

**Axially symmetric Yang-Mills-Higgs solutions in AdS spacetime**Olga Kichakova,<sup>1</sup> Jutta Kunz,<sup>1</sup> Eugen Radu,<sup>1</sup> and Yasha Shnir<sup>2,3</sup><sup>1</sup>*Institut für Physik, Universität Oldenburg, Postfach 2503 D-26111 Oldenburg, Germany*<sup>2</sup>*School of Mathematics, University of Leeds, LS2 9JT Leeds, United Kingdom*<sup>3</sup>*Department of Theoretical Physics and Astrophysics, BSU, Minsk 220080, Belarus*

(Received 19 September 2012; published 27 November 2012)

We consider axially symmetric solutions of  $SU(2)$  Yang-Mills-Higgs theory in globally AdS spacetime and a fixed Schwarzschild-AdS black hole background. The solutions are characterized by two integers  $(m, n)$  where  $m$  is related to the polar angle and  $n$  to the azimuthal angle. Two types of finite-energy, regular configurations are considered: solutions with net magnetic charge  $n > 1$ , and monopole-antimonopole pairs and chains with zero net magnetic charge. The configurations are endowed with an electric charge and also carry a nonvanishing angular momentum density.

DOI: [10.1103/PhysRevD.86.104065](https://doi.org/10.1103/PhysRevD.86.104065)

PACS numbers: 04.40.Nr, 04.70.Bw

**I. INTRODUCTION**

The study of solutions to the  $SU(2)$  Yang-Mills-Higgs (YMH) equations with an adjoint representation Higgs field is a subject of long-standing interest. This model has a nontrivial vacuum structure which gives rise to a variety of regular, nonperturbative finite-energy solutions, such as magnetic monopoles [1,2], multimonopoles [3,4] and composite configurations, containing both monopoles and antimonopoles [5–9]. Moreover, solutions which do not possess any continuous symmetry are also known to exist [10]. (A review of these aspects can be found in Ref. [11].) All these configurations can be generalized to include an electric charge, which further enriches the pattern of the solutions, leading, in the axially symmetric case, to a simple relation between the angular momentum, electric charge and total magnetic charge [12–14].

When including the effects of gravity, a branch of gravitating solutions emerges smoothly from the corresponding flat-space configurations (see the review in Ref. [15]). Moreover, these solutions allow for black hole generalizations. A complicated picture is emerging whose details depend mainly on the presence or not of a global magnetic charge [16–19].

The magnetic monopole and dyon solutions of the YMH system have enjoyed recently some renewed interest in the context of the AdS/CFT conjecture [20]. It has been argued that, if the bulk AdS spacetime contains non-Abelian magnetic monopoles, new interesting phenomena may result, including spontaneous breaking of translational symmetry in the dual theory [21] (see also Ref. [22]). The effects of the inclusion of an electric charge on the spherically symmetric magnetic monopoles have been discussed in Ref. [23]. For black hole solutions with a planar horizon topology and possessing YMH hair, it has been shown in Refs. [24,25] that the dual system defined in  $2 + 1$  dimensions undergoes a second-order phase transition and exhibits the condensation of a composite charge operator. Moreover, it has been suggested in Ref. [26]

that the dual field theory is generally a field theory with a vortex condensate.

These results have motivated us to consider the question of how the inclusion of a negative cosmological constant would affect the properties of the axially symmetric YMH configurations studied so far in a Minkowski spacetime background only. This work is intended as a preliminary study in this direction, since, for simplicity, we are working in the probe limit, i.e., for a fixed (Schwarzschild-)AdS geometry, without including the effects of the backreaction on the spacetime metric. Moreover, we are restricting ourselves to a foliation of the background geometry leading to a  $(2 + 1)$ -dimensional Einstein universe boundary metric.

**II. THE MODEL****A. The action and field equations**

We consider the action of the  $SU(2)$  YMH theory

$$S = -\frac{1}{4\pi} \int d^4x \sqrt{-g} \left( \text{Tr} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} D_\mu \Phi D^\mu \Phi + \frac{\lambda}{8} (\Phi^2 - \eta^2)^2 \right\} \right), \quad (1)$$

with the  $SU(2)$  field strength tensor  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]$  ( $A_\mu$  being the gauge potential) and the covariant derivative of the Higgs field  $\Phi$  in the adjoint representation  $D_\mu \Phi = \nabla_\mu \Phi + ie[A_\mu, \Phi]$ . Here  $e$  denotes the gauge coupling constant,  $\eta$  denotes the vacuum expectation value of the Higgs field, and  $\lambda$  represents the strength of the scalar coupling.

Under  $SU(2)$  gauge transformations  $U$ , the gauge potentials and the Higgs field transform as

$$A'_\mu = U A_\mu U^\dagger + \frac{i}{e} (\partial_\mu U) U^\dagger, \quad \Phi' = U \Phi U^\dagger. \quad (2)$$

Variation of Eq. (1) with respect to the gauge field  $A_\mu$  and the Higgs field  $\Phi$  leads to the field equations of the model

$$D_\mu F^{\mu\nu} = \frac{1}{4} ie[\Phi, D^\nu \Phi], \quad D_\mu D^\mu \Phi = \lambda(\Phi^2 - \eta^2)\Phi, \quad (3)$$

while the variation with respect to the metric  $g_{\mu\nu}$  yields the energy-momentum tensor

$$T_{\mu\nu} = \text{Tr} \left\{ 2 \left( F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) + \frac{1}{2} D_\mu \Phi D_\nu \Phi - \frac{1}{4} g_{\mu\nu} D_\alpha \Phi D^\alpha \Phi - \frac{\lambda}{8} g_{\mu\nu} (\Phi^2 - \eta^2)^2 \right\}. \quad (4)$$

As usual, the nonzero vacuum expectation value of the Higgs field breaks the non-Abelian  $SU(2)$  gauge symmetry to the Abelian  $U(1)$  symmetry. The particle spectrum of the theory then consists of a massless photon, two massive vector bosons of mass  $M_v = e\eta$ , and a massive scalar field  $M_s = \sqrt{2\lambda}\eta$ .

## B. The ansatz

### 1. The metric background

All YMH solutions in this work are studied in the probe limit, in which the backreaction of the matter field on the spacetime geometry is ignored. This approximation, which is valid as long as the dimensionless coupling constant  $\alpha^2 = 4\pi G/\eta^2$  is very small (with  $G$  the Newton's constant), greatly simplifies the problem but retains most of the interesting physics. For example, the nonlinear interaction between gauge fields and scalars is retained; also, the background geometry may possess a horizon.

For the background metric, we take first the AdS<sub>4</sub> spacetime, written in global coordinates

$$ds^2 = \frac{dr^2}{1 + \frac{r^2}{\ell^2}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - \left(1 + \frac{r^2}{\ell^2}\right) dt^2, \quad (5)$$

where  $r, t$  are the radial and time coordinates, respectively (with  $0 \leq r < \infty$ ), while  $\theta$  and  $\varphi$  are angular coordinates with the usual range, parametrizing the two-dimensional sphere  $S^2$ . Also,  $\ell$  is the AdS length scale, which is fixed by the cosmological constant,  $\Lambda = -3/\ell^2$ .

