Anisotropic universe models in $f(T)$ gravity

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We investigate the cosmological reconstruction in an anisotropic universe for both the homogeneous and inhomogeneous content of the universe. Special attention is given to three interesting cases: Bianchi type I, Bianchi type III, and Kantowski-Sachs models. The de Sitter, power-law, and general exponential solutions are assumed for the scale factor in each spatial direction and the corresponding cosmological models are reconstructed. Moreover, for the general exponential solutions—from which the de Sitter and power-law solutions may be obtained—we obtain models which reproduce the early Universe (assuming inflation) and the late-time accelerated expanding Universe. The models obtained for the late-time Universe are consistent with a known result in the literature where a power-law type correction in T is added to a power-law type of $f(T)$ for guaranteeing the avoidance of the big rip and the big freeze.

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I. INTRODUCTION

The probable presence of an unknown form of energy in the Universe, called dark energy, and the evidence supplied by a large number of observations (including the initial data from Type IA supernovae in 1998 [1]) has lead to explorations of the possible theoretical origin of this fluid. Since it is directly responsible for the present accelerating expansion of the Universe, a negative pressure is required, which leads to a negative equation of state parameter. The most popular candidate, the cosmological constant [which possesses a constant equation of state (EOS), $p_\Lambda = -\rho_\Lambda$], can explain the cosmological evolution quite well. However, the open possibility that the EOS is not completely constant but rather evolves dynamically (even crossing the phantom barrier further), and the quite large difference between the observed dark energy density and the vacuum energy density predicted by quantum field theories, has lead to the exploration of other possibilities, such as the existence of scalar fields, vector fields, or modifications of general relativity (GR) (for a review on dark energy candidates, see Ref. [2]).

In the context of modified gravities a wide range of possibilities has been explored, the most popular (due to its simplicity) being $f(R)$ gravity, as it generalizes the Hilbert-Einstein action to a more complex function of the Ricci scalar (for a review of $f(R)$ gravity, see Refs. [3,4]). Nevertheless, other kinds of theories have been suggested which include other curvature invariants, such as the Gauss-Bonnet gravity. In this paper, we study the so-called $f(T)$ gravity, which [analogous to $f(R)$ gravity] consists in a generalization of the action of teleparallel gravity theory (TT)—a theory that assumes the Weitzenböck connection instead of the Levi-Civita connection—and which yields a null curvature and a nonvanishing torsion (for a review, see Ref. [5]). In this gravitational theory, the main field is represented by the so-called tetrads, instead of the metric as in GR. This kind of theory has recently become very popular as it can explain the accelerated expansion of the Universe, and even the inflationary epoch, with no need for dark energy (see Refs. [6–13]). A wide number of aspects have been studied in the context of $f(T)$ gravity, such as its local Lorentz invariance [14], static solutions [15], nondiagonal tetrads [16], and the presence of wormholes [17], as well as other aspects [18,19]. A large effort has also been taken to study cosmological solutions for this class of theories, as well as possible cosmological predictions (see Refs. [20–23]).

In the present work, we are interested in studying some particular cosmological solutions in $f(T)$ gravity, where the appropriate action is reconstructed for each case. Specifically, the Bianchi type I, Kantowski-Sachs (KS), and Bianchi type III models are considered, and in particular some important solutions—such as power-law and de Sitter expansion—and more complex ones—such as exponential functions for the scale factor in each direction of the space—are also studied. Since power-law and de Sitter

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solutions can provide a good description for some specific phases of the Universe's evolution, their reconstruction in $f(T)$ gravity becomes a crucial point in order to consider this class of theories as serious candidates for explaining the whole cosmological history. In addition, here we assume more general cosmological metrics than Friedmann-Lemaître-Robertson-Walker (FLRW) metrics—in particular, anisotropic universes described by the Bianchi type I, KS, and Bianchi type III metrics—in order to provide the most general description of the cosmological evolution in the context of $f(T)$ gravity. Moreover, exponential solutions are also considered: these kinds of expansions have become very popular recently as they may evolve the universe to a nonsingular state where some bounded systems may be broken. Such a state—called a "little rip" (suggested in Ref. [24])—has already been studied in $f(R)$ gravity (see Ref. [25]), as well as in $f(T)$ theories [26]. Moreover, the possible occurrence of a little rip has been also explored in the context of the so-called viable modified gravities (see Ref. [27]). Note that anisotropic cosmological metrics have already been studied in the context of GR with the presence of isotropic and anisotropic fluids, as well as the stability of the solutions [28,29].

Furthermore, the use of an auxiliary scalar field—in analogy to the equivalence of Brans-Dicke theories for $f(R)$ gravity (see, for instance, Ref. [30])—is also implemented, from which may result a useful tool to reconstruct the appropriate action as well as for studying the properties of $f(T)$ gravity.

The main motivations of the assumption of a model with anisotropic geometry are based on several physical aspects: the famous problem of the CMB quadrupole can be solved by considering a universe with planar symmetry [31] where eccentricity in decoupling is generated by a uniform cosmic magnetic field whose current strength, $B(t_0) \sim 10^{-9}$ Gauss, should be close to $e_{\rm dec} \sim 10^{-2}$; the Bianchi type models in loop quantum cosmology [32]; ^{4}He abundance [33]; cosmic parallax [34,35]; small anisotropic pressures [36]; cosmological solutions of the low-energy string effective action [37]; an anisotropic inflationary universe [38]; and others [39]. In $f(R)$ theory there are already some good results [40,41]; therefore, we propose to establish the equations here and achieve the first results in $f(T)$ gravity for the Bianchi type I, type III, and KS models.

The paper is organized as follows. In Sec. II, the basic concepts of $f(T)$ gravity are introduced. In Sec. III, the equations for general Bianchi type I, type III, and KS models are deduced in a particular coordinate system and with diagonal tetrads. Section IV deals with the reconstruction of the $f(T)$ action for some relevant solutions, and several techniques are considered, including a kind of scalar-tensor theory for torsion gravity. Finally, Sec. V is devoted to a conclusion and discussion about the results found in the paper.

II. PRELIMINARY DEFINITIONS AND EQUATIONS OF MOTION

As previously mentioned, the $f(T)$ theory of gravity is defined in the Weitzenböck spacetime, in which the line element is described by

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1}$$

where $g_{\mu\nu}$ are the components of the metric, which is symmetric and possesses ten degrees of freedom. One can describe the theory in the spacetime or in the tangent space, which allows us to rewrite the line element (1) as follows:

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ij}\theta^i\theta^j, \tag{2}$$

$$dx^\mu = e_i{}^\mu \theta^i, \qquad \theta^i = e^i{}_\mu dx^\mu, \tag{3}$$

where $\eta_{ij} = \mathrm{diag}[1, -1, -1, -1]$ and $e_i{}^\mu e^i{}_\nu = \delta^\mu_\nu$ or $e_i{}^\mu e^j{}_\mu = \delta^j_i$. The square root of the metric determinant is given by $\sqrt{-g} = \det[e^i{}_\mu] = e$ and the matrix $e^a{}_\mu$ are called tetrads and represent the dynamic fields of the theory.

