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High-order expansions of the Detweiler-Whiting singular field in Schwarzschild spacetime

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The self-field of a charged particle has a singular component which diverges at the particle. We use both coordinate and covariant approaches to compute an expansion of this singular field for particles in generic geodesic orbits about a Schwarzschild black hole for scalar, electromagnetic and gravitational cases. We check that both approaches yield identical results and give, as an application, the calculation of previously unknown mode-sum regularization parameters. In the so-called *mode-sum regularization* approach to self-force calculations, each mode of the retarded field is finite, while their sum diverges. The sum may be rendered finite and convergent by the subtraction of appropriate *regularization parameters*. Higher-order parameters lead to faster convergence in the mode sum. To demonstrate the significant benefit which they yield, we use our newly derived parameters to calculate a highly accurate value of $F_r = 0.0000137844825756679 59(3)$ for the self-force on a scalar particle in a circular orbit around a Schwarzschild black hole. Finally, as a second example application of our high-order expansions, we compute high-order expressions for use in the effective source approach to self-force calculations.

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I. INTRODUCTION

The notion of a self-force has a long history in physics. Early work led to the derivation of the Abraham-Lorentz-Dirac [1] formula for the radiation reaction force on an accelerating point electric charge moving in a flat spacetime. In the 1960s, DeWitt and Brehme [2] derived the curved spacetime equivalent, and a correction was later provided by Hobbs [3]. For a comprehensive review of the self-force problem, see Refs. [4–6].

With the advent of gravitational wave astronomy, the past two decades have seen a surge in interest in the self-force problem. This has been motivated by the study of so-called extreme mass ratio inspiral systems. These binary systems—consisting of a compact object of ~ 1 solar mass spiralling into a massive black hole of $\sim 10^6$ solar masses—are one of the most proimising candidates for study by planned space-based gravitational wave detectors [7–13]. The nature of interferometric gravitational wave detectors is that unlike traditional electromagnetic spectrum telescopes, they are not directional; instead, they see all directions at once. As a result, a crucial component of any observation is the use of data analysis techniques such as matched filtering to disentangle the desired signal from the noise. This, in turn, requires an accurate model of the signal one expects to see. This is even more true given the recent proposals for the evolved Laser Interferometer Space Antenna (eLISA)/New Gravitational Wave Observatory (NGO) detector [14] (which will have a reduced sensitivity compared to LISA, the previously proposed space-based gravitational wave detector) will mean that accurate waveform models are all the more crucial.

The past two decades have seen new derivations of the equations of motion of a point charge moving in a curved spacetime; this was first done in the gravitational charge case by Mino, Sasaki and Tanaka [15] and Quinn and Wald [16] and in the minimally coupled scalar charge case by Quinn [17]. Since then, there have been significant achievements in putting these derivations on a firmer footing. The use of distributional sources in Einstein's equations is known to be problematic [18]. Nevertheless, with sufficient care, they have proven to be a useful and practical tool in many cases [19,20]. In the case of the self-force problem, recent derivations have avoided the introduction of distributional sources altogether through the use of matched asymptotic expansions and careful limiting procedures [21–26]. At first perturbative order, it is satisfying that these more rigorous derivations reproduce the same equations of motion as one would obtain using distributional sources. Nevertheless, it is likely that these more rigorous methods are required to advance to second perturbative order [27–31] and beyond [32–34].

Several practical self-force computation strategies have developed from these formal derivations, all of which are based on the now-justified assumption that the use of a distributional source is acceptable at first perturbative order:

(i) The *mode-sum* approach [35,36]

(ii) The *effective source* approach [37,38]

(iii) The *matched expansion* approach [39,40]

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The key to all three approaches is the subtraction of an appropriate *singular* component (which only appears due to the use of a distributional source) from the retarded field to leave a finite *regular* field which is solely responsible for the self-force. This singular component must satisfy two criteria:

(1) It has the same singular structure as the full retarded field on the particle's world line.
(2) It does not contribute to the self-force (or its contribution is well-known and can be corrected for).

There are many choices for a singular field which satisfies the above criteria, although not all choices are equal. Detweiler and Whiting [41] identified a particularly appropriate choice. Through a Green function decomposition, they derived a singular field which not only satisfies the above two criteria, but also has the property that when it is subtracted from the full retarded field, it leaves a regularized field which is a solution to the *homogeneous* wave equation. Extensions of this idea of a singular-regular split to extended charge distributions [22,23], to second perturbative order [27–31] and fully nonperturbative contexts, [34] have recently been developed.

In this paper, we develop highly accurate approximations to the Detweiler-Whiting singular field of point scalar and electromagnetic charges as well as that of a point mass. This is achieved through high-order series expansions in a parameter ϵ, which acts as a measure of distance from the particle's world line. As examples, we derive explicit expressions for the case of geodesic motion in Schwarzschild spacetime and show how these may be applied to both the mode-sum and effective source approaches described above. Similar expansions may also be used for the quasilocal component in the matched expansion approach [42–44] whereby the Green function is matched onto a quasinormal mode sum and the retarded field is then computed as the integral of this matched retarded Green function along the world line of the particle.

While the primary focus of this paper is on computing the singular field for Schwarzschild spacetime, many of the expressions we give are valid in more general spacetimes. In particular, where space allows, we do not make any assumptions about the spacetime being Ricci-flat. To make this distinction explicit, we use the Weyl tensor, C_{abcd}, in expressions which are valid only in vacuum and the Riemann tensor, R_{abcd}, in expressions which are also valid for nonvacuum spacetimes. Note that this is done only for space reasons[1]; our raw calculations include all nonvacuum terms in addition to those given in this paper, and we have made the full expressions available in electronic form [45].

[1]The notable exception is the case of the gravitational singular field, as in that case, the equations of motion have not yet been derived for non-Ricci-flat spacetimes.

The layout of this paper is as follows. In Sec. II, we give exact formal expressions for the singular field in scalar, electromagnetic and gravitational cases. In Sec. III, we describe covariant and coordinate approaches to the computation of expansions of the key fundamental bitensors appearing in the formal expressions for the singular field. In Sec. IV, we give explicit expressions for the singular field in the form of covariant series expansions. Our equivalent coordinate expressions are too large to be of use in print; instead, we have made them available electronically [45]. In Sec. V, we use our coordinate expansions to derive high-order regularization parameters for use in the mode-sum method. In doing so, we give the next two previously unknown nonzero regularization parameters in scalar, electromagnetic and gravitational cases. In Sec. VI, we apply our coordinate expansions to the effective source method and compute an effective source for our high-order singular field. This improves on previous effective source calculations by adding an additional four orders, bringing the approximation to eight-from-leading order. In Sec. VII, we summarize our results and discuss further prospects for their application. Finally, in Appendices A and B, we give explicit expressions for many of the expansions in coordinate and covariant form, respectively, along with a more in-depth derivation of our expressions.

Throughout this paper, we use units in which $G = c = 1$ and adopt the sign conventions of Ref. [46]. We denote symmetrization of indices using parenthesis [e.g., (ab)], antisymmetrization using square brackets (e.g., $[ab]$), and we exclude indices from (anti)symmetrization by surrounding them by vertical bars (e.g., $(a|b|c)$, $[a|b|c]$). We denote pairwise (anti)symmetrization using an over bar, e.g., $R_{(\overline{ab}\,\overline{cd})} = \frac{1}{2}(R_{abcd} + R_{cdab})$. Capital letters are used to denote the spinorial/tensorial indices appropriate to the field being considered.

In many of our calculations, we have several spacetime points to be considered. Our convention is that (see Fig. 1):

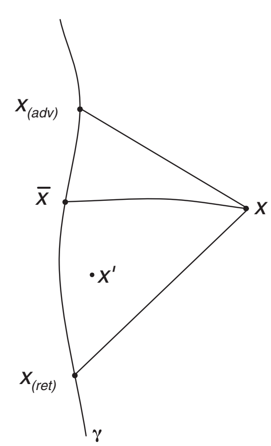

FIG. 1. We expand all bitensors (which are functions of x and x') about the arbitrary point $\bar{x}$ on the world line.

(i) the point x refers to the point where the field is evaluated;
(ii) the point x' refers to an arbitrary spacetime point;
(iii) the point $\bar{x}$ refers to an arbitrary point on the world line;
(iv) the point $x_{(\mathrm{adv})}$ refers to the advanced point of x on the world line;
(v) the point $x_{(\mathrm{ret})}$ refers to the retarded point of x on the world line.

In computing expansions, we use ϵ as an expansion parameter to denote the fundamental scale of separation, so that $x - x' \approx x - \bar{x} = \mathcal{O}(\epsilon)$. Where tensors are to be evaluated at one of these points, we decorate their indices appropriately using $'$ and $\bar{}$, e.g., T^a, $T^{a'}$ and $T^{\bar{a}}$ refer to tensors at x, x' and $\bar{x}$, respectively.

II. SINGULAR FIELD AND SELF-FORCE OF A POINT PARTICLE

In an appropriate gauge, the retarded field, $\varphi^A(x)$, of an arbitrary point particle satisfies the inhomogeneous wave equation with a distributional source,

$$\mathcal{D}^A{}_B \varphi^B = -4\pi \mathcal{Q} \int u^A \delta_4(x, z(\tau')) d\tau', \tag{2.1}$$

where

$$\mathcal{D}^A{}_B = \delta^A{}_B(\Box - m^2) - P^A{}_B \tag{2.2}$$

is the wave operator, $\Box \equiv g^{ab}\nabla_a\nabla_b$, g^{ab} is the (contravariant) metric tensor, ∇_a is the covariant derivative defined by a connection $\mathcal{A}^A{}_{Ba}$: $\nabla_a \varphi^A = \partial_a \varphi^A + \mathcal{A}^A{}_{Ba}\varphi^B$, m is the mass of the field, $\mathcal{Q}$ is the charge of the particle, $P^A{}_B(x)$ is a potential term, $u^A(x)$ is an appropriate tensor constructed from products of the four-velocity on the world line parallel transported to x and $\delta_4(x, z(\tau'))$ is an invariant Dirac functional in a four-dimensional curved spacetime as defined in Eq. (13.1) of Ref. [4]. The retarded solutions to this equation give rise to a field which one might naively expect to exert a self-force

$$F^a = p^a{}_A \varphi^A_{(\mathrm{ret})} \tag{2.3}$$

on the particle, where $p^a{}_A(x)$ is a tensor at x and depends on the type of charge. The distributional nature of the source leads to this self-force being divergent at the location of the particle, and a regularization scheme must be employed. We require a singular field, $\varphi^A_{(\mathrm{S})}$, which captures the singular behavior of φ^A and that, when subtracted from φ^A, leaves a finite *physical* self-force.

Detweiler and Whiting [41] showed how such a singular field can be constructed through a Green function decomposition. In four spacetime dimensions and within a normal neighborhood, the Green function for the retarded/advanced solutions to Eq. (2.1) may be given in Hadamard form,

$$G_{(\mathrm{ret})/(\mathrm{adv})}{}^A{}_{B'}(x, x') = \theta_{-/+}(x, x')\{U^A{}_{B'}(x, x')\delta[\sigma(x, x')] - V^A{}_{B'}(x, x')\theta[-\sigma(x, x')]\}, \tag{2.4}$$

where $\delta[\sigma(x, x')]$ is the covariant form of the Dirac delta function, $\theta[\sigma(x, x')]$ is the Heaviside step function, while $U^A{}_{B'}(x, x')$ and $V^A{}_{B'}(x, x')$ are symmetric bispinors/tensors which are regular for $x' \to x$. The biscalar $\sigma(x, x')$ is the Synge [4] world function; it is equal to one half of the squared geodesic distance between x and x'. The first term here, involving $U^A{}_{B'}(x, x')$, represents the *direct* part of the Green function while the second term, involving $V^A{}_{B'}(x, x')$, is known as the *tail* part of the Green function.

Detweiler and Whiting proposed to define a singular Green function by taking the symmetric Green function, $G_{(\mathrm{sym})}{}^A{}_{B'} = \frac{1}{2}(G_{(\mathrm{ret})}{}^A{}_{B'} + G_{(\mathrm{adv})}{}^A{}_{B'})$ and adding $V^A_{B'}(x, x')$ [a homogeneous solution to Eq. (2.1)]. This leads to the singular Green function,

$$G_{(\mathrm{S})}{}^A{}_{B'}(x, x') = \frac{1}{2}\{U^A{}_{B'}(x, x')\delta[\sigma(x, x')] + V^A{}_{B'}(x, x')\theta[\sigma(x, x')]\}. \tag{2.5}$$

Note that this has support on and *outside* the past and future light cone (i.e., for points x and x' spatially separated) and is only uniquely defined provided x and x' are within a convex normal neighborhood. Given this singular Green function, we may define the Detweiler-Whiting singular field,

$$\varphi^A_{(\mathrm{S})} = \int_{\tau_{(\mathrm{adv})}}^{\tau_{(\mathrm{ret})}} G^A_{(\mathrm{S})B'}(x, z(\tau'))u^{B'} d\tau', \tag{2.6}$$

which also satisfies Eq. (2.1). Subtracting this singular field from the retarded field, we obtain the *regularized* field,

$$\varphi^A_{(\mathrm{R})} = \varphi^A_{(\mathrm{ret})} - \varphi^A_{(\mathrm{S})}, \tag{2.7}$$

which Detweiler and Whiting showed gives the correct finite physical self-force,

$$F^a = p^a{}_A \varphi^A_{(\mathrm{R})}. \tag{2.8}$$

Moreover, this regularized field is a solution of the homogeneous wave equation,

$$\mathcal{D}^A{}_B \varphi^B_{(\mathrm{R})} = 0. \tag{2.9}$$

This holds independently of whether one is considering a scalar or electromagnetically charged point particle or a point mass. To make this more explicit, in the following subsections, we give the form these expressions take in each of the scalar, electromagnetic and gravitational cases.

A. Scalar case

In the scalar case, the singular field, $\Phi^{(\mathrm{S})}$, is a solution of the inhomogeneous scalar wave equation,

$$(\Box - \xi R - m^2)\Phi^{(S)} = q\int\sqrt{-g}\delta_4(x, z(\tau))d\tau, \quad (2.10)$$

where q is the scalar charge, R is the Ricci scalar, g is the determinant of the metric tensor and ξ is the coupling to the background scalar curvature. An expression for $\Phi^{(S)}$ may be found by considering the scalar Green function [obtained by taking $U^A{}_{B'} = U(x, x')$ in Eq. (2.5)],

$$G^{(S)} = \frac{1}{2}\{U(x, x')\delta[\sigma(x, x')] + V(x, x')\theta[\sigma(x, x')]\}, \quad (2.11)$$

with $U(x, x') = \Delta^{1/2}(x, x')$, where $\Delta^{1/2}(x, x')$ is the Van Vleck-Morette determinant as defined in Eq. (7.1) of Ref. [4]. This Green function is a solution of the equation

$$(\Box - \xi R - m^2)G^{(S)} = -4\pi\delta(x, x'). \quad (2.12)$$

Given this expression for the Green function, the scalar singular field is

$$\Phi^{(S)}(x) = q\int_\gamma G^{(S)}(x, z(\tau))d\tau = \frac{q}{2}\left[\frac{U(x, x')}{\sigma_{c'}u^{c'}}\right]_{x'=x_{(\mathrm{ret})}}^{x'=x_{(\mathrm{adv})}} + \frac{q}{2}\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V(x, z(\tau))d\tau, \quad (2.13)$$

and one computes the scalar self-force from the regularized scalar field $\Phi^{(R)} = \Phi^{(\mathrm{ret})} - \Phi^{(S)}$ as

$$F^a = qg^{ab}\Phi^{(R)}_{,b}. \quad (2.14)$$

B. Electromagnetic case

In Lorenz gauge, the electromagnetic singular field satisfies the equation

$$\Box A^{(S)}_a - R_a{}^b A^{(S)}_b = -4\pi e\int g_{aa'}u^{a'}\sqrt{-g}\delta_4(x, z(\tau))d\tau, \quad (2.15)$$

where e is the electric charge, $R_a{}^b$ is the Ricci tensor and $u^{a'}$ is the four-velocity at x'. An expression for $A^{(S)}_a$ may be found by considering the electromagnetic Green function [obtained by taking $U^A{}_{B'} = U(x, x')_{aa'}$ in Eq. (2.5)],

$$G^{(S)}_{aa'}(x, x') = \frac{1}{2}\{U(x, x')_{aa'}\delta(\sigma(x, x')) + V(x, x')_{aa'}\theta(\sigma(x, x'))\}, \quad (2.16)$$

with $U^a{}_{a'} = \Delta^{1/2}g^a{}_{a'}$, where $g^a{}_{a'}$ is the bivector of parallel transport as defined in Eq. (5.11) of Ref. [4]. This Green function is a solution of the equation

$$\Box G^{(S)}_{aa'} - R_a{}^b G^{(S)}_{ba'} = -4\pi g_{aa'}\delta_4(x, x'). \quad (2.17)$$

Given this expression for the Green function, the electromagnetic singular field is

$$A^{(S)}_a = e\int_\gamma G^{(S)}_{aa'}(x, z(\tau'))u^{a'}d\tau' = \frac{e}{2}\left[\frac{u^{a'}U_{aa'}(x, x')}{\sigma_{c'}u^{c'}}\right]_{x'=x_{(\mathrm{ret})}}^{x'=x_{(\mathrm{adv})}} + \frac{e}{2}\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V_{aa'}(x, z(\tau))u^{a'}d\tau. \quad (2.18)$$

One computes the electromagnetic self-force from the electromagnetic regular field, $A^{(R)}_a = A^{(\mathrm{ret})}_a - A^{(S)}_a$, as

$$F^a = eg^{ab}u^c A^{(R)}_{[c,b]}. \quad (2.19)$$

C. Gravitational case

In Lorenz gauge, the trace-reversed singular first-order metric perturbation satisfies the equation

$$\Box\bar{h}^{(S)}_{ab} + 2C_a{}^c{}_b{}^d\bar{h}^{(S)}_{cd} = -16\pi\mu\int g_{a'(a}u^{a'}g_{b)b'}u^{b'}\sqrt{-g}\delta_4(x, z(\tau))d\tau, \quad (2.20)$$

where μ is the mass of the particle and the trace-reversed singular field is related to the non-trace-reversed version by $\bar{h}^{(S)}_{ab} = h^{(S)}_{ab} - \frac{1}{2}h^{(S)}g_{ab}$ with $h^{(S)} = g^{ab}h^{(S)}_{ab}$. An expression for $\bar{h}^{(S)}_{ab}$ may be found by considering the gravitational Green function [obtained by taking $U^A{}_{B'} = U(x, x')^{ab}{}_{a'b'}$ in Eq. (2.5)],

$$G^{(S)}_{aba'b'}(x, x') = \frac{1}{2}\{U(x, x')_{aba'b'}\delta[\sigma(x, x')] + V(x, x')_{aba'b'}\theta[\sigma(x, x')]\}, \quad (2.21)$$

with $U^{ab}{}_{a'b'} = \Delta^{1/2}g^{(a}{}_{a'}g^{b)}{}_{b'}$. This Green function is a solution of the equation

$$\Box G^{(S)}_{aba'b'} + 2C_a{}^p{}_b{}^q G^{(S)}_{pqa'b'} = -4\pi g_{a'(a}g_{b)b'}\delta_4(x, x'). \quad (2.22)$$

Given this expression for the Green function, the trace-reversed singular first-order metric perturbation is

$$\bar{h}^{(S)}_{ab} = 4\mu\int_\gamma G^{(S)}_{aba'b'}(x, z(\tau'))u^{a'}u^{b'}d\tau' = 2\mu\left[\frac{u^{a'}u^{b'}U_{aba'b'}(x, x')}{\sigma_{c'}u^{c'}}\right]_{x'=x_{(\mathrm{ret})}}^{x'=x_{(\mathrm{adv})}} + 2\mu\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V_{aba'b'}(x, z(\tau))u^{a'}u^{b'}d\tau. \quad (2.23)$$

One computes the gravitational self-force from the regularized trace-reversed singular first-order metric perturbation, $\bar{h}^{(R)}_{ab} = \bar{h}^{(\mathrm{ret})}_{ab} - \bar{h}^{(S)}_{ab}$, as

$$F^a = \mu k^{abcd}\bar{h}^{(R)}_{bc;d}, \quad (2.24)$$

where

$$k^{abcd} \equiv \frac{1}{2} g^{ad} u^b u^c - g^{ab} u^c u^d - \frac{1}{2} u^a u^b u^c u^d + \frac{1}{4} u^a g^{bc} u^d + \frac{1}{4} g^{ad} g^{bc}. \tag{2.25}$$

III. COVARIANT AND COORDINATE EXPANSION OF FUNDAMENTAL BITENSORS

In the previous section, we gave expressions for the singular field in terms of the bitensors $U^A{}_{B'}(x, x')$ and $V^A{}_{B'}(x, x')$. The first of these is given by

$$U^{AB'}(x, x') = \Delta^{1/2}(x, x') g^{AB'}(x, x'), \tag{3.1}$$

where $\Delta(x, x')$ is the Van Vleck-Morette determinant [4],

$$\begin{aligned}\Delta(x, x') &= -[-g(x)]^{-1/2} \det(-\sigma_{;ab'}(x, x'))[-g(x')]^{-1/2} \\ &= \det(-g^{a'}{}_a(x, x')\sigma^a{}_{b'}(x, x')),\end{aligned} \tag{3.2}$$

$g^{AB'}$ is the bitensor of parallel transport appropriate to the tensorial nature of the field, e.g.,

$$g^{AB'} = \begin{cases} 1 & \text{(scalar)} \\ g^{ab'} & \text{(electromagnetic)} \\ g^{a'(a} g^{b)b'} & \text{(gravitational),} \end{cases} \tag{3.3}$$

and where the higher spin fields are taken in Lorenz gauge. Here, $g^{\alpha'}{}_\alpha(x, x')$ is the bivector of parallel transport defined by the transport equation

$$\sigma^\alpha g_{ab';\alpha} = 0 = \sigma^{\alpha'} g_{ab';\alpha'}. \tag{3.4}$$

The bitensor $V^{AB'}(x, x')$ may be expressed in terms of a formal expansion in increasing powers of σ [47]:

$$V^{AB'}(x, x') = \sum_{n=0}^{\infty} V_n{}^{AB'}(x, x') \sigma^n(x, x'), \tag{3.5}$$

where the coefficients $V_n^{AB'}(x, x')$ satisfy the recursion relations

$$\sigma^{;\alpha'}(\Delta^{-1/2} V_n^{AB'})_{;\alpha'} + (n+1)\Delta^{-1/2} V_n^{AB'} + \frac{1}{2n} \Delta^{-1/2} \mathcal{D}^{B'}{}_{C'} V_{n-1}^{AC'} = 0, \tag{3.6a}$$

for $n \in \mathbb{N}$, along with the "initial condition"

$$\sigma^{;\alpha'}(\Delta^{-1/2} V_0^{AB'})_{;\alpha'} + \Delta^{-1/2} V_0^{AB'} + \frac{1}{2} \Delta^{-1/2} \mathcal{D}^{B'}{}_{C'} (\Delta^{1/2} g^{AC'}) = 0. \tag{3.6b}$$

Looking at the above equations for $U^{AB'}(x, x')$ and $V_n{}^{AB'}(x, x')$, we see that a key component of the present work involves the computation of several fundamental bitensors, in particular, the world function $\sigma(x, x')$, Van Vleck–Morette determinant $\Delta^{1/2}(x, x')$, four-velocity $u^a(x)$ and bivector of parallel transport $g_a{}^{b'}(x, x')$. This may be achieved by expressing them as expansions about some arbitrary point $\bar{x}$ which is close to x and x'. We derive these here using both covariant and coordinate methods, each of which has its own advantages and disadvantages. The covariant expression is more elegant, allowing for compact formulas; however, these formulas hide complex terms such as high-order derivatives of the Weyl tensor which quickly become extremely time consuming to compute, even using computer tensor algebra packages such as GRTensorII [48] or xAct [49]. The coordinate approach is less elegant but more practical for explicit calculations, and it avoids the need to use tensor algebra. Independently of the approach taken, these expansions may be used to compute expansions of $U^{AB'}(x, x')$ and $V_n{}^{AB'}(x, x')$ (by substituting into the above equations), and hence of the singular field. In the case of covariant expansions, for explicit calculations, one must further expand the covariant expressions in coordinates, yielding an expression which may be directly compared with those obtained from the coordinate approach. The resulting expressions are long but are explicit functions of the coordinates, enabling them to be transformed directly into, for example, C functions; indeed,we give them in such form online [45].

A. Covariant approach

In this subsection, we briefly discuss our method for obtaining covariant expansions for the biscalars appearing in Eqs. (2.13), (2.18), and (2.23). We eventually seek expansions about a point $\bar{x}$ on the world line (which we may treat as fixed in the majority of this paper). In doing so, we follow the strategy of Haas and Poisson [4,50]:

(i) For the generic biscalar $A(x, z(\tau))$, write it as $A(\tau) \equiv A(x, z(\tau))$.

(ii) Compute the expansion about $\tau = \bar{\tau}$. This takes the form

$$A(\tau) = A(\bar{\tau}) + \dot{A}(\bar{\tau})(\tau - \bar{\tau}) + \frac{1}{2}\ddot{A}(\bar{\tau})(\tau - \bar{\tau})^2 + \cdots, \tag{3.7}$$

where $\dot{A}(\bar{\tau}) = A_{;\bar{a}} u^{\bar{a}}$, $\ddot{A}(\bar{\tau}) = A_{;\bar{a}\bar{b}} u^{\bar{a}} u^{\bar{b}}, \cdots$.

(iii) Compute the covariant expansions of the coefficients $\dot{A}(\bar{\tau}), \ddot{A}(\bar{\tau}), \cdots$ about $\bar{\tau}$.

(iv) Evaluate the expansion at the desired point, e.g., $A(x') = A(x, x')$.

(v) The resulting expansion depends on τ through the powers of $\tau - \bar{\tau}$. Replace these by their expansion in ϵ (about $\bar{x}$), the distance between x and the world line.

A key ingredient of this calculation is the expansion of $\Delta\tau \equiv \tau - \bar{\tau}$ in ϵ. The leading orders in this expansion were developed by Haas and Poisson [50] for the particular choices $\Delta\tau_+ \equiv v - \bar{\tau}$ and $\Delta\tau_- \equiv u - \bar{\tau}$. They found

$$\Delta\tau_{\pm} = (\bar{r} \pm \bar{s}) \mp \frac{(\bar{r} \pm \bar{s})^2}{6\bar{s}} R_{u\sigma u\sigma} \mp \frac{(\bar{r} \pm \bar{s})^2}{24\bar{s}} \times [(\bar{r} \pm \bar{s}) R_{u\sigma u\sigma|u} - R_{u\sigma u\sigma|\sigma}] + \mathcal{O}(\epsilon^5), \quad (3.8)$$

where $\bar{r} \equiv \sigma_{\bar{a}} u^{\bar{a}}$ and $\bar{s} \equiv (g^{\bar{a}\bar{b}} + u^{\bar{a}} u^{\bar{b}}) \sigma_{\bar{a}} \sigma_{\bar{b}}$. In Appendix B, we extend their calculation to the higher orders required in the present work. In the same appendix, we also apply the above method to compute covariant expansions of all quantities appearing in the expression for the singular field.

In order to obtain explicit expressions, we substitute in the coordinate expansion for $\sigma_{\bar{a}}$ (as discussed in Sec. IV B) along with the metric, Riemann tensor and 4-velocity (all evaluated at $\bar{x}$). In doing so, we only have to keep terms which contribute up to the required order and truncate any higher-order terms.

B. Coordinate approach

In this subsection, we follow a similar approach as described above, but in coordinates. We will start by considering two arbitrary points x and x' near $\bar{x}$. We will seek expansion where the coefficients are evaluated at $\bar{x}$, so we introduce the notation

$$\Delta x^a = x^a - x^{\bar{a}}, \qquad \delta x^{a'} = x^{a'} - x^a = x^{a'} - \Delta x^a - x^{\bar{a}}, \quad (3.9)$$

where we use the convention that the index carries the information about the point: $\bar{x}^a = x^{\bar{a}}$. In the calculations below, Δx^a and $\delta x^{a'}$ are both assumed to be small, of order ϵ.