Since we want to study the effects of an event horizon on the YMH solutions, we shall consider as well a Schwarzschild-AdS (SAdS) black hole background, with a line element

$$ds^2 = \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - N(r) dt^2, \quad \text{with} \quad (6)$$

$$N(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2},$$

$M > 0$  being a parameter which fixes the Arnowitt-Deser-Misner mass of the solution. For  $\Lambda \rightarrow 0$ , one recovers the usual Schwarzschild solution. This black hole has an event horizon at  $r = r_h$ , with  $r_h$  the solution of the equation

$N(r_h) = 0$ . However, in practice it is more convenient to take  $r_h, \ell$  as input parameters, the function  $N(r)$  being written as

$$N(r) = \left(1 - \frac{r_h}{r}\right) \left(1 + \frac{1}{\ell^2}(r^2 + rr_h + r_h^2)\right). \quad (7)$$

With this parametrization, the mass  $M$ , Hawking temperature  $T_H$  and event horizon area of the SAdS black hole are given by  $M = r_h(r_h^2 + \ell^2)/(2\ell^2)$ ,  $T_H = (1/r_h + 3r_h/\ell^2)/(4\pi)$  and  $A_H = 4\pi r_h^2$ , respectively.

### 2. The matter fields

The axially symmetric YMH configurations we study in this work are less symmetric than the backgrounds of Eqs. (5) and (6), being invariant under the action of the Killing vectors  $\partial/\partial\varphi$  and  $\partial/\partial t$  only. The construction of a YMH ansatz with these symmetries has been discussed by many authors, starting with the pioneering papers by Manton [27] and Rebbi and Rossi [3].

In what follows, we shall use a parametrization of the general ansatz which was employed in the previous studies on axially symmetric YMH solutions in an asymptotically flat spacetime. This ansatz is characterized by two integers  $m$  and  $n$ , where  $m$  is related to the polar angle and  $n$  to the azimuthal angle [6], and it reads

$$A_\mu dx^\mu = \left( B_1 \frac{\tau_r^{(n,m)}}{2e} + B_2 \frac{\tau_\theta^{(n,m)}}{2e} \right) dt + \left( \frac{K_1}{r} dr + (1 - K_2) d\theta \right) \frac{\tau_\varphi^{(n)}}{2e} - n \sin\theta \left( K_3 \frac{\tau_r^{(n,m)}}{2e} + K_4 \frac{\tau_\theta^{(n,m)}}{2e} \right) d\varphi, \quad (8)$$

$$\Phi = \eta(\Phi_1 \tau_r^{(n,m)} + \Phi_2 \tau_\theta^{(n,m)}),$$

where the only  $\varphi$ -dependent terms are the  $SU(2)$  matrices  $\tau_r^{(n,m)}$ ,  $\tau_\theta^{(n,m)}$ , and  $\tau_\varphi^{(n)}$ . These matrices are defined as products of the spatial unit vectors

$$\hat{e}_r^{(n,m)} = (\sin(m\theta) \cos(n\varphi), \sin(m\theta) \sin(n\varphi), \cos(m\theta)),$$

$$\hat{e}_\theta^{(n,m)} = (\cos(m\theta) \cos(n\varphi), \cos(m\theta) \sin(n\varphi), -\sin(m\theta)),$$

$$\hat{e}_\varphi^{(n)} = (-\sin(n\varphi), \cos(n\varphi), 0), \quad (9)$$

with the Pauli matrices  $\tau^a = (\tau_x, \tau_y, \tau_z)$ , i.e.,  $\tau_r^{(n,m)} = \sin(m\theta)(\cos(n\varphi)\tau_x + \sin(n\varphi)\tau_y) + \cos(m\theta)\tau_z$ ,  $\tau_\theta^{(n,m)} = \cos(m\theta)(\cos(n\varphi)\tau_x + \sin(n\varphi)\tau_y) - \sin(m\theta)\tau_z$ , and  $\tau_\varphi^{(n)} = -\sin(n\varphi)\tau_x + \cos(n\varphi)\tau_y$ . As we shall see, this choice of the  $SU(2)$  matrices has the advantage of greatly simplifying the possible boundary conditions. The four magnetic gauge field functions  $K_i$ , two electric gauge functions  $B_i$  and two Higgs field functions  $\Phi_i$  depend on the coordinates  $r$  and  $\theta$  only. All profile functions are even or odd with respect to reflection symmetry,  $\theta \rightarrow \pi - \theta$ .

The symmetry of the gauge field under the spacetime Killing vectors  $\partial/\partial\varphi$  and  $\partial/\partial t$  means that their action can be compensated by a suitable gauge transformation [28,29]. For the time-translational symmetry, we have chosen a natural gauge such that  $\partial A/\partial t = 0$ . However, a rotation around the  $z$  axis can be compensated by a gauge rotation  $\mathcal{L}_\varphi A = D\Psi$  (with  $\Psi = n\tau_z/2e$ ), and therefore  $F_{\mu\varphi} = D_\mu W$ ,  $D_\varphi\Phi = ie[W, \Phi]$ , where  $W = A_\varphi - \Psi$ .

The gauge transformation  $U = \exp\{i\Gamma(r, \theta)\tau_\varphi^{(n)}/2\}$  leaves the ansatz form invariant [30]. Thus, to construct regular solutions we have to fix the gauge. In this work, we have mainly used the modified form of the gauge condition [6], which can be applied in the case of a (Schwarzschild-) AdS background:  $2rN\partial_r K_1 + rK_1\partial_r N - 2\partial_\theta K_2 = 0$ .

With this ansatz and gauge choice, the equations of motion reduce to a set of eight coupled partial differential equations to be solved numerically, subject to the set of boundary conditions discussed below.

### C. Global charges and the vacuum structure

#### 1. The mass-energy, the angular momentum, and the magnetic and electric charges

Let us define the total mass-energy  $E$  of the solutions as the integral over the three-dimensional space of the energy density  $\rho = -T^t_t$ :

$$\begin{aligned} E &= -\frac{1}{4\pi} \int T^t_t \sqrt{-g} d^3x \\ &= \int_{r_0}^{\infty} dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sqrt{-g} \text{Tr} \left\{ F_{r\theta} F^{r\theta} + F_{r\varphi} F^{r\varphi} \right. \\ &\quad \left. + F_{\theta\varphi} F^{\theta\varphi} - F_{rt} F^{rt} - F_{\theta t} F^{\theta t} - F_{\varphi t} F^{\varphi t} \right. \\ &\quad \left. + \frac{1}{4} (D_r \Phi D^r \Phi + D_\theta \Phi D^\theta \Phi + D_\varphi \Phi D^\varphi \Phi - D_t \Phi D^t \Phi) \right. \\ &\quad \left. + \frac{\lambda}{8} (\Phi^2 - \eta^2)^2 \right\}. \end{aligned} \quad (10)$$

In the above relation, one takes  $r_0 = 0$  for solutions in AdS spacetime and  $r_0 = r_h$  for a SAdS black background, in which case  $E$  corresponds to the total mass-energy of the fields outside the horizon.

