda(n_i^* n^i - 1), \tag{2}$$

where λ and A_μ are Lagrange multipliers. λ impose the constraint $n_i^* n^i = 1$, and A_μ are just dummy fields that could be eliminated by equation of motion $A_\mu = in_i^*\partial_\mu n^i$ but make U(1) invariance obvious. All the fields live on a finite interval of length R with Dirichlet boundary conditions:

$$n^1(0) = n^1(R) = 1; \qquad n^i(0) = n^i(R) = 0, \qquad i = 2, \ldots, N. \tag{3}$$

Note that this boundary conditions break translation invariance.

To solve the theory in the large N limit we should integrate over n^k in the path integral to obtain effective action for λ, A_μ,

$$Z = \int \mathcal{D}A\mathcal{D}\lambda\mathcal{D}n^i\mathcal{D}n^{*i} \times \exp\left(i\int d^2x\left(-\frac{N}{g^2}n^i(\partial_\mu + iA_\mu)^2 n^{*i} - \lambda(n_i n^{*i} - 1)\right)\right). \tag{4}$$

It will be useful to separate n^i into $n^1 = \sigma$, $(N-1)$ component n^i and integrate over only the last ones. After rescaling n^i, Gaussian integration leads us to

$$Z = \int \mathcal{D}A\mathcal{D}\lambda\mathcal{D}\sigma \times \exp\Big((-(N-1)\mathrm{Tr}\log((-(\partial_\mu + iA_\mu)^2 - m^2)) + i\int d^2x\Big((\partial_\mu\sigma)^2 - m^2\sigma\sigma^* + \frac{Nm^2}{g^2}\Big)\Big), \tag{5}$$

where $m^2 = \frac{\lambda g^2}{N}$.

Now we will use the steepest descent method with the uniform saddle point: $A_\mu = 0$, $m = \mathrm{const}$, $\sigma = \mathrm{const}$, and in the leading order we can neglect the difference between N and $N-1$. Also, although the translation invariance is broken, it is reasonable to expect that we will describe the

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behavior correctly at least at the qualitative level. Varying action with respect to m^2, σ^*, we obtain saddle-point equations:

$$g^2 \mathrm{Tr} \frac{1}{(-\partial_\mu)^2 - m^2 + i\epsilon} + i \int \left(1 - \frac{g^2 \sigma^2}{N}\right) d^2x = 0, \quad (6)$$

$$m^2 \sigma = 0. \quad (7)$$

The second equation implies that $\sigma = 0$ or $m = 0$. Let us consider the case $\sigma = 0$. Then the first equation reads [the trace should be computed with respect to (3)]:

$$i + g^2 \sum_{n=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{dk}{2\pi R} \frac{1}{k^2 - (\frac{\pi n}{R})^2 - m^2 + i\epsilon} = 0. \quad (8)$$

Using the identity

$$\sum_Z \frac{1}{(\frac{\pi n}{R})^2 + \omega^2} = \frac{2R}{\omega}\left(\frac{1}{2} + \frac{1}{\exp(2R\omega) - 1}\right), \quad (9)$$

and after the Wick rotation, we arrive at

$$1 - \frac{g^2}{2\pi R} \int_0^{+\infty} dk \left(\frac{R}{\sqrt{k^2 + m^2}} + \frac{2R}{\sqrt{k^2 + m^2}} \times \frac{1}{(\exp(2R\sqrt{k^2 + m^2}) - 1)} - \frac{1}{k^2 + m^2} \right) = 0. \quad (10)$$

III. ANALYSIS

Let $x = 1/m$ and

$$Q\left(\frac{x}{R}\right) = \int_0^{+\infty} \frac{2dk}{\sqrt{k^2 + \frac{R^2}{x^2}}} \frac{1}{\left(\exp\left(2\sqrt{k^2 + \frac{R^2}{x^2}}\right) - 1\right)}. \quad (11)$$

If Λ_{uv} is an ultraviolet cutoff, (10) leads to

$$1 - \frac{g^2}{2\pi R}\left(R \log(\Lambda_{uv} x) + RQ(x/R) - \frac{\pi x}{2}\right) = 0. \quad (12)$$

It is more convenient to rewrite it as, recalling (1),

$$\frac{2\pi}{g^2} - \log(\Lambda_{uv} R) = -\log(\Lambda R) = \log(x/R) + Q(x/R) - \frac{\pi x}{2R}. \quad (13)$$