The first item we require for our calculations is a coordinate expansion of the biscalar $\sigma(x, x')$, the Synge world function. We start with a standard coordinate series expansion about x, see, for example, Ref. [43] (note the difference in convention for Δx^a in that paper), to get

$$\sigma(x, x') = \frac{1}{2} g_{ab}(x) \delta x^{a'} \delta x^{b'} + A_{abc}(x) \delta x^{a'} \delta x^{b'} \delta x^{c'} + B_{abcd}(x) \delta x^{a'} \delta x^{b'} \delta x^{c'} \delta x^{d'} + \cdots. \quad (3.10)$$

The coefficients are readily determined in terms of derivatives of the metric at x by use of the defining identity $2\sigma = \sigma_{a'} \sigma^{a'}$; see Ref. [43]. To be explicit, the first few are given by

$$A_{abc}(x) = \frac{1}{4} g_{(ab,c)}(x)$$

$$B_{abcd}(x) = \frac{1}{12} g_{(ab,cd)}(x) - \frac{1}{24} g^{pq}(x) (g_{(ab,|p|}(x) g_{cd),q}(x) - 12 g_{(ab,|p|}(x) g_{|q|c,d)}(x) + 36 g_{p(a,b}(x) g_{|q|c,d)}(x)).$$

We now go one step further by expanding the coefficients about $\bar{x}$ to give a double expansion in Δx^a and $\delta x^{a'}$ with coefficients at $\bar{x}$. The first few terms are

$$\sigma(x, x') = \frac{1}{2} g_{\bar{a}\bar{b}}(\bar{x}) \delta x^{a'} \delta x^{b'} + \left[\frac{1}{2} g_{\bar{a}\bar{b},\bar{c}}(\bar{x}) \delta x^{a'} \delta x^{b'} \Delta x^c + A_{\bar{a}\bar{b}\bar{c}}(\bar{x}) \delta x^{a'} \delta x^{b'} \delta x^{c'}\right] + \left[\frac{1}{4} g_{\bar{a}\bar{b},\bar{c}\bar{d}}(\bar{x}) \delta x^{a'} \delta x^{b'} \Delta x^c \Delta x^d + A_{\bar{a}\bar{b}\bar{c},\bar{d}}(\bar{x}) \delta x^{a'} \delta x^{b'} \delta x^{c'} \Delta x^d + B_{\bar{a}\bar{b}\bar{c}\bar{d}}(\bar{x}) \delta x^{a'} \delta x^{b'} \delta x^{c'} \delta x^{d'}\right] + O(\epsilon^5), \quad (3.11)$$

where now we interpret $\delta x^{a'}$ as $x^{a'} - \Delta x^a - x^{\bar{a}}$ and we use square brackets to distinguish terms of different order in ϵ. Rather than disturb the flow here and throughout this section, we just give the first few terms of each expansion for a general metric to make the structure clear and give explicit expressions in Schwarzschild spacetime to much higher order in Appendix A.

Now that the coefficients are at the fixed point $\bar{x}$, it is straightforward to take derivatives of σ at x and x', for example,

$$\sigma_{a'} = g_{\bar{a}\bar{b}} \delta x^{b'} + [g_{\bar{a}\bar{b},\bar{c}} \delta x^{b'} \Delta x^c + 3A_{\bar{a}\bar{b}\bar{c}} \delta x^{b'} \delta x^{c'}] + \left[\frac{1}{2} g_{\bar{a}\bar{b},\bar{c}\bar{d}} \delta x^{b'} \Delta x^c \Delta x^d + 3A_{\bar{a}\bar{b}\bar{c},\bar{d}} \delta x^{b'} \delta x^{c'} \Delta x^d + 4B_{\bar{a}\bar{b}\bar{c}\bar{d}} \delta x^{b'} \delta x^{c'} \delta x^{d'}\right] + O(\epsilon^4), \quad (3.12)$$

$$\sigma_a = -\sigma_{a'} + \frac{1}{2} g_{\bar{b}\bar{c},\bar{a}} \delta x^{b'} \delta x^{c'} + \left[\frac{1}{2} g_{\bar{b}\bar{c},\bar{d}\bar{a}} \delta x^{b'} \delta x^{c'} \delta x^{d'} + A_{\bar{b}\bar{c}\bar{d},\bar{a}} \delta x^{b'} \delta x^{c'} \delta x^{d'}\right] + O(\epsilon^4), \quad (3.13)$$

$$\sigma_{a'b} = -g_{\bar{a}\bar{b}} - g_{\bar{a}\bar{b},\bar{c}} \Delta x^c + \left[(3A_{\bar{a}\bar{c}\bar{d},\bar{b}} - 12B_{\bar{a}\bar{b}\bar{c}\bar{d}}) \delta x^{c'} \delta x^{d'} - \frac{1}{2} g_{\bar{a}\bar{b},\bar{c}\bar{d}} \Delta x^c \Delta x^d\right] + O(\epsilon^3). \quad (3.14)$$

Likewise, we can calculate the Van Vleck-Morette determinant directly from its definition,

$$\Delta^{\frac{1}{2}}(x, x') = (-[-g(x)]^{-\frac{1}{2}} | -\sigma_{a'b}(x, x') | [-g(x')]^{-\frac{1}{2}})^{\frac{1}{2}}, \quad (3.15)$$

giving

$$\Delta^{\frac{1}{2}}(x, x') = 1 + \left[\frac{1}{2} g^{\bar{c}\bar{d}} (2g_{\bar{a}\bar{c},\bar{b}\bar{d}} - g_{\bar{a}\bar{b},\bar{c}\bar{d}} - g_{\bar{c}\bar{d},\bar{a}\bar{b}}) + \frac{1}{4} g^{\bar{c}\bar{d}} g^{\bar{e}\bar{f}} (g_{\bar{c}\bar{e},\bar{a}} g_{\bar{d}\bar{f},\bar{b}} + 2g_{\bar{c}\bar{a},\bar{b}} (g_{\bar{e}\bar{f},\bar{d}} - 2g_{\bar{d}\bar{e},\bar{f}}) + 2g_{\bar{a}\bar{c},\bar{e}} (g_{\bar{b}\bar{d},\bar{f}} - g_{\bar{b}\bar{f},\bar{d}}) - g_{\bar{a}\bar{b},\bar{c}} (g_{\bar{e}\bar{f},\bar{d}} - 2g_{\bar{d}\bar{e},\bar{f}}))\right] \delta x^a \delta x^b + O(\epsilon^3). \quad (3.16)$$

To obtain an expressions at $x_{(\mathrm{adv})}$ and $x_{(\mathrm{ret})}$, we allow $x^{a'}$ to be on the world line and again give it as an expansion around the point $\bar{x}$, as shown in Fig. 1. Writing $x^{a'}$ in terms of proper time τ gives[2]

$$x^{a'}(\tau) = x^{\bar{a}} + u^{\bar{a}}\Delta\tau + \frac{1}{2!}\dot{u}^{\bar{a}}\Delta\tau^2 + \frac{1}{3!}\ddot{u}^{\bar{a}}\Delta\tau^3 + \cdots, \tag{3.17}$$

where $u^{\bar{a}}$ is the four-velocity at the point $x^{\bar{a}}$, $\Delta\tau = \tau - \bar{\tau}$, and an overdot denotes differentiation with respect to τ.

We are interested in determining the points on the world line which are connected to x by a null geodesic; that is, we want to solve

$$\begin{aligned}\sigma(x^a, x^{a'}(\tau)) &= 0\\ &= \frac{1}{2}g_{\bar{a}\bar{b}}(u^{\bar{a}}\Delta\tau - \Delta x^a)(u^{\bar{b}}\Delta\tau - \Delta x^b) + \Big[\frac{1}{2}g_{\bar{a}\bar{b}}(u^{\bar{a}}\Delta\tau - \Delta x^a)\dot{u}^{\bar{b}}\Delta\tau^2 + \frac{1}{2}g_{\bar{a}\bar{b},\bar{c}}(u^{\bar{a}}\Delta\tau - \Delta x^a)(u^{\bar{b}}\Delta\tau - \Delta x^b)\Delta x^c\\ &\quad + \frac{1}{4}g_{\bar{a}\bar{b},\bar{c}}(\bar{x})(u^{\bar{a}}\Delta\tau - \Delta x^a)(u^{\bar{b}}\Delta\tau - \Delta x^b)(u^{\bar{c}}\Delta\tau - \Delta x^c)\Big] + O(\epsilon^4).\end{aligned} \tag{3.18}$$

By writing $\Delta\tau = \tau_1\epsilon + \tau_2\epsilon^2 + \tau_3\epsilon^3 + \cdots$ and explicitly inserting an ϵ in front of Δx^a, we may equate coefficients of powers of ϵ to obtain

$$\tau_1^2 + 2g_{\bar{a}\bar{b}}u^{\bar{a}}\Delta x^b\tau_1 - g_{\bar{a}\bar{b}}\Delta x^a\Delta x^b = 0, \tag{3.19}$$

$$\begin{aligned}g_{\bar{a}\bar{b}}u^{\bar{a}}(u^{\bar{a}}\tau_1 - \Delta x^a)\tau_2 = -\Big[&\frac{1}{2}g_{\bar{a}\bar{b}}(u^{\bar{a}}\tau_1 - \Delta x^a)\dot{u}^{\bar{b}}\tau_1^2 + \frac{1}{2}g_{\bar{a}\bar{b},\bar{c}}(u^{\bar{a}}\tau_1 - \Delta x^a)(u^{\bar{b}}\tau_1 - \Delta x^b)\Delta x^c\\ &+ \frac{1}{4}g_{\bar{a}\bar{b},\bar{c}}(\bar{x})(u^{\bar{a}}\tau_1 - \Delta x^a)(u^{\bar{b}}\tau_1 - \Delta x^b)(u^{\bar{c}}\tau_1 - \Delta x^c)\Big].\end{aligned} \tag{3.20}$$

Equation (3.19) is a quadratic with two real roots of opposite sign (for x spacelike separated from $\bar{x}$) corresponding to the first approximation to our points $x_{(\mathrm{adv})}$ and $x_{(\mathrm{ret})}$,

$$\tau_{1\pm} = g_{\bar{a}\bar{b}}u^{\bar{a}}\Delta x^b \pm \sqrt{(g_{\bar{a}\bar{b}}u^{\bar{a}}\Delta x^b)^2 + g_{\bar{a}\bar{b}}\Delta x^a\Delta x^b} \equiv \bar{r}_{(1)} \pm \rho, \tag{3.21}$$

where $\bar{r}_{(1)}$ is the leading-order term in the coordinate expansion of the quantity $\bar{r}$ appearing in our covariant expansions. Equation (3.20) is typical of the higher-order equations giving τ_n in terms of lower-order terms.

IV. EXPANSIONS OF THE SINGULAR FIELD

In this section, we list the covariant form of the singular field to order ϵ^4, where ϵ is the fundamental scale of separation, so, for example, $\bar{r}$, $\bar{s}$ and $\sigma(x,\bar{x})^{\bar{a}}$ are all of leading order ϵ. The coordinate forms of these expansions are too long to be useful in print form, so instead, they are available to download [45].

A. Scalar singular field

To $\mathcal{O}(\epsilon^4)$, the scalar singular field is

$$\begin{aligned}\Phi^{(S)} = q\Big\{&\frac{1}{\bar{s}} + \frac{\bar{r}^2 - \bar{s}^2}{6\bar{s}^3}C_{u\sigma u\sigma} + \frac{1}{24\bar{s}^3}[(\bar{r}^2 - 3\bar{s}^2)\bar{r}C_{u\sigma u\sigma;u} - (\bar{r}^2 - \bar{s}^2)C_{u\sigma u\sigma;\sigma}]\\ &+ \frac{1}{360\bar{s}^5}[\Phi^{(S)}]_{(3)} + \frac{1}{4320\bar{s}^5}[\Phi^{(S)}]_{(4)} + \mathcal{O}(\epsilon^5)\Big\},\end{aligned} \tag{4.1}$$

where

[2] In principal, this expression is valid for an arbitrary world line. However, here, we restrict ourselves to the case of geodesic motion and derive the higher derivative terms from the geodesic equations; we will address the case of accelerated motion in a follow-up work [51].

$$
\begin{aligned}
[\Phi^{(\mathrm{S})}]_{(3)} = {}& 15(\bar{r}^2 - \bar{s}^2)^2 C_{u\sigma u\sigma} C_{u\sigma u\sigma} + \bar{s}^2[(\bar{r}^2 - \bar{s}^2)(3C_{u\sigma u\sigma;\sigma\sigma} + 4C_{u\sigma\sigma\bar{a}} C_{u\sigma\sigma}{}^{\bar{a}}) \\
&+ (\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)(4C_{u\sigma u\bar{a}} C_{u\sigma u}{}^{\bar{a}} + 3C_{u\sigma u\sigma;uu}) + \bar{r}(\bar{r}^2 - 3\bar{s}^2)(16C_{u\sigma u}{}^{\bar{a}} C_{u\sigma\sigma\bar{a}} - 3C_{u\sigma u\sigma;u\sigma})] \\
&+ \bar{s}^4\{2C_u{}^{\bar{a}}{}_u{}^{\bar{b}}[(\bar{r}^2 + \bar{s}^2)C_{\sigma\bar{a}\sigma\bar{b}} + 2\bar{r}(\bar{r}^2 + 3\bar{s}^2)C_{u\bar{a}\sigma\bar{b}}] + 2C_u{}^{\bar{a}}{}_\sigma{}^{\bar{b}}[2\bar{r}C_{\sigma\bar{a}\sigma\bar{b}} + (\bar{r}^2 + \bar{s}^2)C_{u\bar{b}\sigma\bar{a}}] \\
&+ (\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)C_{u\bar{a}u\bar{b}} C_u{}^{\bar{a}}{}_u{}^{\bar{b}} + (\bar{r}^2 + \bar{s}^2)2C_{u\bar{a}\sigma\bar{b}} C_u{}^{\bar{a}}{}_\sigma{}^{\bar{b}} + C_{\sigma\bar{a}\sigma\bar{b}} C_\sigma{}^{\bar{a}}{}_\sigma{}^{\bar{b}}\} \qquad (4.2)
\end{aligned}
$$

and

$$
\begin{aligned}
[\Phi^{(\mathrm{S})}]_{(4)} = {}& 30C_{u\sigma u\sigma}[\bar{r}(3\bar{r}^4 - 10\bar{r}^2\bar{s}^2 + 15\bar{s}^4)C_{u\sigma u\sigma;u} - 30(\bar{r}^2 - \bar{s}^2)^2 C_{u\sigma u\sigma;\sigma}] + 2\bar{s}^2\{3C_{u\sigma u\sigma;uuu}\bar{r}(\bar{r}^4 - 10\bar{r}^2\bar{s}^2 - 15\bar{s}^4) \\
&+ 3\bar{r}(\bar{r}^2 - 3\bar{s}^2)C_{u\sigma u\sigma;u\sigma\sigma} - 3(\bar{r}^2 - \bar{s}^2)C_{u\sigma u\sigma;\sigma\sigma\sigma} - 3(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)C_{u\sigma u\sigma;uu\sigma} - 9\bar{r}(\bar{r}^4 - 10\bar{r}^2\bar{s}^2 - 15\bar{s}^4)C_{u\sigma u\bar{a};u} \\
&- C_{u\sigma\sigma}{}^{\bar{a}}[18C_{u\sigma\sigma\bar{a};\sigma}(\bar{r}^2 - \bar{s}^2) - \bar{r}(\bar{r}^2 - 3\bar{s}^2)(10C_{u\sigma\sigma\bar{a};u} - 16C_{u\sigma u\bar{a};\sigma} - 5C_{u\sigma u\sigma;\bar{a}}) - 30C_{u\sigma u\bar{a};u}(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)] \\
&- C_{u\sigma u}{}^{\bar{a}}[36\bar{r}(\bar{r}^2 - 3\bar{s}^2)C_{u\sigma\sigma\bar{a};\sigma} + (\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)(13C_{u\sigma u\bar{a};\sigma} + 5C_{u\sigma u\sigma;\bar{a}} - 25C_{u\sigma\sigma\bar{a};u})]\} \\
&- 12\bar{s}^4\{C_\sigma{}^{\bar{a}}{}_\sigma{}^{\bar{b}}[C_{\sigma\bar{a}\sigma\bar{b};\sigma} - \bar{r}(C_{\sigma\bar{a}\sigma\bar{b};u} - 2C_{u\bar{a}\sigma\bar{b};\sigma}) - (\bar{r}^2 + \bar{s}^2)(2C_{u\bar{a}\sigma\bar{b};u} - C_{u\bar{a}u\bar{b};\sigma}) - \bar{r}(\bar{r}^2 + 3\bar{s}^2)C_{u\bar{a}u\bar{b};u}] \\
&+ 2C_u{}^{\bar{a}}{}_\sigma{}^{\bar{b}}[C_{\sigma\bar{a}\sigma\bar{b};\sigma}\bar{r} + (\bar{r}^2 + \bar{s}^2)(C_{u\bar{a}\sigma\bar{b};\sigma} + C_{u\bar{b}\sigma\bar{a};\sigma} - C_{\sigma\bar{a}\sigma\bar{b};u}) + \bar{r}(\bar{r}^2 + 3\bar{s}^2)(C_{u\bar{a}u\bar{b};\sigma} - C_{u\bar{a}\sigma\bar{b};u} - C_{u\bar{b}\sigma\bar{a};u}) \\
&- C_{u\bar{a}u\bar{b}u}(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)] + C_u{}^{\bar{a}}{}_u{}^{\bar{b}}[(\bar{r}^2 + \bar{s}^2)C_{\sigma\bar{a}\sigma\bar{b};\sigma} - \bar{r}(\bar{r}^2 + 3\bar{s}^2)(C_{\sigma\bar{a}\sigma\bar{b};u} - 2C_{u\bar{a}\sigma\bar{b};\sigma}) \\
&+ (\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)(C_{u\bar{a}u\bar{b};\sigma} - 2C_{u\bar{a}\sigma\bar{b};u}) - C_{u\bar{a}u\bar{b};u}\bar{r}(\bar{r}^4 + 10\bar{r}^2\bar{s}^2 + 5\bar{s}^4)]\}. \qquad (4.3)
\end{aligned}
$$

B. Electromagnetic singular field

To $\mathcal{O}(\epsilon^4)$, the electromagnetic singular field is

$$
\begin{aligned}
A_a^{(\mathrm{S})} = {}& e g_a{}^{\bar{a}}\bigg(\frac{u_{\bar{a}}}{\bar{s}} + \frac{1}{6\bar{s}^3}[3\bar{r}\bar{s}^2 C_{\bar{a}uu\sigma} + C_{u\sigma u\sigma}(\bar{r}^2 - \bar{s}^2)u_{\bar{a}}] + \frac{1}{24\bar{s}^3}\{4\bar{s}^2(C_{\bar{a}uu\sigma;u} - \bar{r}C_{\bar{a}uu\sigma;\sigma}) \\
&+ [\bar{r}(\bar{r}^2 - 3\bar{s}^2)C_{u\sigma u\sigma;u} - (\bar{r}^2 - \bar{s}^2)C_{u\sigma u\sigma\sigma}]u_{\bar{a}}\} + \frac{1}{2880\bar{s}^5}[A_{\bar{a}}^{(\mathrm{S})}]_{(3)} + \frac{1}{25920\bar{s}^5}[A_{\bar{a}}^{(\mathrm{S})}]_{(4)} + \mathcal{O}(\epsilon^5)\bigg), \qquad (4.4)
\end{aligned}
$$

where

$$
\begin{aligned}
[A_{\bar{a}}^{(\mathrm{S})}]_{(3)} = {}& 120C_{u\sigma u\bar{a};u\sigma}\bar{s}^4(\bar{r}^2 + \bar{s}^2) - 120C_{u\sigma u\bar{a};\sigma\sigma}\bar{r}\bar{s}^4 + 120C_u{}^{\bar{c}}{}_u{}^{\bar{d}}C_{\sigma\bar{c}\bar{a}\bar{d}}\bar{r}\bar{s}^6 - 240C_{u\sigma u\bar{a}}C_{u\sigma u\sigma}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) \\
&- 360C_{u\sigma\bar{a}\bar{c}}C_{u\sigma u}{}^{\bar{c}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) - 120C_{u\sigma u\bar{a};uu}\bar{r}\bar{s}^4(\bar{r}^2 + 3\bar{s}^2) + 120\bar{s}^6C_u{}^{\bar{c}}{}_\sigma{}^{\bar{d}}(C_{u\bar{d}\bar{a}\bar{c}}\bar{r} + C_{\sigma\bar{d}\bar{a}\bar{c}}) \\
&+ 40C_{u\bar{c}\bar{a}\bar{d}}C_u{}^{\bar{c}}{}_u{}^{\bar{d}}\bar{s}^6(3\bar{r}^2 + \bar{s}^2) - 120C_{u\bar{a}\sigma\bar{c}}[C_{u\sigma\sigma}{}^{\bar{c}}\bar{r}\bar{s}^4 + 2C_{u\sigma u}{}^{\bar{c}}\bar{s}^4(\bar{r}^2 + \bar{s}^2)] - 120C_{u\bar{a}u\bar{c}}[C_{u\sigma\sigma}{}^{\bar{c}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) \\
&+ C_{u\sigma u}{}^{\bar{c}}\bar{r}\bar{s}^4(\bar{r}^2 + 3\bar{s}^2)] + \{8C_{\sigma\bar{c}\sigma\bar{d}}C_\sigma{}^{\bar{c}}{}_\sigma{}^{\bar{d}}\bar{s}^4 - 5C_{\bar{c}\bar{d}\bar{e}\bar{f}}C^{\bar{c}\bar{d}\bar{e}\bar{f}}\bar{s}^8 - 24C_{u\sigma u\sigma;u\sigma}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) \\
&+ 128C_{u\sigma u}{}^{\bar{c}}C_{u\sigma\sigma\bar{c}}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) + 24C_{u\sigma u\sigma;\sigma\sigma}\bar{s}^2(\bar{r}^2 - \bar{s}^2) + 32C_{u\sigma\sigma\bar{c}}C_{u\sigma\sigma}{}^{\bar{c}}\bar{s}^2(\bar{r}^2 - \bar{s}^2) \\
&+ 120C_{u\sigma u\sigma}C_{u\sigma u\sigma}(\bar{r}^2 - \bar{s}^2)^2 + 16C_u{}^{\bar{c}}{}_u{}^{\bar{d}}C_{\sigma\bar{c}\sigma\bar{d}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) + 24C_{u\sigma u\sigma;uu}\bar{s}^2(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4) \\
&+ 32C_{u\sigma u\bar{c}}C_{u\sigma u}{}^{\bar{c}}\bar{s}^2(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4) + 8C_{u\bar{c}u\bar{d}}C_u{}^{\bar{c}}{}_u{}^{\bar{d}}\bar{s}^4(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4) \\
&+ 16\bar{s}^4C_u{}^{\bar{c}}{}_\sigma{}^{\bar{d}}[2C_{\sigma\bar{c}\sigma\bar{d}}\bar{r} + C_{u\bar{d}\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2)] + 16\bar{s}^4C_{u\bar{c}\sigma\bar{d}}[C_u{}^{\bar{c}}{}_\sigma{}^{\bar{d}}(\bar{r}^2 + \bar{s}^2) + 2C_u{}^{\bar{c}}{}_u{}^{\bar{d}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)]\}u_{\bar{a}} \qquad (4.5)
\end{aligned}
$$

and

$$
\begin{aligned}
[A_{\bar{a}}^{(\mathrm{S})}]_{(4)} = {}& 216C_{u\sigma u\bar{a};\sigma\sigma\sigma}\bar{r}\bar{s}^4 + 540C_{u\sigma u\sigma;\sigma}C_{u\sigma u\bar{a}}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) + 720C_{u\sigma u\bar{a};\sigma}C_{u\sigma u\sigma}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) \\
&- 216C_{u\sigma u\bar{a};u\sigma\sigma}\bar{s}^4(\bar{r}^2 + \bar{s}^2) + 1512C_{u\sigma\bar{a}}{}^{\bar{c}}{}_{;\sigma}C_{u\sigma u\bar{c}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) + 216C_{u\sigma u\bar{a};uu\sigma}\bar{r}\bar{s}^4(\bar{r}^2 + 3\bar{s}^2) \\
&- 864C_{u\sigma\bar{a}}{}^{\bar{c}}{}_{;u}C_{u\sigma u\bar{c}}\bar{r}\bar{s}^4(\bar{r}^2 + 3\bar{s}^2) - 216C_{u\sigma u\bar{a};uuu}\bar{s}^4(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4) - 540C_{u\sigma u\sigma;u}C_{u\sigma u\bar{a}}\bar{s}^2(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4) \\
&+ 720C_{u\sigma u\bar{a};u}C_{u\sigma u\sigma}\bar{s}^2(-\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + 3\bar{s}^4) - 432\bar{s}^6C_\sigma{}^{\bar{c}}{}_{\bar{a}}{}^{\bar{d}}{}_{;\sigma}(C_{u\bar{c}u\bar{d}}\bar{r} + C_{u\bar{d}\sigma\bar{c}}) - 648\bar{s}^6C_u{}^{\bar{c}}{}_\sigma{}^{\bar{d}}{}_{;\sigma}(C_{u\bar{d}\bar{a}\bar{c}}\bar{r} + C_{\sigma\bar{d}\bar{a}\bar{c}}) \\
&+ 16\bar{s}^8C_{u\bar{a}}{}^{\bar{c}\bar{d};\bar{e}}(C_{u\bar{c}\bar{d}\bar{e}}\bar{r} - C_{u\bar{e}\bar{c}\bar{d}}\bar{r} + C_{\sigma\bar{c}\bar{d}\bar{e}} - C_{\sigma\bar{e}\bar{c}\bar{d}}) + 432\bar{s}^4C_{u\sigma\sigma}{}^{\bar{c}}{}_{;\sigma}[C_{u\bar{a}\sigma\bar{c}}\bar{r} + C_{u\bar{a}u\bar{c}}(\bar{r}^2 + \bar{s}^2)] \\
&+ 504\bar{s}^4C_{u\bar{a}\sigma}{}^{\bar{c}}{}_{;\sigma}[C_{u\sigma\sigma\bar{c}}\bar{r} + 2C_{u\sigma u\bar{c}}(\bar{r}^2 + \bar{s}^2)] - 324\bar{s}^4C_{u\sigma\sigma}{}^{\bar{c}}{}_{;u}[C_{u\bar{a}\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) + C_{u\bar{a}u\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)] \\
&+ 108\bar{s}^4C_{u\sigma u}{}^{\bar{c}}{}_{;\sigma}[5C_{u\bar{a}\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) + 6C_{u\sigma\bar{a}\bar{c}}(\bar{r}^2 + \bar{s}^2) + 3C_{u\bar{a}u\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)] - 96\bar{s}^4C_{u\bar{a}\sigma}{}^{\bar{c}}{}_{;u}[3C_{u\sigma\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) \\
&+ 8C_{u\sigma u\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)] + 96\bar{s}^4C_{u\sigma u\bar{a}}{}^{;\bar{c}}[3C_{u\sigma\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) + 4C_{u\sigma u\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)]
\end{aligned}
$$