Our solutions also possess a nonvanishing angular momentum density, since  $T^t_\varphi \neq 0$ . The total angular momentum of the configuration is given by the integral

$$\begin{aligned} J &= \frac{1}{4\pi} \int d^3x \sqrt{-g} T^t_\varphi \\ &= \frac{1}{4\pi} \int d^3x \sqrt{-g} 2 \text{Tr} \left\{ F_{r\varphi} F^{rt} + F_{\theta\varphi} F^{\theta t} + \frac{1}{4} D_\varphi \Phi D^t \Phi \right\}. \end{aligned} \quad (11)$$

As proven in Ref. [12], the angular momentum density can be written as a total derivative in terms of Yang-Mills potentials only:

$$T^t_\varphi = 2 \text{Tr} \left\{ \frac{1}{\sqrt{-g}} \partial_\mu (W F^{\mu t} \sqrt{-g}) \right\}. \quad (12)$$

As a result, the total angular momentum stored in the YMH fields outside the horizon can be expressed as a difference between a term at infinity and an inner boundary contribution:

$$J = \mathcal{J}(r = \infty) - \mathcal{J}(r = r_0), \quad (13)$$

with  $\mathcal{J}(R)$  the field angular momentum inside a sphere of radius  $R$ , defined as

$$\begin{aligned} \mathcal{J}(R) &= \frac{1}{4\pi} \oint_{r=R} 2 \text{Tr} \{ W F^{\mu t} \} dS_\mu \\ &= \frac{1}{2} \int_0^\pi d\theta \sin\theta r^2 [W^{(r)} F^{rt(r)} + W^{(\theta)} F^{rt(\theta)} \\ &\quad + W^{(\varphi)} F^{rt(\varphi)}] \Big|_{r=R}. \end{aligned} \quad (14)$$

The YMH solutions may also possess magnetic and electric charges. A gauge-invariant definition of these quantities is found by employing the Abelian 't Hooft tensor [1]

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= \text{Tr} \left\{ \hat{\Phi} F_{\mu\nu} - \frac{i}{2e} \hat{\Phi} D_\mu \hat{\Phi} D_\nu \hat{\Phi} \right\} \\ &= \hat{\Phi}^a F_{\mu\nu}^a + \frac{1}{e} \epsilon_{abc} \hat{\Phi}^a D_\mu \hat{\Phi}^b D_\nu \hat{\Phi}^c, \end{aligned} \quad (15)$$

where the Higgs field is normalized as  $|\hat{\Phi}|^2 = (1/2) \text{Tr} \hat{\Phi}^2 = \sum_a (\hat{\Phi}^a)^2 = 1$ . Then the 't Hooft tensor yields the electric current  $j_{el}^\nu$ , with  $\nabla_\mu \mathcal{F}^{\mu\nu} = -4\pi j_{el}^\nu$  and the magnetic current  $j_m^\nu$ , with  $\nabla_\mu^* \mathcal{F}^{\mu\nu} = 4\pi j_m^\nu$ , where  $^* \mathcal{F}$  represents the dual field strength tensor. Then  $j_{el}^\nu$  and  $j_m^\nu$  correspond to the electric and magnetic charge densities, respectively. The electric and magnetic charges  $Q_e$ ,  $Q_m$  are given by the integrals

$$Q_m = \frac{1}{4\pi} \oint_{S_\infty^2} ^* \mathcal{F}_{\theta\varphi} d\theta d\varphi, \quad Q_e = \frac{1}{4\pi} \oint_{S_\infty^2} \mathcal{F}_{\theta\varphi} d\theta d\varphi. \quad (16)$$

#### 2. The ground states of the model

In the presence of the Higgs field, the behavior of the matter field as  $r \rightarrow \infty$ , as imposed by finite energy requirements, is similar to the asymptotically flat case.<sup>1</sup> The assumption that the Higgs field asymptotically approaches a constant value  $|\Phi| \rightarrow \eta$ , together with the finite energy condition  $D_\mu \Phi D^\mu \Phi \rightarrow 0$ , leads to two different sets of fundamental solutions which define the ground states of

<sup>1</sup>Note that without a Higgs field, the Yang-Mills-AdS solutions exhibit a very different behavior than in the  $\Lambda \rightarrow 0$  limit (see e.g., Ref. [31]). In particular, one finds finite-mass, stable configurations [32].

the model. (In this discussion, we set  $A_t = 0$  and view the electric fields as excitations.)

For an even value  $m = 2k$  of the winding number with respect to  $\theta$ , the ground state of the model corresponds to a gauge-transformed trivial solution, and the magnetic charge vanishes as

$$\Phi = \eta U \tau_z U^\dagger, \quad A_\mu = \frac{i}{e} (\partial_\mu U) U^\dagger. \quad (17)$$

The situation is different for odd values  $m = 2k + 1 = 1, 3, \dots$  of the winding number with respect to  $\theta$ . The solutions in the sector with topological charge  $n$  will asymptotically approach a ground state with

$$\Phi = U \Phi_\infty^{(1,n)} U^\dagger, \quad A_\mu = U A_{\mu\infty}^{(1,n)} U^\dagger + \frac{i}{e} (\partial_\mu U) U^\dagger, \quad (18)$$

where

$$\begin{aligned} \Phi_\infty^{(1,n)} &= \eta \tau_r^{(1,n)}, \\ A_{\mu\infty}^{(1,n)} dx^\mu &= \frac{\tau_\varphi^{(n)}}{2e} d\theta - n \sin\theta \frac{\tau_\theta^{(1,n)}}{2e} d\varphi \end{aligned} \quad (19)$$

is the solution describing a multimonopole with charge  $n$ . Note that  $U = \exp\{-ik\theta\tau_\varphi^{(n)}\}$  for both even and odd  $m$ .

From these asymptotic behaviors, one gets the magnetic charge  $Q_m$  of the solutions

$$Q_m = \frac{n}{2} [1 - (-1)^m]; \quad (20)$$

i.e., solutions in the topologically trivial sector  $m = 2k$  have no magnetic charge ( $Q_m = 0$ ), whereas solutions in the nontrivial sectors  $m = 2k + 1$  have magnetic charge  $Q_m = n$ .

#### D. The boundary conditions

To obtain finite-energy solutions with the proper symmetries, we must impose appropriate boundary conditions. Without a horizon, these boundary conditions are similar to those used in the previous work in the flat-space limit [6,33].

The large- $r$  behavior of the YMH solutions is fixed by the requirement that the ground states [Eqs. (17) and (18)] be approached asymptotically. In terms of the functions  $K_1$ - $K_4$  and  $\Phi_1$ ,  $\Phi_2$ , these boundary conditions read

$$K_1 = 0, \quad K_2 = 1 - m, \quad K_4 = \frac{\sin(m\theta)}{\sin\theta}, \quad (21)$$

while for  $K_3$  we impose

$$\begin{aligned} K_3 &= \frac{\cos\theta - \cos(m\theta)}{\sin\theta} \quad \text{for odd } m, \quad \text{and} \\ K_3 &= \frac{1 - \cos(m\theta)}{\sin\theta} \quad \text{for even } m. \end{aligned} \quad (22)$$

We further impose the following boundary conditions at infinity for the electric gauge field potentials and the Higgs field functions, respectively:

$$B_1 = V_0, \quad B_2 = 0, \quad \Phi_1 = 1, \quad \Phi_2 = 0, \quad (23)$$

where the constant  $V_0$  corresponds to the electrostatic potential of the configurations.