By using theses fields, one can define the Weitzenböck connection as

$$\Gamma^\alpha_{\mu\nu} = e_i{}^\alpha \partial_\nu e^i{}_\mu = -e^i{}_\mu \partial_\nu e_i{}^\alpha. \tag{4}$$

The main geometrical objects of the spacetime are constructed from this connection. The components of the tensor torsion are defined by the antisymmetric part of this connection,

$$T^\alpha{}_{\mu\nu} = \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu} = e_i{}^\alpha(\partial_\mu e^i{}_\nu - \partial_\nu e^i{}_\mu). \tag{5}$$

The components of the contorsion are defined as

$$K^{\mu\nu}{}_\alpha = -\frac{1}{2}(T^{\mu\nu}{}_\alpha - T^{\nu\mu}{}_\alpha - T_\alpha{}^{\mu\nu}). \tag{6}$$

In order to make more clear the definition of the scalar equivalent to the curvature scalar of GR, we first define a new tensor $S_\alpha{}^{\mu\nu}$, constructed from the components of the tensor's torsion and contorsion as

$$S_\alpha{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_\alpha + \delta^\mu_\alpha T^{\beta\nu}{}_\beta - \delta^\nu_\alpha T^{\beta\mu}{}_\beta). \tag{7}$$

We can now define the torsion scalar by the following contraction:

$$T = T^\alpha{}_{\mu\nu} S_\alpha{}^{\mu\nu}. \tag{8}$$

The action of the theory is defined by generalizing the teleparallel theory as

$$S = \int e[f(T) + \mathcal{L}_{\rm Matter}] d^4x, \tag{9}$$

where $f(T)$ is an algebraic function of the torsion scalar T. Making the functional variation of the action (9) with respect to the tetrads, we get the following equations of motion [14,15,21]:

$$S_\mu{}^{\nu\rho}\partial_\rho T f_{TT} + [e^{-1}e^i_\mu \partial_\rho(e e_i{}^\alpha S_\alpha{}^{\nu\rho}) + T^\alpha{}_{\lambda\mu}S_\alpha{}^{\nu\lambda}]f_T + \frac{1}{4}\delta^\nu_\mu f = 4\pi \mathcal{T}^\nu_\mu, \tag{10}$$

where $\mathcal{T}^\nu_\mu$ is the energy-momentum tensor, $f_T = df(T)/dT$ and $f_{TT} = d^2f(T)/dT^2$. By setting $f(T) = a_1T + a_0$, the equations of motion (10) are the same as that of the teleparallel theory with a cosmological constant, and this is dynamically equivalent to the GR. These equations clearly depend on the choice of the set of tetrads [19].

The contribution of the interaction with the matter fields is given by the energy-momentum tensor which, in this case, is defined as

$$\mathcal{T}_\mu{}^\nu = \mathrm{diag}(1, -\omega_x, -\omega_y, -\omega_z)\rho, \tag{11}$$

where the ω_i ($i = x, y, z$) are the parameters of equations of state related to the pressures p_x, p_y, and p_z.

III. FIELD EQUATIONS FOR BIANCHI TYPE I, TYPE III, AND KANTOWSKI-SACHS MODELS

Let us first establish the equations of motion of a set of diagonal tetrads using the Cartesian coordinate metric for describing Bianchi type I, type III, and KS models. We propose starting with the Bianchi type III case, from which the Bianchi type I and KS cases can be recovered. For the Bianchi type III case, the metric reads

$$dS^2 = dt^2 - A^2(t)dx^2 - e^{-2\alpha x}B^2(t)dy^2 - C^2(t)dz^2, \tag{12}$$

where α is a constant parameter. Note that the Bianchi type I case is recovered by setting $\alpha = 0$ in the Bianchi type III case, while the KS case is recovered when one takes $\alpha = 0$ and $B(t) = C(t)$. Let us choose the following set of diagonal tetrads related to the metric (12):

$$[e^a{}_\mu] = \mathrm{diag}[1, A, e^{-\alpha x}B, C]. \tag{13}$$

The determinant of the matrix (13) is $e = e^{-\alpha x}ABC$. The components of the tensor torsion (5) for the tetrads (13) are given by

$$T^1{}_{01} = \frac{\dot A}{A}, \qquad T^2{}_{02} = \frac{\dot B}{B}, \qquad T^2{}_{21} = \alpha, \qquad T^3{}_{03} = \frac{\dot C}{C}, \tag{14}$$

and the components of the corresponding tensor contorsion are

$$K^{01}{}_1 = \frac{\dot A}{A}, \qquad K^{02}{}_2 = \frac{\dot B}{B}, \qquad K^{12}{}_2 = \frac{\alpha}{A^2}, \qquad K^{03}{}_3 = \frac{\dot C}{C}. \tag{15}$$

The components of the tensor $S_\alpha{}^{\mu\nu}$ in Eq. (7) are given by

$$S_0{}^{01} = S_3{}^{31} = \frac{\alpha}{2A^2}, \qquad S_1{}^{10} = \frac{1}{2}\left(\frac{\dot B}{B} + \frac{\dot C}{C}\right), \qquad S_2{}^{20} = \frac{1}{2}\left(\frac{\dot A}{A} + \frac{\dot C}{C}\right), \qquad S_3{}^{30} = \frac{1}{2}\left(\frac{\dot A}{A} + \frac{\dot B}{B}\right). \tag{16}$$

By using the components (14) and (16), the torsion scalar (8) is given by

$$T = -2\left(\frac{\dot A\dot B}{AB} + \frac{\dot A\dot C}{AC} + \frac{\dot B\dot C}{BC}\right). \tag{17}$$

The equations of motion are given by

$$16\pi\rho = f + 4f_T\left[\frac{\dot A\dot B}{AB} + \frac{\dot A\dot C}{AC} + \frac{\dot B\dot C}{BC} - \frac{\alpha^2}{2A^2}\right], \tag{18}$$

$$-16\pi p_x = f + 2f_T\left[\frac{\ddot B}{B} + \frac{\ddot C}{C} + \frac{\dot A\dot B}{AB} + \frac{\dot A\dot C}{AC} + 2\frac{\dot B\dot C}{BC}\right] + 2\left(\frac{\dot B}{B} + \frac{\dot C}{C}\right)\dot T f_{TT}, \tag{19}$$

$$-16\pi p_y = f + 2f_T\left[\frac{\ddot A}{A} + \frac{\ddot C}{C} + \frac{\dot A\dot B}{AB} + 2\frac{\dot A\dot C}{AC} + \frac{\dot B\dot C}{BC}\right] + 2\left(\frac{\dot A}{A} + \frac{\dot C}{C}\right)\dot T f_{TT}, \tag{20}$$

$$-16\pi p_z = f + 2f_T\left[\frac{\ddot A}{A} + \frac{\ddot B}{B} + 2\frac{\dot A\dot B}{AB} + \frac{\dot A\dot C}{AC} + \frac{\dot B\dot C}{BC} - \frac{\alpha^2}{A^2}\right] + 2\left(\frac{\dot A}{A} + \frac{\dot B}{B}\right)\dot T f_{TT}, \tag{21}$$

$$\frac{\alpha}{2A^2}\left[\left(\frac{\dot A}{A} - \frac{\dot B}{B}\right)f_T - \dot T f_{TT}\right] = 0, \tag{22}$$

$$\alpha\left(\frac{\dot A}{A} - \frac{\dot B}{B}\right)f_T = 0. \tag{23}$$

In the particular case where $f(T) = T - 2\Lambda$, the Eqs. (18)–(23) are identical to that of GR [42]. The equation of constraint (23) appears in both GR and in $f(R)$ gravity [41], but here we have a second equation of constraint (22), which appears as a generalization of the previous one, because here we have a contribution of a term of the second derivative of the function $f(T)$ with respect to T.