$$
\begin{aligned}
&+24\bar{s}^4C_{u\bar{a}u}{}^{\bar{c}}{}_{;\sigma}[9C_{u\sigma\sigma\bar{c}}(\bar{r}^2+\bar{s}^2)+17C_{u\sigma u\bar{c}}\bar{r}(\bar{r}^2+3\bar{s}^2)]-216\bar{s}^4C_{u\sigma u}{}^{\bar{c}}{}_{;u}[3C_{u\bar{a}\sigma\bar{c}}\bar{r}(\bar{r}^2+3\bar{s}^2)+6C_{u\sigma\bar{a}\bar{c}}\bar{r}(\bar{r}^2+3\bar{s}^2)\\
&+2C_{u\bar{a}u\bar{c}}\bar{s}^4(\bar{r}^4+6\bar{r}^2\bar{s}^2+\bar{s}^4)]-72\bar{s}^4C_{u\bar{a}u}{}^{\bar{c}}{}_{;u}[8C_{u\sigma\sigma\bar{c}}\bar{r}(\bar{r}^2+3\bar{s}^2)+7C_{u\sigma u\bar{c}}(\bar{r}^4+6\bar{r}^2\bar{s}^2+\bar{s}^4)]\\
&-216\bar{s}^6C_{u}{}^{\bar{c}}{}_{u}{}^{\bar{d}}{}_{;\sigma}[3C_{\sigma\bar{c}\bar{a}\bar{d}}\bar{r}+C_{u\bar{c}\bar{a}\bar{d}}(3\bar{r}^2+\bar{s}^2)]-144\bar{s}^6C_{u}{}^{\bar{c}}{}_{\bar{a}}{}^{\bar{d}}{}_{;\sigma}[3C_{u\bar{d}\sigma\bar{c}}\bar{r}+C_{u\bar{c}u\bar{d}}(3\bar{r}^2+\bar{s}^2)]\\
&+48\bar{s}^6C_{u\bar{a}\sigma}{}^{\bar{c};\bar{d}}[3C_{u\bar{c}\sigma\bar{d}}\bar{r}+3C_{u\bar{d}\sigma\bar{c}}\bar{r}+3C_{\sigma\bar{c}\sigma\bar{d}}+C_{u\bar{c}u\bar{d}}(3\bar{r}^2+\bar{s}^2)]+216\bar{s}^6C_{\sigma}{}^{\bar{c}}{}_{\bar{a}}{}^{\bar{d}}{}_{;u}[3C_{u\bar{d}\sigma\bar{c}}\bar{r}+C_{u\bar{c}u\bar{d}}(3\bar{r}^2+\bar{s}^2)]\\
&+144\bar{s}^6C_{u}{}^{\bar{c}}{}_{\sigma}{}^{\bar{d}}{}_{;u}[3C_{\sigma\bar{d}\bar{a}\bar{c}}\bar{r}+C_{u\bar{d}\bar{a}\bar{c}}(3\bar{r}^2+\bar{s}^2)]+48\bar{s}^6C_{u\bar{a}u}{}^{\bar{c};\bar{d}}[3\bar{r}C_{\sigma\bar{c}\sigma\bar{d}}+3\bar{r}C_{u\bar{c}u\bar{d}}(\bar{r}^2+\bar{s}^2)+C_{u\bar{c}\sigma\bar{d}}(3\bar{r}^2+\bar{s}^2)\\
&+C_{u\bar{d}\sigma\bar{c}}(3\bar{r}^2+\bar{s}^2)]+216\bar{s}^6C_{u}{}^{\bar{c}}{}_{\bar{a}}{}^{\bar{d}}{}_{;u}[3C_{u\bar{c}u\bar{d}}\bar{r}(\bar{r}^2+\bar{s}^2)+C_{u\bar{d}\sigma\bar{c}}(3\bar{r}^2+\bar{s}^2)]+144\bar{s}^6C_{u}{}^{\bar{c}}{}_{u}{}^{\bar{d}}{}_{;u}[3C_{u\bar{c}\bar{a}\bar{d}}\bar{r}(\bar{r}^2+\bar{s}^2)\\
&+C_{\sigma\bar{c}\bar{a}\bar{d}}(3\bar{r}^2+\bar{s}^2)]+\{45C^{\bar{c}\bar{d}\bar{e}\bar{f}}{}_{;\sigma}C_{\bar{c}\bar{d}\bar{e}\bar{f}}\bar{s}^8-45C^{\bar{c}\bar{d}\bar{e}\bar{f}}{}_{;u}C_{\bar{c}\bar{d}\bar{e}\bar{f}}\bar{r}\bar{s}^8+36C_{u\sigma u\sigma;u\sigma\sigma}\bar{r}\bar{s}^2(\bar{r}^2-3\bar{s}^2)\\
&-540C_{u\sigma u\sigma;\sigma}C_{u\sigma u\sigma}(\bar{r}^2-\bar{s}^2)^2-36C_{u\sigma u\sigma;\sigma\sigma\sigma}\bar{s}^2(\bar{r}^2-\bar{s}^2)+36C_{u\sigma u\sigma;uuu}\bar{r}\bar{s}^2(\bar{r}^4-10\bar{r}^2\bar{s}^2-15\bar{s}^4)\\
&-36C_{u\sigma u\sigma;uu\sigma}\bar{s}^2(\bar{r}^4-6\bar{r}^2\bar{s}^2-3\bar{s}^4)+180C_{u\sigma u\sigma;u}C_{u\sigma u\sigma}(3\bar{r}^5-10\bar{r}^3\bar{s}^2+15\bar{r}\bar{s}^4)-216\bar{s}^2C_{u\sigma\sigma}{}^{\bar{c}}{}_{;\sigma}[2\bar{r}C_{u\sigma u\bar{c}}(\bar{r}^2-3\bar{s}^2)\\
&-2C_{u\sigma\sigma\bar{c}}(-\bar{r}^2+\bar{s}^2)]-72\bar{s}^4C_{\sigma}{}^{\bar{c}}{}_{\sigma}{}^{\bar{d}}{}_{;\sigma}[2C_{u\bar{c}\sigma\bar{d}}\bar{r}+C_{\sigma\bar{c}\sigma\bar{d}}+C_{u\bar{c}u\bar{d}}(\bar{r}^2+\bar{s}^2)]-144\bar{s}^4C_{u}{}^{\bar{c}}{}_{\sigma}{}^{\bar{d}}{}_{;\sigma}[C_{\sigma\bar{c}\sigma\bar{d}}\bar{r}+C_{u\bar{c}\sigma\bar{d}}(\bar{r}^2+\bar{s}^2)\\
&+C_{u\bar{d}\sigma\bar{c}}(\bar{r}^2+\bar{s}^2)+C_{u\bar{c}u\bar{d}}\bar{r}(\bar{r}^2+3\bar{s}^2)]+72\bar{s}^4C_{\sigma}{}^{\bar{c}}{}_{\sigma}{}^{\bar{d}}{}_{;u}[C_{\sigma\bar{c}\sigma\bar{d}}\bar{r}+2C_{u\bar{c}\sigma\bar{d}}(\bar{r}^2+\bar{s}^2)+C_{u\bar{c}u\bar{d}}\bar{r}(\bar{r}^2+3\bar{s}^2)]\\
&+60\bar{s}^2C_{u\sigma\sigma}{}^{\bar{c}}{}_{;u}[2C_{u\sigma\sigma\bar{c}}\bar{r}(\bar{r}^2-3\bar{s}^2)+5C_{u\sigma u\bar{c}}(\bar{r}^4-6\bar{r}^2\bar{s}^2-3\bar{s}^4)]+72\bar{s}^2C_{u\sigma u}{}^{\bar{c}}{}_{;u}[3C_{u\sigma u\bar{c}}\bar{r}(\bar{r}^4-10\bar{r}^2\bar{s}^2-15\bar{s}^4)\\
&+5C_{u\sigma\sigma\bar{c}}(\bar{r}^4-6\bar{r}^2\bar{s}^2-3\bar{s}^4)]-72\bar{s}^4C_{u}{}^{\bar{c}}{}_{u}{}^{\bar{d}}{}_{;\sigma}[C_{\sigma\bar{c}\sigma\bar{d}}(\bar{r}^2+\bar{s}^2)+2C_{u\bar{c}\sigma\bar{d}}\bar{r}(\bar{r}^2+3\bar{s}^2)+C_{u\bar{c}u\bar{d}}(\bar{r}^4+6\bar{r}^2\bar{s}^2+\bar{s}^4)]\\
&+144\bar{s}^4C_{u}{}^{\bar{c}}{}_{\sigma}{}^{\bar{d}}{}_{;u}[C_{\sigma\bar{c}\sigma\bar{d}}(\bar{r}^2+\bar{s}^2)+C_{u\bar{c}\sigma\bar{d}}\bar{r}(\bar{r}^2+3\bar{s}^2)+C_{u\bar{d}\sigma\bar{c}}\bar{r}(\bar{r}^2+3\bar{s}^2)+C_{u\bar{c}u\bar{d}}(\bar{r}^4+6\bar{r}^2\bar{s}^2+\bar{s}^4)]\\
&-60\bar{s}^2C_{u\sigma u\sigma}{}^{;\bar{c}}[C_{u\sigma\sigma\bar{c}}\bar{r}(\bar{r}^2-3\bar{s}^2)+C_{u\sigma u\bar{c}}(\bar{r}^4-6\bar{r}^2\bar{s}^2-3\bar{s}^4)]-12\bar{s}^2C_{u\sigma u}{}^{\bar{c}}{}_{;\sigma}[16C_{u\sigma\sigma\bar{c}}\bar{r}(\bar{r}^2-3\bar{s}^2)\\
&+13C_{u\sigma u\bar{c}}(\bar{r}^4-6\bar{r}^2\bar{s}^2-3\bar{s}^4)]+72\bar{s}^4C_{u}{}^{\bar{c}}{}_{u}{}^{\bar{d}}{}_{;u}[C_{\sigma\bar{c}\sigma\bar{d}}\bar{r}(\bar{r}^2+3\bar{s}^2)+2C_{u\bar{c}\sigma\bar{d}}(\bar{r}^4+6\bar{r}^2\bar{s}^2+\bar{s}^4)\\
&+C_{u\bar{c}u\bar{d}}\bar{r}(\bar{r}^4+10\bar{r}^2\bar{s}^2+5\bar{s}^4)]\}u_{\bar{a}}. \qquad (4.6)
\end{aligned}
$$

C. Gravitational singular field

To $\mathcal{O}(\epsilon^4)$, the gravitational singular field is

$$
\begin{aligned}
\bar{h}^{(\mathrm{S})}_{ab} = {}& 4\mu g_a{}^{\bar{a}}g_b{}^{\bar{b}}\bigg(\frac{u_{\bar{a}}u_{\bar{b}}}{\bar{s}}+\frac{1}{6\bar{s}^3}[(\bar{r}^2-\bar{s}^2)C_{u\sigma u\sigma}u_{\bar{a}}u_{\bar{b}}-6\bar{r}\bar{s}^2C_{u\sigma u(\bar{a}}u_{\bar{b})}-6\bar{s}^4C_{\bar{a}u\bar{b}u}]+\frac{1}{24\bar{s}^3}\{12\bar{s}^4(C_{\bar{a}u\bar{b}u;\sigma}-\bar{r}C_{\bar{a}u\bar{b}u;u})\\
&+8\bar{s}^2[u_{(\bar{a}}C_{\bar{b})uu\sigma;u}(\bar{r}^2+\bar{s}^2)-\bar{r}u_{(\bar{a}}C_{\bar{b})uu\sigma;\sigma}]+u_{\bar{a}}u_{\bar{b}}[\bar{r}(\bar{r}^2-3\bar{s}^2)C_{u\sigma u\sigma;u}-(\bar{r}^2-\bar{s}^2)C_{u\sigma u\sigma;\sigma}]\}\\
&+\frac{1}{1440\bar{s}^5}[\bar{h}^{(\mathrm{S})}_{\bar{a}\bar{b}}]_{(3)}+\frac{1}{6480\bar{s}^5}[\bar{h}^{(\mathrm{S})}_{\bar{a}\bar{b}}]_{(4)}+\mathcal{O}(\epsilon^5)\bigg), \qquad (4.7)
\end{aligned}
$$

where

$$
\begin{aligned}
[\bar{h}^{(\mathrm{S})}_{\bar{a}\bar{b}}]_{(3)} = {}& -240C_{\bar{a}u\bar{b}u;\sigma\sigma}\bar{s}^6+240C_{\bar{a}u\bar{b}u;u\sigma}\bar{r}\bar{s}^6+480C_{\bar{a}}{}^{\bar{c}}{}_{u\sigma}C_{\bar{b}uu\bar{c}}\bar{r}\bar{s}^6-120C_{\bar{a}u\sigma}{}^{\bar{c}}C_{\bar{b}u\sigma\bar{c}}\bar{s}^6+960C_{\bar{a}u\bar{b}}{}^{\bar{c}}C_{u\sigma u\bar{c}}\bar{r}\bar{s}^6\\
&+40C_{\bar{a}u\bar{b}u}{}^{;\bar{c}}{}_{\bar{c}}\bar{s}^8+20C_{\bar{a}u}{}^{\bar{c}\bar{d}}C_{\bar{b}u\bar{c}\bar{d}}\bar{s}^8+240C_{\bar{a}}{}^{\bar{c}}{}_{\bar{b}}{}^{\bar{d}}C_{u\bar{c}u\bar{d}}\bar{s}^8+360C_{\bar{a}uu\sigma}C_{\bar{b}uu\sigma}\bar{s}^4(\bar{r}^2+\bar{s}^2)\\
&+240C_{\bar{a}u\bar{b}u}C_{u\sigma u\sigma}\bar{s}^4(\bar{r}^2+\bar{s}^2)-80C_{\bar{a}u\bar{b}u;uu}\bar{s}^6(3\bar{r}^2+\bar{s}^2)-40C_{\bar{a}uu}{}^{\bar{c}}\bar{s}^6[6C_{\bar{b}u\sigma\bar{c}}\bar{r}+C_{\bar{b}uu\bar{c}}(3\bar{r}^2+\bar{s}^2)]\\
&+\{120C_{\bar{b}uu\sigma;\sigma\sigma}\bar{r}\bar{s}^4+240C_{\bar{b}uu\sigma}C_{u\sigma u\sigma}\bar{r}\bar{s}^2(\bar{r}^2-3\bar{s}^2)-120C_{\bar{b}uu\sigma;u\sigma}\bar{s}^4(\bar{r}^2+\bar{s}^2)-360C_{\bar{b}}{}^{\bar{c}}{}_{u\sigma}C_{u\sigma u\bar{c}}\bar{s}^4(\bar{r}^2+\bar{s}^2)\\
&+120C_{\bar{b}uu\sigma;uu}\bar{r}\bar{s}^4(\bar{r}^2+3\bar{s}^2)+120C_{\bar{b}}{}^{\bar{c}}{}_{\sigma}{}^{\bar{d}}(C_{u\bar{c}u\bar{d}}\bar{r}+C_{u\bar{c}\sigma\bar{d}})\bar{s}^6+120C_{\bar{b}u\sigma}{}^{\bar{c}}[C_{u\sigma\sigma\bar{c}}\bar{r}+2C_{u\sigma u\bar{c}}(\bar{r}^2+\bar{s}^2)]\bar{s}^4\\
&+40C_{\bar{b}}{}^{\bar{c}}{}_{u}{}^{\bar{d}}[3C_{u\bar{c}\sigma\bar{d}}\bar{r}+C_{u\bar{c}u\bar{d}}(3\bar{r}^2+\bar{s}^2)]\bar{s}^6+120C_{\bar{b}uu}{}^{\bar{c}}[C_{u\sigma\sigma\bar{c}}(\bar{r}^2+\bar{s}^2)+C_{u\sigma u\bar{c}}\bar{r}(\bar{r}^2+3\bar{s}^2)]\bar{s}^4\}u_{\bar{a}}\\
&+\{4C_{\sigma\bar{c}\sigma\bar{d}}C_{\sigma}{}^{\bar{c}}{}_{\sigma}{}^{\bar{d}}\bar{s}^4-5C_{\bar{c}\bar{d}\bar{e}\bar{f}}C^{\bar{c}\bar{d}\bar{e}\bar{f}}\bar{s}^8+12C_{u\sigma u\sigma;\sigma\sigma}\bar{s}^2(\bar{r}^2-\bar{s}^2)+16C_{u\sigma\sigma\bar{c}}C_{u\sigma\sigma}{}^{\bar{c}}\bar{s}^2(\bar{r}^2-\bar{s}^2)\\
&-12C_{u\sigma u\sigma;u\sigma}\bar{r}\bar{s}^2(\bar{r}^2-3\bar{s}^2)+64C_{u\sigma u}{}^{\bar{c}}C_{u\sigma\sigma\bar{c}}\bar{r}\bar{s}^2(\bar{r}^2-3\bar{s}^2)+60C^2_{u\sigma u\sigma}(\bar{r}^2-\bar{s}^2)^2+8C_{u}{}^{\bar{c}}{}_{u}{}^{\bar{d}}C_{\sigma\bar{c}\sigma\bar{d}}\bar{s}^4(\bar{r}^2+\bar{s}^2)\\
&+12C_{u\sigma u\sigma;uu}\bar{s}^2(\bar{r}^4-6\bar{r}^2\bar{s}^2-3\bar{s}^4)+16C_{u\sigma u\bar{c}}C_{u\sigma u}{}^{\bar{c}}\bar{s}^2(\bar{r}^4-6\bar{r}^2\bar{s}^2-3\bar{s}^4)+4C_{u\bar{c}u\bar{d}}C_{u}{}^{\bar{c}}{}_{u}{}^{\bar{d}}\bar{s}^4(\bar{r}^4+6\bar{r}^2\bar{s}^2+\bar{s}^4)\\
&+8C_{u}{}^{\bar{c}}{}_{\sigma}{}^{\bar{d}}[2C_{\sigma\bar{c}\sigma\bar{d}}\bar{r}+C_{u\bar{d}\sigma\bar{c}}(\bar{r}^2+\bar{s}^2)]\bar{s}^4+8C_{u\bar{c}\sigma\bar{d}}[C_{u}{}^{\bar{c}}{}_{\sigma}{}^{\bar{d}}(\bar{r}^2+\bar{s}^2)+2C_{u}{}^{\bar{c}}{}_{u}{}^{\bar{d}}\bar{r}(\bar{r}^2+3\bar{s}^2)]\bar{s}^4\}u_{\bar{a}}u_{\bar{b}} \qquad (4.8)
\end{aligned}
$$

and

$$
\begin{aligned}
[\bar h^{(S)}_{\bar a\bar b}]_{(4)} ={}& 270(C_{\bar a u\bar b u;\sigma\sigma\sigma} - C_{\bar a u\bar b u;u\sigma\sigma}\bar r)\bar s^6 - 1080(C_{u\sigma u}{}^{\bar c}{}_{;\sigma}C_{\bar a u\bar b\bar c} + C_{\bar a}{}^{\bar c}{}_{u\sigma;\sigma}C_{\bar b uu\bar c})\bar r\bar s^6 - 2700C_{\bar a u\bar b}{}^{\bar c}{}_{;\sigma}\bar r C_{u\sigma u\bar c}\bar s^6 \\
&- 90C_{\bar a u\bar b u}{}^{;\bar c}{}_{\bar c\sigma}\bar s^8 + 90C_{\bar a u\bar b u}{}^{;\bar c}{}_{\bar c u}\bar r\bar s^8 - 360C_u{}^{\bar c}{}_u{}^{\bar d}{}_{;\sigma}C_{\bar a\bar c\bar b\bar d}\bar s^8 + 720C_u{}^{\bar c}{}_u{}^{\bar d}{}_{;u}\bar r C_{\bar a\bar c\bar b\bar d}\bar s^8 + 180C_u{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;\bar d}C_{\bar a u\bar b\bar c}\bar s^8 \\
&+ 180C_u{}^{\bar c}{}_u{}^{\bar d}{}_{;\bar c}\bar r C_{\bar a u\bar b\bar d}\bar s^8 - 120C_{\bar a u}{}^{\bar c\bar d}{}_{;\sigma}C_{\bar b u\bar c\bar d}\bar s^8 - 60C_{\bar a u\sigma}{}^{\bar c;\bar d}C_{\bar b u\bar c\bar d}\bar s^8 + 60C_{\bar a u}{}^{\bar c\bar d}{}_{;u}\bar r C_{\bar b u\bar c\bar d}\bar s^8 - 180C_{\bar a}{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;\bar d}C_{\bar b uu\bar c}\bar s^8 \\
&- 180C_{\bar a}{}^{\bar c}{}_u{}^{\bar d}{}_{;\bar d}\bar r C_{\bar b uu\bar c}\bar s^8 - 720C_{\bar a}{}^{\bar c}{}_{\bar b}{}^{\bar d}{}_{;\sigma}C_{u\bar c u\bar d}\bar s^8 + 360C_{\bar a}{}^{\bar c}{}_{\bar b}{}^{\bar d}{}_{;u}\bar r C_{u\bar c u\bar d}\bar s^8 - 270C_{u\sigma u\sigma;\sigma}C_{\bar a u\bar b u}\bar s^4(\bar r^2 + \bar s^2) \\
&- 1080C_{\bar a uu\sigma;\sigma}C_{\bar b uu\sigma}\bar s^4(\bar r^2 + \bar s^2) - 540C_{\bar a u\bar b u;\sigma}C_{u\sigma u\sigma}\bar s^4(\bar r^2 + \bar s^2) - 270C_{\bar a u\bar b u;uuu}\bar r\bar s^6(\bar r^2 + \bar s^2) \\
&+ 270(C_{u\sigma u\sigma;u}C_{\bar a u\bar b u} + 4C_{\bar a uu\sigma;u}C_{\bar b uu\sigma} + 2C_{\bar a u\bar b u;u}C_{u\sigma u\sigma})\bar r\bar s^4(\bar r^2 + 3\bar s^2) + 540C_{\bar a u\sigma}{}^{\bar c}{}_{;\sigma}(C_{\bar b uu\bar c}\bar r + C_{\bar b u\sigma\bar c})\bar s^6 \\
&+ 90(C_{\bar a u\bar b u;uu\sigma} + 6C_{u\sigma u}{}^{\bar c}{}_{;u}C_{\bar a u\bar b\bar c} + 2C_{\bar a}{}^{\bar c}{}_{u\sigma;u}C_{\bar b uu\bar c} + 6C_{\bar a u\bar b}{}^{\bar c}{}_{;u}C_{u\sigma u\bar c})(3\bar r^2 + \bar s^2)\bar s^6 + 60C_{\bar a uu}{}^{\bar c;\bar d}(3C_{\bar b\bar c u\bar d}\bar r \\
&+ 3C_{\bar b\bar c\sigma\bar d} - C_{\bar b u\bar c\bar d}\bar r)\bar s^8 + 180C_{\bar a u\bar b}{}^{\bar c;\bar d}(C_{u\bar c u\bar d}\bar r + C_{u\bar c\sigma\bar d})\bar s^8 - 180C_{\bar a u\sigma}{}^{\bar c}{}_{;u}[3C_{\bar b u\sigma\bar c}\bar r + C_{\bar b uu\bar c}(3\bar r^2 + \bar s^2)]\bar s^6 \\
&+ 180C_{\bar a uu}{}^{\bar c}{}_{;\sigma}[-3\bar r C_{\bar b\bar c u\sigma} + 3\bar r C_{\bar b u\sigma\bar c} + C_{\bar b uu\bar c}(3\bar r^2 + \bar s^2)]\bar s^6 + 90C_{\bar a u\bar b u}{}^{;\bar c}[3C_{u\sigma\sigma\bar c}\bar r + 2C_{u\sigma u\bar c}(3\bar r^2 + \bar s^2)]\bar s^6 \\
&+ 180C_{\bar a uu}{}^{\bar c}{}_{;u}[-3C_{\bar b uu\bar c}\bar r(\bar r^2 + \bar s^2) + 3C_{\bar b\bar c u\sigma}(3\bar r^2 + \bar s^2) - C_{\bar b u\sigma\bar c}(3\bar r^2 + \bar s^2)]\bar s^6 + \{-108C_{\bar b uu\sigma;\sigma\sigma\sigma}\bar r\bar s^4 \\
&+ 36C_u{}^{\bar c}{}_\sigma{}^{\bar d;\bar e}C_{\bar b\bar c\bar d\bar e}\bar s^8 + 36C_u{}^{\bar c}{}_u{}^{\bar d;\bar e}\bar r C_{\bar b\bar c\bar d\bar e}\bar s^8 + 12C_{\bar b}{}^{\bar c\bar d\bar e}{}_{;\sigma}C_{u\bar c\bar d\bar e}\bar s^8 + 24C_{\bar b}{}^{\bar c}{}_\sigma{}^{\bar d;\bar e}C_{u\bar c\bar d\bar e}\bar s^8 + 12C_{\bar b}{}^{\bar c\bar d\bar e}{}_{;u}C_{u\bar c\bar d\bar e}\bar r\bar s^8 \\
&+ 24C_{\bar b}{}^{\bar c}{}_u{}^{\bar d;\bar e}C_{u\bar c\bar d\bar e}\bar r\bar s^8 - 90(3C_{u\sigma u\sigma;\sigma}C_{\bar b uu\sigma} + 4C_{\bar b uu\sigma;\sigma}C_{u\sigma u\sigma})\bar r\bar s^2(\bar r^2 - 3\bar s^2) + 108(C_{\bar b uu\sigma;u\sigma\sigma} \\
&+ 7C_{\bar b}{}^{\bar c}{}_{u\sigma;\sigma}C_{u\sigma u\bar c})\bar s^4(\bar r^2 + \bar s^2) - 108(C_{\bar b uu\sigma;uu\sigma} + 4C_{\bar b}{}^{\bar c}{}_{u\sigma;u}C_{u\sigma u\bar c})\bar r\bar s^4(\bar r^2 + 3\bar s^2) + 90(3C_{u\sigma u\sigma;u}C_{\bar b uu\sigma} \\
&+ 4C_{\bar b uu\sigma;u}C_{u\sigma u\sigma})\bar s^2(\bar r^4 - 6\bar r^2\bar s^2 - 3\bar s^4) + 108C_{\bar b uu\sigma;uuu}\bar s^4(\bar r^4 + 6\bar r^2\bar s^2 + \bar s^4) - 216C_u{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;\sigma}(C_{\bar b\bar c u\bar d}\bar r + C_{\bar b\bar c\sigma\bar d})\bar s^6 \\
&- 324C_{\bar b}{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;\sigma}(C_{u\bar c u\bar d}\bar r + C_{u\bar c\sigma\bar d})\bar s^6 + 8C_{\bar b u}{}^{\bar c\bar d;\bar e}(C_{u\bar c\bar d\bar e}\bar r - C_{u\bar e\bar c\bar d}\bar r + C_{\sigma\bar c\bar d\bar e} - C_{\sigma\bar e\bar c\bar d})\bar s^8 \\
&- 216C_{u\sigma\sigma}{}^{\bar c}{}_{;\sigma}[C_{\bar b u\sigma\bar c}\bar r + C_{\bar b uu\bar c}(\bar r^2 + \bar s^2)]\bar s^4 - 252C_{\bar b u\sigma}{}^{\bar c}{}_{;\sigma}[\bar r C_{u\sigma\sigma\bar c} + 2C_{u\sigma u\bar c}(\bar r^2 + \bar s^2)]\bar s^4 \\
&+ 54C_{u\sigma u}{}^{\bar c}{}_{;\sigma}[6C_{\bar b\bar c u\sigma}(\bar r^2 + \bar s^2) - 5C_{\bar b u\sigma\bar c}(\bar r^2 + \bar s^2) - 3\bar r C_{\bar b uu\bar c}(\bar r^2 + 3\bar s^2)]\bar s^4 + 162C_{u\sigma\sigma}{}^{\bar c}{}_{;u}[C_{\bar b u\sigma\bar c}(\bar r^2 + \bar s^2) \\
&+ C_{\bar b uu\bar c}\bar r(\bar r^2 + 3\bar s^2)]\bar s^4 + 12C_{\bar b uu}{}^{\bar c}{}_{;\sigma}[-9C_{u\sigma\sigma\bar c}(\bar r^2 + \bar s^2) - 17\bar r C_{u\sigma u\bar c}(\bar r^2 + 3\bar s^2)]\bar s^4 - 48C_{\bar b uu\sigma}{}^{;\bar c}[3C_{u\sigma\sigma\bar c}(\bar r^2 + \bar s^2) \\
&+ 4\bar r C_{u\sigma u\bar c}(\bar r^2 + 3\bar s^2)]\bar s^4 + 48C_{\bar b u\sigma}{}^{\bar c}{}_{;u}[3C_{u\sigma\sigma\bar c}(\bar r^2 + \bar s^2) + 8C_{u\sigma u\bar c}\bar r(\bar r^2 + 3\bar s^2)]\bar s^4 + 108C_{u\sigma u}{}^{\bar c}{}_{;u}[3\bar r C_{\bar b u\sigma\bar c}(\bar r^2 + 3\bar s^2) \\
&- 6C_{\bar b\bar c u\sigma}\bar r(\bar r^2 + 3\bar s^2) + 2C_{\bar b uu\bar c}(\bar r^4 + 6\bar r^2\bar s^2 + \bar s^4)]\bar s^4 + 36C_{\bar b uu}{}^{\bar c}{}_{;u}[8C_{u\sigma\sigma\bar c}\bar r(\bar r^2 + 3\bar s^2) \\
&+ 7C_{u\sigma u\bar c}(\bar r^4 + 6\bar r^2\bar s^2 + \bar s^4)]\bar s^4 - 72C_u{}^{\bar c}{}_u{}^{\bar d}{}_{;\sigma}[3C_{\bar b\bar c\sigma\bar d}\bar r + C_{\bar b\bar c u\bar d}(3\bar r^2 + \bar s^2)]\bar s^6 + 108C_u{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;u}[3C_{\bar b\bar c\sigma\bar d}\bar r \\
&+ C_{\bar b\bar c u\bar d}(3\bar r^2 + \bar s^2)]\bar s^6 + 108C_u{}^{\bar c}{}_u{}^{\bar d}{}_{;u}[3C_{\bar b\bar c u\bar d}\bar r(\bar r^2 + \bar s^2) + C_{\bar b\bar c\sigma\bar d}(3\bar r^2 + \bar s^2)]\bar s^6 \\
&- 108C_{\bar b}{}^{\bar c}{}_u{}^{\bar d}{}_{;\sigma}[3C_{u\bar c\sigma\bar d}\bar r + C_{u\bar c u\bar d}(3\bar r^2 + \bar s^2)]\bar s^6 + 24C_{\bar b u\sigma}{}^{\bar c;\bar d}[3C_{u\bar c\sigma\bar d}\bar r + 3C_{u\bar d\sigma\bar c}\bar r + 3C_{\sigma\bar c\sigma\bar d} + C_{u\bar c u\bar d}(3\bar r^2 + \bar s^2)]\bar s^6 \\
&+ 72C_{\bar b}{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;u}[3\bar r C_{u\bar c\sigma\bar d} + C_{u\bar c u\bar d}(3\bar r^2 + \bar s^2)]\bar s^6 + 72C_{\bar b}{}^{\bar c}{}_u{}^{\bar d}{}_{;u}[3C_{u\bar c u\bar d}\bar r(\bar r^2 + \bar s^2) + C_{u\bar c\sigma\bar d}(3\bar r^2 + \bar s^2)]\bar s^6 \\
&+ 24C_{\bar b uu}{}^{\bar c;\bar d}[3C_{\sigma\bar c\sigma\bar d}\bar r + 3C_{u\bar c u\bar d}\bar r(\bar r^2 + \bar s^2) + C_{u\bar c\sigma\bar d}(3\bar r^2 + \bar s^2) + C_{u\bar d\sigma\bar c}(3\bar r^2 + \bar s^2)]\bar s^6\}u_{\bar a} \\
&+ \{-18(C_\sigma{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;\sigma} - C_\sigma{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;u}\bar r)C_{\sigma\bar c\sigma\bar d}\bar s^4 + 27C^{\bar c\bar d\bar e\bar f}{}_{;\sigma}C_{\bar c\bar d\bar e\bar f}\bar s^8 - 18C^{\bar c\bar d\bar e\bar f}{}_{;u}\bar r C_{\bar c\bar d\bar e\bar f}\bar s^8 + 9C_{u\sigma u\sigma;u\sigma\sigma}\bar r\bar s^2(\bar r^2 - 3\bar s^2) \\
&- 15C_{u\sigma u\sigma}{}^{;\bar c}\bar r C_{u\sigma\sigma\bar c}\bar s^2(\bar r^2 - 3\bar s^2) - 135C_{u\sigma u\sigma;\sigma}C_{u\sigma u\sigma}(\bar r^2 - \bar s^2)^2 + 9C_{u\sigma u\sigma;uuu}\bar r\bar s^2(\bar r^4 - 10\bar r^2\bar s^2 - 15\bar s^4) \\
&+ 9C_{u\sigma u\sigma;uu\sigma}\bar s^2(-\bar r^4 + 6\bar r^2\bar s^2 + 3\bar s^4) + 15C_{u\sigma u\sigma}{}^{;\bar c}C_{u\sigma u\bar c}\bar s^2(-\bar r^4 + 6\bar r^2\bar s^2 + 3\bar s^4) + C_{u\sigma u\sigma;\sigma\sigma\sigma}(-9\bar r^2\bar s^2 + 9\bar s^4) \\
&+ 45C_{u\sigma u\sigma;u}C_{u\sigma u\sigma}(3\bar r^5 - 10\bar r^3\bar s^2 + 15\bar r\bar s^4) - 54C_{u\sigma\sigma}{}^{\bar c}{}_{;\sigma}[2C_{u\sigma u\bar c}\bar r(\bar r^2 - 3\bar s^2) + C_{u\sigma\sigma\bar c}(\bar r^2 - \bar s^2)]\bar s^2 \\
&+ 36C_{u\bar c\sigma\bar d}\bar s^4[-C_\sigma{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;\sigma}\bar r + C_\sigma{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;u}(\bar r^2 + \bar s^2)] - 18C_{u\bar c u\bar d}\bar s^4[C_\sigma{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;\sigma}(\bar r^2 + \bar s^2) - C_\sigma{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;u}\bar r(\bar r^2 + 3\bar s^2)] \\
&- 36C_u{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;\sigma}[C_{\sigma\bar c\sigma\bar d}\bar r + C_{u\bar c\sigma\bar d}(\bar r^2 + \bar s^2) + C_{u\bar d\sigma\bar c}(\bar r^2 + \bar s^2) + C_{u\bar c u\bar d}\bar r(\bar r^2 + 3\bar s^2)]\bar s^4 \\
&+ 15C_{u\sigma\sigma}{}^{\bar c}{}_{;u}[2C_{u\sigma\sigma\bar c}\bar r(\bar r^2 - 3\bar s^2) + 5C_{u\sigma u\bar c}(\bar r^4 - 6\bar r^2\bar s^2 - 3\bar s^4)]\bar s^2 + 18C_{u\sigma u}{}^{\bar c}{}_{;u}[3\bar r C_{u\sigma u\bar c}(\bar r^4 - 10\bar r^2\bar s^2 - 15\bar s^4) \\
&+ 5C_{u\sigma\sigma\bar c}(\bar r^4 - 6\bar r^2\bar s^2 - 3\bar s^4)]\bar s^2 - 18C_u{}^{\bar c}{}_u{}^{\bar d}{}_{;\sigma}[C_{\sigma\bar c\sigma\bar d}(\bar r^2 + \bar s^2) + 2C_{u\bar c\sigma\bar d}\bar r(\bar r^2 + 3\bar s^2) \\
&+ 18C_{u\bar c u\bar d}(\bar r^4 + 6\bar r^2\bar s^2 + \bar s^4)]\bar s^4 + 36C_u{}^{\bar c}{}_\sigma{}^{\bar d}{}_{;u}[C_{\sigma\bar c\sigma\bar d}(\bar r^2 + \bar s^2) + C_{u\bar c\sigma\bar d}\bar r(\bar r^2 + 3\bar s^2) + C_{u\bar d\sigma\bar c}\bar r(\bar r^2 + 3\bar s^2) \\
&+ C_{u\bar c u\bar d}(\bar r^4 + 6\bar r^2\bar s^2 + \bar s^4)]\bar s^4 + C_{u\sigma u}{}^{\bar c}{}_{;\sigma}[-48C_{u\sigma\sigma\bar c}\bar r(\bar r^2 - 3\bar s^2) + 39C_{u\sigma u\bar c}(-\bar r^4 + 6\bar r^2\bar s^2 + 3\bar s^4)]\bar s^2 \\
&+ 18C_u{}^{\bar c}{}_u{}^{\bar d}{}_{;u}[C_{\sigma\bar c\sigma\bar d}\bar r(\bar r^2 + 3\bar s^2) + 2C_{u\bar c\sigma\bar d}(\bar r^4 + 6\bar r^2\bar s^2 + \bar s^4) + C_{u\bar c u\bar d}\bar r(\bar r^4 + 10\bar r^2\bar s^2 + 5\bar s^4)]\bar s^4\}u_{\bar a}u_{\bar b}.
\end{aligned}
\tag{4.9}
$$