The boundary conditions of the matter fields on the event horizon result from the requirement of regularity at  $r = r_h$ , together with an expansion of the field variables as a power series in  $(r - r_h)$ . Here one should remark that the numerical study of solutions in a fixed SAdS background is simplified by introducing a new radial coordinate

$\bar{r} = \sqrt{r^2 - r_h^2}$ , such that the event horizon resides at  $\bar{r} = 0$ . [The corresponding expression of the SAdS line element follows directly from Eqs. (6) and (7).] Also, the boundary conditions at the horizon take a simpler form in this case: the magnetic functions  $K_i$  and the Higgs functions  $\Phi_1$ ,  $\Phi_2$  are requested to satisfy the Neumann boundary conditions

$$\begin{aligned} \partial_r K_1|_{\bar{r}=0} &= \partial_r K_2|_{\bar{r}=0} = \partial_r K_3|_{\bar{r}=0} = \partial_r K_4|_{\bar{r}=0} \\ &= \partial_r \Phi_1|_{\bar{r}=0} = \partial_r \Phi_2|_{\bar{r}=0} = 0, \end{aligned} \quad (24)$$

while the electric field functions satisfy the Dirichlet boundary conditions

$$B_1(\bar{r} = 0, \theta) = B_2(\bar{r} = 0, \theta) = 0. \quad (25)$$

The corresponding sets of boundary conditions in the AdS background limit  $r_h \rightarrow 0$  are more complicated and do not result directly from Eqs. (24) and (25). Regularity of the solutions at the origin ( $r = 0$ ) requires that the magnetic gauge field functions  $K_i$  satisfy

$$K_1(0, \theta) = K_3(0, \theta) = K_4(0, \theta) = 0, \quad K_2(0, \theta) = 1. \quad (26)$$

For the electric components of the gauge field, we impose instead

$$\begin{aligned} \sin(m\theta)B_1(0, \theta) + \cos(m\theta)B_2(0, \theta) &= 0, \\ \partial_r[\cos(m\theta)B_1(r, \theta) - \sin(m\theta)B_2(r, \theta)]|_{r=0} &= 0, \end{aligned} \quad (27)$$

the Higgs field functions  $\Phi_i$  satisfying a similar set of boundary conditions,

$$\begin{aligned} \sin(m\theta)\Phi_1(0, \theta) + \cos(m\theta)\Phi_2(0, \theta) &= 0, \\ \partial_r[\cos(m\theta)\Phi_1(r, \theta) - \sin(m\theta)\Phi_2(r, \theta)]|_{r=0} &= 0. \end{aligned} \quad (28)$$

The boundary conditions along the  $z$  axis ( $\theta = 0$  and  $\theta = \pi$ ) are determined by the symmetries and regularity requirements, and read

$$\begin{aligned} K_1 = K_3 = \Phi_2 = 0, \quad \partial_\theta K_2 = \partial_\theta K_4 = \partial_\theta \Phi_1 = 0, \\ \partial_\theta B_1(r, 0) = 0, \quad B_2(r, 0) = 0. \end{aligned} \quad (29)$$

Additionally, regularity on the symmetry axis requires that  $K_2(r, \theta = 0) = K_4(r, \theta = 0)$ . We use this condition as an additional test of the numerical output.

### III. THE RESULTS

Axially symmetric solutions of the YMH equations have been extensively studied in a Minkowski spacetime background, starting with the pioneering work of Ref. [3]. However, relatively little is known about the AdS counterparts of the  $\Lambda = 0$  configurations. Unlike the flat spacetime case, in an AdS spacetime there are no analytic solutions that might be used as a guiding line. Moreover, when a cosmological constant is included (no matter how small  $|\Lambda|$  is), no solution close to the BPS configuration can be found [23]. Some numerical results supporting the existence of multimonopoles and monopole-antimonopole pairs with  $Q_e = 0$  in an AdS background are reported in Ref. [34], while a discussion of the basic properties of electrically charged  $m = 1$  solutions is given in Ref. [12]. However, the effects induced by the presence of the event horizon have not yet been studied in the literature, except for  $m = 1, n = 1$  spherically symmetric configurations [25].

Before discussing the properties of the solutions, let us mention that the dependence on the massive boson vector mass  $M_v = e\eta$  is removed by using the rescaling  $r \rightarrow r/(e\eta)$ ,  $\Lambda \rightarrow \Lambda e^2 \eta^2$ ,  $r_h \rightarrow r_h/(e\eta)$  and  $B_i \rightarrow B_i/(e\eta)$ . Also, to simplify the picture, we restrict our study in this work to the Prasad-Sommerfield limit  $\lambda = 0$ . Thus, apart from the value of the event horizon radius  $r_h$ , the configurations will depend on the winding numbers  $m$  and  $n$ , on the parameter  $V_0$  which fixes the value of the electric potential of the gauge field at spatial infinity, and on the value of the cosmological constant  $\Lambda$ . The numerical calculations are performed with the help of the package FIDISOL, based on the Newton-Raphson iterative procedure [35].

Our numerical results indicate that all known axially symmetric YMH solutions admit generalizations in an AdS background. Moreover, one can also put a small SAdS black hole inside these configurations. Qualitatively, the Higgs field and Yang-Mills field behavior is very similar to that corresponding to Minkowski spacetime monopoles. In particular, we notice a similar shape for the functions  $K_i$ ,  $B_i$  and  $\Phi_i$ , and also for the energy density.

The most important new feature of the AdS solutions is that the magnitude  $V_0$  of the electric potential  $B_1$  at infinity is no longer restricted. In an asymptotically Minkowski spacetime, this constant is restricted to  $V_0 \leq 1$ , i.e.,  $|A_i| \leq |\Phi|$ . For  $V_0 > 1$ , some gauge field functions become oscillating instead of asymptotically decaying, which leads to an infinite mass in the solutions. However, in an AdS spacetime, finite-energy solutions with arbitrary  $V_0$  are allowed; i.e., there are no limits on the value of the electrostatic potential.

For any value of  $\Lambda \leq 0$ , the electric charge  $Q_e$  can be read from the asymptotics of the electric potential  $B_1$ :

$$B_1 = V_0 - \frac{Q_e}{r} + \dots \quad (30)$$

After replacing the asymptotic expressions of the solutions in the general expression [Eq. (13)], one finds that the contribution to the total angular momentum from the boundary integral at infinity can be written as

$$\mathcal{J}_\infty = [1 + (-1)^m] \frac{nQ_e}{2}. \quad (31)$$

Therefore, the total angular momentum [Eq. (13)] of  $m = 2k + 1$  solutions, i.e., with a net magnetic charge, is given entirely by the contribution of the inner boundary term,  $J = -\mathcal{J}_{(\text{e.h.})}$ . However, that term vanishes as  $r_h \rightarrow 0$ . As a result, the solutions in an AdS background with  $Q_m \neq 0$  have zero total angular momentum.<sup>2</sup> The situation is different for configurations with  $m = 2k$ , i.e., for a vanishing magnetic charge,  $Q_m = 0$ . The total angular momentum of such solutions in an AdS spacetime is proportional to the electric charge,  $J = nQ_e$ . However, due to the event horizon contribution to the general relation [Eq. (13)], this simple relation does not hold for their generalization in a SAdS background.