If $x \ll R$, Q could be calculated using a saddle-point approximation, with $k = 0$ as a saddle-point,

$$Q(x/R) \approx \frac{\sqrt{\pi x} e^{-\frac{2R}{x}}}{\sqrt{R}}, \qquad x \ll R, \quad (14)$$

so Q is exponentially suppressed and so negligible. In the limit $R \to +\infty$, $\frac{\pi x}{2R}$ is also negligible and we repeat the well-known result [2,3]:

$$\frac{2\pi}{g^2} = \log(\Lambda_{uv} x_0). \quad (15)$$

It is interesting to find $1/R$ corrections. If $x_0 = 1/m_0$, λ_0 are solutions for $R = +\infty$, then trivial calculation yields

$$x = x_0 + \frac{\pi x_0^2}{2R} + \frac{3\pi^2 x_0^3}{8R^2} + O(1/R^3). \quad (16)$$

Therefore,

$$m^2 = \frac{g^2 \lambda}{N} = \frac{1}{x^2} = \frac{1}{x_0^2} - \frac{\pi}{x_0 R} + O(1/R^3), \quad (17)$$

$$m = m_0 - \frac{\pi}{2R} - \frac{\pi^2}{8 m_0 R^2} + O(1/R^3). \quad (18)$$

In the next section, we will use this expansion to calculate $1/R$ corrections to vacuum energy.

Another mode is $x \gg R$. $Q(+\infty) = +\infty$, because the integral is divergent at lower bound. This mode is much more difficult to deal with; therefore, we calculated the right side of (13) numerically. The result is shown in the Fig. 1. The thick curve is the right side of (13), while the thin one is without $Q(x/R)$. At large x/R it has an asymptotic value -1.26, so

$$Q(x/R) \approx \frac{\pi x}{2R} - \log\left(\frac{x}{R}\right) - 1.26 + \cdots, \qquad x \gg R. \quad (19)$$

It is possible to calculate the next order term:

$$Q(x/R) = \frac{\pi x}{2R} + \log\left(\frac{R}{x}\right) - (\log(2\pi) - \gamma) - \frac{\zeta(3)}{2\pi^2}\left(\frac{R}{x}\right)^2 + O((R/x)^3), \quad (20)$$

where $\gamma \approx 0.577\ldots$—the Euler-Mascheroni constant. Recalling that $1/x = m$,

$$m^2 = \frac{2\pi^2}{R^2 \zeta(3)} (\log(\Lambda R) - (\log(2\pi) - \gamma)). \quad (21)$$

Note that the gap equation has a solution only when R is large enough.

Let us consider the other case: $m = 0$, $\sigma \neq 0$. Then (6) reads

$$\frac{-g^2}{\pi R} \sum_{n=1} \int_0^{+\infty} \frac{dk}{k^2 + (\frac{\pi n}{R})^2} - \frac{g^2 |\sigma|^2}{N} + 1 = 0. \quad (22)$$

Again using (9), we obtain

$$\frac{g^2 |\sigma|^2}{N} = 1 - \frac{g^2}{2\pi R} \int_0^{+\infty} dk \left(\frac{2R}{k}\left(\frac{1}{2} + \frac{1}{\exp(2Rk) - 1}\right) - \frac{1}{k^2} \right). \quad (23)$$

Note that the integral is not divergent in infrared, as one might expect recalling the Mermin-Wagner-Coleman theorem. Indeed, there is no spontaneous symmetry breaking at all: boundary conditions break $SU(N)$ to $SU(N-1)$ from the very beginning and $SU(N-1)$ remains unbroken in all phases. Because of Dirichlet boundary conditions, we have

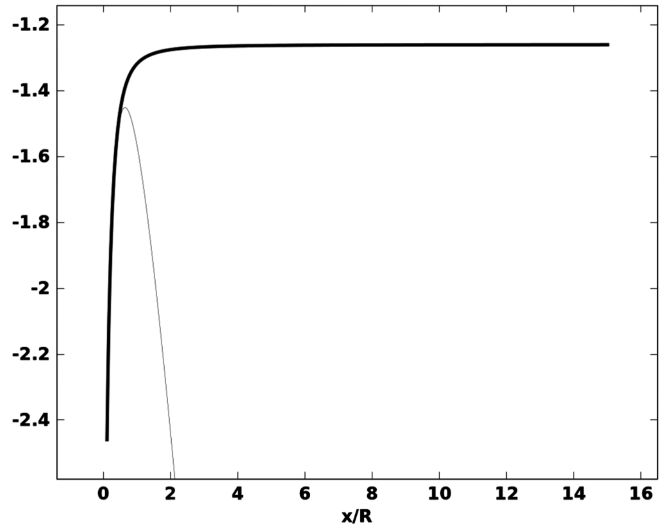

FIG. 1. The thick curve is the right side of Eq. (13); the thin curve is with $Q(x/R)$ omitted.