V. MODE-SUM REGULARIZATION PARAMETERS

The singular field expansions derived in the previous sections have several applications in explicit self-force calculations. One of the most successful computational approaches to date is the *mode-sum* scheme of Barack and Ori [35,36]; the majority of existing calculations are based on it in one form or another [50,52–74]. The basic idea is to decompose the singular retarded field into spherical harmonic modes which are continuous and finite in general for the scalar case and in Lorenz gauge for the electromagnetic and gravitational cases. A key component of the calculation involves the subtraction of so-called regularization parameters—analytically derived expressions which render the formally divergent sum over spherical harmonic modes finite. In this section, we derive these parameters from our singular field expressions and show how they may be used to compute the self-force with extremely high accuracy.

A. Rotated coordinates

In order to obtain expressions which are readily written as mode sums, previous calculations [35,50,54] found it useful to work in a rotated coordinate frame. We found it most efficient to carry out this rotation prior to doing any calculations. To this end, we introduce Riemann normal coordinates on the 2-sphere at $\bar{x}$ in the form

$$w_1 = 2\sin\left(\frac{\alpha}{2}\right)\cos\beta, \qquad w_2 = 2\sin\left(\frac{\alpha}{2}\right)\sin\beta, \tag{5.1}$$

where α and β are rotated angular coordinates given by

$$\sin\theta\cos\phi = \cos\alpha, \tag{5.2}$$

$$\sin\theta\sin\phi = \sin\alpha\cos\beta, \tag{5.3}$$

$$\cos\theta = \sin\alpha\sin\beta. \tag{5.4}$$

In these coordinates, the Schwarzschild metric is given by the line element

$$\begin{aligned} ds^2 = &-\left(\frac{r-2M}{r}\right)dt^2 + \left(\frac{r}{r-2M}\right)dr^2 \\ &+ r^2\left\{\left[\frac{16 - w_2^2(8 - w_1^2 - w_2^2)}{4(4 - w_1^2 - w_2^2)}\right]dw_1^2 \right. \\ &+ 2dw_1dw_2\left[\frac{w_1w_2(8 - w_1^2 - w_2^2)}{4(4 - w_1^2 - w_2^2)}\right] \\ &\left. + \left[\frac{16 - w_1^2(8 - w_1^2 - w_2^2)}{4(4 - w_1^2 - w_2^2)}\right]dw_2^2\right\}. \end{aligned} \tag{5.5}$$

The algebraic form of the metric makes it very suitable for using with computer algebra programmes such as MATHEMATICA. The apparent complexity of having a non-diagonal metric on S^2 is in fact minimal since the determinant of that metric is simply 1.

B. Mode decomposition

The method of regularization of the self-force through l-mode decomposition is by now standard, see, for example, Refs. [35,50,54]. Having calculated the singular field, it is straightforward to calculate the component of the self-force which arises from the singular field,[3] F_a, for scalar, electromagnetic and gravitational cases using Eqs. (2.14), (2.19), and (2.24), with the singular field substituted for the regular field. We study the multipole decomposition of F_a by writing

$$F_a(r, t, \alpha, \beta) = \sum_{lm} F_a^{lm}(r, t) Y^{lm}(\alpha, \beta), \tag{5.6}$$

where $Y^{lm}(\theta, \phi)$ are scalar spherical harmonics, and accordingly,

$$F_a^{lm}(r, t) = \int F_a(r, t, \alpha, \beta) Y^{lm*}(\alpha, \beta) d\Omega. \tag{5.7}$$

To calculate the l-mode contribution at $\bar{x} = (t_0, r_0, \alpha_0, \beta_0)$, we have

$$F_a^l(r_0, t_0) = \lim_{\Delta r \to 0} \sum_m F_a^{lm}(r_0 + \Delta r, t_0) Y^{lm}(\alpha_0, \beta_0). \tag{5.8}$$

In previous calculations, Eq. (5.6) has naturally arisen in Schwarzschild coordinates with $\theta_0 = \frac{\pi}{2}$, and it was necessary to perform a rotation to move the coordinate location of the particle from the equatorial plane to a pole in the new coordinate system. However, by choosing to work in an S^2 Riemann normal coordinate system from the start, our particle is already located on the pole. This saves us from further transformation and expansions at this stage. With the particle on the pole, $Y^{lm}(\alpha_0 = 0, \beta_0) = 0$ for all $m \neq 0$. This also allows us, without loss of generality, to take $\beta_0 = 0$. Taking α_0, β_0 and m all to be equal to zero in Eq. (5.8) gives us

$$\begin{aligned} F_a^l(r_0, t_0) &= \lim_{\Delta r \to 0} \sqrt{\frac{2l+1}{4\pi}} F_a^{l,m=0}(r_0 + \Delta r, t_0) \\ &= \frac{2l+1}{4\pi} \lim_{\Delta r \to 0} \int F_a(r_0 + \Delta r, t_0, \alpha, \beta) P_l(\cos\alpha) d\Omega. \end{aligned} \tag{5.9}$$

For each spin field, the singular self-force, $F_a(r, t, \alpha, \beta)$, has the form

$$F_a(r, t, \alpha, \beta) = \sum_{n=1}^{\infty} \frac{B_a^{(3n-2)}}{\rho^{2n+1}} \epsilon^{n-3}, \tag{5.10}$$

where $B_a^{(k)} = b_{a_1 a_2 \ldots a_k}(\bar{x}) \Delta x^{a_1} \Delta x^{a_2} \ldots \Delta x^{a_k}$. On identifying $\tau_1 = \bar{r}_{(1)} \pm \rho$, this form can be easily seen to follow from the coordinate representation of the above expressions for

[3] In this section, for notational convenience, we drop the implied (S) superscript denoting "singular" as we are always referring to the singular component.

the singular field. In using Eq. (5.10) to determine the regularization parameters, we only need to take the sum to the appropriate order: $n = 1$ for $F_{a[-1]}$, $n = 2$ for $F_{a[0]}$, etc.

Explicitly, in our coordinates $\rho = \sqrt{(g_{\bar{a}\bar{b}}u^{\bar{a}}\Delta x^b)^2 + g_{\bar{a}\bar{b}}\Delta x^a \Delta x^b}$ takes the form

$$\begin{aligned}\rho(r,t,\alpha,\beta)^2 = {}& \frac{(E^2 r_0^3 - L^2(r_0-2M))}{r_0(r_0-2M)^2}\Delta r^2 \\ &+ (L^2 + r_0^2)\Delta w_1^2 \\ &- \left(\frac{2Er_0\dot{r}_0}{r_0-2M}\Delta r + 2EL\Delta w_1\right)\Delta t \\ &+ \frac{2Lr_0\dot{r}_0}{r_0-2M}\Delta r \Delta w_1 \\ &+ \left(E^2 + \frac{2M}{r_0} - 1\right)\Delta t^2 + r_0^2 \Delta w_2^2, \qquad (5.11)\end{aligned}$$

where the α, β dependence is contained exclusively in Δw_1 and Δw_2. $E = -u_t$ and $L = u_\phi$ are the energy per unit mass and angular momentum along the axis of symmetry, respectively. In particular, taking $t = t_0$ ($\Delta t = 0$) allows us to write

$$\begin{aligned}\rho(r,t_0,\alpha,\beta)^2 = {}& \frac{E^2 r_0^4}{(L^2+r_0^2)(r_0-2M)^2}\Delta r^2 + (L^2 + r_0^2) \\ &\times \left(\Delta w_1 + \frac{L r_0 \dot{r}_0}{(r_0-2M)(L^2+r_0^2)}\Delta r\right)^2 \\ &+ r_0^2 \Delta w_2^2. \qquad (5.12)\end{aligned}$$

We further define

$$\rho_0(\alpha,\beta)^2 = \rho(r_0,t_0,\alpha,\beta)^2 = (L^2 + r_0^2)\Delta w_1^2 + r_0^2 \Delta w_2^2, \qquad (5.13)$$

and this allows us to rewrite our Δw's in the alternate forms

$$\begin{aligned}\Delta w_1^2 &= 2(1-\cos\alpha)\cos^2\beta = \frac{\rho_0^2}{(r_0^2+L^2)\chi}\cos^2\beta \\ &= \frac{\rho_0^2}{L^2\chi}(k - (1-\chi)), \qquad (5.14)\end{aligned}$$

$$\begin{aligned}\Delta w_2^2 &= 2(1-\cos\alpha)\sin^2\beta = \frac{\rho_0^2}{(r_0^2+L^2)\chi}\sin^2\beta \\ &= \frac{\rho_0^2}{L^2\chi}(1-\chi), \qquad (5.15)\end{aligned}$$

where

$$\chi(\beta) = 1 - k\sin^2\beta, \qquad k = \frac{L^2}{r_0^2 + L^2}. \qquad (5.16)$$

Suppose, for the moment, that we may take the limit in Eq. (5.9) through the integral sign; then, using our alternate forms we have

$$\begin{aligned}\lim_{\Delta r\to 0}\frac{B_a^{(3n-2)}}{\rho^{2n+1}}\epsilon^{n-3} &= \frac{b_{i_1 i_2 \ldots i_{3n-2}}(r_0)\Delta w^{i_1}\Delta w^{i_2}\ldots\Delta w^{i_{3n-2}}}{\rho_0^{2n+1}}\epsilon^{n-3} \\ &= \rho_0^{n-3}\epsilon^{n-3}c_{a(n)}(r_0,\chi). \qquad (5.17)\end{aligned}$$

In Ref. [75], it was shown that the integral and limit in Eq. (5.9) are indeed interchangeable for all orders except the leading order, $n = 1$ term, where the limiting ρ_0^{-3} would not be integrable. Thus, we find the singular self-force now has the form

$$\begin{aligned}F_a^l(r_0,t_0,\alpha,\beta) = {}& \frac{2l+1}{4\pi}\bigg[\epsilon^{-2}\lim_{\Delta r\to 0}\int\frac{B_a^{(1)}}{\rho^3}P_l(\cos\alpha)d\Omega \\ &+ \sum_{n=2}\epsilon^{n-3}\int\rho_0^{n-3}c_{a(n)}(r_0,\chi)P_l(\cos\alpha)d\Omega\bigg] \\ \equiv{}& F^l_{a[-1]}(r_0,t_0)\epsilon^{-2} + F^l_{a[0]}(r_0,t_0)\epsilon^{-1} \\ &+ F^l_{a[2]}(r_0,t_0)\epsilon^{1} + F^l_{a[4]}(r_0,t_0)\epsilon^{3} \\ &+ F^l_{a[6]}(r_0,t_0)\epsilon^{5} + \ldots, \qquad (5.18)\end{aligned}$$

where the β dependence in the c_n's are hidden in χ, while the α and β dependence of $F_a(r, t_0, \alpha, \beta)$ is hidden in both the ρ's and c_n's. Note here that we use the convention that a subscript in square brackets denotes the term which will contribute at that order in $1/l$. Furthermore, the integrand in the summation is odd or even under $\Delta w_i \to -\Delta w_i$ according to whether n (and so $3n - 2$) is odd or even. This means only the even terms are nonvanishing, while $F^l_{a[1]}(r_0,t_0) = F^l_{a[3]}(r_0,t_0) = F^l_{a[5]}(r_0,t_0) = 0$, etc.

Some care is required in dealing with taking the limit in the first term. As this has been addressed previously [50,54,75,76] and our main interest is in the higher orders, we omit the details here and only discuss the calculation of the higher-order terms.

In the higher-order terms in Eq. (5.18), we may immediately work with $\rho_0^2 = 2\chi(L^2 + r_0^2)(1 - \cos\alpha)$, so

$$\begin{aligned}\rho_0(r_0,t_0,\alpha,\beta)^n &= [2\chi(L^2+r_0^2)(1-\cos\alpha)]^{n/2} \\ &= [2\chi(L^2+r_0^2)]^{n/2}\sum_{l=0}\mathcal{A}_l^{n/2}(0)P_l(\cos\alpha), \qquad (5.19)\end{aligned}$$

where $\mathcal{A}_l^{-(1/2)}(0) = \sqrt{2}$ from the generating function of the Legendre polynomials and, as derived in Appendix D of Ref. [54], for $(n+1)/2 \in \mathbb{N}$,

$$\mathcal{P}_{n/2} = (-1)^{(n+1)/2} 2^{1+n/2} (n!!)^2,$$
$$\mathcal{A}_l^{n/2}(0) = \frac{\mathcal{P}_{n/2}(2l+1)}{(2l-n)(2l-n+2)\dots(2l+n)(2l+n+2)}. \tag{5.20}$$

In this case, the angular integrals involve

$$\frac{1}{2\pi}\int \frac{d\beta}{\chi(\beta)^{n/2}} = \langle \chi^{-n/2}(\beta)\rangle = {}_2F_1\left(\frac{n}{2}, \frac{1}{2}, 1, k\right), \tag{5.21}$$

where $(n+1)/2 \in \mathbb{N}\cup\{0\}$. The resulting equations can then be tidied up using the following special cases of hypergeometric functions

$$\langle \chi^{-\frac{1}{2}}\rangle = \mathcal{F}_{\frac{1}{2}}(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k\right) = \frac{2}{\pi}\mathcal{K}, \tag{5.22}$$

$$\langle \chi^{\frac{1}{2}}\rangle = \mathcal{F}_{-\frac{1}{2}}(k) = {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k\right) = \frac{2}{\pi}\mathcal{E}, \tag{5.23}$$

where

$$\mathcal{K} \equiv \int_0^{\pi/2} (1 - k\sin^2\beta)^{-1/2} d\beta,$$
$$\mathcal{E} \equiv \int_0^{\pi/2} (1 - k\sin^2\beta)^{1/2} d\beta \tag{5.24}$$

are complete elliptic integrals of the first and second kinds, respectively. All other powers of χ can be integrated to hypergeometric functions which can then be manipulated to be one of the above by the use of the recurrence relation in Eq. (15.2.10) of Ref. [77]. That is

$$\mathcal{F}_{p+1}(k) = \frac{p-1}{p(k-1)}\mathcal{F}_{p-1}(k) + \frac{1-2p+(p-\frac{1}{2})k}{p(k-1)}\mathcal{F}_p(k). \tag{5.25}$$

In the next sections, we give the results of applying this calculation to each of scalar, electromagnetic and gravitational cases in turn. In doing so, we omit the explicit dependence on l which in each case is

$$F^l_{a[-1]} = (2l+1)F_{a[-1]}, \qquad F^l_{a[0]} = F_{a[0]},$$
$$F^l_{a[2]} = \frac{F_{a[2]}}{(2l-1)(2l+3)},$$
$$F^l_{a[4]} = \frac{F_{a[4]}}{(2l-3)(2l-1)(2l+3)(2l+5)},$$
$$F^l_{a[6]} = \frac{F_{a[6]}}{(2l-5)(2l-3)(2l-1)(2l+3)(2l+5)(2l+7)}. \tag{5.26}$$

TABLE I. Relation between notational choices for the regularization parameters. The most common choices are those of either Barack and Ori [35] or Detweiler, Messaritaki, and Whiting [54].

RP	BO	DMW
$F_{a[-1]}$	A_a	A_a
$F_{a[0]}$	B_a	B_a
$F_{a[1]}$	C_a	C_a
$F_{a[2]}$	...	D_a
$F_{a[4]}$	...	E^1_a
$F_{a[6]}$	...	E^2_a

It is also worth pointing out that there exists in the literature several different notations for the regularization parameters. We have adopted a notation which is readily extensible to other orders and which makes the dependence on l explicit. To avoid confusion, in Table I, we give the relation between our notation and other common notations.

C. Scalar case

In the scalar case, the regularization parameters are given by

$$F_{t[-1]} = \frac{\dot{r}_0 \mathrm{sgn}(\Delta r)}{2(L^2 + r_0^2)},$$
$$F_{r[-1]} = -\frac{E r_0 \mathrm{sgn}(\Delta r)}{2(r_0 - 2M)(L^2 + r_0^2)}, \tag{5.27}$$
$$F_{\theta[-1]} = 0, \qquad F_{\phi[-1]} = 0,$$

$$F_{t[0]} = -\frac{E r_0 \dot{r}_0}{\pi(L^2 + r_0^2)^{3/2}}(2\mathcal{E} - \mathcal{K}), \tag{5.28}$$

$$F_{r[0]} = \frac{1}{\pi r_0 (r_0 - 2M)(L^2 + r_0^2)^{3/2}}\{[2E^2 r_0^3 - (r_0 - 2M) \times (L^2 + r_0^2)]\mathcal{E} - [E^2 r_0^3 + (r_0 - 2M) \times (L^2 + r_0^2)]\mathcal{K}\},$$
$$F_{\theta[0]} = 0, \qquad F_{\phi[0]} = -\frac{r_0 \dot{r}_0}{L\pi(L^2 + r_0^2)^{1/2}}(\mathcal{E} - \mathcal{K}), \tag{5.29}$$

$$F_{t[2]} = \frac{E\dot{r}_0}{2\pi r_0^4 (L^2 + r_0^2)^{7/2}}(F^{\mathcal{E}}_{t[2]}\mathcal{E} + F^{\mathcal{K}}_{t[2]}\mathcal{K}), \tag{5.30}$$

where

$$F^{\mathcal{E}}_{t[2]} = 8E^2(L^2 - r_0^2)r_0^7 - (L^2 + r_0^2)(36L^6M + 104L^4Mr_0^2 + 98L^2Mr_0^4 + L^2r_0^5 + 46Mr_0^6 - 7r_0^7),$$

$$F^{\mathcal{K}}_{t[2]} = -E^2r_0^7(3L^2 - 5r_0^2) + 2r_0^2(L^2 + r_0^2)(9L^4M + 18L^2Mr_0^2 + 13Mr_0^4 - 2r_0^5),$$

$$F_{r[2]} = \frac{1}{2\pi r_0^6(r_0 - 2M)(L^2 + r_0^2)^{7/2}}(F^{\mathcal{E}}_{r[2]}\mathcal{E} + F^{\mathcal{K}}_{r[2]}\mathcal{K}), \tag{5.31}$$

where

$$F^{\mathcal{E}}_{r[2]} = -8E^4r_0^{10}(L^2 - r_0^2) + 4E^2r_0^3(L^2 + r_0^2)(9L^6M + 26L^4Mr_0^2 + 23L^2Mr_0^4 + L^2r_0^5 + 14Mr_0^6 - 3r_0^7)$$
$$- (r_0 - 2M)(L^2 + r_0^2)^2(28L^6M + 82L^4Mr_0^2 + 82L^2Mr_0^4 - L^2r_0^5 + 32Mr_0^6 - 3r_0^7),$$

$$F^{\mathcal{K}}_{r[2]} = E^4r_0^{10}(3L^2 - 5r_0^2) - E^2r_0^5(L^2 + r_0^2)(18L^4M + 34L^2Mr_0^2 + L^2r_0^3 + 32Mr_0^4 - 7r_0^5)$$
$$+ (r_0 - 2M)r_0^2(L^2 + r_0^2)^2(14L^4M + 28L^2Mr_0^2 + 16Mr_0^4 - r_0^5),$$

$$F_{\theta[2]} = 0, \tag{5.32}$$

$$F_{\phi[2]} = \frac{\dot{r}_0}{2\pi L r_0^4(L^2 + r_0^2)^{5/2}}(F^{\mathcal{E}}_{\phi[2]}\mathcal{E} + F^{\mathcal{K}}_{\phi[2]}\mathcal{K}), \tag{5.33}$$

where

$$F^{\mathcal{E}}_{\phi[2]} = E^2r_0^7(7L^2 - r_0^2) + (L^2 + r_0^2)(28L^6M + 58L^4Mr_0^2 + 34L^2Mr_0^4 - L^2r_0^5 + r_0^7),$$

$$F^{\mathcal{K}}_{\phi[2]} = -E^2r_0^7(3L^2 - r_0^2) - r_0^2(L^2 + r_0^2)(14L^4M + 16L^2Mr_0^2 + r_0^5),$$

$$F_{t[4]} = \frac{3E\dot{r}_0}{40\pi r_0^{11}(L^2 + r_0^2)^{11/2}}(F^{\mathcal{E}}_{t[4]}\mathcal{E} + F^{\mathcal{K}}_{t[4]}\mathcal{K}), \tag{5.34}$$

where

$$F^{\mathcal{E}}_{t[4]} = -30E^4r_0^{16}(23L^4 - 82L^2r_0^2 + 23r_0^4) + 2E^2r_0^5(L^2 + r_0^2)(44800L^{12}M + 219136L^{10}Mr_0^2 + 428252L^8Mr_0^4$$
$$+ 418776L^6Mr_0^6 + 206374L^4Mr_0^8 + 45L^4r_0^9 + 45188L^2Mr_0^{10} - 1230L^2r_0^{11} - 166Mr_0^{12} + 645r_0^{13})$$
$$- 2(L^2 + r_0^2)^2(20480L^{14}M^2 - 97280L^{12}M^2r_0^2 + 85120L^{12}Mr_0^3 - 700832L^{10}M^2r_0^4 + 388480L^{10}Mr_0^5$$
$$- 1426472L^8M^2r_0^6 + 704552L^8Mr_0^7 - 1358276L^6M^2r_0^8 + 635226L^6Mr_0^9 - 635180L^4M^2r_0^{10} + 286498L^4Mr_0^{11}$$
$$- 15L^4r_0^{12} - 124540L^2M^2r_0^{12} + 54086L^2Mr_0^{13} - 90L^2r_0^{14} - 2796M^2r_0^{14} + 182Mr_0^{15} + 285r_0^{16}),$$

$$F^{\mathcal{K}}_{t[4]} = 15E^4r_0^{16}(15L^4 - 82L^2r_0^2 + 31r_0^4) - 4E^2r_0^7(L^2 + r_0^2)(11200L^{10}M + 44984L^8Mr_0^2 + 68227L^6Mr_0^4$$
$$+ 46849L^4Mr_0^6 + 13493L^2Mr_0^8 - 270L^2r_0^9 + 127Mr_0^{10} + 210r_0^{11}) + r_0^2(L^2 + r_0^2)^2(20480L^{12}M^2$$
$$- 115200L^{10}M^2r_0^2 + 85120L^{10}Mr_0^3 - 599072L^8M^2r_0^4 + 314000L^8Mr_0^5 - 908104L^6M^2r_0^6 + 433792L^6Mr_0^7$$
$$- 589164L^4M^2r_0^8 + 268648L^4Mr_0^9 - 151484L^2M^2r_0^{10} + 66380L^2Mr_0^{11} - 15L^2r_0^{12}$$
$$- 5592M^2r_0^{12} + 1204Mr_0^{13} + 345r_0^{14}),$$

$$F_{r[4]} = \frac{3}{40\pi r_0^{13}(r_0 - 2M)(L^2 + r_0^2)^{11/2}}(F^{\mathcal{E}}_{r[4]}\mathcal{E} + F^{\mathcal{K}}_{r[4]}\mathcal{K}), \tag{5.35}$$

where

$$F^{\mathcal{E}}_{r[4]} = 30E^6 r_0^{19}(23L^4 - 82L^2 r_0^2 + 23r_0^4) - E^4 r_0^8(L^2 + r_0^2)(89600L^{12}M + 438272L^{10}Mr_0^2 + 856504L^8Mr_0^4 + 837552L^6Mr_0^6 + 411938L^4Mr_0^8 + 495L^4r_0^9 + 92836L^2Mr_0^{10} - 3690L^2r_0^{11} - 902Mr_0^{12} + 1575r_0^{13}) + 8E^2r_0^3(L^2 + r_0^2)^2(5120L^{14}M^2 - 35200L^{12}M^2r_0^2 + 26720L^{12}Mr_0^3 - 227368L^{10}M^2r_0^4 + 123200L^{10}Mr_0^5 - 456300L^8M^2r_0^6 + 225979L^8Mr_0^7 - 434510L^6M^2r_0^8 + 206277L^6Mr_0^9 - 203983L^4M^2r_0^{10} + 94211L^4Mr_0^{11} - 40376L^2M^2r_0^{12} + 18367L^2Mr_0^{13} - 135L^2r_0^{14} - 571M^2r_0^{14} - 146Mr_0^{15} + 135r_0^{16}) - (r_0 - 2M)(L^2 + r_0^2)^3 \times (40960L^{14}M^2 - 86016L^{12}M^2r_0^2 + 116480L^{12}Mr_0^3 - 860224L^{10}M^2r_0^4 + 510080L^{10}Mr_0^5 - 1780112L^8M^2r_0^6 + 882400L^8Mr_0^7 - 1657392L^6M^2r_0^8 + 752340L^6Mr_0^9 - 743164L^4M^2r_0^{10} + 316100L^4Mr_0^{11} + 30L^4r_0^{12} - 136236L^2M^2r_0^{12} + 53200L^2Mr_0^{13} + 75L^2r_0^{14} - 3120M^2r_0^{14} + 160Mr_0^{15} + 165r_0^{16}),$$