#### A. Axially symmetric solutions in globally AdS spacetime

Let us start by recalling that the configurations we are discussing here are characterized by two integers,  $m$  and  $n$ , where  $m$  is related to the polar angle and  $n$  to the azimuthal angle. For both Minkowski and AdS backgrounds, the regular solutions of the system [Eq. (3)] correspond to electrically charged (multi)monopoles ( $m = 1, n \geq 1$ ) with magnetic charge  $Q_m = n$ , while the configurations with  $m = 2, n = 1, 2$  correspond to electrically charged monopole-antimonopole pairs with zero net magnetic charge. In general, configurations with  $m \geq 2, n = 1, 2$  correspond to chains of  $m$  monopoles and antimonopoles with a net electric charge. Here, the Higgs field vanishes at  $m$  isolated points along the symmetry axis  $z = r \cos \theta$ .

The topological reason for the appearance of these saddle-point solutions of the YMH equations is related to the existence of noncontractible loops in the configuration space of the model; minimization of the energy functional along such a loop yields an equilibrium state in the middle of the loop, where the constituents are in stationary equilibrium [36]. Since in this case we have a complicated pattern of short-range interactions between the constituent monopoles, it is instructive to use a simplified picture of the effective electromagnetic interaction between the poles and electric current rings which, according to the equations of the 't Hooft field tensor [Eq. (15)], both generate the

<sup>2</sup>Note that these solutions possess a nonvanishing angular momentum density,  $T_\phi^t \neq 0$ . (This holds for configurations in the Prasad-Sommerfield limit as well.)

Abelian magnetic field which is supporting equilibrium. (See Ref. [37] for a detailed discussion of these aspects.)

The results for  $\Lambda = 0$  show that, as the winding number  $n$  increases further, i.e.,  $n \geq 3$ , instead of isolated nodes on the symmetry axis in the flat-space limit, vortex ring solutions arise, where the Higgs field vanishes on one or more rings centered around the symmetry axis [6–8]. However, the coupling of the matter fields to gravity and/or an increase of the electric charge of the configurations [9,18,33] can change the situation drastically, since the structure of the nodes of the Higgs fields depends strongly on the scalar coupling, the magnitude of the electric charge of the system, and the gravitational constant. Therefore, the information about the structure of the nodes becomes less important. Furthermore, new branches of solutions may appear at critical values of the parameters [18,33,38,39].

We have found that all basic features of these nongravitating solutions with  $\Lambda = 0$  repeat in the case of an AdS background. As the winding number  $n$  increases, the attraction between the constituents increases, so for  $n \geq 2$  the energy density distribution is deformed to the system of  $m$  tori centered around the symmetry axis. A negative cosmological constant introduces an attractive

force, which reduces the typical size of the configurations. Then, for large enough  $|\Lambda|$ , the additional AdS gravitational attraction makes it difficult to classify the individual constituents according to the position of the nodes of the scalar field, as is possible in flat space.

We illustrate the AdS solutions with a few examples in Fig. 1. There we exhibit the distribution of the electric and magnetic charge densities  $j_{el}^t, j_m^t$ , and the energy isosurfaces for the spherically symmetric dyon ( $m = n = 1$ ), the two-dyon pair ( $m = 2, n = 1$ ), and a chain of three dyons ( $m = 3, n = 1$ ). In these solutions, the  $m$  individual constituents are located on the symmetry axis, with roughly equal distances between them. (There we fix the values  $\Lambda = -1/3$  and  $V_0 = 1$  for the cosmological constant and the electric potential, respectively; a similar picture has been found, however, for other values of these parameters.)

In Fig. 2, we show the mass-energy of ( $m = 3, 4; n = 1$ ) YMH chain solutions with fixed electric charge (left) and fixed electric potential (right) as a function of the cosmological constant. One can notice the existence of AdS solutions with  $V_0 > 1$ ; however, as expected, the mass-energy of such configurations diverges as  $\Lambda \rightarrow 0$ . A similar picture has been found for solutions with  $m = 1$  and  $m = 2$  (see also Refs. [13,34]).