a natural IR cutoff π/R [see Eq. (8)]. Using (20) (if $m=0$ then $x=\infty$), we can write explicitly

$$\frac{g^2|\sigma|^2}{N} = 1 - \frac{g^2}{2\pi}(\log(\Lambda_{uv}R) + \gamma - \log(2\pi)), \quad (24)$$

or

$$\frac{|\sigma|^2}{N} = -\log(\Lambda R) + \log(2\pi) - \gamma. \quad (25)$$

IV. VACUUM ENERGY

Above we found the following effective action:

$$S_{\text{eff}} = iN\,\mathrm{Tr}\log\left(-\partial^2 - \frac{\lambda g^2}{N}\right) + \int d^2x\lambda. \quad (26)$$

From now on, we will work in Euclidian space, so

$$S_{\text{eff,Eucl}} = N\,\mathrm{Tr}\log\left(-\partial^2 + \frac{\lambda g^2}{N}\right) - \int d^2x\lambda. \quad (27)$$

However, (3) breaks translation invariance and so $\langle 0|T_{\mu\nu}|0\rangle \neq \epsilon\eta_{\mu\nu}$, and to calculate vacuum energy we will just calculate the effective action. Using Pauli-Villars regularization [8]

$$S^{\text{reg}}_{\text{eff,Eucl}} = N\sum_{i=0}^{2} c_i\,\mathrm{Tr}\log(-\partial^2 + m^2 + m_i^2) - \int d^2x\lambda, \quad (28)$$

$$m_0 = 0; \quad c_0 = 1; \quad c_1 = \frac{m_2^2}{m_1^2 - m_2^2}; \quad c_2 = \frac{-m_1^2}{m_1^2 - m_2^2}. \quad (29)$$

At the end we should take limits $m_1 \to +\infty$, $m_2 \to +\infty$.

Regularized action should be stationary for λ found above, so

$$\int d^2x \frac{1}{g^2} = \sum_{i=0}^{2} c_i\,\mathrm{Tr}\frac{1}{-\partial^2 + m_i^2 + m^2}. \quad (30)$$

Similar traces appeared above [Eq. (6)] and they contained a nasty integral such as (11). From now on, we will consider the case $R \to +\infty$ in which the calculation is simplified significantly. In this case (14) is correct and the nasty integral is of no interest due to the $\exp(-2\sqrt{m_i^2 + m^2}R)$ factor. After these remarks, the trivial calculation yields

$$\frac{1}{g^2} = \frac{1}{2\pi R}\left(\frac{R}{2}\log\left(\frac{m^2+m_2^2}{m^2}\right) + \frac{Rm_2^2}{2(m_1^2-m_2^2)}\log\left(\frac{m^2+m_2^2}{m^2+m_1^2}\right) - \frac{\pi}{2m} - \frac{\pi m_2^2}{2(m_1^2-m_2^2)}\frac{1}{\sqrt{m^2+m_1^2}} + \frac{\pi m_1^2}{2(m_1^2-m_2^2)}\frac{1}{\sqrt{m^2+m_2^2}}\right). \quad (31)$$

Setting $m_1^2 = xM^2$, $m_2^2 = M^2$ and taking

$$x \to 1, \qquad M \to +\infty, \quad (32)$$

we obtain

$$\frac{1}{g^2} = \frac{1}{2\pi R}\left(-\frac{R}{2} - \frac{\pi}{2m}\right) = -\frac{1}{4\pi} - \frac{1}{4mR}. \quad (33)$$

The regularized action (28) contains $\mathrm{Tr}\log(-\partial^2 + m^2)$. It is well known that this is the Casimir energy for a massive complex scalar field [9]. In 1 + 1

$$E = -\frac{m}{2} - \frac{Rm^2}{\pi}\sum_{n=1}^{+\infty}\frac{K_1(2Rmn)}{Rmn}, \quad (34)$$

where K_1 is the modified Bessel function.