$$F^{\mathcal{K}}_{r[4]} = -15E^6r_0^{19}(15L^4 - 82L^2r_0^2 + 31r_0^4) + E^4r_0^{10}(L^2 + r_0^2)(44800L^{10}M + 179936L^8Mr_0^2 + 272908L^6Mr_0^4 + 187126L^4Mr_0^6 + 135L^4r_0^7 + 55232L^2Mr_0^8 - 1710L^2r_0^9 + 118Mr_0^{10} + 1035r_0^{11}) - E^2r_0^5(L^2 + r_0^2)^2 \times (20480L^{12}M^2 - 158720L^{10}M^2r_0^2 + 106880L^{10}Mr_0^3 - 769632L^8M^2r_0^4 + 399280L^8Mr_0^5 - 1159632L^6M^2r_0^6 + 559556L^6Mr_0^7 - 755876L^4M^2r_0^8 + 352004L^4Mr_0^9 - 196524L^2M^2r_0^{10} + 89680L^2Mr_0^{11} - 405L^2r_0^{12} - 5528M^2r_0^{12} + 512Mr_0^{13} + 675r_0^{14}) + (r_0 - 2M)r_0^2(L^2 + r_0^2)^3(20480L^{12}M^2 - 60928L^{10}M^2r_0^2 + 58240L^{10}Mr_0^3 - 375840L^8M^2r_0^4 + 204080L^8Mr_0^5 - 564472L^6M^2r_0^6 + 265360L^6Mr_0^7 - 350956L^4M^2r_0^8 + 152380L^4Mr_0^9 - 84276L^2M^2r_0^{10} + 33740L^2Mr_0^{11} + 15L^2r_0^{12} - 3120M^2r_0^{12} + 640Mr_0^{13} + 75r_0^{14}),$$

$$F_{\theta[4]} = 0, \tag{5.36}$$

$$F_{\phi[4]} = \frac{3\dot{r}_0}{40\pi L r_0^{11}(L^2 + r_0^2)^{9/2}}(F^{\mathcal{E}}_{\phi[4]}\mathcal{E} + F^{\mathcal{K}}_{\phi[4]}\mathcal{K}), \tag{5.37}$$

where

$$F^{\mathcal{E}}_{\phi[4]} = -15E^4r_0^{16}(43L^4 - 82L^2r_0^2 + 3r_0^4) - 10E^2r_0^5(L^2 + r_0^2)(4352L^{12}M + 16512L^{10}Mr_0^2 + 22948L^8Mr_0^4 + 13346L^6Mr_0^6 + 2136L^4Mr_0^8 - 9L^4r_0^9 - 710L^2Mr_0^{10} + 126L^2r_0^{11} - 9r_0^{13}) + (L^2 + r_0^2)^2(40960L^{14}M^2 - 96256L^{12}M^2r_0^2 + 116480L^{12}Mr_0^3 - 704064L^{10}M^2r_0^4 + 429440L^{10}Mr_0^5 - 1134992L^8M^2r_0^6 + 595040L^8Mr_0^7 - 755632L^6M^2r_0^8 + 372500L^6Mr_0^9 - 194724L^4M^2r_0^{10} + 94940L^4Mr_0^{11} + 30L^4r_0^{12} - 6276L^2M^2r_0^{12} + 4040L^2Mr_0^{13} + 105L^2r_0^{14} + 480M^2r_0^{14} - 45r_0^{16}),$$

$$F^{\mathcal{K}}_{\phi[4]} = 15E^4r_0^{16}(L^2 - 3r_0^2)(15L^2 - r_0^2) + 10E^2r_0^7(L^2 + r_0^2)(2176L^{10}M + 6352L^8Mr_0^2 + 6018L^6Mr_0^4 + 1666L^4Mr_0^6 - 320L^2Mr_0^8 + 63L^2r_0^9 - 9r_0^{11}) - r_0^2(L^2 + r_0^2)^2(20480L^{12}M^2 - 66048L^{10}M^2r_0^2 + 58240L^{10}Mr_0^3 - 293280L^8M^2r_0^4 + 163760L^8Mr_0^5 - 314392L^6M^2r_0^6 + 156960L^6Mr_0^7 - 114876L^4M^2r_0^8 + 55420L^4Mr_0^9 - 6516L^2M^2r_0^{10} + 3740L^2Mr_0^{11} + 15L^2r_0^{12} + 480M^2r_0^{12} - 45r_0^{14}),$$

$$F_{t[6]} = \frac{-3E\dot{r}_0}{560\pi r_0^{16}(L^2 + r_0^2)^{15/2}}(F^{\mathcal{E}}_{t[6]}\mathcal{E} + F^{\mathcal{K}}_{t[6]}\mathcal{K}), \tag{5.38}$$

where

$$
\begin{aligned}
F^{\mathcal{E}}_{t[6]} = {} & 28000E^6 r_0^{23}(r_0 - L)(L + r_0)(11L^4 - 74L^2 r_0^2 + 11r_0^4) - 25E^4 r_0^8 (L^2 + r_0^2)(-16056320L^{18}M \\
& - 107151360L^{16}Mr_0^2 - 302586880L^{14}Mr_0^4 - 464979968L^{12}Mr_0^6 - 412568652L^{10}Mr_0^8 - 201055024L^8 Mr_0^{10} \\
& - 39268410L^6 Mr_0^{12} - 1575L^6 r_0^{13} + 5226426L^4 Mr_0^{14} + 99435L^4 r_0^{15} + 3185118L^2 Mr_0^{16} - 186165L^2 r_0^{17} \\
& + 19662Mr_0^{18} + 35385r_0^{19}) + 2E^2 r_0^3 (L^2 + r_0^2)^2(-1007616000L^{20}M^2 - 2885324800L^{18}M^2 r_0^2 \\
& - 1548288000L^{18}Mr_0^3 + 5271990272L^{16}M^2 r_0^4 - 9940582400L^{16}Mr_0^5 + 35832487264L^{14}M^2 r_0^6 \\
& - 27145052800L^{14}Mr_0^7 + 69571689904L^{12}M^2 r_0^8 - 40793731200L^{12}Mr_0^9 + 69887626312L^{10}M^2 r_0^{10} \\
& - 36329433800L^{10}Mr_0^{11} + 39015325900L^8 M^2 r_0^{12} - 19063343950L^8 Mr_0^{13} + 11166709052L^6 M^2 r_0^{14} \\
& - 5373108900L^6 Mr_0^{15} + 7875L^6 r_0^{16} + 1046817944L^4 M^2 r_0^{16} - 575985300L^4 Mr_0^{17} + 94500L^4 r_0^{18} \\
& - 118281276L^2 M^2 r_0^{18} + 29440500L^2 Mr_0^{19} - 1178625L^2 r_0^{20} - 8271468M^2 r_0^{20} + 479850Mr_0^{21} + 414750r_0^{22}) \\
& + (L^2 + r_0^2)^3(-5775360000L^{20}M^3 + 2580480000L^{20}M^2 r_0 - 18980904960L^{18}M^3 r_0^2 + 750796800L^{18}M^2 r_0^3 \\
& + 3429888000L^{18}Mr_0^4 + 10876463104L^{16}M^3 r_0^4 - 53063915520L^{16}M^2 r_0^5 + 21396480000L^{16}Mr_0^6 \\
& + 143196789568L^{14}M^3 r_0^6 - 191859546624L^{14}M^2 r_0^7 + 56793312800L^{14}Mr_0^8 + 292841560608L^{12}M^3 r_0^8 \\
& - 317782413664L^{12}M^2 r_0^9 + 83096000800L^{12}Mr_0^{10} + 297880915104L^{10}M^3 r_0^{10} - 295661821784L^{10}M^2 r_0^{11} \\
& + 72364880400L^{10}Mr_0^{12} + 168534399040L^8 M^3 r_0^{12} - 159472848000L^8 M^2 r_0^{13} + 37560515600L^8 Mr_0^{14} \\
& + 50707761864L^6 M^3 r_0^{14} - 46826640820L^6 M^2 r_0^{15} + 10839698800L^6 Mr_0^{16} + 7000L^6 r_0^{17} \\
& + 6272875728L^4 M^3 r_0^{16} - 5878415984L^4 M^2 r_0^{17} + 1398754050L^4 Mr_0^{18} + 41125L^4 r_0^{19} - 86931352L^2 M^3 r_0^{18} \\
& + 212436L^2 M^2 r_0^{19} + 21357300L^2 Mr_0^{20} + 124250L^2 r_0^{21} - 29620256M^3 r_0^{20} + 16081176M^2 r_0^{21} \\
& - 597150Mr_0^{22} - 245875r_0^{23}),
\end{aligned}
$$

$$
\begin{aligned}
F^{\mathcal{K}}_{t[6]} = {} & -875E^6 r_0^{23}(-105L^6 + 1189L^4 r_0^2 - 1531L^2 r_0^4 + 247r_0^6) + 50E^4 r_0^{10}(L^2 + r_0^2)(-4014080L^{16}M \\
& - 23275520L^{14}Mr_0^2 - 55468800L^{12}Mr_0^4 - 68718512L^{10}Mr_0^6 - 45183275L^8 Mr_0^8 - 13052460L^6 Mr_0^{10} \\
& + 362358L^4 Mr_0^{12} + 18900L^4 r_0^{13} + 919476L^2 Mr_0^{14} - 49560L^2 r_0^{15} + 15501Mr_0^{16} + 12180r_0^{17}) \\
& - E^2 r_0^5 (L^2 + r_0^2)^2(-1007616000L^{18}M^2 - 2003660800L^{16}M^2 r_0^2 - 1548288000L^{16}Mr_0^3 \\
& + 6977961472L^{14}M^2 r_0^4 - 8585830400L^{14}Mr_0^5 + 29653513376L^{12}M^2 r_0^6 - 19705027200L^{12}Mr_0^7 \\
& + 43981240344L^{10}M^2 r_0^8 - 23922541200L^{10}Mr_0^9 + 32635169200L^8 M^2 r_0^{10} - 16162950800L^8 Mr_0^{11} \\
& + 12004692860L^6 M^2 r_0^{12} - 5726827800L^6 Mr_0^{13} + 1578230992L^4 M^2 r_0^{14} - 803087700L^4 Mr_0^{15} + 7875L^4 r_0^{16} \\
& - 113069964L^2 M^2 r_0^{16} + 24092400L^2 Mr_0^{17} - 1118250L^2 r_0^{18} - 13324056M^2 r_0^{18} + 1526700Mr_0^{19} + 553875r_0^{20}) \\
& + 2r_0^2 (L^2 + r_0^2)^3(1443840000L^{18}M^3 - 645120000L^{18}M^2 r_0 + 3481866240L^{16}M^3 r_0^2 + 376780800L^{16}M^2 r_0^3 \\
& - 857472000L^{16}Mr_0^4 - 5698068736L^{14}M^3 r_0^4 + 12906055680L^{14}M^2 r_0^5 - 4598832000L^{14}Mr_0^6 \\
& - 30679784768L^{12}M^3 r_0^6 + 36702979536L^{12}M^2 r_0^7 - 10214544200L^{12}Mr_0^8 - 46687160912L^{10}M^3 r_0^8 \\
& + 47920180732L^{10}M^2 r_0^9 - 12034259400L^{10}Mr_0^{10} - 34910457500L^8 M^3 r_0^{10} + 33450421620L^8 M^2 r_0^{11} \\
& - 7955786400L^8 Mr_0^{12} - 13231827540L^6 M^3 r_0^{12} + 12237529680L^6 M^2 r_0^{13} - 2830778675L^6 Mr_0^{14} \\
& - 2081061396L^4 M^3 r_0^{14} + 1920152080L^4 M^2 r_0^{15} - 448289925L^4 Mr_0^{16} - 1750L^4 r_0^{17} \\
& - 4481148L^2 M^3 r_0^{16} + 24248796L^2 M^2 r_0^{17} - 11278125L^2 Mr_0^{18} - 8750L^2 r_0^{19} + 11067088M^3 r_0^{18} \\
& - 6431148M^2 r_0^{19} + 440325Mr_0^{20} + 77000r_0^{21}),
\end{aligned}
$$

$$
F_{r[6]} = \frac{-3}{560\pi r_0^{18}(L^2 + r_0^2)^{15/2}(r_0 - 2M)}(F^{\mathcal{E}}_{r[6]}\mathcal{E} + F^{\mathcal{K}}_{r[6]}\mathcal{K}), \tag{5.39}
$$

where

$$\begin{aligned}
F^{\mathcal{E}}_{r[6]} = {} & -28000E^8(r_0 - L)(L + r_0)(11L^4 - 74r_0^2L^2 + 11r_0^4)r_0^{26} + 50E^6(L^2 + r_0^2)(19565r_0^{19} + 6086Mr_0^{18} \\
& - 113785L^2r_0^{17} + 1633964L^2Mr_0^{16} + 76615L^4r_0^{15} + 2559418L^4Mr_0^{14} - 5075L^6r_0^{13} - 19625630L^6Mr_0^{12} \\
& - 100527512L^8Mr_0^{10} - 206284326L^{10}Mr_0^8 - 232489984L^{12}Mr_0^6 - 151293440L^{14}Mr_0^4 - 53575680L^{16}Mr_0^2 \\
& - 8028160L^{18}M)r_0^{11} - E^4(L^2 + r_0^2)^2(1094625r_0^{22} + 545450Mr_0^{21} - 4202625L^2r_0^{20} - 16774936M^2r_0^{20} \\
& + 110101000L^2Mr_0^{19} + 1414875L^4r_0^{18} - 331621052L^2M^2r_0^{18} - 822447000L^4Mr_0^{17} - 7875L^6r_0^{16} \\
& + 1429685188L^4M^2r_0^{16} - 9480504900L^6Mr_0^{15} + 19802086804L^6M^2r_0^{14} - 35145688050L^8Mr_0^{13} \\
& + 72068652100L^8M^2r_0^{12} - 68030273700L^{10}Mr_0^{11} + 130518064824L^{10}M^2r_0^{10} - 76873222400L^{12}Mr_0^9 \\
& + 129714899808L^{12}M^2r_0^8 - 51274604800L^{14}Mr_0^7 + 65633972928L^{14}M^2r_0^6 - 18785894400L^{16}Mr_0^5 \\
& + 8353439744L^{16}M^2r_0^4 - 2924544000L^{18}Mr_0^3 - 6114713600L^{18}M^2r_0^2 - 2015232000L^{20}M^2)r_0^6 \\
& + 2E^2(L^2 + r_0^2)^3(242375r_0^{23} + 138500Mr_0^{22} - 419125L^2r_0^{21} - 10992820M^2r_0^{21} + 20399392M^3r_0^{20} \\
& + 6959650L^2Mr_0^{20} + 9625L^4r_0^{19} - 81665756L^2M^2r_0^{19} + 138887552L^2M^3r_0^{18} - 822884550L^4Mr_0^{18} \\
& - 875L^6r_0^{17} + 3518901164L^4M^2r_0^{17} - 3802035608L^4M^3r_0^{16} - 7067551050L^6Mr_0^{16} + 30648837404L^6M^2r_0^{15} \\
& - 33234129320L^6M^3r_0^{14} - 25352340750L^8Mr_0^{14} + 106617027800L^8M^2r_0^{13} - 111740075320L^8M^3r_0^{12} \\
& - 49660036200L^{10}Mr_0^{12} + 197830827680L^{10}M^2r_0^{11} - 195029907128L^{10}M^3r_0^{10} - 57553509200L^{12}Mr_0^{10} \\
& + 209550665904L^{12}M^2r_0^9 - 183717663248L^{12}M^3r_0^8 - 39556896400L^{14}Mr_0^8 + 121346652064L^{14}M^2r_0^7 \\
& - 77791192288L^{14}M^3r_0^6 - 14956032000L^{16}Mr_0^6 + 28755231744L^{16}M^2r_0^5 + 7146388480L^{16}M^3r_0^4 \\
& - 2403072000L^{18}Mr_0^4 - 3619737600L^{18}M^2r_0^3 + 18731642880L^{18}M^3r_0^2 - 2297856000L^{20}M^2r_0 \\
& + 4902912000L^{20}M^3)r_0^3 + (2M - r_0)(L^2 + r_0^2)^4(53375r_0^{23} + 173600Mr_0^{22} + 29750L^2r_0^{21} - 5212944M^2r_0^{21} \\
& + 10317184M^3r_0^{20} - 14183750L^2Mr_0^{20} + 25375L^4r_0^{19} + 62499436L^2M^2r_0^{19} - 76153328L^2M^3r_0^{18} \\
& - 679641900L^4Mr_0^{18} + 7000L^6r_0^{17} + 3345821696L^4M^2r_0^{17} - 4111469784L^4M^3r_0^{16} - 5518261350L^6Mr_0^{16} \\
& + 25823273900L^6M^2r_0^{15} - 30032966752L^6M^3r_0^{14} - 20150264400L^8Mr_0^{14} + 88797134680L^8M^2r_0^{13} \\
& - 96877958296L^8M^3r_0^{12} - 40738068000L^{10}Mr_0^{12} + 166165277616L^{10}M^2r_0^{11} - 165473393280L^{10}M^3r_0^{10} \\
& - 48857572400L^{12}Mr_0^{10} + 177451524416L^{12}M^2r_0^9 - 149256330784L^{12}M^3r_0^8 - 34733020000L^{14}Mr_0^8 \\
& + 101711045376L^{14}M^2r_0^7 - 51263318208L^{14}M^3r_0^6 - 13563648000L^{16}Mr_0^6 + 21232814080L^{16}M^2r_0^5 \\
& + 21391316992L^{16}M^3r_0^4 - 2247168000L^{18}Mr_0^4 - 5520998400L^{18}M^2r_0^3 + 23777402880L^{18}M^3r_0^2 \\
& - 2580480000L^{20}M^2r_0 + 5775360000L^{20}M^3),
\end{aligned}$$

$$\begin{aligned}
F^{\mathcal{K}}_{r[6]} = {} & 875E^8(-105L^6 + 1189r_0^2L^4 - 1531r_0^4L^2 + 247r_0^6)r_0^{26} - 25E^6(L^2 + r_0^2)(27055r_0^{17} + 25612Mr_0^{16} \\
& - 123235L^2r_0^{15} + 1887182L^2Mr_0^{14} + 62125L^4r_0^{13} + 676066L^4Mr_0^{12} - 2625L^6r_0^{11} - 26099670L^6Mr_0^{10} \\
& - 90366550L^8Mr_0^8 - 137437024L^{10}Mr_0^6 - 110937600L^{12}Mr_0^4 - 46551040L^{14}Mr_0^2 - 8028160L^{16}M)r_0^{13} \\
& + 2E^4(L^2 + r_0^2)^2(370125r_0^{20} + 697975Mr_0^{19} - 1065750L^2r_0^{18} - 6904028M^2r_0^{18} + 27977700L^2Mr_0^{17} \\
& + 244125L^4r_0^{16} - 86371482L^2M^2r_0^{16} - 315079050L^4Mr_0^{15} + 615225146L^4M^2r_0^{14} - 2582807100L^6Mr_0^{13} \\
& + 5441132830L^6M^2r_0^{12} - 7522573725L^8Mr_0^{11} + 15199781250L^8M^2r_0^{10} - 11253096600L^{10}Mr_0^9 \\
& + 20574272172L^{10}M^2r_0^8 - 9303284800L^{12}Mr_0^7 + 13728299088L^{12}M^2r_0^6 - 4056729600L^{14}Mr_0^5 \\
& + 3016609536L^{14}M^2r_0^4 - 731136000L^{16}Mr_0^3 - 1087846400L^{16}M^2r_0^2 - 503808000L^{18}M^2)r_0^8 \\
& - E^2(L^2 + r_0^2)^3(314125r_0^{21} + 970000Mr_0^{20} - 358750L^2r_0^{19} - 18222440M^2r_0^{19} + 31216064M^3r_0^{18} \\
& - 3780450L^2Mr_0^{18} - 875L^4r_0^{17} - 36108732L^2M^2r_0^{17} + 87877152L^2M^3r_0^{16} - 1092352500L^4Mr_0^{16} \\
& + 4756636984L^4M^2r_0^{15} - 5211718840L^4M^3r_0^{14} - 7494234450L^6Mr_0^{14} + 32418083300L^6M^2r_0^{13}
\end{aligned}$$

$$- 35018994560L^6M^3r_0^{12} - 21674793600L^8Mr_0^{12} + 89821855320L^8M^2r_0^{11} - 92610055960L^8M^3r_0^{10}$$
$$- 33236721600L^{10}Mr_0^{10} + 127823271040L^{10}M^2r_0^9 - 120667329776L^{10}M^3r_0^8 - 28422864400L^{12}Mr_0^8$$
$$+ 95019791488L^{12}M^2r_0^7 - 72612130368L^{12}M^3r_0^6 - 12853344000L^{14}Mr_0^6 + 30055494144L^{14}M^2r_0^5$$
$$- 5260183040L^{14}M^3r_0^4 - 2403072000L^{16}Mr_0^4 - 1609113600L^{16}M^2r_0^3 + 14441594880L^{16}M^3r_0^2$$
$$- 2297856000L^{18}M^2r_0 + 4902912000L^{18}M^3)r_0^5 + (r_0 - 2M)(L^2 + r_0^2)^4(27125r_0^{21} + 299600Mr_0^{20}$$
$$+ 9625L^2r_0^{19} - 4164624M^2r_0^{19} + 7521664M^3r_0^{18} - 12312300L^2Mr_0^{18} + 3500L^4r_0^{17} + 60136468L^2M^2r_0^{17}$$
$$- 77649520L^2M^3r_0^{16} - 435265950L^4Mr_0^{16} + 2136130980L^4M^2r_0^{15} - 2610250432L^4M^3r_0^{14}$$
$$- 2918350050L^6Mr_0^{14} + 13485739400L^6M^2r_0^{13} - 15471992072L^6M^3r_0^{12} - 8700997200L^8Mr_0^{12}$$
$$+ 37444962000L^8M^2r_0^{11} - 39767055768L^8M^3r_0^{10} - 13875397200L^{10}Mr_0^{10} + 54025898376L^{10}M^2r_0^9$$
$$- 50546368608L^{10}M^3r_0^8 - 12345326000L^{12}Mr_0^8 + 40319920928L^{12}M^2r_0^7 - 27561410368L^{12}M^3r_0^6$$
$$- 5798688000L^{14}Mr_0^6 + 11983523840L^{14}M^2r_0^5 + 2639284736L^{14}M^3r_0^4 - 1123584000L^{16}Mr_0^4$$
$$- 1631539200L^{16}M^2r_0^3 + 9361981440L^{16}M^3r_0^2 - 1290240000L^{18}M^2r_0 + 2887680000L^{18}M^3)r_0^2,$$

$$F_{\theta[6]} = 0, \tag{5.40}$$

$$F_{\phi[6]} = \frac{-3\dot{r}_0}{560\pi L r_0^{16}(L^2 + r_0^2)^{13/2}}(F^{\mathcal{E}}_{\phi[6]}\mathcal{E} + F^{\mathcal{K}}_{\phi[6]}\mathcal{K}), \tag{5.41}$$

where

$$F^{\mathcal{E}}_{\phi[6]} = 875E^6r_0^{23}(1773L^4r_0^2 - 337L^6 - 947L^2r_0^4 + 15r_0^6) + 175E^4r_0^8(L^2 + r_0^2)(983040L^{18}M + 7045120L^{16}Mr_0^2$$
$$+ 21570560L^{14}Mr_0^4 + 36582912L^{12}Mr_0^6 + 37139572L^{10}Mr_0^8 + 22617566L^8Mr_0^{10} + 7708830L^6Mr_0^{12}$$
$$+ 225L^6r_0^{13} + 1199730L^4Mr_0^{14} - 9375L^4r_0^{15} + 4926L^2Mr_0^{16} + 9375L^2r_0^{17} - 225r_0^{19}) + E^2r_0^3(L^2 + r_0^2)^2$$
$$\times (2015232000L^{20}M^2 + 8400691200L^{18}M^2r_0^2 + 1376256000L^{18}Mr_0^3 + 11063078912L^{16}M^2r_0^4$$
$$+ 7276953600L^{16}Mr_0^5 + 82626752L^{14}M^2r_0^6 + 15631481600L^{14}Mr_0^7 - 12481032128L^{12}M^2r_0^8$$
$$+ 17141443200L^{12}Mr_0^9 - 10632497080L^{10}M^2r_0^{10} + 9613116800L^{10}Mr_0^{11} - 2120762600L^8M^2r_0^{12}$$
$$+ 2046335900L^8Mr_0^{13} + 1035946292L^6M^2r_0^{14} - 316454600L^6Mr_0^{15} + 15750L^6r_0^{16} + 431941952L^4M^2r_0^{16}$$
$$- 166655300L^4Mr_0^{17} + 133875L^4r_0^{18} + 17346492L^2M^2r_0^{18} - 2290400L^2Mr_0^{19} - 850500L^2r_0^{20}$$
$$- 312480M^2r_0^{20} + 39375r_0^{22}) - (L^2 + r_0^2)^3(2580480000L^{20}M^2r_0 - 5775360000L^{20}M^3$$
$$- 21381242880L^{18}M^3r_0^2 + 4448665600L^{18}M^2r_0^3 + 2247168000L^{18}Mr_0^4 - 16436058112L^{16}M^3r_0^4$$
$$- 20228515840L^{16}M^2r_0^5 + 12144384000L^{16}Mr_0^6 + 37967372736L^{14}M^3r_0^6 - 78865174016L^{14}M^2r_0^7$$
$$+ 27288335200L^{14}Mr_0^8 + 91117248928L^{12}M^3r_0^8 - 114866020480L^{12}M^2r_0^9 + 32738062000L^{12}Mr_0^{10}$$
$$+ 80974789248L^{10}M^3r_0^{10} - 86565871136L^{10}M^2r_0^{11} + 22304895200L^{10}Mr_0^{12} + 35112838392L^8M^3r_0^{12}$$
$$- 34540540744L^8M^2r_0^{13} + 8392988800L^8Mr_0^{14} + 6757023136L^6M^3r_0^{14} - 6373891596L^6M^2r_0^{15}$$
$$+ 1516716950L^6Mr_0^{16} - 7000L^6r_0^{17} + 273916728L^4M^3r_0^{16} - 286552800L^4M^2r_0^{17} + 80573500L^4Mr_0^{18}$$
$$- 28875L^4r_0^{19} - 24711184L^2M^3r_0^{18} + 14372004L^2M^2r_0^{19} - 855050L^2Mr_0^{20} - 50750L^2r_0^{21}$$
$$+ 309120M^3r_0^{20} - 245280M^2r_0^{21} + 13125r_0^{23}),$$

$$F^{\mathcal{K}}_{\phi[6]} = -875E^6r_0^{23}(-105L^6 + 829L^4r_0^2 - 587L^2r_0^4 + 15r_0^6) + 175E^4r_0^{10}(L^2 + r_0^2)(-491520L^{16}M - 3092480L^{14}Mr_0^2$$
$$- 8102400L^{12}Mr_0^4 - 11336736L^{10}Mr_0^6 - 8972282L^8Mr_0^8 - 3855372L^6Mr_0^{10} - 750282L^4Mr_0^{12}$$
$$+ 3825L^4r_0^{13} - 12696L^2Mr_0^{14} - 5550L^2r_0^{15} + 225r_0^{17}) - E^2r_0^5(L^2 + r_0^2)^2(1007616000L^{18}M^2$$
$$+ 3318681600L^{16}M^2r_0^2 + 688128000L^{16}Mr_0^3 + 2674925056L^{14}M^2r_0^4 + 3036364800L^{14}Mr_0^5$$
$$- 2164346848L^{12}M^2r_0^6 + 5191177600L^{12}Mr_0^7 - 4277576360L^{10}M^2r_0^8 + 4156658800L^{10}Mr_0^9$$
$$- 1698491920L^8M^2r_0^{10} + 1358613200L^8Mr_0^{11} + 281096500L^6M^2r_0^{12} - 46924500L^6Mr_0^{13}$$
$$+ 248366216L^4M^2r_0^{14} - 97522600L^4Mr_0^{15} + 7875L^4r_0^{16} + 15819372L^2M^2r_0^{16} - 3686900L^2Mr_0^{17}$$

$$- 456750L^2r_0^{18} - 312480M^2r_0^{18} + 39375r_0^{20}) + r_0^2(L^2 + r_0^2)^3(-2887680000L^{18}M^3 + 1290240000L^{18}M^2r_0 - 8163901440L^{16}M^3r_0^2 + 1095372800L^{16}M^2r_0^3 + 1123584000L^{16}Mr_0^4 - 1209975296L^{14}M^3r_0^4 - 11012229120L^{14}M^2r_0^5 + 5089056000L^{14}Mr_0^6 + 19718951872L^{12}M^3r_0^6 - 29772000928L^{12}M^2r_0^7 + 9243911600L^{12}Mr_0^8 + 28381407744L^{10}M^3r_0^8 - 31906051368L^{10}M^2r_0^9 + 8496115600L^{10}Mr_0^{10} + 16529207256L^8M^3r_0^{10} - 16531893472L^8M^2r_0^{11} + 4060456400L^8Mr_0^{12} + 4051974744L^6M^3r_0^{12} - 3820351368L^6M^2r_0^{13} + 903318850L^6Mr_0^{14} + 233659760L^4M^3r_0^{14} - 228732252L^4M^2r_0^{15} + 59520650L^4Mr_0^{16} - 3500L^4r_0^{17} - 17332624L^2M^3r_0^{16} + 10640724L^2M^2r_0^{17} - 891800L^2Mr_0^{18} - 11375L^2r_0^{19} + 309120M^3r_0^{18} - 245280M^2r_0^{19} + 13125r_0^{21}).$$

