

Can we distinguish between black holes and wormholes by their Einstein-ring systems?

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For the past decade, gravitational lensing in the strong gravitational field has been studied eagerly. It is well known that, for the lensing by a black hole, an infinite number of Einstein rings are formed by the light rays which wind around the black hole nearly on the photon sphere, which are called relativistic Einstein rings. This is also the case for the lensing by a wormhole. In this paper, we study the Einstein ring and relativistic Einstein rings for the Schwarzschild black hole and the Ellis wormhole, the latter of which is an example of traversable wormholes of the Morris-Thorne class. Given the configuration of the gravitational lensing and the radii of the Einstein ring and relativistic Einstein rings, we can distinguish between a black hole and a wormhole in principle. We conclude that we can detect the relativistic Einstein rings by wormholes which have the radii of the throat $a \approx 0.5$ pc at a Galactic center with the distance 10 Mpc and which have $a \approx 10$ AU in our Galaxy using the most powerful modern instruments which have the resolution of 10^{-2} arcsecond such as a 10-meter optical-infrared telescope. The black holes which make the Einstein rings of the same size as the ones by the wormholes are galactic supermassive black holes and the relativistic Einstein rings by the black holes are too small to measure with the current technology. We may test the hypotheses of astrophysical wormholes by using the Einstein ring and relativistic Einstein rings in the future.

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I. INTRODUCTION

Gravitational lensing is a very useful tool for astrophysics and cosmology. At first the gravitational lensing mainly was investigated on a theoretical basis in the weak gravitational field. Using the gravitational lensing, we determine the cosmological constant, the distribution of dark matter and the Hubble constant, the existence of extrasolar planets, and so on (see Schneider *et al.* [1] and Perlick [2,3] for the detail of the gravitational lens, and references therein).

For the past decade, gravitational lensing in the strong gravitational field has been studied eagerly (see Virbhadra and Keeton [4], Virbhadra [5], Bozza [6], Bozza and Mancini [7], and references therein). Frittelli *et al.* [8], Virbhadra and Ellis [9,10], and Bozza *et al.* [11] studied the gravitational lensing in the strong field with the Schwarzschild spacetime and found the infinite Einstein rings which are too close to each other to separately resolve. In this paper, we call these rings relativistic Einstein rings. The gravitational lensing in the strong field on the spherically symmetric static spacetime was investigated by Bozza [12], Hasse and Perlick [13], and Perlick [14]. They showed that the relativistic Einstein rings are formed not only in the Schwarzschild spacetime but also in the other spherically symmetric static spacetime.

General relativity permits nontrivial topology of the spacetime such as wormhole spacetimes (see Visser [15] for the details of wormholes). Some hypotheses of astrophysical wormholes have been investigated [16–18]. For example, Kardashev *et al.* suggest that some active galactic nuclei and other compact astrophysical objects may be

explained as wormholes [19]. We may test these hypotheses by using the gravitational lensing in the future.

Kim and Cho [20] and Cramer *et al.* [21] pioneered gravitational lensing effects by wormholes. Since then, the gravitational lensing effects by various wormholes have been investigated [21–27].

The Ellis spacetime which was investigated by Ellis [28] is an example of traversable wormholes of the Morris-Thorne class [29,30]. The deflection angle of light in the Ellis wormhole geometry was studied by Chetouani and Clément [31] and recently Nakajima and Asada [32]. The gravitational lensing on the Ellis geometry was studied by Dey and Sen [33], Abe [34], and Toki *et al.* [35] in the weak gravitational field and Perlick [14], Nandi *et al.* [22], and Tejeiro and Larranaga [36] in the strong gravitational field.

Perlick [14], Nandi *et al.* [22], and Tejeiro and Larranaga [36] pointed out that the qualitative features of the gravitational lensing in the Ellis spacetime are very similar to the ones in the Schwarzschild spacetime for their photon spheres and their asymptotic flatness.