The first term corresponds to the energy of boundary excitations. Usually it is omitted and the second term is called "the Casimir energy," but in our case m depends on R, so the first term is important. If $mR \gg 1$, then the sum has the asymptotic behavior $\exp(-2mR)$ and so is negligible. Expressions (28) and (33) are free of divergences. Trlog in (28) could be calculated exactly via Schwinger proper-time representation, but the expression is rather long and we will not give it here. After taking (32), we obtain $-\frac{Nm}{2}$ [the $\exp(-2mR)$ term is dropped]. Therefore,

$$E_{\text{vac}} = -\frac{Nm}{2} + \frac{NRm^2}{4\pi} + \frac{Nm}{4} = \frac{NRm^2}{4\pi} - \frac{Nm}{4}, \quad (35)$$

where (33) was used. There is no "interference" between the two terms in (28), and the limit (32) can be taken separately.

Note that there is no mass parameter in the original Lagrangian. The mass is dynamically generated. Therefore, to study R dependence in full, we should take into account that m depends on R. We will return to this fact in the next section. Substituting (17) and (18) into (35), we arrive at

$$E_{\rm vac} = \frac{Nm_0^2R}{4\pi} - \frac{m_0N}{2} + \frac{N\pi}{8R} + O(1/R^2), \qquad R \to +\infty. \tag{36}$$

V. DISCUSSION

In Ref. [6], Shifman and Yung argued that for the CP($N-1$) model, the Lüscher coefficient follows a rich pattern of behavior, equal to $\frac{\pi N}{12}$ when $R \ll \Lambda^{-1}$ because n^i could be considered massless, and approaches a value of 0 because n^i are massive when $R \gg \Lambda^{-1}$. Indeed, we have seen that there is phase transition when $R \sim 1/\Lambda$ [$R_{\rm crit} = \exp(\log(2\pi) - \gamma)/\Lambda$ to be precise] and below this value n^i are massless. But above $1/\Lambda$ we explicitly see Casimir-type behavior despite the existence of the mass gap.

However, in this situation, the mass depends on R and the Lüscher term comes not from a modified Bessel function in (34) (as in the massless case) but from the first term that is often of no physical meaning, but not in this case. The considerations above led us to $-\frac{\pi N}{8}$ when $R \gg \Lambda^{-1}$. Note that the sign is opposite to the one in a usual Casimir energy expression [9].

In recent works [7,10], Thomas and Zhitnitsky studied deformed QCD [11] on $S^1 \times S^3$. By means of the monopole gas and the Sine-Gordon representations, they argued that despite the existence of the mass gap the vacuum energy obeys Casimir-type behavior $\sim 1/\mathbb{L}$ ($\mathbb{L}$ is the radius of a 3-sphere) also with opposite sign. They relate it with the fact that the mass is not present in the theory from the very beginning but emerges as a result of some dynamics. Obviously, it is the case of the CP($N-1$) model.

One can also analyze Neumann boundary conditions $\partial n^i(0) = \partial n^i(R) = 0$. In this case, the sign of $\pi x/2R$ in the gap equation (13) is different because now we have to sum from $n = 0$ in Tr log, so the equation always has a unique solution; therefore, there is no phase transition. This is consistent with the Mermin-Wagner-Coleman theorem because boundary conditions do not break $SU(N) \to SU(N-1)$. A simple calculation shows that in this case we have again $N\pi/8R$ in the vacuum energy. Indeed, one can obtain $m(R)$ just by changing the sign of a π in (18). Also, one has to change the sign before $m/2$ in (34) and before $1/4mR$ in (33) due to the sum from $n = 0$ in Tr log. So we obtain $+Nm/4$ in (35).

The Casimir scaling with the "wrong sign" might be related with the topological properties of the theory, as, for example, the wrong sign in topological susceptibility of the 2d pure U(1) can be related with the existence of the different topological sectors [10]. However, the question of instanton configurations in the model is rather subtle and needs an individual paper for the thorough discussion. We will note only that one may add the θ term to the action: $i\theta \int d^2x \epsilon_{\mu\nu}\partial_\mu A_\nu = i\theta q$. In infinite space $\langle q\rangle_\theta \propto m^2\theta$ [2] so the θ-dependent portion of the vacuum energy also receives $1/R$ corrections.

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