D. Electromagnetic case

In the electromagnetic case, an ambiguity arises in the definition of u^a in the angular directions *away* from the world line. In Eq. (2.19), one is free to define $u^a(x)$ as they wish, provided $\lim_{x->\bar{x}} u^a(x) = u^{\bar{a}}$. A natural covariant choice would be to define this through parallel transport, $u^a(x) = g^a_{\bar{b}} u^{\bar{b}}$. However, in reality, it is more practical in numerical calculations to define u^a such that its components in Schwarzschild coordinates are equal to the components of $u^{\bar{a}}$ in Schwarzschild coordinates. Doing so, the regularization parameters are given by

$$F_{t[-1]} = -\frac{\dot{r}_0 \,\mathrm{sgn}(\Delta r)}{2(L^2 + r_0^2)}, \qquad F_{r[-1]} = \frac{E r_0 \,\mathrm{sgn}(\Delta r)}{2(r_0 - 2M)(L^2 + r_0^2)}, \qquad F_{\theta[-1]} = 0, \qquad F_{\phi[-1]} = 0, \tag{5.42}$$

$$F_{t[0]} = -\frac{E\dot{r}_0}{\pi r_0(r_0^2 + L^2)^{3/2}}(r_0^2\mathcal{K} + 2L^2\mathcal{E}), \tag{5.43}$$

$$F_{r[0]} = \frac{1}{\pi r_0^3(r_0^2 + L^2)^{3/2}(r_0 - 2M)}(F^{\mathcal{E}}_{r[0]}\mathcal{E} + F^{\mathcal{K}}_{r[0]}\mathcal{K}), \tag{5.44}$$

where

$$F^{\mathcal{E}}_{r[0]} = 2E^2L^2r_0^3 + (L^2 + r_0^2)(2L^2 + r_0^2)(2M - r_0), \qquad F^{\mathcal{K}}_{r[0]} = E^2r_0^5 + r_0^2(L^2 + r_0^2)(r_0 - 2M),$$

$$F_{\theta[0]} = 0, \tag{5.45}$$

$$F_{\phi[0]} = \frac{\dot{r}_0}{\pi L r_0\sqrt{L^2 + r_0^2}}[\mathcal{E}(2L^2 + r_0^2) - \mathcal{K}r_0^2], \tag{5.46}$$

$$F_{t[2]} = -\frac{E\dot{r}_0}{2\pi r_0^4(L^2 + r_0^2)^{7/2}}(F^{\mathcal{E}}_{t[2]}\mathcal{E} + F^{\mathcal{K}}_{t[2]}\mathcal{K}), \tag{5.47}$$

where

$$F^{\mathcal{E}}_{t[2]} = 2E^2r_0^5(-L^4 + 10L^2r_0^2 + 3r_0^4) + (L^2 + r_0^2)(60L^6M + 168L^4Mr_0^2 + 182L^2Mr_0^4 - 13L^2r_0^5 + 58Mr_0^6 - 5r_0^7),$$

$$F^{\mathcal{K}}_{t[2]} = 2r_0^2(L^2 + r_0^2)(-21L^4M - 48L^2Mr_0^2 + 3L^2r_0^3 - 23Mr_0^4 + r_0^5) - E^2r_0^7(11L^2 + 3r_0^2),$$

$$F_{r[2]} = -\frac{1}{2\pi r_0^6(L^2 + r_0^2)^{7/2}(r_0 - 2M)}(F^{\mathcal{E}}_{r[2]}\mathcal{E} + F^{\mathcal{K}}_{r[2]}\mathcal{K}), \tag{5.48}$$

where

$$F^{\mathcal{E}}_{r[2]} = -2E^4r_0^8(-L^4 + 10L^2r_0^2 + 3r_0^4) - (L^2 + r_0^2)^2(2M - r_0)(44L^6M + 94L^4Mr_0^2 + 54L^2Mr_0^4 + L^2r_0^5 + 3r_0^7) + 2E^2r_0^3(L^2 + r_0^2)(-30L^6M - 86L^4Mr_0^2 + L^4r_0^3 - 98L^2Mr_0^4 + 10L^2r_0^5 - 26Mr_0^6 + r_0^7),$$

$$F^{\mathcal{K}}_{r[2]} = E^4r_0^{10}(11L^2 + 3r_0^2) - r_0^2(L^2 + r_0^2)^2(r_0 - 2M)(22L^4M + 24L^2Mr_0^2 + 2L^2r_0^3 + 3r_0^5) + E^2r_0^5(L^2 + r_0^2)(42L^4M + 98L^2Mr_0^2 - 7L^2r_0^3 + 40Mr_0^4 + r_0^5),$$

$$F_{\theta[2]} = 0, \tag{5.49}$$

$$F_{\phi[2]} = -\frac{\dot{r}_0}{2\pi L r_0^4 (L^2 + r_0^2)^{5/2}} (F^{\mathcal{E}}_{\phi[2]}\mathcal{E} + F^{\mathcal{K}}_{\phi[2]}\mathcal{K}), \tag{5.50}$$

where

$$F^{\mathcal{E}}_{\phi[2]} = E^2 r_0^5(-2L^4 - 7L^2 r_0^2 + 3r_0^4) - (L^2 + r_0^2)(44L^6 M + 94L^4 M r_0^2 + 54L^2 M r_0^4 + L^2 r_0^5 + 3r_0^7),$$
$$F^{\mathcal{K}}_{\phi[2]} = r_0^2(L^2 + r_0^2)(22L^4 M + 24L^2 M r_0^2 + 2L^2 r_0^3 + 3r_0^5) - E^2 r_0^7(3r_0^2 - L^2),$$

$$F_{t[4]} = \frac{3E\dot{r}_0}{40\pi r_0^{11}(L^2 + r_0^2)^{11/2}} (F^{\mathcal{E}}_{t[4]}\mathcal{E} + F^{\mathcal{K}}_{t[4]}\mathcal{K}), \tag{5.51}$$

where

$$\begin{aligned}F^{\mathcal{E}}_{t[4]} = {}& -30E^4 r_0^{14}(3L^6 - 102L^4 r_0^2 + 43L^2 r_0^4 + 20r_0^6) + 2E^2 r_0^5(L^2 + r_0^2)(34560L^{12}M + 169728L^{10}M r_0^2 + 333564L^8 M r_0^4 \\ & + 328912L^6 M r_0^6 + 167074L^4 M r_0^8 - 1245L^4 r_0^9 + 32948L^2 M r_0^{10} + 1230L^2 r_0^{11} + 230M r_0^{12} + 555r_0^{13}) \\ & - 4(L^2 + r_0^2)^2(11520L^{14}M^2 - 18240L^{12}M^2 r_0^2 + 33600L^{12}M r_0^3 - 226624L^{10}M^2 r_0^4 + 153960L^{10}M r_0^5 \\ & - 496164L^8 M^2 r_0^6 + 280764L^8 M r_0^7 - 485652L^6 M^2 r_0^8 + 255197L^6 M r_0^9 - 230930L^4 M^2 r_0^{10} + 116771L^4 M r_0^{11} \\ & - 30L^4 r_0^{12} - 45200L^2 M^2 r_0^{12} + 21487L^2 M r_0^{13} + 270L^2 r_0^{14} - 1342M^2 r_0^{14} + 229M r_0^{15} + 120r_0^{16}),\end{aligned}$$

$$\begin{aligned}F^{\mathcal{K}}_{t[4]} = {}& 15E^4 r_0^{16}(-87L^4 + 66L^2 r_0^2 + 25r_0^4) - 4E^2 r_0^7(L^2 + r_0^2)(8640L^{10}M + 34872L^8 M r_0^2 + 53283L^6 M r_0^4 \\ & + 37555L^4 M r_0^6 - 225L^4 r_0^7 + 9809L^2 M r_0^8 + 420L^2 r_0^9 + 265M r_0^{10} + 165r_0^{11}) + r_0^2(L^2 + r_0^2)^2(23040L^{12}M^2 \\ & - 56640L^{10}M^2 r_0^2 + 67200L^{10}M r_0^3 - 402608L^8 M^2 r_0^4 + 249120L^8 M r_0^5 - 643736L^6 M^2 r_0^6 + 346608L^6 M r_0^7 \\ & - 427796L^4 M^2 r_0^8 + 217192L^4 M r_0^9 - 110916L^2 M^2 r_0^{10} + 52580L^2 M r_0^{11} + 615L^2 r_0^{12} - 5368M^2 r_0^{12} \\ & + 1516M r_0^{13} + 255r_0^{14}),\end{aligned}$$

$$F_{r[4]} = \frac{3}{40\pi r_0^{13}(L^2 + r_0^2)^{11/2}(r_0 - 2M)} (F^{\mathcal{E}}_{r[4]}\mathcal{E} + F^{\mathcal{K}}_{r[4]}\mathcal{K}), \tag{5.52}$$

where

$$\begin{aligned}F^{\mathcal{E}}_{r[4]} = {}& 30E^6 r_0^{17}(3L^6 - 102L^4 r_0^2 + 43L^2 r_0^4 + 20r_0^6) - E^4 r_0^8(L^2 + r_0^2)(69120L^{12}M + 339456L^{10}M r_0^2 \\ & + 667128L^8 M r_0^4 + 657884L^6 M r_0^6 - 30L^6 r_0^7 + 334658L^4 M r_0^8 \\ & - 2745L^4 r_0^9 + 62656L^2 M r_0^{10} + 4080L^2 r_0^{11} + 610M r_0^{12} + 1035r_0^{13}) + (L^2 + r_0^2)^3(2M - r_0)(46080L^{14}M^2 \\ & + 89856L^{12}M^2 r_0^2 + 53760L^{12}M r_0^3 - 86336L^{10}M^2 r_0^4 + 211520L^{10}M r_0^5 - 344128L^8 M^2 r_0^6 + 317600L^8 M r_0^7 \\ & - 306808L^6 M^2 r_0^8 + 221140L^6 M r_0^9 - 98676L^4 M^2 r_0^{10} + 66220L^4 M r_0^{11} + 60L^4 r_0^{12} - 5244L^2 M^2 r_0^{12} \\ & + 4440L^2 M r_0^{13} + 105L^2 r_0^{14} + 160M^2 r_0^{14} - 75r_0^{16}) + 2E^2 r_0^3(L^2 + r_0^2)^2(23040L^{14}M^2 - 36480L^{12}M^2 r_0^2 \\ & + 67200L^{12}M r_0^3 - 445056L^{10}M^2 r_0^4 + 303824L^{10}M r_0^5 \\ & - 959952L^8 M^2 r_0^6 + 545340L^8 M r_0^7 - 922996L^6 M^2 r_0^8 + 486240L^6 M r_0^9 - 428644L^4 M^2 r_0^{10} + 216604L^4 M r_0^{11} \\ & + 105L^4 r_0^{12} - 78428L^2 M^2 r_0^{12} + 35368L^2 M r_0^{13} + 1350L^2 r_0^{14} - 2684M^2 r_0^{14} + 608M r_0^{15} + 165r_0^{16}),\end{aligned}$$

$$\begin{aligned}F^{\mathcal{K}}_{r[4]} = {}& -15E^6 r_0^{19}(-87L^4 + 66L^2 r_0^2 + 25r_0^4) + E^4 r_0^{10}(L^2 + r_0^2)(34560L^{10}M + 139488L^8 M r_0^2 + 213132L^6 M r_0^4 \\ & + 150250L^4 M r_0^6 - 915L^4 r_0^7 + 37496L^2 M r_0^8 + 2550L^2 r_0^9 + 1210M r_0^{10} + 585r_0^{11}) \\ & - r_0^2(L^2 + r_0^2)^3(r_0 - 2M)(63760L^8 M^2 r_0^4 - 23040L^{12}M^2 - 24768L^{10}M^2 r_0^2 - 26880L^{10}M r_0^3 - 82240L^8 M r_0^5 \\ & + 115848L^6 M^2 r_0^6 - 87920L^6 M r_0^7 + 56084L^4 M^2 r_0^8 - 36340L^4 M r_0^9 + 5324L^2 M^2 r_0^{10} - 3540L^2 M r_0^{11} \\ & + 15L^2 r_0^{12} - 160M^2 r_0^{12} + 75r_0^{14}) - E^2 r_0^5(L^2 + r_0^2)^2(23040L^{12}M^2 - 56640L^{10}M^2 r_0^2 + 67200L^{10}M r_0^3 \\ & - 394416L^8 M^2 r_0^4 + 245024L^8 M r_0^5 - 618528L^6 M^2 r_0^6 + 334004L^6 M r_0^7 - 400636L^4 M^2 r_0^8 + 203252L^4 M r_0^9 \\ & + 180L^4 r_0^{10} - 97892L^2 M^2 r_0^{10} + 44568L^2 M r_0^{11} + 1365L^2 r_0^{12} - 5368M^2 r_0^{12} + 1816M r_0^{13} + 105r_0^{14}),\end{aligned}$$

$$F_{\theta[4]} = 0, \tag{5.53}$$

$$F_{\phi[4]} = \frac{3\dot{r}_0}{40\pi L r_0^{11}(L^2 + r_0^2)^{9/2}}(F^{\mathcal{E}}_{\phi[4]}\mathcal{E} + F^{\mathcal{K}}_{\phi[4]}\mathcal{K}), \tag{5.54}$$

where

$$\begin{aligned}
F^{\mathcal{E}}_{\phi[4]} = {}& -15E^4 r_0^{14}(2L^6 + 17L^4 r_0^2 - 108L^2 r_0^4 + 5r_0^6) + 2E^2 r_0^7(L^2 + r_0^2)(4096L^{10}M + 16188L^8 M r_0^2 + 24154L^6 M r_0^4 \\
& + 16608L^4 M r_0^6 - 165L^4 r_0^7 + 5986L^2 M r_0^8 - 810L^2 r_0^9 + 75r_0^{11}) + (L^2 + r_0^2)^2(46080L^{14}M^2 \\
& + 89856L^{12}M^2 r_0^2 + 53760L^{12}M r_0^3 - 86336L^{10}M^2 r_0^4 + 211520L^{10}M r_0^5 - 344128L^8 M^2 r_0^6 \\
& + 317600L^8 M r_0^7 - 306808L^6 M^2 r_0^8 + 221140L^6 M r_0^9 - 98676L^4 M^2 r_0^{10} + 66220L^4 M r_0^{11} \\
& + 60L^4 r_0^{12} - 5244L^2 M^2 r_0^{12} + 4440L^2 M r_0^{13} + 105L^2 r_0^{14} + 160M^2 r_0^{14} - 75r_0^{16}), \\
F^{\mathcal{K}}_{\phi[4]} = {}& 15E^4 r_0^{16}(L^4 - 58L^2 r_0^2 + 5r_0^4) - 2E^2 r_0^9(L^2 + r_0^2)(2048L^8 M + 6302L^6 M r_0^2 + 6790L^4 M r_0^4 \\
& - 90L^4 r_0^5 + 3256L^2 M r_0^6 - 375L^2 r_0^7 + 75r_0^9) + r_0^2(L^2 + r_0^2)^2(-23040L^{12}M^2 - 24768L^{10}M^2 r_0^2 \\
& - 26880L^{10}M r_0^3 + 63760L^8 M^2 r_0^4 - 82240L^8 M r_0^5 + 115848L^6 M^2 r_0^6 - 87920L^6 M r_0^7 + 56084L^4 M^2 r_0^8 \\
& - 36340L^4 M r_0^9 + 5324L^2 M^2 r_0^{10} - 3540L^2 M r_0^{11} + 15L^2 r_0^{12} - 160M^2 r_0^{12} + 75r_0^{14}).
\end{aligned}$$

E. Gravitational case

1. Self-force regularization

The self-force on a gravitational particle is given by

$$F^a = k^{abcd}\bar{h}^{\rm R}_{bc;d}, \tag{5.55}$$

where

$$k^{abcd} \equiv \frac{1}{2}g^{ad}u^b u^c - g^{ab}u^c u^d - \frac{1}{2}u^a u^b u^c u^d + \frac{1}{4}u^a g^{bc}u^d + \frac{1}{4}g^{ad}g^{bc}. \tag{5.56}$$

Note that, as in the electromagnetic case, an ambiguity arises here due to the presence of terms involving the four-velocity at x. One is free to arbitrarily choose how to define this, provided $\lim_{x\to\bar{x}} u^a = u^{\bar{a}}$. Following Barack and Sago [64], we choose to take the Schwarzschild components of the four-velocity at x to be exactly those at $\bar{x}$. The regularization parameters in the gravitational case are given by

$$F^t_{[-1]} = -\mathrm{sgn}(\Delta r)\frac{r_0\dot{r}_0}{2(L^2+r_0^2)(r_0-2M)}, \quad F^r_{[-1]} = -\mathrm{sgn}(\Delta r)\frac{E}{2(L^2+r_0^2)}, \quad F^\theta_{[-1]} = 0, \quad F^\phi_{[-1]} = 0, \tag{5.57}$$

$$F^t_{[0]} = -\frac{\dot{r}_0 E}{\pi(r_0 - 2M)(L^2 + r_0^2)^{3/2}}(2L^2\mathcal{E} + r_0^2\mathcal{K}), \tag{5.58}$$

$$\begin{aligned}
F^r_{[0]} &= \frac{1}{\pi r_0^4(L^2 + r_0^2)^{3/2}}\{[(r_0 - 2M)(L^2 + r_0^2)(2L^2 + r_0^2) - 2E^2 L^2 r_0^3]\mathcal{E} - r_0^2[(r_0 - 2M)(L^2 + r_0^2) + E^2 r_0^3]\mathcal{K}\}, \\
F^\theta_{[0]} &= 0, \\
F^\phi_{[0]} &= -\frac{\dot{r}_0}{\pi L r_0^3(L^2 + r_0^2)^{1/2}}[(2L^2 + r_0^2)\mathcal{E} - r_0^2\mathcal{K}],
\end{aligned} \tag{5.59}$$

$$F^t_{[2]} = \frac{E\dot{r}_0}{2\pi r_0^3(r_0 - 2M)(L^2 + r^2)^{7/2}}(F^t_{\mathcal{E}[2]}\mathcal{E} + D^t_{\mathcal{K}[2]}\mathcal{K}), \tag{5.60}$$

where

$$\begin{aligned}
F^t_{\mathcal{E}[2]} &= -2E^2 r_0^5(11L^4 + 34L^2 r_0^2 + 15r_0^4) - (L^2 + r_0^2)(276L^6 M + 768L^4 M r_0^2 + 782L^2 M r_0^4 - 37L^2 r_0^5 + 274M r_0^6 - 29r_0^7), \\
F^t_{\mathcal{K}[2]} &= E^2 r_0^5(12L^4 + 35L^2 r_0^2 + 15r_0^4) + 2r_0^2(L^2 + r_0^2)(93L^4 M + 204L^2 M r_0^2 - 9L^2 r_0^3 + 107M r_0^4 - 7r_0^5),
\end{aligned}$$

$$F^r_{[2]} = \frac{1}{2\pi r_0^7(L^2 + r_0^2)^{7/2}}(F^r_{\mathcal{E}[2]}\mathcal{E} + F^r_{\mathcal{K}[2]}\mathcal{K}), \tag{5.61}$$

where

$$F^r_{\mathcal{E}[2]} = (r_0 - 2M)(L^2 + r_0^2)^2(188L^6M + 406L^4Mr_0^2 + 222L^2Mr_0^4 + 13L^2r_0^5 + 15r_0^7) - 2E^4r_0^8(11L^4 + 34L^2r_0^2 + 15r_0^4)$$
$$- 2E^2r_0^3(L^2 + r_0^2)(138L^6M + 422L^4Mr_0^2 - 19L^4r_0^3 + 422L^2Mr_0^4 - 34L^2r_0^5 + 122Mr_0^6 - 7r_0^7),$$
$$F^r_{\mathcal{K}[2]} = -r_0^2(r_0 - 2M)(L^2 + r_0^2)^2(94L^4M + 96L^2Mr_0^2 + 14L^2r_0^3 + 15r_0^5) + E^4r_0^8(12L^4 + 35L^2r_0^2 + 15r_0^4)$$
$$+ E^2r_0^5(L^2 + r_0^2)(210L^4M - 12L^4r_0 + 410L^2Mr_0^2 - 19L^2r_0^3 + 184Mr_0^4 + r_0^5)],$$

$$F^\theta_{[2]} = 0, \tag{5.62}$$

$$F^\phi_{[2]} = \frac{\dot r_0}{2\pi L^3 r_0^6(L^2 + r_0^2)^{5/2}}(F^\phi_{\mathcal{E}[2]}\mathcal{E} + F^\phi_{\mathcal{K}[2]}\mathcal{K}), \tag{5.63}$$

where

$$F^\phi_{\mathcal{E}[2]} = -(L^2 + r_0^2)(188L^8M + 406L^6Mr_0^2 - 64L^6r_0^3 + 222L^4Mr_0^4 - 163L^4r_0^5 - 145L^2r_0^7 - 48r_0^9)$$
$$- E^2L^2r_0^5(38L^4 + 31L^2r_0^2 - 15r_0^4),$$
$$F^\phi_{\mathcal{K}[2]} = E^2L^2r_0^5(12L^4 + L^2r_0^2 - 15r_0^4) + r_0^2(L^2 + r_0^2)(94L^6M + 96L^4Mr_0^2 - 74L^4r_0^3 - 121L^2r_0^5 - 48r_0^7),$$

$$F^t_{[4]} = \frac{3E\dot r_0}{40\pi r_0^{10}(r_0 - 2M)(L^2 + r^2)^{11/2}}(F^t_{\mathcal{E}[4]}\mathcal{E} + F^t_{\mathcal{K}[4]}\mathcal{K}), \tag{5.64}$$

where

$$F^t_{\mathcal{E}[4]} = 30E^4r_0^{10}(64L^{10} + 384L^8r_0^2 + 989L^6r_0^4 + 1222L^4r_0^6 + 437L^2r_0^8 + 12r_0^{10}) + 2E^2r_0^5(L^2 + r_0^2)(92160L^{12}M$$
$$+ 445008L^{10}Mr_0^2 + 859044L^8Mr_0^4 - 1920L^8r_0^5 + 838312L^6Mr_0^6 - 9780L^6r_0^7 + 433114L^4Mr_0^8 - 19065L^4r_0^9$$
$$+ 102188L^2Mr_0^{10} - 9870L^2r_0^{11} + 5030Mr_0^{12} - 585r_0^{13}) + 4(L^2 + r_0^2)^2(46080L^{14}M^2 + 403200L^{12}M^2r_0^2$$
$$- 48000L^{12}Mr_0^3 + 1231984L^{10}M^2r_0^4 - 219840L^{10}Mr_0^5 + 1841004L^8M^2r_0^6 - 411324L^8Mr_0^7$$
$$+ 1490772L^6M^2r_0^8 - 406397L^6Mr_0^9 + 480L^6r_0^{10} + 668810L^4M^2r_0^{10} - 232331L^4Mr_0^{11} + 2040L^4r_0^{12}$$
$$+ 161840L^2M^2r_0^{12} - 75367L^2Mr_0^{13} + 1590L^2r_0^{14} + 18382M^2r_0^{14} - 10669Mr_0^{15} + 210r_0^{16}),$$
$$F^t_{\mathcal{K}[4]} = -15E^4r_0^{12}(64L^8 + 328L^6r_0^2 + 495L^4r_0^4 + 22L^2r_0^6 - 81r_0^8) - 4E^2r_0^7(L^2 + r_0^2)(25920L^{10}M + 102372L^8Mr_0^2$$
$$+ 152523L^6Mr_0^4 - 480L^6r_0^5 + 103375L^4Mr_0^6 - 1575L^4r_0^7 + 25889L^2Mr_0^8 - 120L^2r_0^9 - 455Mr_0^{10} + 495r_0^{11})$$
$$- r_0^2(L^2 + r_0^2)^2(92160L^{12}M^2 + 766080L^{10}M^2r_0^2 - 130560L^{10}Mr_0^3 + 2014208L^8M^2r_0^4 - 497760L^8Mr_0^5$$
$$+ 2443016L^6M^2r_0^6 - 743088L^6Mr_0^7 + 1527236L^4M^2r_0^8 - 552712L^4Mr_0^9 + 960L^4r_0^{10} + 496596L^2M^2r_0^{10}$$
$$- 206180L^2Mr_0^{11} - 135L^2r_0^{12} + 73528M^2r_0^{12} - 30796Mr_0^{13} - 735r_0^{14}),$$

$$F^r_{[4]} = \frac{3}{40\pi r_0^{14}(L^2 + r_0^2)^{11/2}}(F^r_{\mathcal{E}[4]}\mathcal{E} + F^r_{\mathcal{K}[4]}\mathcal{K}), \tag{5.65}$$

where

$$F^r_{\mathcal{E}[4]} = 30E^6r_0^{13}(64L^{10} + 384L^8r_0^2 + 989L^6r_0^4 + 1222L^4r_0^6 + 437L^2r_0^8 + 12r_0^{10}) + E^4r_0^8(L^2 + r_0^2)(184320L^{12}M$$
$$+ 893856L^{10}Mr_0^2 - 1920L^{10}r_0^3 + 1746888L^8Mr_0^4 - 18240L^8r_0^5 + 1744604L^6Mr_0^6 - 53550L^6r_0^7$$
$$+ 907298L^4Mr_0^8 - 58665L^4r_0^9 + 197536L^2Mr_0^{10} - 16320L^2r_0^{11} + 9010Mr_0^{12} - 645r_0^{13}) + 2E^2r_0^3(L^2 + r_0^2)^2$$
$$\times (92160L^{14}M^2 + 1036800L^{12}M^2r_0^2 - 211200L^{12}Mr_0^3 + 3363456L^{10}M^2r_0^4 - 889424L^{10}Mr_0^5$$
$$+ 5003472L^8M^2r_0^6 - 1487220L^8Mr_0^7 + 1920L^8r_0^8 + 3866356L^6M^2r_0^8 - 1272840L^6Mr_0^9 + 9780L^6r_0^{10}$$
$$+ 1570564L^4M^2r_0^{10} - 594724L^4Mr_0^{11} + 10875L^4r_0^{12} + 321308L^2M^2r_0^{12} - 146848L^2Mr_0^{13} + 1830L^2r_0^{14}$$
$$+ 36764M^2r_0^{14} - 20288Mr_0^{15} - 105r_0^{16}) - (r_0 - 2M)(L^2 + r_0^2)^3(184320L^{14}M^2 + 1711104L^{12}M^2r_0^2$$
$$- 245760L^{12}Mr_0^3 + 4872896L^{10}M^2r_0^4 - 884480L^{10}Mr_0^5 + 6311728L^8M^2r_0^6 - 1185120L^8Mr_0^7$$
$$+ 4083688L^6M^2r_0^8 - 721620L^6Mr_0^9 + 1920L^6r_0^{10} + 1299396L^4M^2r_0^{10} - 198700L^4Mr_0^{11} + 2460L^4r_0^{12}$$
$$+ 209484L^2M^2r_0^{12} - 28120L^2Mr_0^{13} - 105L^2r_0^{14} + 28640M^2r_0^{14} - 5120Mr_0^{15} - 525r_0^{16}),$$