By setting $\alpha = 0$, the Bianchi type I case is recovered and the equations of motion read

$$16\pi\rho = f + 4f_T\left[\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right], \quad (24)$$

$$-16\pi p_x = f + 2f_T\left[\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + 2\frac{\dot{B}\dot{C}}{BC}\right] + 2\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\dot{T}f_{TT}, \quad (25)$$

$$-16\pi p_y = f + 2f_T\left[\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right] + 2\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\dot{T}f_{TT}, \quad (26)$$

$$-16\pi p_z = f + 2f_T\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + 2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right] + 2\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{T}f_{TT}. \quad (27)$$

The equations of motion corresponding to the KS model are obtained by setting $\alpha = 0$ and $B = C$, yielding

$$16\pi\rho = f + 4f_T\left[\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB}\right], \quad (28)$$

$$-16\pi p_x = f + 4f_T\left[\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 + \frac{\dot{A}\dot{B}}{AB}\right] + 4\frac{\dot{B}}{B}\dot{T}f_{TT}, \quad (29)$$

$$-16\pi p_y = f + 2f_T\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 + 3\frac{\dot{A}\dot{B}}{AB}\right] + 2\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{T}f_{TT}, \quad (30)$$

$$p_y = p_z.$$

In the next section we will perform the reconstruction scheme of the action of the system for some particular cases.

IV. RECONSTRUCTING $f(T)$ GRAVITY IN INHOMOGENEOUS UNIVERSES

Let us now consider the reconstruction of the $f(T)$ action for some particular solutions of the class of metrics explored in the previous section. Specifically, we consider de Sitter, power-law, and general exponential solutions. Note that de Sitter and power-law solutions have been widely explored in other contexts of modified gravity, such as $f(R)$ and Gauss-Bonnet gravities (see Ref. [43]), since they can provide a good description of the cosmological evolution along its particular phases.

We start by considering for simplicity Bianchi type I and Kantowski-Sachs ($\alpha = 0$) metrics. Then, the conservation equation for the energy-momentum tensor (11) can be easily obtained:

$$\dot{\rho} + (H_x + H_y + H_z)\rho + H_x p_x + H_y p_y + H_z p_z = 0, \quad (31)$$

where we have defined $H_x = \frac{\dot{A}}{A}$ $H_y = \frac{\dot{B}}{B}$ $H_z = \frac{\dot{C}}{C}$. We can now analyze de Sitter and power-law solutions and exponential expansion in the Bianchi type I metric on one side, and in the Kantowski-Sachs metric on the other, where $B = C$, which implies that $p_y = p_z$.

A. de Sitter solutions

de Sitter solutions are well known in the context of cosmology since the current epoch, where the Universe's expansion is being accelerated, can be described approximately with a de Sitter solution. These kinds of solutions consist of an exponential expansion of the scale factor, which yields a constant Hubble parameter. In the case of the Bianchi type I and Kantowski-Sachs metrics ($\alpha = 0$) in Eq. (12), we may assume an exponential expansion for each spatial direction,

$$A = A_0 e^{at} \qquad B = B_0 e^{bt} \qquad C = C_0 e^{ct}, \quad (32)$$

and the rates of the expansion for each direction can be defined as,

$$H_x = \frac{\dot{A}}{A} = H_{x0}, \qquad H_y = \frac{\dot{B}}{B} = H_{y0}, \qquad H_c = \frac{\dot{C}}{C} = H_{z0}, \quad (33)$$

where $\{H_{x0} = a, H_{y0} = b, H_{z0} = c\}$ are constants. The torsion scalar defined in Eq. (17) is given by

$$T_0 = -2(H_{x0}H_{y0} + H_{x0}H_{z0} + H_{y0}H_{z0}). \quad (34)$$

Then, by assuming $p_x = p_y = p_z = p$ and an equation of state $p = w\rho$, the conservation equation (31) can be easily solved for the ansatz (32):

$$\rho = \rho_0 e^{-(H_{x0}+H_{y0}+H_{z0})(1+w)t}. \quad (35)$$

Hence the field equations (24)–(27) become

$$16\pi\rho_0 e^{-(H_{x0}+H_{y0}+H_{z0})(1+w)t} = f(T_0) + 4[H_{x0}H_{y0} + H_{z0}(H_{x0} + H_{y0})]f_T(T_0), \quad (36)$$

$$-16\pi w\rho_0 e^{-(H_{x0}+H_{y0}+H_{z0})(1+w)t} = f(T_0) + 2(H_{y0} + H_{z0})(H_{x0} + H_{y0} + H_{z0})f_T(T_0), \quad (37)$$

$$-16\pi w\rho_0 e^{-(H_{x0}+H_{y0}+H_{z0})(1+w)t} = f(T_0) + 2(H_{x0} + H_{z0})(H_{x0} + H_{y0} + H_{z0})f_T(T_0), \quad (38)$$

$$-16\pi w\rho_0 e^{-(H_{x0}+H_{y0}+H_{z0})(1+w)t} = f(T_0) + 2(H_{x0}+H_{y0})(H_{x0}+H_{y0}+H_{z0})f_T(T_0). \tag{39}$$