In this paper, we will consider the Einstein ring and relativistic Einstein rings in the Ellis spacetime and the Schwarzschild spacetime, both of which are static and spherically symmetric ones. We ask whether we can distinguish the Einstein-ring systems on the Schwarzschild spacetime and on the Ellis spacetime. To answer this question, we focus on the relations between the Einstein ring and the relativistic Einstein rings.

This paper is organized as follows. In Sec. II we will review the deflection angle on the Ellis wormhole spacetime. In Sec. III we give the radii of the Einstein ring and relativistic Einstein rings in the Ellis spacetime. In Sec. IV

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we will compare the Einstein ring and the relativistic Einstein rings in the Ellis spacetime to the ones in the Schwarzschild spacetime. In Sec. V we summarize and discuss our result. In this paper we use the units in which $c = 1$.

II. ELLIS WORMHOLE SPACETIME AND DEFLECTION ANGLE

In this section, we review the deflection angle on the Ellis wormhole spacetime [31,32]. The line element in the Ellis wormhole solution is written in the following form:

$$ds^2 = -dt^2 + dr^2 + (r^2 + a^2)d\Omega^2, \quad (2.1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and a is a positive constant. Introducing $\rho^2 = r^2 + a^2$, we can rewrite this into

$$ds^2 = -dt^2 + \left(1 - \frac{a^2}{\rho^2}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2, \quad (2.2)$$

where $\rho = \pm a$ corresponds to the wormhole throat. The spacetime has the Killing vectors $t^\mu \partial_\mu = \partial_t$ and $\phi^\mu \partial_\mu = \partial_\phi$ for stationarity and axial symmetry.

We can concentrate ourselves on the equatorial plane because of spherical symmetry. Using the conservation of the energy $E \equiv -g_{\mu\nu} k^\mu t^\nu$ and angular momentum $L \equiv g_{\mu\nu} k^\mu \phi^\nu$ and $k^\mu k_\mu = 0$, where k^μ is the photon wave number, the photon trajectory is then given by

$$\frac{1}{\rho^4} \left(\frac{d\rho}{d\phi}\right)^2 = \frac{1}{b^2} \left(1 - \frac{a^2}{\rho^2}\right) \left(1 - \frac{b^2}{\rho^2}\right), \quad (2.3)$$

where $b \equiv L/E$ is the impact parameter of the photon.

We can see that the photon is scattered if $|b| > a$, while it reaches the throat if $|b| < a$. Since we are interested in the scattering problem, we assume $|b| > a$. Using $u = 1/\rho$, we find

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{b^2} (1 - a^2 u^2)(1 - b^2 u^2). \quad (2.4)$$

Putting

$$G(u) = a^2(a^{-2} - u^2)(b^{-2} - u^2), \quad (2.5)$$

the azimuthal angle ϕ can be given as a function of u by

$$\phi = \pm \int_u^{b^{-1}} \frac{du}{\sqrt{G(u)}}. \quad (2.6)$$

Here we have set $\phi(b^{-1}) = 0$. The deflection angle α is then calculated to give

$$\alpha = 2 \int_0^{b^{-1}} \frac{du}{\sqrt{G(u)}} - \pi. \quad (2.7)$$

In the present case, we find

$$\int_0^{b^{-1}} \frac{du}{\sqrt{G(u)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \left(\frac{a}{b}\right)^2 \sin^2 \theta}} = K\left(\frac{a}{b}\right), \quad (2.8)$$

where we have transformed $u = b^{-1} \sin\theta$ and $K(k)$ denotes the complete elliptic integral of the first kind (for example, see Ref. [37]), which is defined as

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}. \quad (2.9)$$

Hence, the deflection angle is given by

$$\alpha = 2K\left(\frac{a}{b}\right) - \pi. \quad (2.10)$$

Since $K(k)$ admits a power series

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!} \right]^2 k^{2n}, \quad (2.11)$$

where $n!!$ denotes the double factorial of n and $(-1)!! = 1$, we get the deflection angle

$$\alpha = \pi \sum_{n=1}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!} \right]^2 \left(\frac{a}{b}\right)^{2n}. \quad (2.12)$$

Thus, the deflection angle is approximately given in the weak-field regime $|b| \gg a$ by

$$\alpha \simeq \frac{\pi}{4} \left(\frac{a}{b}\right)^2. \quad (2.13)$$

In general, the deflection angle is always greater than its weak-field approximation and is diverging as $|b| \rightarrow a$.