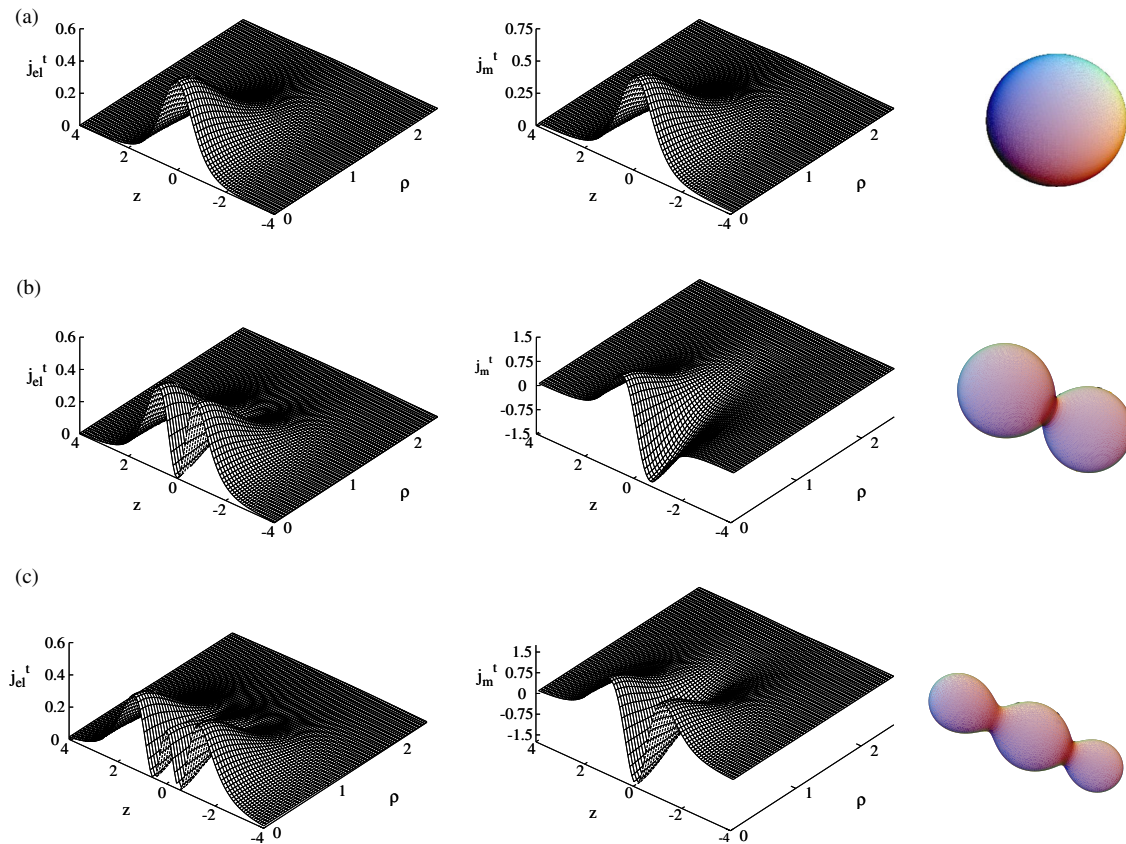


FIG. 1 (color online). The electric charge density (left panels) and the magnetic charge densities (middle panels) are shown at  $V_0 = 1$ ,  $\Lambda = -1/3$  as functions of the coordinates  $\rho = r \sin\theta$ ,  $z = r \cos\theta$  for the following: (a)  $n = 1$ ,  $m = 1$  spherically symmetric AdS dyon (first set); (b)  $n = 1$ ,  $m = 2$  AdS dyon pair regular solution (middle set); (c)  $n = 1$ ,  $m = 3$  three dyon chain. Energy isosurfaces of these configurations are exhibited in the right panels.

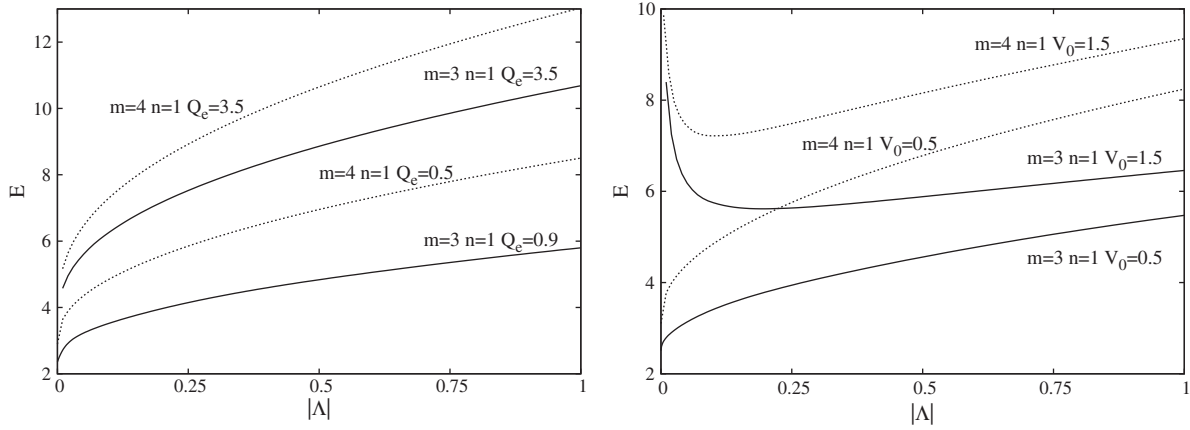


FIG. 2. The mass-energy  $E$  is shown as a function of the cosmological constant for  $m = 3, 4$  AdS YMH chains with a fixed electric charge (left) and a fixed electric potential (right).

### B. Axially symmetric solutions in the Schwarzschild-AdS background

According to the standard arguments, one expects these solutions to possess generalizations in a SAdS background. This is indeed confirmed by the numerical analysis. However, for all solutions reported in this work, there is an upper bound on the event horizon radius, typically with  $r_h^{(\max)} < \ell$ . The possible existence of large SAdS black holes with YMH hair is an open question and may require considering a region of the parameter space not covered by our numerical study (e.g., very large values of the electric charge).

Let us start with a discussion of the fundamental configurations with  $m = 1$ . In this case, an important role is played by the Abelian dyon solution (with  $V_0 = Q_e/r_h$ ):

$$\begin{aligned} K_1 = K_2 = K_3 = K_4 = 0, \quad B_1 = V_0 - \frac{Q_e}{r}, \\ B_2 = 0, \quad \Phi_1 = 1, \quad \Phi_2 = 0. \end{aligned} \quad (32)$$

This solution exists for all values of  $r_h$  and has a finite mass-energy  $E = (n^2 + Q_e^2)/2r_h$  and a vanishing angular momentum  $J = 0$ .

We have found numerical evidence that, for a given value of the electric charge, a branch of  $m = 1, n \geq 1$  non-Abelian solutions bifurcates from the Abelian configuration of Eq. (32), for a critical value of the event horizon radius  $r_h$ . Close to the bifurcating point, the YM potentials and the Higgs fields are written as a sum of the solution [Eq. (32)] and a small perturbation  $\delta F = (\delta K_i, \delta \Phi_i, \delta B_i)$ . Linearizing the YMH equations with respect to these perturbations leads to an eigenvalue matrix equation for  $r_h$  of the form

$$\left( \left( \partial_{rr} + \frac{1}{r^2 N(r)} \partial_{\theta\theta} \right) I_{8 \times 8} + M_r \partial_r + M_\theta \partial_\theta + M_0 \right) \delta F = 0, \quad (33)$$

where  $M_r, M_\theta$  and  $M_0$  are  $8 \times 8$  matrices, depending only on the Abelian solution [Eq. (32)] and the metric function  $N(r)$ , with a complicated expression. The above equation

simplifies only for the spherically symmetric solutions, in which case  $\delta F = 0$  (the higher-order terms, however, are not zero), except for  $\delta K_2 = \delta K_4 = w_1(r)$ , which solves the equation

$$w_1'' + \frac{N'}{N} w_1' + \frac{1}{r^2 N} \left( 1 - r^2 + \frac{r^2 (\frac{Q_e}{r_h} - \frac{Q_e}{r})^2}{N} \right) w_1 = 0, \quad (34)$$

with the boundary conditions  $w_1(r_h) = b > 0$ ,  $w_1(\infty) = 0$ . The critical values of  $r_h$  found in this way for given electric charges (or, equivalently, for given electrostatic potentials  $V_0$ ) are in very good agreement with those found by directly solving the set of YMH equations.

The respective branch of non-Abelian solutions continues inwards in  $r_h$ , joining smoothly for  $r_h = 0$  the corresponding  $(m = 1, n)$  dyonic solution in a fixed AdS background discussed in the previous subsection. As expected, along this branch, the mass of YMH solutions is smaller than the mass of the corresponding Abelian configurations, i.e., they are thermodynamically favored.<sup>3</sup> This feature appears to be universal, being recovered for all considered values of  $\Lambda$ .

To illustrate this behavior, we exhibit in Fig. 3 (left) the mass-energy and the value of the magnetic gauge potential  $K_2$  at  $r = r_h, \theta = 0$  for  $m = 1$  dyons with  $n = 1, 2$  versus the horizon radius  $r_h$ . (These results are found for fixed values of the electric charge and cosmological constant,  $Q_e = 1.5$  and  $\Lambda = -1$ , respectively.) The dotted curve corresponds to the branch of Abelian solutions with the same values of  $Q_e, n$  and  $r_h$ . One can see that a fundamental branch of dyons emerges from the corresponding AdS solution with  $r_h = 0$  and extends up to a maximal value of the horizon radius  $r_h^{(\max)}$ , where it merges with the Abelian branch. Similar results are found when studying instead solutions with a fixed electric potential  $V_0$  (i.e., in a grand canonical ensemble), with

<sup>3</sup>Here we are comparing solutions in a canonical ensemble, i.e., with the same electric charge.