Note that the only possible solution in the presence of a perfect fluid is one with $w = -1$, as the rhs of Eqs. (36)–(39) is independent of time—according to the expression of the scalar torsion for a pure de Sitter solution (34)—unless $H_{x0} + H_{y0} + H_{z0} = 0$, which would imply a decelerating expansion in a particular direction, namely $H_{i0} < 0$. Moreover, for a particular $f(T)$ action, the system of equations (34)–(39) reduces to an algebraic system of equations for the variables $\{H_{x0}, H_{y0}, H_{z0}\}$. Since the system of equations (34)–(39) is composed of four equations while there are only three variables, the above four-equation system has to be reduced. However, even in the case of the Kantowski-Sachs metric, where $B(t) = C(t) \to H_{y0} = H_{z0}$, the system (34)–(39) still possesses three independent equations with two variables. Hence, the only possible solution imposes the condition that

$$A(t) = B(t) = C(t) \to H_{x0} = H_{y0} = H_{z0} = H_0, \tag{40}$$

and the metric (12) reduces to the well known Friedmann-Lemaître-Robertson-Walker (FLRW) metric with an exponential expansion, $A(t) = A_0 e^{H_0 t}$. Hence, the only solution for a pure de Sitter expansion in Bianchi type I and Kantowski-Sachs metrics gives an FLRW universe [44], and the system of equations (34)–(39) now reduces to a unique independent equation,

$$16\pi\rho_0 = f(T_0) + 12H_0^2 f_T(T_0). \tag{41}$$

The roots of the algebraic equation (41) then give the de Sitter points of a particular $f(T)$ action. In order to illustrate such possibility, let us consider the action

$$f(T) = (-T)^n, \tag{42}$$

where n is a real constant. Then Eq. (41) is rewritten as

$$16\pi\rho_0 = (1-2n)(6H_0^2)^n, \tag{43}$$

whose solution is given by

$$H_0^2 = \frac{1}{6}\left(\frac{16\pi\rho_0}{1-2n}\right)^{1/n}. \tag{44}$$

Hence the only physical solution (ρ_0, $H_0^2 \geq 0$) imposes $n \leq 1/2$. The de Sitter solution is then a direct consequence of the energy density ρ_0, which can be interpreted as a cosmological constant according to the condition imposed above for its equation of state, $w = -1$. Nevertheless, in vacuum Eq. (43) reduces to $0 = (1-2n)(6H_{x0}^2)^n$, whose only solution is given by $n = 1/2$, resulting in the action $f(T) = \sqrt{-T}$, which posses an infinite number of de Sitter points. Moreover, we may consider in vacuum the action

$$f(T) = C_1 T + C_2(-T)^n, \tag{45}$$

where $\{C_1, C_2\}$ are the coupling constants and n is a real constant. The field equation (41) in vacuum yields

$$0 = C_1 6H_0^2 + C_2(1-2n)(6H_0^2)^n. \tag{46}$$

So the roots of this equation give the de Sitter points allowed by the class of theories expressed in Eq. (45). Note that now the exponential expansion is a direct consequence of the action instead of the contribution of a kind of cosmological constant, as in the case shown above. For instance, $n = 2$ yields the solution

$$H_0 = \sqrt{\frac{C_1}{18C_2}}, \tag{47}$$

while for higher powers of n, more de Sitter points can be obtained for the action (45). Note that in $f(R)$ theories, de Sitter points constitute the critical points of the dynamical system, which may be (un)stable, and could explain both the inflationary and dark energy epochs (see Ref. [45]), which may also be the case in $f(T)$ gravity.

B. Power-law solutions

Now let us explore a cosmological evolution described by a power law in each direction of the space expansion. In such a case, the scale parameters for the Bianchi type I and Kantowski-Sachs metric (12), where we set ($\alpha = 0$), can be expressed as

$$A(t) = A_0 t^a, \qquad B(t) = B_0 t^b, \qquad C(t) = C_0 t^c, \tag{48}$$

where $\{a, b, c\}$ and $\{A_0, B_0, C_0\}$ are constants to be determined by the field equations and initial conditions, respectively. The expansion rates are given by

$$H_x = \frac{a}{t}, \qquad H_y = \frac{b}{t}, \qquad H_z = \frac{c}{t}, \tag{49}$$

while the expression for the torsion scalar (17) yields

$$T = -2\left(\frac{ab}{t^2} + \frac{ac}{t^2} + \frac{bc}{t^2}\right). \tag{50}$$

Then, introducing the above quantities into the field equations (24)–(27), we get the following system of differential equations in $f(T)$:

$$16\pi\rho(T) = f(T) - 2Tf_T(T), \tag{51}$$

$$-16\pi w\rho(T) = f(T) + \frac{(b+c)(1-a-b-c)}{bc + a(b+c)}Tf_T(T) + 2\frac{(b+c)}{bc+a(b+c)}T^2 f_{TT}(T), \tag{52}$$

$$-16\pi w\rho(T) = f(T) + \frac{(a+c)(1-a-b-c)}{bc+a(b+c)}Tf_T(T) + 2\frac{(a+c)}{bc+a(b+c)}T^2 f_{TT}(T), \quad (53)$$

$$-16\pi w\rho(T) = f(T) + \frac{(a+b)(1-a-b-c)}{bc+a(b+c)}Tf_T(T) + 2\frac{(a+b)}{bc+a(b+c)}T^2 f_{TT}(T), \quad (54)$$

where we have assumed for simplicity that $p_x = p_y = p_z = p$ and an EOS $p = w\rho$. Hence, the system (51)–(54) is a set of differential equations in $f(T)$ with the torsion scalar T as the independent variable.

Firstly, let us consider the vacuum case, or in other words, the homogeneous part of the first Eq. (51), which becomes $f(T) - 2Tf_T(T) = 0$ and whose solution is given by

$$f(T) = C_1\sqrt{-T}, \quad (55)$$

where C_1 is an integration constant. In order to satisfy the remaining Eqs. (52)–(54), the condition $a = b = c$ must be imposed so that the Hubble parameters (49) reduce to the usual FLRW-cosmology-reproducing power-law solution.

In the presence of a kind of isotropic perfect fluid $p = w\rho$, we can first solve the continuity equation (31) in order to obtain $\rho = \rho(T)$, which yields

$$\rho = \rho_0 t^{-(a+b+c)(1+w)} = \rho_0\left(-\frac{T}{2(ab+ac+bc)}\right)^{\frac{(a+b+c)(1+w)}{2}}. \quad (56)$$

Hence, the solution for the set of equations (51)–(54) is given by $f(T) = f_h(T) + f_p(T)$, where $f_h(T)$ is the solution of the homogeneous equation, which coincides with the vacuum solution (55), and $f_p(T)$ is the particular solution. Then, by using Eq. (56) in Eq. (51), the particular solution can be easily found:

$$f_p(T) = \chi T^{\frac{(1+w)(a+b+c)}{2}}, \quad (57)$$

where χ is a constant given by

$$\chi = \frac{2^{4-(1+w)(a+b+c)/2}\pi\rho_0}{[-1+(1+w)(a+b+c)][-bc-a(b+c)]^{\frac{(1+w)(a+b+c)}{2}}}. \quad (58)$$

Note that the condition $(1+w)(a+b+c) = 2n$, with n being any natural number, has to be imposed in order to avoid a complex gravitational action that would lack any physical sense [recall that $T < 0$ for an expanding universe according to Eq. (50)]. In order to satisfy the complete set of equations (51)–(54), we introduce the solution (57) into the field equations (52)–(54), and the following solutions for the parameters $\{a, b, c\}$ are found:

(i) $c = \frac{1-w(a+b)}{w}$, where $w \neq 0$. This provides an anisotropic solution in $f(T)$ gravity with $A(t)$, $B(t)$, and $C(t)$ being different functions in Eq. (48), and recalling that the perfect fluid assumed is an isotropic fluid. This does not hold in GR or teleparallel theory, but it is possible here due to the presence of second derivatives of the function $f(T)$ with respect to the torsion scalar T in Eqs. (51)–(54). Note that field equations may be rewritten as the usual equations in teleparallel theory:

$$\begin{aligned} &H_xH_y + H_xH_z + H_yH_z = 16\pi(\rho + \rho_{f(T)}),\\ &-\dot{H}_y - H_y^2 - \dot{H}_z - H_z^2 - H_yH_z = 8\pi(w\rho + p^x_{f(T)}),\\ &-\dot{H}_x - H_x^2 - \dot{H}_z - H_z^2 - H_xH_z = 8\pi(w\rho + p^y_{f(T)}),\\ &-\dot{H}_x - H_x^2 - \dot{H}_y - H_y^2 - H_xH_y = 8\pi(w\rho + p^z_{f(T)}). \end{aligned} \quad (59)$$

Here, the extra terms coming from $f(T)$ are defined as an energy density $\rho_{f(T)}$ and pressures $\{p^x_{f(T)}, p^y_{f(T)}, p^z_{f(T)}\}$, which are the origin of the anisotropic evolution. In this case, we have to fix $C_1 = 0$ in Eq. (55) in order to satisfy the whole system.

(ii) $a = b = c$. The cosmological evolution expressed by Eq. (48) reduces to an FLRW metric, as in the homogeneous part of the equations, so that $C_1 \neq 0$.

(iii) $c = -ab/(a+b)$. Despite that fact that this satisfies Eqs. (51)–(54), once Eq. (57) is substituted into the equations, it gives $T = 0$, and the rhs of Eqs. (51)–(54) becomes null while the lhs is not [since $\rho = \rho(t)$, as given in Eq. (56)], so this is not a real solution.

Therefore, we have obtained a complete set of power-law solutions for the Bianchi type I and Kantowski-Sachs metrics in the context of $f(T)$ gravity. Nevertheless, the action is clearly dependent on the EOS parameter w. Note also that in vacuum the only possible solution reduces to an FLRW metric.

C. General exponential solutions

In this subsection we consider a more general exponential expansion for each spatial direction:

$$A = A_0 e^{g_x(t)}, \qquad B = B_0 e^{g_y(t)}, \qquad C = C_0 e^{g_z(t)}, \quad (60)$$

where the function $g_i(t)$ is assumed as

$$g_i(t) = h_i(t)\ln(t), \qquad i = x, y, z, \quad (61)$$

and A_0, B_0, and C_0 are positive constants. Note that the de Sitter solutions and power-law solutions can be recovered from this one by setting $h_i(t) = a_i t/(\ln(t))$ and $h_i(t) = a_i$, respectively, where $\{a_i\} = \{a, b, c\}$. In what follows, we will use an adiabatic approximation for the expansion in each spatial direction and neglect the derivatives of $h_i(t)$, i.e., we set ($\dot{h}_i \sim \ddot{h}_i \sim 0$). The expansion rates in this case are given by

$$H_x = \frac{h_x(t)}{t}, \qquad H_y = \frac{h_y(t)}{t}, \qquad H_z = \frac{h_z(t)}{t}. \tag{62}$$

Thus the torsion scalar (17) becomes

$$T = -2\left[\frac{h_x(t)h_y(t)}{t^2} + \frac{h_x(t)h_z(t)}{t^2} + \frac{h_y(t)h_z(t)}{t^2}\right]. \tag{63}$$

The acceleration in each direction is given by

$$\ddot{A} = \frac{h_x(h_x - 1)}{t^2}A, \quad \ddot{B} = \frac{h_y(h_y - 1)}{t^2}B, \quad \ddot{C} = \frac{h_z(h_z - 1)}{t^2}C. \tag{64}$$

Since A, B, and C are positive, the acceleration is guaranteed in each direction when $h_i > 1$, while for $0 < h_i < 1$ the universe is in deceleration.

The simplest example of $h_i(t)$ is

$$h_i(t) = \frac{h_{i\mathrm{in}} + h_{i\mathrm{out}} q t^2}{1 + qt^2}, \tag{65}$$

where $h_{i\mathrm{in}}$, $h_{i\mathrm{out}}$, and q are positive constants, and q is assumed to be small enough to make $h_i(t)$ vary slowly. Thus, the torsion scalar is always negative. From Eq. (65) we see that at early time $t = 0$, $h_i \to h_{i\mathrm{in}}$, and for the late universe $h_i \to h_{i\mathrm{out}}$. By using Eqs. (65) and (63), one gets the following equation:

$$\begin{aligned} &q^2 T t^6 + (2qT + 2q^2 X)t^4 + (T + 2qY)t^2 + 2Z = 0,\\ &X = h_{x\mathrm{out}}h_{y\mathrm{out}} + h_{x\mathrm{out}}h_{z\mathrm{out}} + h_{y\mathrm{out}}h_{z\mathrm{out}},\\ &Y = h_{x\mathrm{in}}h_{y\mathrm{out}} + h_{x\mathrm{out}}h_{y\mathrm{in}} + h_{x\mathrm{in}}h_{z\mathrm{out}}\\ &\qquad + h_{x\mathrm{out}}h_{z\mathrm{in}} + h_{y\mathrm{in}}h_{z\mathrm{out}} + h_{y\mathrm{out}}h_{z\mathrm{in}},\\ &Z = h_{x\mathrm{in}}h_{y\mathrm{in}} + h_{x\mathrm{in}}h_{z\mathrm{in}} + h_{y\mathrm{in}}h_{z\mathrm{in}}, \end{aligned} \tag{66}$$

whose solutions read

$$\begin{aligned} t^2 &= \{\Psi_0(T), \Psi_\pm(T)\},\\ \Psi_0(T) &\equiv \alpha^{1/3} + \beta_0\alpha^{-1/3} + \beta_1 \qquad \Psi_\pm(T) \equiv e^{\mp 2i\pi/3}\alpha^{1/3} + e^{\pm 2i\pi/3}\alpha^{-1/3} + \beta_1,\\ \alpha &= \frac{\sqrt{\alpha_1}}{q^2T^2} - \alpha_2, \qquad \beta_0 = \frac{qT(8X - 6Y) + 4q^2X^2 + T^2}{9q^2T^2}, \qquad \beta_1 = -\frac{2qX + 2T}{3qT},\\ \alpha_1 &= \frac{1}{27qT}[27qT^2Z^2 + [(-36q^2TX - 36qT^2)Y + 16q^3X^3 + 48q^2TX^2 + 30qT^2X - 2T^3]Z\\ &\quad + 8q^2TY^3 + (-4q^3X^2 - 8q^2TX + 8qT^2)Y^2 + (-4q^2TX^2 - 8qT^2X + 2T^3)Y - qT^2X^2 - 2T^3X],\\ \alpha_2 &= \frac{qT^2(27Z - 18Y + 15X) + q^2T(24X^2 - 18XY) + 8q^3X^3 - T^3}{27q^3T^3}. \end{aligned} \tag{67}$$