III. EINSTEIN RING AND RELATIVISTIC EINSTEIN RINGS OF ELLIS WORMHOLE

In this section, we examine the diameter angles of the Einstein ring and the relativistic Einstein rings on the Ellis spacetime. Now we will consider the case that both the observer and the source object are far from the lensing object, or $D_l \gg b$ and $D_{ls} \gg b$, where D_l and D_{ls} are the separations between the observer and lens and between the lens and source, respectively. The configuration of the gravitational lensing is given in Fig. 1. Then, the lens equation is given by

$$D_{ls} \bar{\alpha} = D_s (\theta - \phi), \quad (3.1)$$

where $\bar{\alpha} = (\alpha \bmod 2\pi)$ is the effective deflection angle, θ and ϕ are the angles of the lensed image and the real image from the observer, respectively, and $D_s = D_l + D_{ls}$ is the separation between the observer and source. Note that we have assumed $|\bar{\alpha}| \ll 1$, $|\theta| \ll 1$, and $|\phi| \ll 1$. The deflection angle can be expressed $\alpha = \bar{\alpha} + 2\pi n$, where n is a non-negative integer, denoting the winding number of the light ray.

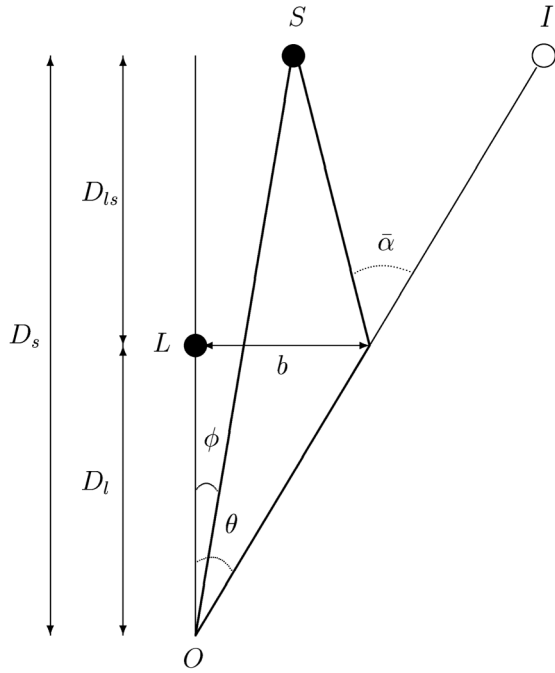


FIG. 1. The configuration of the gravitational lensing. The light rays emitted by the source S are deflected by the lens L (a wormhole or a black hole) and reach the observer O with the angle of the lensed image θ , instead of the real angle ϕ . b and $\bar{\alpha}$ are the impact parameter and the effective deflection angle, respectively. D_l and D_{ls} are the separations between the observer and the lens and between the lens and the source, respectively.

The ring image corresponds to the image angle θ for vanishing real angle $\phi = 0$. By the symmetry, the image is necessarily a ring with the diameter angle θ .

Since $b = D_l \theta$, we find that the ring image is given by

$$\theta_n = \frac{a}{D_l k_n}, \quad (3.2)$$

where $k_n \in (0, 1)$ is a unique root of the transcendental equation,

$$2K(k) - \frac{\eta}{k} = (2n + 1)\pi, \quad (3.3)$$

$$\eta = \frac{D_s}{D_l D_{ls}} a. \quad (3.4)$$

We should note that $2K(k) - \eta/k$ is monotonically increasing with respect to k and changes from $-\infty$ to ∞ as k increases from 0 to 1. The uniqueness of the root follows from the monotonicity. Moreover, we can conclude that k_n monotonically increases and approaches 1 as $n \rightarrow \infty$ and hence the image angle θ_n monotonically decreases and approaches a/D_l .