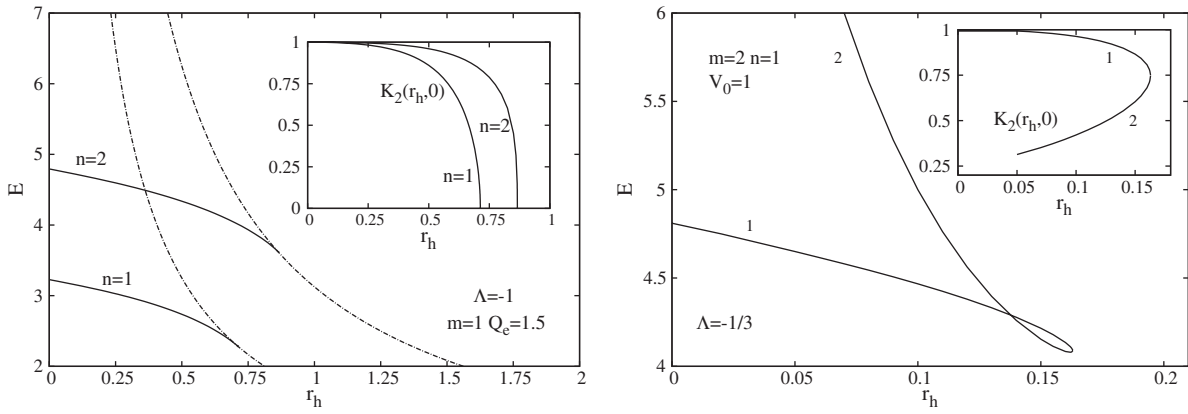


FIG. 3. The mass-energy  $E$  is shown as a function of the event horizon radius  $r_h$  for electrically charged YMH solutions in a fixed Schwarzschild-AdS background with a net magnetic charge,  $m = 1$  (left), and with a vanishing total magnetic charge,  $m = 2$  (right). The insets show the value of the magnetic gauge potential  $K_2$  at  $\theta = 0$  on the horizon. For  $m = 1$ , the branch of Abelian solutions as given by Eq. (32) is also shown (dotted curves).

the YMH configurations emerging again as perturbations of the Abelian solution [Eq. (32)].

A different picture is found for solutions with  $m = 2$ , i.e., electrically charged monopole-antimonopole pairs. In this case, there are no branches of YMH solutions emerging as perturbations of a critical (electrically charged and magnetically neutral) Abelian configuration. As a result, one finds a different dependence on the horizon radius, which is illustrated in the right-hand side of Fig. 3. Again, a lower branch of non-Abelian solutions (labeled “1” on the right of Fig. 3) emerges in the limit of small  $r_h$  from the regular AdS chain configuration. This branch extends in  $r_h$  and bifurcates with a second branch at some maximal value of the horizon radius. (For example, for the  $n = 1$  pair of dyons with  $V_0 = 1$ ,  $\Lambda = -1/3$ , one finds  $r_h^{(\max)} \simeq 0.16$ .) This second branch extends backwards in  $r_h$ . Since the upper two-dyon branch is axially symmetric, it is not linked to the trivial Abelian branch anymore; instead, as the horizon radius decreases to zero, it appears to approach a solution of the Bartnik-McKinnon type [15], i.e., with a trivial Higgs field and nonvanishing non-Abelian potentials. (However, this upper limit is rather difficult to study numerically.) Note also that the mass of these solutions in a fixed SAdS background exhibits a loop close to  $r_h^{(\max)}$  when plotted in terms of  $r_h$ . Outside the loop, the second branch possesses a higher mass than the first branch.

We have also constructed solutions with  $m = 3, 4$  and  $n = 1, 2, 3$ , although with a lower numerical accuracy. Not completely unexpectedly, these solutions follow the pattern found for the  $m = 2$  case above. Working again with a fixed electric potential  $V_0$ , there is always a lower branch of configurations smoothly emerging from the corresponding  $(m, n)$  solutions in a fixed AdS background, which extends up to a maximal value of the horizon radius  $r_h$ . There it joins a secondary branch of solutions, which extends backwards in  $r_h$ .

The reason why the  $m \geq 2$  solutions show a different pattern can be understood heuristically by noticing that

they are composite, saddle-point configurations. Thus, they are unstable, and we cannot expect a (single-component) Abelian solution to decay into them.

#### IV. FURTHER REMARKS AND CONCLUSIONS

We have given numerical evidence that, when the global AdS spacetime replaces Minkowski spacetime as the background geometry, all known YMH axially symmetric solutions admit generalizations with rather similar properties. However, a more complicated picture emerges when the solutions are studied in a fixed SAdS black hole background, with the configurations with the lowest winding number  $m = 1$  emerging as perturbations of some critical Abelian dyonic solutions.

One may ask if these solutions play some role in the conjectured AdS/CFT correspondence [20]. Indeed, the  $(m = 1, n = 1)$  spherically symmetric dyonic solutions of the YMH model have found an interesting interpretation in this context [24–26,40].

Since we have not started with a consistent truncation of string theory, we do not have a detailed microscopic description of the dual theory for the YMH action [Eq. (1)]. Nevertheless, some basic elements of the gravity/gauge duality dictionary [20] still allow us to say the following: First, the dual theory is defined in an Einstein universe in  $d = 3$  dimensions, with a line element  $ds^2 = \ell^2(d\theta^2 + \sin^2\theta d\varphi^2) - dt^2$ . In this approach, the Hawking temperature of the SAdS black hole corresponds to the temperature of the  $d = 3$  system. Also, the  $SU(2)$  gauge symmetry of the bulk action corresponds to a global  $SU(2)$  symmetry in the dual field theory. As usual in models with an electric field, the chemical potential and the electric charge density of the three-dimensional system are defined from the asymptotics [Eq. (30)] of the bulk electric gauge potential, the charge density operator being proportional to  $Q_e$ . Moreover, we have seen that, for odd



values of  $m$  and any winding number  $n$ , the magnetic gauge potential does not trivialize as  $r \rightarrow \infty$ . In an AdS/CFT context, this boundary value plays the role of a magnetic source for the dual field theory. Concerning the Higgs field, we notice that for the solutions in this work without a scalar potential  $\lambda = 0$ , its generic asymptotic behavior is

$$\Phi^{(a)} = v^{(a)} + \frac{\beta^{(a)}}{r^3} + \dots, \quad (35)$$

with  $v^{(a)}$  a source for a triplet of operators  $\Psi^{(a)}$  in the dual theory. However, we can always choose a gauge such that  $v^{(a)}$  is constant on the boundary. In such a gauge, the magnetic field on the boundary corresponds to the field of vortices. For example, the axially symmetric multimonopole corresponds to a vortex of magnetic flux  $nQ_m$ , whereas the monopole-antimonopole pair in the bulk corresponds to a system of vortices with opposite fluxes. Therefore, the transition between Abelian and non-Abelian branches in the bulk at some critical value of the horizon radius, i.e., Hawking temperature, is considered to be dual to the phase transition on the boundary, related with the condensation of vortices.