We see from Eq. (67) that there is one real positive solution, $\Psi_0(T)$, and two complex solutions, $\Psi_\pm(T)$. By using Eqs. (62)–(64) the system of equations of motion (24)–(27) becomes

$$16\pi\rho = f - 2Tf_T, \tag{68}$$

$$-16\pi p_x = f + 2f_T\left[\frac{(h_y + h_z)^2 - (h_y + h_z) - h_y h_z}{\Psi_0(T)} - \frac{T}{2}\right] + 4\left(\frac{h_y + h_z}{\Psi_0(T)}\right)Tf_{TT}, \tag{69}$$

$$-16\pi p_y = f + 2f_T\left[\frac{(h_x + h_z)^2 - (h_x + h_z) - h_x h_z}{\Psi_0(T)} - \frac{T}{2}\right] + 4\left(\frac{h_x + h_z}{\Psi_0(T)}\right)Tf_{TT}, \tag{70}$$

$$-16\pi p_z = f + 2f_T\left[\frac{(h_x + h_y)^2 - (h_x + h_y) - h_x h_y}{\Psi_0(T)} - \frac{T}{2}\right] + 4\left(\frac{h_x + h_y}{\Psi_0(T)}\right)Tf_{TT}, \tag{71}$$

$$h_i = \frac{h_{i\mathrm{in}} + h_{i\mathrm{out}} q\Psi_0(T)}{1 + q\Psi_0(T)}. \tag{72}$$

1. A special case

Here, we assume that the expansion rates are equal in the three spatial directions ($h_x = h_y = h_z$), and the system (68)–(71) reduces to

$$16\pi\rho = f - 2Tf_T, \tag{73}$$

$$-16\pi p_x = f - 2f_T\left(T + \sqrt{\frac{-2T}{3\Psi_0(T)}}\right) + 4T\sqrt{\frac{-2T}{3\Psi_0(T)}} f_{TT}, \tag{74}$$

$$p_x = p_y = p_z,$$

which means that the assumption of having the same rate in the three spatial directions leads to an isotropic matter content. By combining Eqs. (73) and (74), one gets

$$4T\sqrt{\frac{-2T}{3\Psi_0(T)}} f_{TT} - 2f_T\left[(1+\omega)T + \sqrt{\frac{-2T}{3\Psi_0(T)}}\right] + (1+\omega)f = 0, \tag{75}$$

where we have used the barotropic equation $p_x = \omega\rho$. Let us consider an asymptotic analysis, looking for the early- (small time) and late-time universe (large time), for which the function $h_i(t)$ yields $h_{i\mathrm{in}}$ and $h_{i\mathrm{out}}$, respectively. Thus, for the early universe Eq. (75) reduces to

$$4T^2 f_{TT} + 6Tf_T[3h_{i\mathrm{in}}(\omega+1) - 1] - 3h_{i\mathrm{in}}(\omega+1)f = 0, \quad h_{i\mathrm{in}} = h_{x\mathrm{in}} = h_{y\mathrm{in}} = h_{z\mathrm{in}}, \tag{76}$$

whose general solution reads

$$f(T) = C_3 T^{\gamma_+} + C_4 T^{\gamma_-}, \qquad \gamma_\pm = \frac{5 - 9h_{x\mathrm{in}}(1+\omega) \pm \sqrt{25 - 78h_{x\mathrm{in}}(1+\omega) + 81h_{x\mathrm{in}}^2(1+\omega)^2}}{4}, \tag{77}$$

where C_3 and C_4 are integration constants. From Eq. (77), by writing the radicand as $[5 - 9h_{x\mathrm{in}}(1+\omega)]^2 + 12h_{x\mathrm{in}}(1+\omega)$, one observes that for any $\omega > -1$, $\gamma_+ > 0$ and $\gamma_- < 0$. Moreover, in this context of asymptotic analysis, we observe from Eq. (63) that for small t, the torsion scalar T is large, while for large t, the torsion is small. Thus, for small t with $h_{x\mathrm{in}} > 1$, corresponding to the inflation, the algebraic expression of $f(T)$ is given by

$$f(T) = C_3 T^{\gamma_+}. \tag{78}$$

Since $h_x(t)$ reduces to $h_{x\mathrm{out}}$ in the late universe, the model corresponding to the late accelerated universe can be obtained by replacing $h_{x\mathrm{in}}$ by $h_{x\mathrm{out}}$. Precisely, for large t, the torsion scalar is small, and for $h_{x\mathrm{out}} > 1$, the dominant term in Eq. (77), corresponding to the model of the late-time universe, is

$$f(T) = C_4 T^{\gamma'_-}, \qquad \gamma'_\pm = \frac{5 - 9h_{x\mathrm{out}}(1+\omega) \pm \sqrt{25 - 78h_{x\mathrm{out}}(1+\omega) + 81h_{x\mathrm{out}}^2(1+\omega)^2}}{4}. \tag{79}$$

This model is equivalent to teleparallel gravity for $C_4 = 1$ and $h_{x\mathrm{out}} = 2/(5+5\omega)$. It is easy to see from this that, for any ordinary matter, i.e., $\omega > 0$, one gets $h_{x\mathrm{out}} < 1$, meaning that teleparallel gravity without a cosmological constant cannot provide the late acceleration of the universe (remembering that the acceleration is guaranteed for $h_{x\mathrm{out}} > 1$, and that $0 < h_{x\mathrm{out}} < 1$, characterizing a decelerated expanding universe). Thus, the contribution of the $f(T)$ terms play the role of the dark energy.

Looking for the expression of $f(T)$ for large cosmic time t, i.e., the expression (77) (replacing $h_{x\mathrm{in}}$ by $h_{x\mathrm{out}}$), a similarity can be observed with a result of Bamba *et al.* in Ref. [26]. In that work, they studied $f(T)$ theory in the FLRW metric, first assuming a power-law expression for $f(T)$ [their Eq. (4.6)] for investigating what type of finite future-time singularities may appear. Later, they introduced a correction term, still in the form of a power law [their Eq. (4.22)], in order to analyze the possible avoidance of the singularities. Then, they obtained the global expression (4.23) of Ref. [26]. Note that this expression is equivalent to our Eq. (77). Moreover, they showed in their "TABLE II" that the big rip and the big freeze can be removed if the product of the exponents is negative. This is exactly our result, since γ'_+ and γ'_- are always positive and negative, respectively ($\gamma'_\pm$ are obtained from $\gamma_\pm$ by replacing $h_{x\mathrm{in}}$ by $h_{x\mathrm{out}}$). This means that for both Bianchi type I and KS, if the expansion of the universe occurs with the same rate in all directions, models that can realize the late-time accelerated expansion of the universe, and that are able to prevent the big rip and the big freeze, can be reconstructed.