In the weak-field regime $|b| \gg a$, the winding number n should be $n = 0$. Using the deflection angle (2.13), we can solve the equation (3.3) approximately and get the diameter angle of the Einstein ring:

$$\begin{aligned} \theta_0 &\simeq \left(\frac{\pi}{4} \frac{D_{ls}}{D_s D_l^2} a^2 \right)^{\frac{1}{3}} \\ &\simeq 2.0 \text{ arcsecond} \left(\frac{D_{ls}}{10 \text{ Mpc}} \right)^{\frac{1}{3}} \left(\frac{20 \text{ Mpc}}{D_s} \right)^{\frac{1}{3}} \left(\frac{10 \text{ Mpc}}{D_l} \right)^{\frac{2}{3}} \left(\frac{a}{0.5 \text{ pc}} \right)^{\frac{2}{3}}. \end{aligned} \quad (3.5)$$

This approximation is good for $D_l \gg a$ and $D_{ls} \gg a$. The relative error is $\sim 10^{-2}$ for $a = 0.5 \text{ pc}$ and $D_l = D_{ls} = 10 \text{ Mpc}$.

In the especially strong-field regime, where the winding number n becomes $n \geq 1$, we can easily check that $a \simeq b$ or $k_n \simeq 1$ satisfies the transcendental equation (3.3) in numerical calculations. Physically this means that the light rays which wind around the wormhole nearly on the photon sphere make the relativistic Einstein rings [12,14]. Then the diameter angles of the relativistic Einstein rings are approximately given by

$$\theta_{n \geq 1} \simeq \frac{a}{D_l} \simeq 1.0 \times 10^{-2} \text{ arcsecond} \left(\frac{10 \text{ Mpc}}{D_l} \right) \left(\frac{a}{0.5 \text{ pc}} \right). \quad (3.6)$$

Regardless of the values of D_{ls} , D_l , and a , the relative error of the above approximation to the direct numerical solution of the outermost relativistic Einstein ring ($n = 1$) is $\sim 10^{-3}$ and those of the other relativistic Einstein rings ($n \geq 2$) are smaller than 10^{-5} . This implies that it is difficult to resolve each relativistic Einstein ring separately.

Thus, we conclude that there is one Einstein-ring image and countably infinite relativistic Einstein-ring images, the latter of which accumulate to form the apparently single ring image of the throat with the diameter a/D_l . This conclusion does not depend on the value of η .

If we are given the distance D_s to the source from the observer, the distance D_l to the lens from the observer, and the radius θ_0 of the Einstein ring, we can determine the radius of the throat a from Eq. (3.5). Then, we can use $\theta_{n \geq 1}$ (3.6) to test the assumption that the lens object is a wormhole.

From Eqs. (3.5) and (3.6), we obtain the relation between θ_0 and $\theta_{n \geq 1}$ by

$$\theta_{n \geq 1} \simeq \left(\frac{4}{\pi} \frac{D_s}{D_{ls}} \right)^{\frac{1}{3}} \theta_0^{\frac{3}{2}}. \quad (3.7)$$

This relation generally holds in astrophysical situations, as long as $a \ll D_l$ and $a \ll D_{ls}$ are satisfied. If the lens is identified with a wormhole, we can even estimate the radius of the wormhole throat in terms of D_s , θ_0 , and θ_n through

$$a \simeq \theta_{n \geq 1} \left(1 - \frac{4}{\pi} \theta_{n \geq 1}^{-2} \theta_0^3 \right) D_s. \quad (3.8)$$

This follows from Eqs. (3.6) and (3.7).

IV. COMPARISON BETWEEN WORMHOLES AND BLACK HOLES

In this section we compare the Einstein ring and the relativistic Einstein rings in the Ellis spacetime to the ones in the Schwarzschild spacetime and show that we can distinguish between black holes and wormholes.