$$F^r_{\mathcal{K}[4]} = -15E^6 r_0^{15}(64L^8 + 328L^6 r_0^2 + 495L^4 r_0^4 + 22L^2 r_0^6 - 81 r_0^8) - E^4 r_0^{10}(L^2 + r_0^2)(103680L^{10}M + 415248L^8 M r_0^2 - 2880L^8 r_0^3 + 627612L^6 M r_0^4 - 10680L^6 r_0^5 + 418570L^4 M r_0^6 - 8835L^4 r_0^7 + 93896L^2 M r_0^8 + 4350L^2 r_0^9 - 2870 M r_0^{10} + 2505 r_0^{11}) - E^2 r_0^5 (L^2 + r_0^2)^2 (92160L^{12}M^2 + 1019520L^{10}M^2 r_0^2 - 257280L^{10} M r_0^3 + 2806176L^8 M^2 r_0^4 - 893744L^8 M r_0^5 + 3309168L^6 M^2 r_0^6 - 1180004L^6 M r_0^7 + 1920L^6 r_0^8 + 1878796L^4 M^2 r_0^8 - 724772L^4 M r_0^9 - 900L^4 r_0^{10} + 517652L^2 M^2 r_0^{10} - 205608L^2 M r_0^{11} - 5685L^2 r_0^{12} + 73528M^2 r_0^{12} - 28696 M r_0^{13} - 1785 r_0^{14}) + r_0^2(r_0 - 2M)(L^2 + r_0^2)^3(92160L^{12}M^2 + 815232L^{10}M^2 r_0^2 - 157440L^{10} M r_0^3 + 1939840L^8 M^2 r_0^4 - 468160L^8 M r_0^5 + 1942488L^6 M^2 r_0^6 - 494560L^6 M r_0^7 + 892484L^4 M^2 r_0^8 - 217780L^4 M r_0^9 - 960L^4 r_0^{10} + 195164L^2 M^2 r_0^{10} - 38820L^2 M r_0^{11} - 1545L^2 r_0^{12} + 28640M^2 r_0^{12} - 5120 M r_0^{13} - 525 r_0^{14}),$$

$$F^\theta_{[4]} = 0, \tag{5.66}$$

$$F^\phi_{[4]} = \frac{3\dot{r}_0}{40\pi r_0^{13} L^5 (L^2 + r^2)^{9/2}}(F^\phi_{\mathcal{E}[4]}\mathcal{E} + F^\phi_{\mathcal{K}[4]}\mathcal{K}), \tag{5.67}$$

where

$$F^\phi_{\mathcal{E}[4]} = 15E^4 L^4 r_0^{10}(128L^{10} + 960L^8 r_0^2 + 2266L^6 r_0^4 + 1369L^4 r_0^6 - 228L^2 r_0^8 - 35 r_0^{10}) + 2E^2 L^2 r_0^5 (L^2 + r_0^2)(115200L^{14}M + 449744L^{12} M r_0^2 + 660732L^{10} M r_0^4 - 5760L^{10} r_0^5 + 442406L^8 M r_0^6 - 21300L^8 r_0^7 + 116472L^6 M r_0^8 - 18475L^6 r_0^9 - 1186L^4 M r_0^{10} + 4310L^4 r_0^{11} + 10765L^2 r_0^{13} + 4240 r_0^{15}) + (L^2 + r_0^2)^2(184320L^{18}M^2 + 1711104L^{16}M^2 r_0^2 - 245760L^{16} M r_0^3 + 4872896L^{14}M^2 r_0^4 - 884480L^{14} M r_0^5 + 6311728L^{12}M^2 r_0^6 - 1090720L^{12} M r_0^7 + 4083688L^{10}M^2 r_0^8 - 436180L^{10} M r_0^9 - 5760L^{10} r_0^{10} + 1299396L^8 M^2 r_0^{10} + 73140L^8 M r_0^{11} - 83700L^8 r_0^{12} + 209484L^6 M^2 r_0^{12} + 15720L^6 M r_0^{13} - 236585L^6 r_0^{14} + 28640L^4 M^2 r_0^{14} - 63200L^4 M r_0^{15} - 271325L^4 r_0^{16} - 21120L^2 M r_0^{17} - 138400L^2 r_0^{18} - 25600 r_0^{20}),$$

$$F^\phi_{\mathcal{K}[4]} = -15E^4 L^4 r_0^{12}(192L^8 + 584L^6 r_0^2 + 169L^4 r_0^4 - 322L^2 r_0^6 - 35 r_0^8) - 2E^2 L^2 r_0^7 (L^2 + r_0^2)(63360L^{12}M + 197992L^{10} M r_0^2 + 216538L^8 M r_0^4 - 4800L^8 r_0^5 + 87890L^6 M r_0^6 - 7110L^6 r_0^7 + 5264L^4 M r_0^8 + 2455L^4 r_0^9 + 8645L^2 r_0^{11} + 4240 r_0^{13}) - r_0^2 (L^2 + r_0^2)^2 (92160L^{16}M^2 + 815232L^{14}M^2 r_0^2 - 157440L^{14} M r_0^3 + 1939840L^{12}M^2 r_0^4 - 422080L^{12} M r_0^5 + 1942488L^{10}M^2 r_0^6 - 309120L^{10} M r_0^7 + 892484L^8 M^2 r_0^8 + 9580L^8 M r_0^9 - 39360L^8 r_0^{10} + 195164L^6 M^2 r_0^{10} + 22780L^6 M r_0^{11} - 152745L^6 r_0^{12} + 28640L^4 M^2 r_0^{12} - 52640L^4 M r_0^{13} - 213325L^4 r_0^{14} - 21120L^2 M r_0^{15} - 125600L^2 r_0^{16} - 25600 r_0^{18}).$$

2. *huu* regularization

The quantity

$$H^{(R)} = \frac{1}{2} h^{(R)}_{ab} u^a u^b \tag{5.68}$$

was first proposed by Detweiler [5] as a tool for constructing gauge-invariant measurements from self-force calculations. It has since proven invaluable in extracting gauge-invariant results from gauge-dependent self-force calculations [62,78].

Much the same as with self-force calculations, the calculation of $H^{(R)}$ requires the subtraction of the appropriate singular piece, $H^{(S)} = \frac{1}{2} h^{(S)}_{ab} u^a u^b$ from the full retarded field. In this section, we give this subtraction in the form of mode-sum regularization parameters. In doing so, we keep with our convention that the term proportional to $l + \frac{1}{2}$ is denoted by $H_{[-1]}$ (= 0 in this case), the constant term is denoted by $H_{[0]}$ and so on.

Note that, as in the self-force case, an ambiguity arises here due to the presence of terms involving the four-velocity at x. One is free to arbitrarily choose how to define this, provided $\lim_{x\to\bar{x}} u^a = u^{\bar{a}}$. As before, we choose this in such a way that the Schwarzschild components of the four-velocity at x are exactly those at $\bar{x}$. The regularization parameters are then given by

$$H_{[0]} = \frac{2\mathcal{K}}{\pi\sqrt{L^2 + r_0^2}}, \tag{5.69}$$

$$H_{[2]} = \frac{H^{\mathcal{E}}_{[2]}\mathcal{E} + H^{\mathcal{K}}_{[2]}\mathcal{K}}{\pi r_0^3 (L^2 + r_0^2)^{3/2}}, \tag{5.70}$$

where

$$H^{\mathcal{E}}_{[2]} = 2E^2 r_0^5 + (L^2 + r_0^2)(36L^2M - 8L^2 r_0 + 38Mr_0^2 - 9r_0^3),$$
$$H^{\mathcal{K}}_{[2]} = -E^2 r_0^3(16L^2 + 17r_0^2) - 2(L^2 + r_0^2)(16L^2M - 4L^2 r_0 + 33Mr_0^2 - 12r_0^3),$$
$$H_{[4]} = \frac{3(H^{\mathcal{E}}_{[4]}\mathcal{E} + H^{\mathcal{K}}_{[4]}\mathcal{K})}{20\pi r^{10}(L^2 + r^2)^{7/2}}, \tag{5.71}$$

where

$$\begin{aligned} H^{\mathcal{E}}_{[4]} = {} & -120E^4 r_0^{12}(8L^4 + 17L^2 r_0^2 + 7r_0^4) + 2E^2 r_0^5(L^2 + r_0^2)(3584L^8M + 12712L^6Mr_0^2 + 15516L^4Mr_0^4 + 120L^4 r_0^5 \\ & + 6182L^2Mr_0^6 + 735L^2 r_0^7 + 34Mr_0^8 + 495r_0^9) + 2(L^2 + r_0^2)^2(1536L^{10}M^2 + 13888L^8M^2 r_0^2 - 1600L^8Mr_0^3 \\ & + 40584L^6M^2 r_0^4 - 9440L^6Mr_0^5 + 46888L^4M^2 r_0^6 - 14100L^4Mr_0^7 + 120L^4 r_0^8 + 18936L^2M^2 r_0^8 \\ & - 5350L^2Mr_0^9 + 15L^2 r_0^{10} + 340M^2 r_0^{10} + 850Mr_0^{11} - 90r_0^{12}), \\ H^{\mathcal{K}}_{[4]} = {} & 15E^4 r_0^{10}(64L^6 + 224L^4 r_0^2 + 259L^2 r_0^4 + 91r_0^6) - 4E^2 r_0^7(L^2 + r_0^2)(1376L^6M + 3174L^4Mr_0^2 + 420L^4 r_0^3 \\ & + 1965L^2Mr_0^4 + 960L^2 r_0^5 + 227Mr_0^6 + 510r_0^7) - r_0^2(L^2 + r_0^2)^2(1536L^8M^2 + 15904L^6M^2 r_0^2 - 7360L^6Mr_0^3 \\ & + 36160L^4M^2 r_0^4 - 19320L^4Mr_0^5 + 22412L^2M^2 r_0^6 - 11040L^2Mr_0^7 - 720L^2 r_0^8 + 680M^2 r_0^8 + 860Mr_0^9 - 705r_0^{10}). \end{aligned}$$

F. Example

As an example application of our high-order regularization parameters, we consider the case of a scalar particle on a circular geodesic of the Schwarzschild spacetime. In this case, the retarded field may be computed using the frequency domain method described in Ref. [54], along with improved asymptotics for the boundary conditions (by expanding inside the exponential rather than outside) and with the use of the arbitrary precision differential equation solving support in MATHEMATICA [79]. These improvements allowed us to substantially increase the accuracy of the computed retarded field. We found this to be necessary to get the full benefit from the higher-order regularization parameters.

In Fig. 2, we show the effect of subtracting in turn the cumulative sums of the regularization parameters from the full retarded field. Shown in the figure in order from top to bottom are $F_r^{\rm ret}$ and the result of subtracting from it in turn the cumulative sum of the regularization terms $F^l_{r[-1]}$, $F^l_{r[0]}$, $F^l_{r[2]}$, $F^l_{r[4]}$, $F^l_{r[6]}$, $F^l_{r[8]}$, $F^l_{r[10]}$ and $F^l_{r[12]}$. The parameters $F_{r[-1]}$, $F_{r[0]}$, $F_{r[2]}$, $F_{r[4]}$ and $F_{r[6]}$ are the analytically derived ones given in Sec. V C. The parameters $F_{r[8]}$, $F_{r[10]}$ and $F_{r[12]}$ were determined through a numerical fit to the data. The resulting rapid convergence with l enables the calculation of an extremely accurate value for the self-force. Summing over l, we find $F_r = 0.0000137844825756679 59(3)$, where the uncertainty in the last digit is estimated by assuming that the error comes purely from the fact that the sum is only done up to a finite $l_{\max} = 80$.

In addition to providing a highly accurate benchmark, this example may be used to assess the benefits which can be obtained from the use of higher-order regularization parameters. The most obvious benefit is that with fixed computational resources (i.e., fixed number of spherical harmonic modes), one can obtain a much more accurate value for the self-force. This is highlighted by comparison of our value for F_r with that of the previous benchmark given in Ref. [54]. Both calculations consider the same case of a scalar charge in a circular orbit of radius $10M$ around a Schwarzschild black hole. Using 40 l-modes and regularization parameters up to $F^l_{r[2]}$, Ref. [54] obtained a value for the self-force with a fractional accuracy of 10^{-9}. The inclusion of the next two regularization parameters improves this to a fractional error of 10^{-12} which increases to a fractional error of 10^{-17} when 80 modes are used.

This example represents a somewhat extreme case: it uses highly accurate frequency domain methods combined

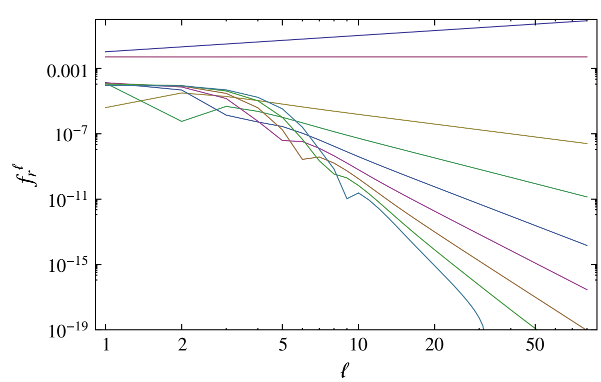

FIG. 2 (color online). Regularization of the radial component of the self-force for the case of a scalar particle on a circular geodesic of radius $r_0 = 10M$ in Schwarzschild spacetime. In decreasing slope, the above lines represent the unregularized self-force, self-force regularized by subtracting from it in turn the cumulative sum of $F^l_{r[-1]}$, $F^l_{r[0]}$, $F^l_{r[2]}$, $F^l_{r[4]}$, $F^l_{r[6]}$, $F^l_{r[8]}$, $F^l_{r[10]}$, $F^l_{r[12]}$.

with high-precision numerical integration and a relatively large number of spherical harmonic modes. In more typical time-domain calculations, numerical data up to $l \sim 15$ is used, and it is common that the dominant source of error comes from the tail fit. While it may seem that one merely needs to compute more modes to reduce this error, this is not a realistic solution. In a mode-sum calculation, the number of spherical harmonic modes required for each l scales as l^2, meaning that simply running simulations for larger and larger l rapidly becomes prohibitively expensive in terms of computational cost. Additionally, the improvement with each additional mode falls off as an inverse power in l, meaning that many more l modes are required for an increasingly small benefit. In this case, the inclusion of higher-order regularization parameters essentially eliminates this problem: without them, the tail fit is the dominant source of error; with them, sufficiently accurate results may be obtained without even fitting for a tail.

It should be pointed out that there is one caveat to our conclusions. The use of high-order regularization parameters requires the subtraction of increasingly (relatively) large numbers to obtain a small regularized remainder. It is therefore essential that any numerically provided data for the retarded field must be of sufficient accuracy for the subtraction to yield meaningful results. As a result, calculations which were previously deemed sufficient would not necessarily gain an immediate benefit from higher-order regularization parameters.

VI. EFFECTIVE SOURCE

As another application of our high-order expansions of the Detweiler-Whiting singular field, we consider its use in the effective source approach to calculating the self-force. The effective source approach—independently proposed by Barack and Golbourn [37] and by Vega and Detweiler [38]—relies on knowledge of the singular field

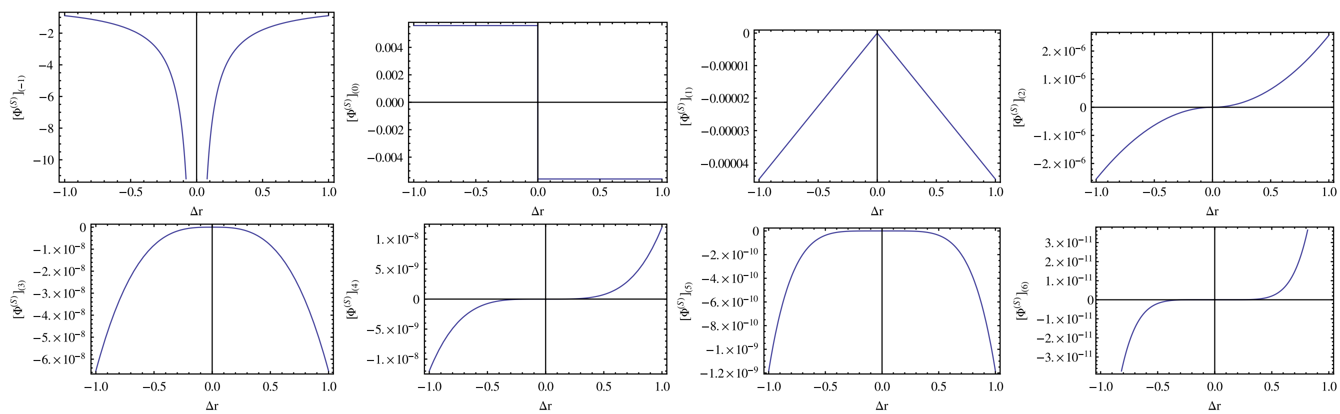

FIG. 3 (color online). Terms in the coordinate expansion of the singular field for $\mathcal{O}(\epsilon^{-1})$ (top left) to $\mathcal{O}(\epsilon^6)$ (bottom right). Shown is the case of a circular geodesic of radius $r_0 = 10M$ in Schwarzschild spacetime.

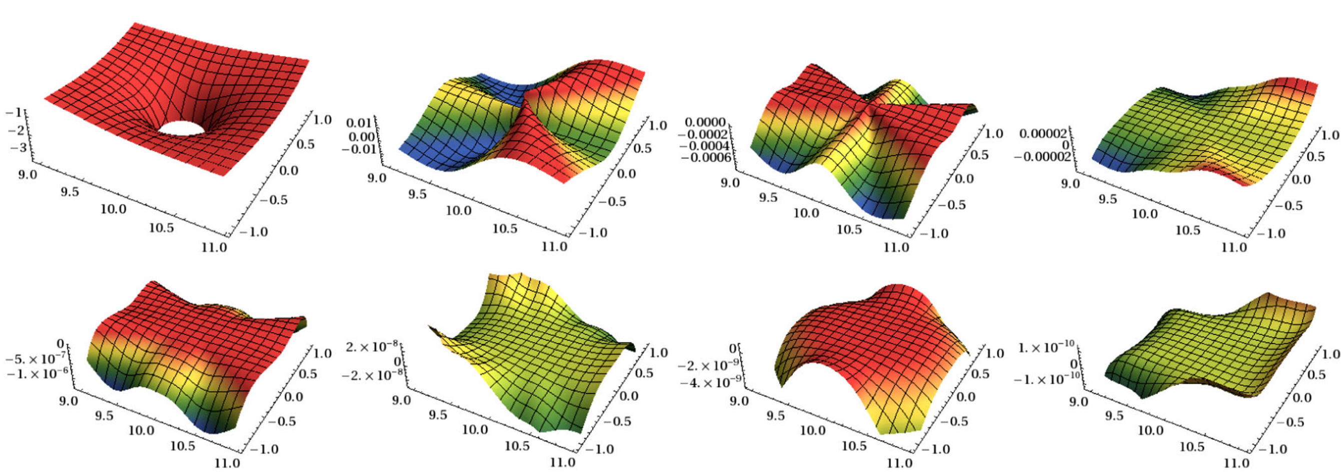

FIG. 4 (color online). Terms in the coordinate expansion of the singular field, $[\Phi^{(S)}]_{(n)}$, in the region of the particle for $\mathcal{O}(\epsilon^{-1})$ (top left) to $\mathcal{O}(\epsilon^6)$ (bottom right). Shown is the case of a circular geodesic of radius $r_0 = 10M$ in Schwarzschild spacetime.

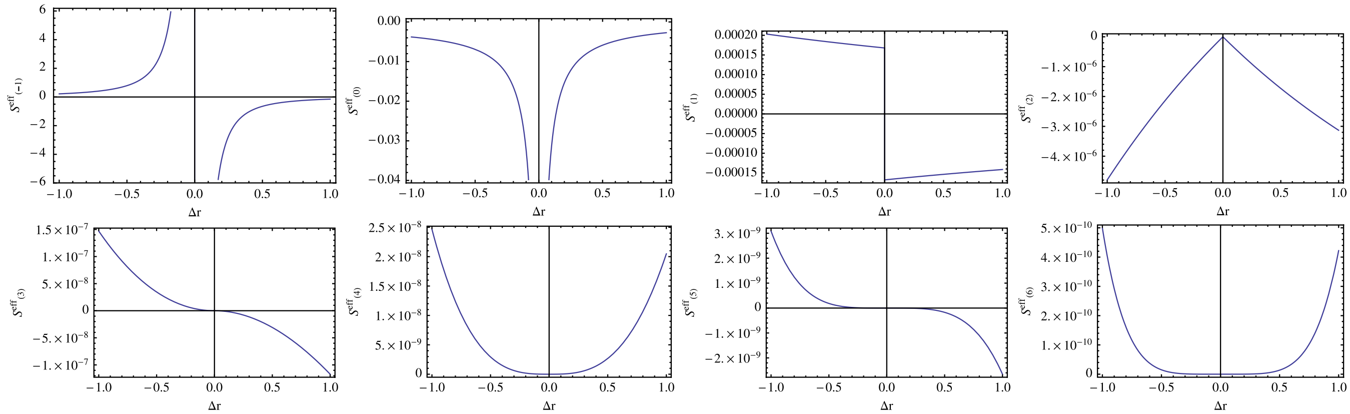

FIG. 5 (color online). Effective source for the approximate singular field of order $\mathcal{O}(\epsilon^{-1})$ (top left) to $\mathcal{O}(\epsilon^{6})$ (bottom right). Shown is the case of a circular geodesic of radius $r_0 = 10M$ in Schwarzschild spacetime.

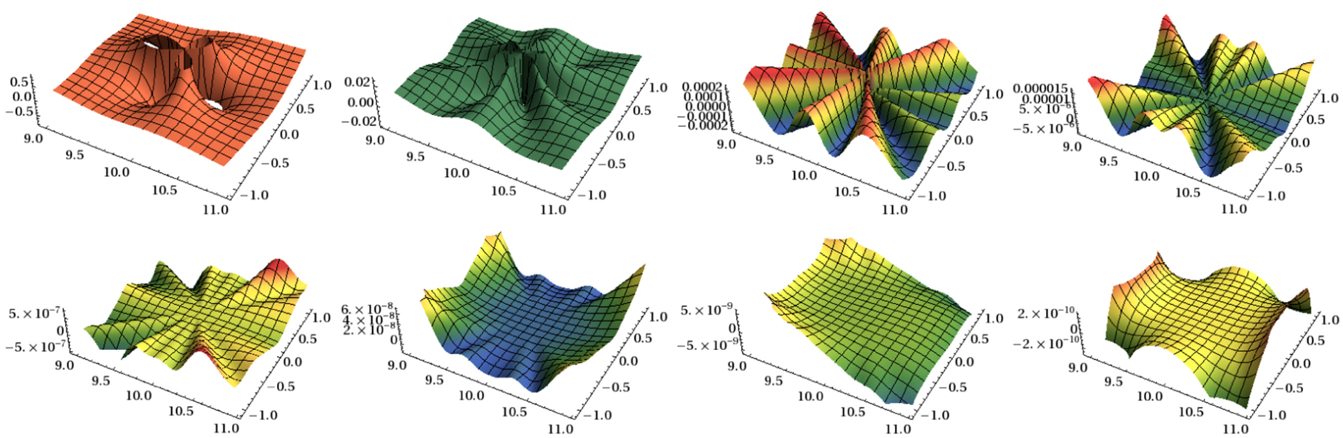

FIG. 6 (color online). Effective source for the approximate singular field, $S^{\mathrm{eff}}{}_{(n)}$, in the region of the particle of order $\mathcal{O}(\epsilon^{-1})$ (top left) to $\mathcal{O}(\epsilon^{6})$ (bottom right). Shown is the case of a circular geodesic of radius $r_0 = 10M$ in Schwarzschild spacetime.

to derive an equation for a regularized field which gives the self-force without any need for post-processed regularization. If the singular field is known exactly, then the regularized field is totally regular and is a solution of the homogeneous wave equation. In reality, exact expressions for the singular field can only be obtained for very simple spacetimes. More generally, the best one can do is an approximation such as that given in Sec. III. Splitting the retarded field into approximate singular and regularized parts,

$$\varphi^A_{(\mathrm{ret})} = \tilde{\varphi}^A_{(\mathrm{S})} + \tilde{\varphi}^A_{(\mathrm{R})}, \tag{6.1}$$

substituting into the wave equation, Eq. (2.1) and rearranging, we obtain an equation for the regularized field,

$$\mathcal{D}^A{}_B \tilde{\varphi}^B_{(\mathrm{R})} = S^A_{\mathrm{eff}}, \tag{6.2}$$

with an effective source,

$$S^A_{\mathrm{eff}} = -\mathcal{D}^A{}_B \tilde{\varphi}^B_{(\mathrm{S})} - 4\pi\mathcal{Q}\int u^A \delta_4(x, z(\tau'))d\tau'. \tag{6.3}$$

For sufficiently good approximations to the singular field, $\tilde{\varphi}^A_{(\mathrm{R})}$ and S^A are finite everywhere, in particular, on the world line. As a result, one never encounters problematic singularities or δ-functions, making the approach particularly suitable for use in time domain numerical simulations. A detailed review of this approach can be found in Refs. [80,81].

In Figs. 3–6, we show the result of applying our expansions to the case of a scalar particle on a circular geodesic of radius $r_0 = 10M$ in Schwarzschild spacetime. Similar plots can be obtained for the electromagnetic case, gravitational case and for more generic motion. However, the general structure does not change and is best illustrated by this simple example.

VII. DISCUSSION

In this paper, we have developed high-order expansions of the Detweiler-Whiting singular field of a point scalar or electromagnetic particle and of a point mass. Many of our expressions are very general, not necessarily being restricted to any particular choice of spacetime. In our explicit coordinate calculations, however, for simplicity, we chose to limit ourselves to the Schwarzschild geometry. A logical extension of this work would be to consider, instead, the Kerr spacetime. This has already been done to $\mathcal{O}(\epsilon^2)$ for the case of a scalar charge in a circular geodesic orbit in Kerr spacetime, and spherical-harmonic regularization parameters have been computed [82]. We will present a more general version of this calculation in forthcoming work [83]. However, the use of spherical harmonics is not well-suited to the Kerr spacetime. A more appropriate choice of basis functions are the spheroidal harmonics; it may be more sensible to compute regularization parameters in this spheroidal harmonic basis. Nevertheless, the full singular field is the same, independently of the choice of basis functions, and the majority of our calculations can easily be carried over to the Kerr case with little modification.

In the cases of electromagnetic and gravitational fields, there is an arbitrariness in the choice of gauge. For this work, we have focused only on the case of Lorenz gauge, $A^a{}_{;a} = 0$ and $h_{ab}{}^{;b} = 0$, where the singular field is best understood. However, other gauges such as Regge-Wheeler and radiation gauge are better suited to time domain numerical calculations. Indeed, there are many outstanding problems in the time-domain evolution of metric perturbations in Lorenz gauge [84]. Given recent developments in understanding the singular behavior of the retarded field in other gauges [73,74,85,86], it may soon be possible to implement calculations similar to what we have presented here for other gauges. This could be achieved, for example, by either taking our Lorenz gauge expressions and performing the gauge transformation to a different gauge or by directly computing the singular field in the other gauge. It should be noted, however, that there are many nontrivial issues which can arise when considering other gauges; for example, the gauge tranformation itself may be singular or the singular field may be even more singular[4] than in Lorenz gauge. Any calculations will require considerable care to correctly deal with such issues.

In Sec. VI, we gave plots of an effective source which may be used in a numerical evolution. We have not yet, however, fed this effective source into an actual numerical code. Existing results from effective source calculations have used a source obtained by taking the first four terms in the expansion of the singular field (to $\mathcal{O}(\epsilon^2)$) which yields a source which is C^0. This is sufficient to obtain correct results, but limits the convergence of any numerical scheme with resolution or number of modes required. By including the next four terms in the expansion of the singular field (to $\mathcal{O}(\epsilon^6)$), the highest-order effective source shown in Sec. VI is now C^4, allowing for the use of high-order finite differencing to potentially obtain a significant increase in numerical accuracy.

We have limited the focus of the present work to linear order. However, for calculations with sufficient accuracy to match the requirements of gravitational wave detection, it will be necessary to go beyond first order and to compute a second order self-force. Rapid progress is being made in addressing the difficult problem of deriving the second-order equations of motion [29–31] and the computation of a second-order effective source. The next step is to implement this second-order prescription in a numerical calculation. As this involves an effective source which contains terms involving the first-order singular field, it seems likely that many of the calculations done here may be of use for second order.

[4]A specific example of this is the case of metric perturbations in Regge-Wheeler gauge, where there are additional delta-function divergences in the singular field which are not smoothed out by a mode decomposition [72].

ACKNOWLEDGMENTS

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APPENDIX A: COORDINATE EXPANSIONS IN SCHWARSCHILD SPACETIME

In this appendix, we give coordinate expansions of the key quantities appearing in the singular field, Eqs. (2.13), (2.18), and (2.23).