2. Using an auxiliary scalar field

In this subsection, we would like to use the reconstruction scheme, for which an auxiliary scalar field is introduced in the action of the theory. By this method, the functional form of $f(T)$ can be found through two other scalar functions P and Q (for more clarification on the method, see the cases of $f(R)$ gravity [46] and $f(T)$ gravity [16]). Here, we return to the general system (68)–(72), where we introduce the auxiliary scalar field ϕ by defining the algebraic function $f(T)$ as

$$f(T) = P(\phi)T + Q(\phi). \quad (80)$$

By using Eq. (80) and varying the action (9) with respect to the scalar field ϕ, one gets

$$P'(\phi)T + Q'(\phi) = 0, \quad (81)$$

which may be solved with respect to ϕ, yielding $\phi = \phi(T)$. Here, the prime (′) denotes the derivative with respect to ϕ. By using Eq. (80), one obtains

$$f_T(T) = P(\phi(T)), \qquad f_{TT}(T) = P_T(\phi(T)). \quad (82)$$

Making use of Eq. (82), one can rewrite the system (68)–(71) as

$$16\pi\rho = P(\phi)T + Q(\phi) - 2TP(\phi), \quad (83)$$

$$-16\pi p_x = P(\phi)T + Q(\phi) + 2P(\phi)\left[\frac{(h_y+h_z)^2 - (h_y+h_z) - h_y h_z}{\Psi_0(T)} - \frac{T}{2}\right] + 4\left(\frac{h_y+h_z}{\Psi_0(T)}\right)TP_T(\phi), \quad (84)$$

$$-16\pi p_y = P(\phi)T + Q(\phi) + 2P(\phi)\left[\frac{(h_x+h_z)^2 - (h_x+h_z) - h_x h_z}{\Psi_0(T)} - \frac{T}{2}\right] + 4\left(\frac{h_x+h_z}{\Psi_0(T)}\right)TP_T(\phi), \quad (85)$$

$$-16\pi p_z = P(\phi)T + Q(\phi) + 2P(\phi)\left[\frac{(h_x+h_y)^2 - (h_x+h_y) - h_x h_y}{\Psi_0(T)} - \frac{T}{2}\right] + 4\left(\frac{h_x+h_y}{\Psi_0(T)}\right)TP_T(\phi). \quad (86)$$

Let us consider that $p_y = p_z$. Then, by equating Eq. (85) with Eq. (86), one gets

$$TP_T(\phi) = -\frac{1}{2}P(\phi)(h_x + h_y + h_z - 1). \quad (87)$$

Using Eq. (87), the system (83)–(86) reduces to

$$16\pi\rho = Q(\phi) - TP(\phi), \quad (88)$$

$$-16\pi p_x = Q(\phi) + TP(\phi), \quad (89)$$

$$p_x = p_y = p_z, \qquad \omega_x = \omega_y = \omega_z. \quad (90)$$

By using Eq. (88), one can determine $Q(\phi)$ as

$$Q(\phi) = 16\pi\rho + TP(\phi), \quad (91)$$

which, substituted into Eqs. (89) and (90), yields

$$-16\pi(\omega_x + 1)\rho = 2TP(\phi). \quad (92)$$

As one can redefine the scalar field properly, we may choose $\phi = t$; then $P(\phi) = P(t) = \bar{P}(T)$. Note that in the case where $\omega_x = -1$, one gets $\bar{P}(T) = 0$, and the algebraic function $f(T) = Q(t) = 16\pi\rho = \bar{Q}(T)$. But this requires one to have the complete expression of ρ depending on T, i.e., solving the equation of continuity. Let us consider in general the case where $\omega_x \neq -1$ and try to solve the equation of continuity, which will help us to determine $\bar{P}(T)$ and $\bar{Q}(T)$, leading to the reconstruction of the algebraic function $f(T)$. We choose the case of KS geometry ($h_y = h_z$), where the equation of continuity is written as

$$\dot{\rho} + \rho(1 + \omega_x)(H_x + 2H_y) = 0, \quad (93)$$

which can be solved, giving

$$\rho(t) = C_5 \exp[G_1(t) + G_2(t)], \quad (94)$$

$$G_1(t) = \frac{(1+\omega_x)(h_{x\text{in}} + 2h_{y\text{in}})}{t}, \quad (95)$$

$$G_2(t) = \sqrt{q}[(1+\omega_x)(h_{x\text{in}} - h_{x\text{out}} + 2h_{y\text{in}} - 2h_{y\text{out}})] \times \arctan(\sqrt{q}t), \quad (96)$$

where C_5 is an integration constant. Making use of Eq. (67), one can cast $\rho(t)$ in terms of T. Also, from Eq. (92), one obtains $P(t)$ in terms of T, and from Eq. (91) one gets $Q(t)$ in terms of T. Therefore, we can reconstruct the algebraic function $f(T)$, given in Eq. (80), as

$$f(T) = -16\pi C_3\omega_x \exp\left[\frac{g_1}{\sqrt{\Psi_0(T)}} + g_2 \arctan\left(\sqrt{q\Psi_0(T)}\right)\right], \quad (97)$$

where $g_1 = (1+\omega_x)(h_{x\text{in}} + 2h_{y\text{in}})$ and $g_2 = \sqrt{q}(1+\omega_x)\times(h_{x\text{in}} - h_{x\text{out}} + 2h_{y\text{in}} - 2h_{y\text{out}})$.

In principle, with Eq. (97) some cosmological models can now be reproduced. We focus our attention on the early universe (which may be characterized by the inflation), and the late-time universe (characterized by its accelerated expansion) in the three spatial directions.

At early time, i.e., for small t (or large T), $\Psi_0(T)$ is very small, and the corresponding algebraic function is

$$f(T) = -16\pi C_3\omega_x \exp\left[\frac{g_1}{\sqrt{\Psi_0(T)}}\right]. \quad (98)$$

At late time, the time t is large [corresponding to a small torsion scalar and large $\Psi_0(T)$], and the algebraic function reads

$$f(T) = -16\pi C_3\omega_x \exp\left[g_2 \arctan\left(\sqrt{q\Psi_0(T)}\right)\right]. \quad (99)$$

Thus we see that models corresponding to the inflation and the late-time accelerated universe can be reconstructed within KS metrics where the matter content is partially isotropic ($p_y = p_z$). Since in this work we fall in the situation where $\omega_x = \omega_y = \omega_y$ one could use the WMAP data and try to check if they fit with this anisotropic model. Because ultimately $p_x = p_y = p_z$ and $h_y = h_z$, one can just use the first two equations of motion of the KS case, i.e., Eqs. (28) and (29). In order to cancel the contribution of the modified part of the algebraic function $f(T)$, we cast it into the form $f(T) = T + j(T)$ [the teleparallel term plus the modified additive algebraic function $j(T)$]. Thus, Eqs. (28) and (30) become

$$8\pi\rho_{\rm eff} = \left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB}, \tag{100}$$

$$-8\pi p_{x\rm eff} = 2\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2, \tag{101}$$

where $\rho_{\rm eff}$ and $p_{x\rm eff}$ are the effective energy density and effective pressure, respectively, and are defined by

$$\rho_{\rm eff} = \rho - \frac{1}{16\pi}\left\{j + 4j_T\left[\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB}\right]\right\}, \tag{102}$$

$$p_{x\rm eff} = p_x + \frac{1}{16\pi}\left\{j + 4j_T\left[\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 + \frac{\dot{A}\dot{B}}{AB}\right] + 4\frac{\dot{B}}{B}\dot{T}j_{TT}\right\}. \tag{103}$$