Now, we will briefly review the deflection angle, Einstein ring, and relativistic Einstein rings for the Schwarzschild spacetime [9–11, 13, 38, 39] and present the relation between θ_0 and $\theta_{n \geq 1}$.

In the weak-field regime $b \gg r_g$, where $r_g = 2GM$ is the Schwarzschild radius of the black hole of mass M , the deflection angle is approximately given by

$$\alpha \simeq \frac{2r_g}{b}. \quad (4.1)$$

The diameter angle of the Einstein ring is given by

$$\begin{aligned} \theta_0 &\simeq \sqrt{\frac{2D_{ls}}{D_l D_s}} r_g \\ &\simeq 2.0 \text{ arcsecond} \left(\frac{D_{ls}}{10 \text{ Mpc}} \right)^{\frac{1}{2}} \left(\frac{M}{10^{10} M_\odot} \right)^{\frac{1}{2}} \\ &\quad \times \left(\frac{10 \text{ Mpc}}{D_l} \right)^{\frac{1}{2}} \left(\frac{20 \text{ Mpc}}{D_s} \right)^{\frac{1}{2}}. \end{aligned} \quad (4.2)$$

We can determine r_g in the same way as the radius of the throat a .

In the especially strong-field regime, where the winding number n becomes $n \geq 1$, the impact parameter b that satisfies the lens equation should be nearly the critical impact parameter $b \simeq (3\sqrt{3}/2)r_g$ (see Refs. [9, 11, 12]). Then the diameter angles of the inseparable relativistic Einstein rings $\theta_{n \geq 1}$ are given by

$$\begin{aligned} \theta_{n \geq 1} &\simeq \frac{3\sqrt{3}}{2} \frac{r_g}{D_l} \\ &\simeq 5.1 \times 10^{-5} \text{ arcsecond} \left(\frac{M}{10^{10} M_\odot} \right) \left(\frac{10 \text{ Mpc}}{D_l} \right). \end{aligned} \quad (4.3)$$

It is useful to remember that the leading term of the deflection angle in the weak-field regime is the second order of the small amount a/b on the Ellis geometry (2.13), while it is the first order of the small amount r_g/b on the Schwarzschild geometry (4.1). So the relation between θ_0 and $\theta_{n \geq 1}$ for the Schwarzschild spacetime,

$$\theta_{n \geq 1} \simeq \frac{3\sqrt{3}}{4} \frac{D_s}{D_{ls}} \theta_0^2, \quad (4.4)$$

is different from that on the Ellis spacetime (3.7). Figure 2 shows the angle of the relativistic Einstein ring $\theta_{n \geq 1}$ versus the angle of the Einstein ring θ_0 for $D_l = D_{ls} = 10$ Mpc. Thus, we can distinguish between black holes and wormholes in principle if we are given D_s , D_l , θ_0 , and $\theta_{n \geq 1}$.

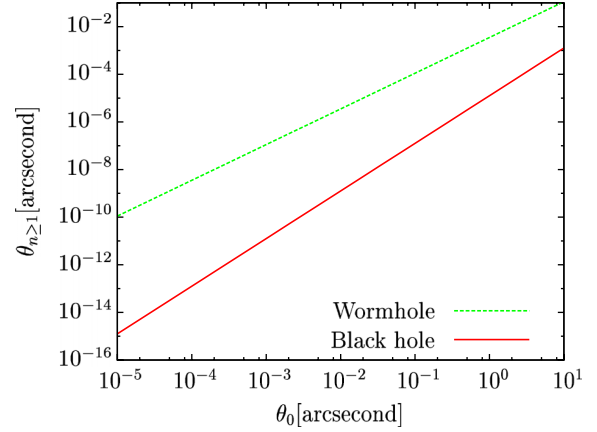


FIG. 2 (color online). The angle of the relativistic Einstein ring $\theta_{n \geq 1}$ versus the angle of the Einstein ring θ_0 for $D_l = D_{ls} = 10$ Mpc. The broken (green) and solid (red) lines plot the cases where the lens objects are a wormhole and a black hole, respectively.