As avenues for further research, it would be interesting to extend the solutions in this work by including the effects of the backreaction on the spacetime geometry, especially in the black hole case. Here we expect that some features revealed in Sec. III, in particular the existence of an instability of the Abelian dyon configurations, will remain valid, translating into instabilities (and corresponding new non-Abelian branches) of the Kerr-Newman-AdS black hole. Another interesting possible extension of our solutions would be to construct their counterparts for a Poincaré patch of AdS, in which case the dual theory will be defined in a boundary metric corresponding to  $2 + 1$ -dimensional Minkowski spacetime.

## ACKNOWLEDGMENTS

We would like to thank Burkhard Kleihaus for useful discussions. O. Kichakova, J. Kunz and E. Radu gratefully acknowledge support by the DFG; in particular, within DFG Research Training Group 1620, ‘‘Models of Gravity.’’ The work of Ya. Shnir is supported by the grants of DAAD and EPSRC.

- 
- [1] G. 't Hooft, *Nucl. Phys.* **B79**, 276 (1974).
  - [2] A. M. Polyakov, *Pis'ma Zh. Eksp. Teor. Fiz.* **20**, 430 (1974) [*JETP Lett.* **20**, 194 (1974)].
  - [3] C. Rebbi and P. Rossi, *Phys. Rev. D* **22**, 2010 (1980).
  - [4] R. S. Ward, *Commun. Math. Phys.* **79**, 317 (1981); P. Forgacs, Z. Horvath, and L. Palla, *Phys. Lett.* **99B**, 232 (1981); **101B**, 457(E) (1981); M. K. Prasad, *Commun. Math. Phys.* **80**, 137 (1981).
  - [5] B. Kleihaus and J. Kunz, *Phys. Rev. D* **61**, 025003 (1999).
  - [6] B. Kleihaus, J. Kunz, and Y. Shnir, *Phys. Lett. B* **570**, 237 (2003).
  - [7] B. Kleihaus, J. Kunz, and Y. Shnir, *Phys. Rev. D* **68**, 101701 (2003).
  - [8] B. Kleihaus, J. Kunz, and Y. Shnir, *Phys. Rev. D* **70**, 065010 (2004).
  - [9] B. Kleihaus, J. Kunz, and Y. Shnir, *Phys. Rev. D* **71**, 024013 (2005).
  - [10] E. Corrigan and P. Goddard, *Commun. Math. Phys.* **80**, 575 (1981); P. M. Sutcliffe, *Int. J. Mod. Phys. A* **12**, 4663 (1997).
  - [11] Y. Shnir, *Magnetic Monopoles* (Springer, New York, 2005).
  - [12] J. J. Van der Bij and E. Radu, *Int. J. Mod. Phys. A* **17**, 1477 (2002).
  - [13] J. J. van der Bij and E. Radu, *Int. J. Mod. Phys. A* **18**, 2379 (2003).
  - [14] M. S. Volkov and E. Wonnert, *Phys. Rev. D* **67**, 105006 (2003).
  - [15] M. S. Volkov and D. V. Gal'tsov, *Phys. Rep.* **319**, 1 (1999).
  - [16] B. Hartmann, B. Kleihaus, and J. Kunz, *Phys. Rev. D* **65**, 024027 (2001).
  - [17] B. Kleihaus and J. Kunz, *Phys. Lett. B* **494**, 130 (2000).
  - [18] B. Kleihaus, J. Kunz, F. Navarro-Lerida, and U. Neemann, *Gen. Relativ. Gravit.* **40**, 1279 (2008).
  - [19] B. Kleihaus, J. Kunz, and F. Navarro-Lerida, *Phys. Lett. B* **599**, 294 (2004).
  - [20] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998); *Int. J. Theor. Phys.* **38**, 1113 (1999).
  - [21] S. Bolognesi and D. Tong, *J. High Energy Phys.* 01 (2011) 153.
  - [22] P. Sutcliffe, *J. High Energy Phys.* 08 (2011) 032.
  - [23] A. R. Lugo and F. A. Schaposnik, *Phys. Lett. B* **467**, 43 (1999).
  - [24] A. R. Lugo, E. F. Moreno, and F. A. Schaposnik, *J. High Energy Phys.* 03 (2010) 013.
  - [25] A. R. Lugo, E. F. Moreno, and F. A. Schaposnik, *J. High Energy Phys.* 11 (2010) 081.
  - [26] D. Allahbakhshi, *J. High Energy Phys.* 09 (2011) 085.
  - [27] N. S. Manton, *Nucl. Phys.* **B135**, 319 (1978).
  - [28] M. Heusler, *Helv. Phys. Acta* **69**, 501 (1996).
  - [29] P. Forgacs and N. S. Manton, *Commun. Math. Phys.* **72**, 15 (1980); P. G. Bergmann and E. J. Flaherty, *J. Math. Phys. (N.Y.)* **19**, 212 (1978).
  - [30] Y. Brihaye and J. Kunz, *Phys. Rev. D* **50**, 4175 (1994).
  - [31] E. Winstanley, *Classical Quantum Gravity* **16**, 1963 (1999); J. Bjoraker and Y. Hosotani, *Phys. Rev. D* **62**, 043513 (2000); R. B. Mann, E. Radu, and D. H. Tchrakian, *Phys. Rev. D* **74**, 064015 (2006).
  - [32] P. Breitenlohner, D. Maison, and G. Lavrelashvili, *Classical Quantum Gravity* **21**, 1667 (2004).
  - [33] B. Kleihaus, J. Kunz, and U. Neemann, *Phys. Lett. B* **623**, 171 (2005).

- [34] E. Radu and D.H. Tchrakian, [Phys. Rev. D \*\*71\*\*, 064002 \(2005\)](#).
- [35] W. Schönauer and R. Weiß, [J. Comput. Appl. Math. \*\*27\*\*, 279 \(1989\)](#); M. Schauder, R. Weiß, and W. Schönauer, The CADSOL Program Package, Universität Karlsruhe, Interner Bericht Nr. 46/92 (1992).
- [36] C.H. Taubes, [Commun. Math. Phys. \*\*86\*\*, 257 \(1982\)](#); [86](#), 299 (1982); [97](#), 473 (1985).
- [37] Y. Shnir, [Phys. Rev. D \*\*72\*\*, 055016 \(2005\)](#).
- [38] J. Kunz, U. Neemann, and Y. Shnir, [Phys. Rev. D \*\*75\*\*, 125008 \(2007\)](#).
- [39] J. Kunz, U. Neemann, and Y. Shnir, [Phys. Lett. B \*\*640\*\*, 57 \(2006\)](#).
- [40] D. Allahbakhshi and F. Ardalan, [J. High Energy Phys. \*\*10\*\* \(2010\) 114](#).