By dividing Eq. (101) by Eq. (100), and using $\omega_{\rm eff} = p_{x\rm eff}/\rho_{\rm eff}$, one gets

$$\omega_{\rm eff} = \frac{2 - 3h_y}{h_y + 2h_x}. \tag{104}$$

Since the observational data from the WMAP project do not depend on the spacial direction, i.e., they are not based on an anisotropic geometry, we have to impose $h_x = h_y$ in Eq. (104) in order to get a suitable cosmological feature. Thus, assuming that the accelerated expansion of the universe is realized with the same rate in all directions, $h_x = h_y = h_z > 1$, and the universe is essentially filled by the dark energy, where we can neglect the usual matter content such that $\omega_{\rm eff} \sim \omega_{\rm DE}$. Therefore, using Eq. (104) we have the possibility of regaining the well-known range of values allowed by the current seven-year WMAP data for the parameter of the equation of state of the dark energy, $\omega_{\rm DE} = -1.1 \pm 0.14$ WMAP [47].

D. On Bianchi type III solutions

In this section, we present some comments on Bianchi type III solutions. This case is quite exceptional since we do not have the freedom of making cosmological reconstruction, as in the case of Bianchi type I and KS, due to the constraints equations (22) and (23).

From Eq. (23), since the parameter is different from zero and the algebraic function cannot be a constant, one gets

$$\frac{\dot{A}}{A} = \frac{\dot{B}}{B}, \tag{105}$$

which, when inserted into Eq. (22), leads to

$$\dot{T}f_{TT} = 0, \tag{106}$$

meaning that one has $\dot{T} = 0$ or $f_{TT} = 0$. The first case, $\dot{T} = 0$, implies that one has a constant torsion scalar, i.e.,

$$\frac{\dot{A}^2}{A^2} + 2\frac{\dot{A}\dot{C}}{AC} = K, \tag{107}$$

where K is a positive constant. Let us consider $A = C^n$, with n bigger than zero or less than -2. Thus, Eq. (107) can be solved, yielding

$$C(t) = C_0' \exp\left(\pm\sqrt{\frac{K}{n(n+2)}}t\right), \tag{108}$$

leading to

$$A(t) = B(t) = (C_0')^n \exp\left(\pm n\sqrt{\frac{K}{n(n+2)}}t\right), \tag{109}$$

where C_0' is a positive constant. It is important to note that in order to guarantee the expansion of the universe, one needs to have

$$A(t) = B(t) = \begin{cases} (C_0')^n \exp\left(-n\sqrt{\frac{K}{n(n+2)}}t\right) & \text{for } n < -2, \\ (C_0')^n \exp\left(n\sqrt{\frac{K}{n(n+2)}}t\right) & \text{for } n > 0. \end{cases} \tag{110}$$

In this case, we see that the rate of expansion is constant for the three spatial directions: this is the de Sitter universe.

Now we can perform the reconstruction of the algebraic function $f(T)$. One can cast Eqs. (18)–(21) in the following system:

$$16\pi\rho = f + 4f_T\left(K - \frac{\alpha^2}{2A^2}\right), \tag{111}$$

$$-16\pi\omega_x\rho = f + 2Kf_T\left[\frac{2n^2 + 3n + 1}{n(n+2)}\right]. \tag{112}$$

$$-16\pi\omega_z\rho = f + 4Kf_T\left(\frac{2n+1}{n+2}\right) - 2\frac{\alpha^2}{A^2}f_T, \tag{113}$$

$$p_x = p_y. \tag{114}$$

By combining Eq. (111) with Eq. (113), one can eliminate the term containing α, obtaining

$$-16\pi(\omega_z + 1)\rho = \frac{4K(n-1)}{n+2}f_T. \tag{115}$$

The energy density ρ can be eliminated by combining Eq. (112) with Eq. (115), yielding the differential equation

$$2K[2n(n-1)\omega_x - (\omega_z+1)(2n^2+3n+1)]f_T - n(n+2)(\omega_z+1)f = 0, \tag{116}$$

whose general solution is

$$f(T) = C_6 \exp[R(n)T], \quad R(n) = \frac{n(n+2)(\omega_z+1)}{2K[2n(n-1)\omega_x - (\omega_z+1)(2n^2+3n+1)]}, \tag{117}$$

where C_6 is an integration constant. Note that for $n = 1$ and $\omega_x = \omega_z$, Eq. (40) is recovered.

The second case from Eq. (106), $f_{TT} = 0$, implies that f_T is constant: if we choose it to be minus two times the cosmological constant Λ, then $f(T)$ is written as

$$f(T) = T - 2\Lambda, \tag{118}$$

which is teleparallel gravity with a cosmological constant.

V. CONCLUSION

In this paper the Bianchi type I, Kantowski-Sachs, and Bianchi type III metrics have been studied in the context of $f(T)$ gravities. Particularly, we have shown the reconstruction of some important cosmological solutions, obtaining the corresponding $f(T)$ action. We have initially assumed a particular choice of coordinates and tetrads: specifically, Cartesian coordinates and a diagonal set of tetrads have been imposed in order to avoid the well-known constraint $f_{TT} = 0$, which reduces trivially to the action of teleparallel gravity (see Ref. [48]).

Then, several important cosmological solutions have been considered. In particular, de Sitter solutions, where the scale factor is an exponential function of the cosmic time, have been considered for Bianchi type I and Kantowski-Sachs metrics by imposing a particular exponential expansion in each direction of the space. We have shown that the only possible solution turns out to be the FLRW metric, such that no possible de Sitter anisotropic evolution can be found in $f(T)$ unless one considers an anisotropic fluid. Nevertheless, in the case of power-law solutions, we have found that in the presence of a perfect isotropic fluid, an anisotropic cosmological evolution can be found for a particular choice of the action $f(T)$, while in vacuum the action reduces to the FLRW metric.

Moreover, we have extended the cosmological reconstruction scheme to general exponential solutions, from which the above de Sitter and power-law solutions are particular cases. We have assumed an adiabatic approximation for the expansion in each spatial direction. We studied two cases: a special case, and a second case where an auxiliary field is used. In the both cases, we have shown that the models can realize the early accelerated universe, characterized by the inflation, and the late-time acceleration of our current universe. In the special case, the model presents an interesting aspect because it ensures the avoidance of the big rip and the big freeze. In the case where the auxiliary field is used, the model corresponding to the late-time accelerated universe fits with the seven-year WMAP data, confirming the consistency of the result.

The Bianchi type III case presents some constraints from which only two forms of the algebraic function $f(T)$ can be obtained. The first is the well-known teleparallel gravity with a cosmological constant, and the second is a de Sitter type solution.

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