If the lens is identified with a black hole, we can estimate the Schwarzschild radius and, hence, the black hole mass by

$$r_g \simeq \frac{2}{3\sqrt{3}} \theta_{n \geq 1} \left(1 - \frac{3\sqrt{3}}{4} \theta_0^2 \theta_{n \geq 1}^{-1} \right) D_s \quad (4.5)$$

in terms of θ_0 , $\theta_{n \geq 1}$, and D_s . This follows from Eqs. (4.3) and (4.4).

V. DISCUSSION AND CONCLUSION

It is well known that the qualitative features of the gravitational lensing on the Ellis spacetime are very similar to the ones on the Schwarzschild spacetime for their photon spheres and their asymptotic flatness [14, 22, 36]. However, we realize that their quantitative features are very different due to their different weak-field behaviors.

We consider the experimental situation where we know the separation D_s between the observer and the source and the separation D_l between the observer and the lens. We assume that we do not know whether the lens object is a black hole or a wormhole and do not know its parameter, i.e., the mass M or the radius a of the throat in advance.

We need at least two observable quantities to determine whether the lens object is a black hole or wormhole since the lens system has one parameter in this situation. First, we observe an Einstein ring and determine the parameter for both possibilities. Second, we observe relativistic Einstein rings and tell the wormhole from the black hole. If the predicted relativistic ring angles by the black hole and by the wormhole were of similar size, we could not discern the difference. However, Eqs. (3.7) and (4.4) and Fig. 2 show that we do not confuse them.

We conclude that we can detect the relativistic Einstein rings by wormholes which have $a \simeq 0.5$ pc at a Galactic center with the distance $D_l = D_{ls} = 10$ Mpc and which

have $a \simeq 10$ AU in our Galaxy with the distance $D_l = D_{ls} = 10$ kpc using the most powerful modern instruments which have the resolution of 10^{-2} arcsecond such as a 10-meter optical-infrared telescope. Note that the corresponding black holes which have the Einstein rings of the same size are galactic supermassive black holes with $10^{10}M_\odot$ and 10^7M_\odot , respectively, and that the relativistic Einstein rings by these black holes are too small to measure with the current technology.

In fact, our results imply that we can distinguish between slowly rotating Kerr-Newmann black holes and the Ellis wormholes with their Einstein-ring systems. This is because the leading term of the deflection angle for the lensing by the Kerr-Newmann black holes in the weak-field regime is equal to the one for the lensing by the Schwarzschild black holes, while the black hole charge and small spin only slightly change the radii of the relativistic Einstein rings [12,40–43]. Moreover, this also suggests that it is much more challenging to determine the charge and/or small spin of black holes than to distinguish between black holes and the Ellis wormholes.

We assumed that the observer, the lensing object, and the source object are directly aligned, though such a configuration is fairly rare. In general the strong gravitational lensing effect is observed as broken-ring images which are called relativistic images [9]. Therefore, a more realistic problem is to size the relativistic images. Our result suggests that we can distinguish black holes and wormholes by using the relativistic images. To observe the relativistic images is one of the challenging works with many difficulties. Bozza *et al.* pointed out that relativistic images are always very faint with respect to the weak field images [11]. The very large telescope interferometer has high resolution [44,45] but it will not work because of this demagnifying effect.

We also assumed pointlike sources, although astrophysical sources have their own size. If the source object is a galaxy, it may conceal the relativistic Einstein rings, especially in the case that the lens object is a black hole. Testing some hypotheses of astrophysical wormholes by using the relativistic Einstein rings and the Einstein ring is left as future work.

Tejeiro and Larranaga [36] investigated the gravitational lensing effect of the wormhole solution obtained by connecting the Ellis solution as an interior region and the Schwarzschild solution as an exterior region [46]. They concluded that we cannot distinguish the Schwarzschild black hole and the wormhole unless the discontinuity of the magnification curve at the boundary is observed. This does not contradict our results because their wormhole solution behaves as the Schwarzschild solution in the weak-field regime.

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