1. Synge world function

Letting $\cos\gamma = \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos(\phi - \phi')$ so that $2(1 - \cos\gamma) = \Delta w_1^2 + \Delta w_2^2$, the expansion of the world function to the order required in this paper is

$$\sigma(x, x') = \sum_{i,j,k=0}^{i+j+2k\le 9} \sigma_{ijk}(t' - t)^i (r' - r)^j (1 - \cos\gamma)^k + \mathcal{O}(\epsilon^{10}), \tag{A1}$$

where the nonzero coefficients are

$$\sigma_{001}=r^2,\qquad \sigma_{002}=\frac{Mr}{3},\qquad \sigma_{003}=\frac{1}{90}M(9r-2M),\qquad \sigma_{004}=\frac{M(14M^2-19Mr+30r^2)}{840r},\qquad \sigma_{011}=r,$$

$$\sigma_{012}=\frac{M}{6},\qquad \sigma_{013}=\frac{M}{20},\ \sigma_{014}=\frac{1}{840}M\left(15-\frac{7M^2}{r^2}\right),\qquad \sigma_{020}=-\frac{r}{4M-2r},\qquad \sigma_{021}=\frac{M}{12M-6r},$$

$$\sigma_{022}=\frac{M(r-M)}{60r(2M-r)},\qquad \sigma_{023}=\frac{M(-14M^2+6Mr+3r^2)}{840r^2(2M-r)},\qquad \sigma_{030}=-\frac{M}{2(r-2M)^2},\qquad \sigma_{031}=\frac{M}{12(r-2M)^2},$$

$$\sigma_{032}=\frac{M(2M^2-2Mr+r^2)}{120r^2(r-2M)^2},\qquad \sigma_{033}=\frac{M(56M^3-54M^2r+12Mr^2+3r^3)}{1680r^3(r-2M)^2},\qquad \sigma_{040}=-\frac{M(M-8r)}{24r(r-2M)^3},$$

$$\sigma_{041}=\frac{M(5M^2-3Mr-6r^2)}{120r^2(r-2M)^3},\qquad \sigma_{042}=\frac{M(42M^3-70M^2r+39Mr^2-16r^3)}{3360r^3(r-2M)^3},\qquad \sigma_{050}=-\frac{M(M^2-2Mr+6r^2)}{24r^2(r-2M)^4},$$

$$\sigma_{051}=\frac{M(20M^3-31M^2r+12Mr^2+8r^3)}{240r^3(r-2M)^4},\qquad \sigma_{052}=\frac{M(7M^4-21M^3r+23M^2r^2-11Mr^3+5r^4)}{1680r^4(r-2M)^4},$$

$$\sigma_{060}=-\frac{M(35M^3-86M^2r+86Mr^2-144r^3)}{720r^3(r-2M)^5},\qquad \sigma_{061}=\frac{M(245M^4-532M^3r+421M^2r^2-120Mr^3-40r^4)}{1680r^4(r-2M)^5},$$

$$\sigma_{070}=\frac{M(-15M^4+44M^3r-54M^2r^2+36Mr^3-40r^4)}{240r^4(r-2M)^6},$$

$$\sigma_{071}=\frac{M(280M^5-763M^4r+832M^3r^2-444M^2r^3+100Mr^4+20r^5)}{1120r^5(r-2M)^6},$$

$$\sigma_{080}=\frac{M(385M^5-1316M^4r+1928M^3r^2-1576M^2r^3+788Mr^4-640r^5)}{4480r^5(2M-r)^7},$$

$$\sigma_{090}=\frac{M(-5005M^6+19558M^5r-33394M^4r^2+32584M^3r^3-19960M^2r^4+7984Mr^5-5040r^6)}{40320r^6(r-2M)^8},$$

$$\sigma_{200}=\frac{M}{r}-\frac{1}{2},\qquad \sigma_{201}=\frac{M(r-2M)}{6r^2},\qquad \sigma_{202}=\frac{M(10M^2-11Mr+3r^2)}{60r^3},$$

$$\sigma_{203}=\frac{M(-92M^3+142M^2r-78Mr^2+15r^3)}{840r^4},\qquad \sigma_{210}=-\frac{M}{2r^2},\qquad \sigma_{211}=\frac{M(4M-r)}{12r^3},$$

$$\sigma_{212}=-\frac{M(30M^2-22Mr+3r^2)}{120r^4},\ \sigma_{213}=\frac{M(368M^3-426M^2r+156Mr^2-15r^3)}{1680r^5},\qquad \sigma_{220}=\frac{M(5M-4r)}{12r^3(2M-r)},$$

$$\sigma_{221}=-\frac{M(23M^2-20Mr+3r^2)}{60r^4(2M-r)},\qquad \sigma_{222}=\frac{M(686M^3-802M^2r+281Mr^2-24r^3)}{1680r^5(2M-r)},\qquad \sigma_{230}=-\frac{M(M-r)^2}{4r^4(r-2M)^2},$$

$$\sigma_{231}=\frac{M(24M^3-55M^2r+32Mr^2-4r^3)}{120r^5(r-2M)^2},\qquad \sigma_{232}=\frac{M(-630M^4+1234M^3r-832M^2r^2+218Mr^3-15r^4)}{1680r^6(r-2M)^2},$$

$$\sigma_{240}=-\frac{M(M^3-78M^2r+116Mr^2-48r^3)}{240r^5(r-2M)^3},\qquad \sigma_{241}=-\frac{M(553M^4-269M^3r-444M^2r^2+322Mr^3-40r^4)}{1680r^6(r-2M)^3},$$

$$\sigma_{250}=\frac{M(75M^4-84M^3r-51M^2r^2+104Mr^3-40r^4)}{240r^6(r-2M)^4},$$

$$\sigma_{251}=\frac{M(-4396M^5+8227M^4r-4760M^3r^2+350M^2r^3+412Mr^4-60r^5)}{3360r^7(r-2M)^4},$$

$$\sigma_{260}=\frac{M(2317M^5-5560M^4r+4220M^3r^2-146M^2r^3-1254Mr^4+480r^5)}{3360r^7(r-2M)^5},$$

$$\sigma_{270}=\frac{M(3759M^6-12402M^5r+16015M^4r^2-9336M^3r^3+1347M^2r^4+1048Mr^5-420r^6)}{3360r^8(r-2M)^6},$$

$$
\begin{aligned}
\sigma_{400} &= \frac{M^2(2M-r)}{24r^5}, \qquad \sigma_{401} = -\frac{M^2(54M^2-49Mr+11r^2)}{360r^6},\\
\sigma_{402} &= \frac{M^2(1956M^3-2702M^2r+1228Mr^2-183r^3)}{10080r^7}, \qquad \sigma_{410} = \frac{M^2(2r-5M)}{24r^6},\\
\sigma_{411} &= \frac{M^2(324M^2-245Mr+44r^2)}{720r^7}, \qquad \sigma_{412} = \frac{M^2(-3423M^3+4053M^2r-1535Mr^2+183r^3)}{5040r^8},\\
\sigma_{420} &= \frac{M^2(429M^2-394Mr+86r^2)}{720r^7(2M-r)}, \sigma_{421} = \frac{M^2(-7509M^3+9170M^2r-3575Mr^2+438r^3)}{5040r^8(2M-r)},\\
\sigma_{430} &= \frac{M^2(-301M^3+460M^2r-228Mr^2+36r^3)}{240r^8(r-2M)^2},\\
\sigma_{431} &= \frac{M^2(35472M^4-63269M^3r+40912M^2r^2-11240Mr^3+1088r^4)}{10080r^9(r-2M)^2},\\
\sigma_{440} &= -\frac{M^2(42507M^4-94876M^3r+77976M^2r^2-27740Mr^3+3546r^4)}{20160r^9(r-2M)^3},\\
\sigma_{450} &= \frac{M^2(-57987M^5+178306M^4r-214952M^3r^2+126744M^2r^3-36328Mr^4+3992r^5)}{20160r^{10}(r-2M)^4},\\
\sigma_{600} &= \frac{M^3(26M^2-25Mr+6r^2)}{720r^9}, \qquad \sigma_{601} = \frac{M^3(-1818M^3+2475M^2r-1115Mr^2+166r^3)}{15120r^{10}},\\
\sigma_{610} &= -\frac{M^3(117M^2-100Mr+21r^2)}{720r^{10}}, \qquad \sigma_{611} = \frac{M^3(18180M^3-22275M^2r+8920Mr^2-1162r^3)}{30240r^{11}},\\
\sigma_{620} &= \frac{M^3(23931M^3-31560M^2r+13652Mr^2-1930r^3)}{30240r^{11}(2M-r)},\\
\sigma_{630} &= \frac{M^3(-27687M^4+51002M^3r-34745M^2r^2+10344Mr^3-1131r^4)}{10080r^{12}(r-2M)^2},\\
\sigma_{800} &= \frac{M^4(978M^3-1393M^2r+660Mr^2-104r^3)}{40320r^{13}}, \qquad \sigma_{810} = \frac{M^4(-6357M^3+8358M^2r-3630Mr^2+520r^3)}{40320r^{14}}.
\end{aligned} \tag{A2}
$$

2. Van Vleck determinant

Inserting the above expansion for $\sigma(x, x')$ into the definition of the Van Vleck-Morette determinant, Eq. (3.2), gives

$$
\Delta^{1/2}(x, x') = 1 + \sum_{i,j,k=0}^{i+j+2k\le 7} \Delta^{1/2}_{ijk}(t'-t)^i(r'-r)^j(1-\cos\gamma)^k + \mathcal{O}(\epsilon^{10}), \tag{A3}
$$

where the nonzero coefficients are

$$
\begin{aligned}
\Delta^{1/2}_{002} &= \frac{M^2}{15r^2}, \qquad \Delta^{1/2}_{003} = \frac{M^2(27r-34M)}{378r^3}, \qquad \Delta^{1/2}_{012} = -\frac{M^2}{15r^3},\\
\Delta^{1/2}_{013} &= \frac{M^2(17M-9r)}{126r^4}, \qquad \Delta^{1/2}_{021} = \frac{M^2}{60Mr^3-30r^4}, \qquad \Delta^{1/2}_{022} = \frac{M^2(322M-177r)}{2520r^4(2M-r)},\\
\Delta^{1/2}_{031} &= \frac{M^2(2r-3M)}{30r^4(r-2M)^2}, \qquad \Delta^{1/2}_{032} = \frac{M^2(-308M^2+332Mr-93r^2)}{1260r^5(r-2M)^2}, \qquad \Delta^{1/2}_{040} = \frac{M^2}{60r^4(r-2M)^2},\\
\Delta^{1/2}_{041} &= -\frac{M^2(910M^2-1268Mr+459r^2)}{5040r^5(r-2M)^3}, \qquad \Delta^{1/2}_{050} = \frac{M^2(4M-3r)}{60r^5(r-2M)^3},\\
\Delta^{1/2}_{051} &= \frac{M^2(-1190M^3+2664M^2r-2035Mr^2+537r^3)}{5040r^6(r-2M)^4}, \qquad \Delta^{1/2}_{060} = \frac{M^2(5432M^2-7720Mr+2943r^2)}{30240r^6(r-2M)^4},
\end{aligned}
$$

$$\Delta^{1/2}_{070} = \frac{M^2(1036M^3 - 2120M^2r + 1524Mr^2 - 393r^3)}{2520r^7(r-2M)^5}, \qquad \Delta^{1/2}_{201} = \frac{M^2(r-2M)}{30r^5},$$
$$\Delta^{1/2}_{202} = \frac{M^2(460M^2 - 428Mr + 99r^2)}{2520r^6}, \qquad \Delta^{1/2}_{211} = \frac{M^2(5M-2r)}{30r^6}, \qquad \Delta^{1/2}_{212} = -\frac{M^2(690M^2 - 535Mr + 99r^2)}{1260r^7},$$
$$\Delta^{1/2}_{220} = -\frac{M^2}{30r^6}, \qquad \Delta^{1/2}_{221} = -\frac{M^2(443M^2 - 412Mr + 90r^2)}{1260r^7(2M-r)}, \Delta^{1/2}_{230} = \frac{M^2}{10r^7},$$
$$\Delta^{1/2}_{231} = \frac{M^2(161M^3 - 464M^2r + 317Mr^2 - 60r^3)}{1260r^8(r-2M)^2}, \qquad \Delta^{1/2}_{240} = \frac{M^2(-3526M^2 + 3746Mr - 981r^2)}{5040r^8(r-2M)^2},$$
$$\Delta^{1/2}_{250} = -\frac{M^2(9392M^3 - 15628M^2r + 8631Mr^2 - 1572r^3)}{5040r^9(r-2M)^3}, \qquad \Delta^{1/2}_{400} = \frac{M^2(r-2M)^2}{60r^8},$$
$$\Delta^{1/2}_{401} = \frac{M^2(-1300M^3 + 1682M^2r - 714Mr^2 + 99r^3)}{5040r^9}, \qquad \Delta^{1/2}_{410} = -\frac{M^2(16M^2 - 14Mr + 3r^2)}{60r^9},$$
$$\Delta^{1/2}_{411} = \frac{M^2(5850M^3 - 6728M^2r + 2499Mr^2 - 297r^3)}{5040r^{10}}, \qquad \Delta^{1/2}_{420} = \frac{M^2(5648M^2 - 4800Mr + 981r^2)}{10080r^{10}},$$
$$\Delta^{1/2}_{430} = -\frac{M^2(2020M^2 - 1872Mr + 393r^2)}{2520r^{11}}, \qquad \Delta^{1/2}_{600} = \frac{M^3(199M - 89r)(r-2M)^2}{7560r^{12}},$$
$$\Delta^{1/2}_{610} = \frac{M^3(-3184M^3 + 4224M^2r - 1850Mr^2 + 267r^3)}{5040r^{13}}. \tag{A4}$$

3. Expansions of an arbitrary point on the world line about $x^{\bar{a}}$

The four-velocity of a general geodesic orbit taken to lie in the equatorial plane is given by the standard expressions [87]

$$\dot{t}(\tau) = \frac{Er(\tau)}{r(\tau) - 2M}, \qquad \dot{r}(\tau) = \sqrt{E^2 - \left(1 - \frac{2M}{r(\tau)}\right)\left(1 + \frac{L^2}{r(\tau)^2}\right)}, \qquad \dot{\theta}(\tau) = 0, \qquad \dot{\phi}(\tau) = \frac{L}{r(\tau)^2}. \tag{A5}$$

It is straightforward to calculate the higher-order proper time derivatives of these expressions and evaluate both the four-velocity and its higher derivatives at $x^{\bar{a}}$, giving, for example,

$$\dot{t}_0 = \frac{Er_0}{r_0 - 2M}, \qquad \dot{r}_0 = \sqrt{E^2 - \left(1 - \frac{2M}{r_0}\right)\left(1 + \frac{L^2}{r_0^2}\right)}, \qquad \dot{\theta}_0 = 0, \qquad \dot{\phi}_0 = \frac{L}{r_0^2},$$
$$\ddot{t}_0 = -\frac{2EM\dot{r}_0}{(r_0 - 2M)^2}, \qquad \ddot{r}_0 = \frac{L^2r_0 - Mr_0^2 - 3L^2M}{r_0^4}, \qquad \ddot{\theta}_0 = 0, \qquad \ddot{\phi}_0 = -\frac{2L\dot{r}_0}{r_0^3},$$
$$\dddot{t}_0 = \frac{2EM[2(E^2-1)r_0^4 - r_0^2(3L^2 + 2M^2) + 9L^2Mr_0 + 5Mr_0^3 - 6L^2M^2]}{r_0^4(r_0 - 2M)^3},$$
$$\dddot{r}_0 = \frac{\dot{r}_0(-3L^2r_0 + 2Mr_0^2 + 12L^2M)}{r_0^5}, \qquad \dddot{\theta}_0 = 0, \qquad \dddot{\phi}_0 = \frac{2L[3(E^2-1)r_0^3 - 4L^2r_0 + 7Mr_0^2 + 9L^2M]}{r_0^7}. \tag{A6}$$

Combining Eq. (A6) with Eq. (3.17), we can express $x^{a'}$ in terms of $x^{\bar{a}}$ and $\Delta\tau$:

$$t' = \frac{Er_0}{r_0 - 2M}\Delta\tau - \frac{EM\dot{r}_0}{(r_0 - 2M)^2}\Delta\tau^2 + \cdots,$$
$$r' = r_0 + \dot{r}_0\Delta\tau - \frac{(-L^2r_0 + Mr_0^2 + 3L^2M)}{2r_0^4}\Delta\tau^2 + \cdots,$$
$$\theta' = \frac{\pi}{2}, \qquad \phi' = \frac{L}{r_0^2}\Delta\tau - \frac{L\dot{r}_0}{r_0^3}\Delta\tau^2 + \cdots. \tag{A7}$$

It is also straightforward to obtain $\delta x^{a'}$, in terms of Δx^a, $x^{\bar{a}}$ and $\Delta\tau$ by noting that $\delta x^{a'} = x^{a'} - \Delta x^a - x^{\bar{a}}$. Finally, we can calculate $u^{a'}$ in terms of $x^{\bar{a}}$ and $\Delta\tau$ by inserting r' from Eq. (A7) into our equations for the four-velocity:

$$u^{t'} = \frac{Er_0}{r_0 - 2M} - \frac{2EMr_0\dot{r}_0}{(r_0-2M)^2}\Delta\tau + \frac{EM\{6L^2M^2 - 9L^2Mr_0 + r_0^2[3L^2 + 2M^2 - 2(E^2-1)r_0^2 - 5Mr_0]\}}{r_0^4(2M-r_0)^3}\Delta\tau^2 + \cdots,$$
$$u^{r'} = \dot{r}_0 + \frac{r_0(L^2 - Mr_0) - 3L^2M}{r_0^4}\Delta\tau + \frac{\dot{r}_0(12L^2M - 3L^2r_0 + 2Mr_0^2)}{2r_0^5}\Delta\tau^2 + \cdots,$$
$$u^{\theta'} = 0, \qquad u^{\phi'} = \frac{L}{r_0^2} - \frac{2L\dot{r}_0}{r_0^3}\Delta\tau + \frac{L[3(E^2-1)r_0^3 - 4L^2r_0 + 7Mr_0^2 + 9L^2M]}{r_0^7}\Delta\tau^2 + \cdots. \tag{A8}$$

4. Expansions of retarded and advanced points

Taking $\Delta\tau$ to have leading order ϵ, the same leading order of our Δx terms, we can further expand it in orders of ϵ, giving

$$\Delta\tau = \tau_1\epsilon + \tau_2\epsilon^2 + \tau_3\epsilon^3 + \tau_4\epsilon^4 + \cdots. \tag{A9}$$

Substituting $\delta x^{a'}$ obtained from Eq. (A7) and $\Delta\tau$ from Eq. (A9) into $\sigma(x,x')$, Eq. (3.10), gives $\sigma(x,x')$ as a function of Δx^a, $x^{\bar{a}}$ and the τ_n's:

$$\sigma(x,x') = \frac{1}{2}\bigg[\frac{(2M-r_0)\Delta t^2}{r_0} + \frac{r_0(\Delta r - 2\dot{r}_0\tau_1)\Delta r}{r_0-2M} + r_0^2(\Delta\theta^2 + \Delta\phi^2) - (2L\Delta\phi + \tau_1 - 2E\Delta t)\tau_1\bigg]$$
$$+ \frac{1}{2}\bigg\{\frac{2}{r_0}\bigg(\frac{EM\Delta t}{r_0-2M} - L\Delta\phi\bigg)\tau_1\Delta r + \frac{M(\dot{r}_0\tau_1 - \Delta r)\Delta r^2}{(r_0-2M)^2} + r_0\bigg[(\dot{r}_0\tau_1 + \Delta r)(\Delta\theta^2 + \Delta\phi^2) - \frac{2\dot{r}_0\tau_2\Delta r}{r_0-2M}\bigg]$$
$$- \frac{M(\dot{r}_0\tau_1 + \Delta r)\Delta t^2}{r_0^2} - 2(L\Delta\phi - E\Delta t + \tau_1)\tau_2\bigg\} + \cdots. \tag{A10}$$

If we now specify that $x^{a'}$ coincides with the point where the world line intersects with the light cone of x, we can use the equation $\sigma(x,x') = 0$ to solve for the τ_n's in terms of Δx^a and $x^{\bar{a}}$. This gives us

$$\tau_1 = E\Delta t - L\Delta\phi + \frac{r_0\dot{r}_0\Delta r}{2M - r_0} \pm \rho,$$
$$\tau_2 = \frac{\pm 1}{8\rho}\bigg(\frac{\{L^2[\Delta r^2 + 4M^2(\Delta\theta^2 + 3\Delta\phi^2)] - 4LM^2(E\Delta t \pm 2\rho)\Delta\phi - 2EM^2(3E\Delta t \pm 2\rho)\Delta t\}\Delta r}{M^2r_0}$$
$$+ \frac{4LM^2\dot{r}_0\Delta t^2\Delta\phi - 4M^2[\dot{r}_0(E\Delta t \pm \rho) + 2\Delta r]\Delta t^2 + L^2\Delta r^3}{Mr_0^2}$$
$$+ \frac{[4LM^2(E\Delta t - 2\dot{r}_0\Delta r)\Delta\phi + 8E^2M^3r_0(\Delta\theta^2 + \Delta\phi^2) + L^2\Delta r^2 - 2EM^2(3E\Delta t \pm 2\rho)\Delta t]\Delta r}{M^2(2M - r_0)}$$
$$- \frac{\{(r_0 - 2M)[4LM^2\dot{r}_0\Delta\phi + 4M^2(E\dot{r}_0\Delta t \mp \dot{r}_0\rho + E^2\Delta r) - L^2\Delta r] + 8E^2M^3\Delta r\}\Delta r^2}{M(r_0 - 2M)^3}$$
$$- \frac{4L^2M\Delta r\Delta t^2}{r_0^4} - 4r_0(\Delta\theta^2 + \Delta\phi^2)[(E^2 - 2)\Delta r - \dot{r}_0(E\Delta t - L\Delta\phi \pm \rho)]\bigg), \tag{A11}$$

with the higher-order terms following in the same manner.

Using our equations for $\Delta\tau$, Eqs. (A9) and (A11), we rewrite $x^{a'}$ (and consequently $\delta x^{a'}$), the four-velocity, $u^{a'}$, $\Delta^{\frac{1}{2}}(x,x')$ and $\sigma_{a'}$ (Eqs. (A7) and (A8), (3.14) and (3.16), respectively), in terms of Δx^a and $x^{\bar{a}}$.

5. Bivector of Parallel Transport

To calculate the bivector of parallel transport, $g^a{}_{b'}(x,x')$, we first write it in terms of a coordinate expansion about x,

$$g^a{}_{b'}(x,x') = \delta^a{}_{b'} + G^a{}_{bc}(x)\delta x^{c'} + G^a{}_{bcd}(x)\delta x^{c'}\delta x^{d'} + G^a{}_{bcde}(x)\delta x^{c'}\delta x^{d'}\delta x^{e'} + \ldots, \tag{A12}$$

where the coefficients $G^a{}_{b\ldots}(x)$ are functions of x^a written in terms of Δx^a and $x^{\bar{a}}$. Calculating $g^a{}_{b',c'}(x,x')$ is straightforward:

$$g^a{}_{b',c'}(x,x') = G^a{}_{bc}(x) + 2G^a{}_{bcd}(x)\delta x^{d'} + 3G^a{}_{bcde}(x)\delta x^{d'}\delta x^{e'} + 4G^a{}_{bcde}(x)\delta x^{d'}\delta x^{e'}\delta x^{f'} + \cdots. \tag{A13}$$

Using the identity $g^a{}_{b';c'}\sigma^{c'} = g^a{}_{b',c'}\sigma^{c'} - \Gamma^{d'}_{b'c'}g^a{}_{d'}\sigma^{c'} = 0$ with Eqs. (3.12), (A12), and (A13), and our expression for $\delta x^{a'}$ (obtained from the previous section), one can calculate the above coefficients and hence obtain the bivector of parallel transport, $g^a{}_{b'}(x,x')$, in terms of Δx^a and $x^{\bar{a}}$.

6. Scalar Singular Field

Combining Eqs. (2.13) and (3.1), the scalar singular field can be written as

$$\Phi^{(S)}(x) = \frac{q}{2}\left[\frac{\Delta^{\frac{1}{2}}(x,x')}{\sigma_{c'}(x,x')u^{c'}(x')}\right]^{x'=x_{(\mathrm{adv})}}_{x'=x_{(\mathrm{ret})}} + \frac{q}{2}\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V(x,x(\tau'))d\tau. \tag{A14}$$

We already have everything required for the first term here, which gives the direct part of the scalar singular field. It should be noted that $x' = x_{(\mathrm{ret})}$ and $x' = x_{(\mathrm{adv})}$ are the equivalent of setting $\pm\rho = -\rho$ and $\pm\rho = +\rho$, respectively, when substituting the τ_n, Eq. (A11), into $\Delta^{1/2}$, $\sigma_{a'}$ and $u^{a'}$.

In the scalar case, Eq. (3.5) for the scalar tail part becomes

$$V(x,x') = \sum_{n=0}^{\infty} V_n(x,x')\sigma^n(x,x'). \tag{A15}$$

To calculate coordinate expansions of the V_n, first, we require a coordinate expansion for V_0 about x of the form

$$V_0(x,x') = v_0(x) + v_{0a}(x)\delta x^{a'} + v_{0ab}(x)\delta x^{a'}\delta x^{b'} + v_{0abc}(x)\delta x^{a'}\delta x^{b'}\delta x^{c'} + \cdots. \tag{A16}$$

The "initial condition" described by Eq. (3.6b) in the scalar case then becomes

$$2\sigma^{;a'}V_{0;a'} - 2V_0\Delta^{-\frac{1}{2}}\sigma^{;a'}\Delta^{\frac{1}{2}}_{;a'} + 2V_0 + (\Box' - m^2 - \xi R)\Delta^{\frac{1}{2}} = 0, \tag{A17}$$

and from this, it is quite simple to read off expressions for the coefficients $v_{0a\ldots}$. Once we have V_0 to the desired order, we compute a coordinate expansion for V_n $(n > 0)$ of the form

$$V_n(x,x') = v_n(x) + v_{na}(x)\delta x^{a'} + v_{nab}(x)\delta x^{a'}\delta x^{b'} + v_{nabc}(x)\delta x^{a'}\delta x^{b'}\delta x^{c'} + \cdots. \tag{A18}$$

The recursion relation for V_n, Eq. (3.6a), in the scalar case is then

$$2n\sigma^{;a'}V_{n;a'} - 2nV_n\Delta^{-\frac{1}{2}}\sigma^{;a'}\Delta^{\frac{1}{2}}_{;a'} + 2n(n+1)V_n + (\Box' - m^2 - \xi R)V_{n-1} = 0, \tag{A19}$$

from which we can obtain expressions for the coefficients, $v_{na\ldots}$. Here, the number of terms which must be computed is determined by the accuracy to which we require the singular field. For the present calculation, we require up to v_{0abcde}, v_{1abc} and v_{2a}.

Once we have V_n to the required n, using Eqs. (3.10) and (A15), along with our expression for $\delta x^{a'}$ obtained from Eq. (A7), we get $V(x,x')$ in terms of $\Delta\tau$, Δx^a and $x^{\bar{a}}$. This can be easily integrated over τ as required by Eq. (A14). Our final expression for $\Phi^{(S)}(x)$ is then obtained by using Eqs. (A9) and (A11) to remove the $\Delta\tau$ dependence. As before, $\tau_{(\mathrm{ret})}$ and $\tau_{(\mathrm{adv})}$ are obtained by allowing $\pm\rho = -\rho$ and $\pm\rho = +\rho$, respectively.

7. Electromagnetic singular field

For the electromagnetic singular field, we use Eqs. (2.18) and (3.1) to give

$$A_a^{(S)} = \frac{e}{2}\left[\frac{\Delta^{\frac{1}{2}}(x,x')g_{aa'}(x,x')u^{a'}(x')}{\sigma_{c'}(x,x')u^{c'}(x')}\right]^{x'=x_{(\mathrm{adv})}}_{x'=x_{(\mathrm{ret})}} + \frac{e}{2}\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V_{aa'}(x,z(\tau))u^{a'}(x')d\tau, \tag{A20}$$

where $V_{aa'}(x,z(\tau'))$ is given by Eq. (3.5),

$$V^{aa'}(x,x') = \sum_{n=0}^{\infty} V_n^{aa'}(x,x')\sigma^n(x,x'), \tag{A21}$$

and the relevant metrics at x and x' can be used to lower indices. We require a coordinate expansion of $V_0^{aa'}$ of the form,

$$V_0^{aa'}(x,x') = v_0^{aa'}(x) + v_0^{aa'}{}_b(x)\delta x^{b'} + v_0^{aa'}{}_{bc}(x)\delta x^{b'}\delta x^{c'} + v_0^{aa'}{}_{bcd}(x)\delta x^{b'}\delta x^{c'}\delta x^{d'} + \cdots. \tag{A22}$$

Substituting this into the initial condition in Eq. (3.6b), which in the electromagnetic case is

$$2\sigma^{;b'}V_0^{aa'}{}_{;b'} - 2V_0^{aa'}\Delta^{-\frac{1}{2}}\sigma^{;b'}\Delta^{\frac{1}{2}}{}_{;b'} + 2V_0^{aa'} + (\delta^{a'}{}_{b'}\Box' - R^{a'}{}_{b'})(\Delta^{\frac{1}{2}}g^{ab'}) = 0, \tag{A23}$$

the coefficients of Eq. (A22), $v_0^{aa'}{}_{b\ldots}$, can easily be recursively obtained. It should be noted that the covariant derivatives do require the appropriate Christoffel symbols, which can be obtained from the suitable metric at x'. Next, we construct coordinate expansions for the $V_n^{aa'}$. These have the form

$$V_n^{aa'}(x,x') = v_n^{aa'}(x) + v_n^{aa'}{}_b(x)\delta x^{b'} + v_n^{aa'}{}_{bc}(x)\delta x^{b'}\delta x^{c'} + v_n^{aa'}{}_{bcd}(x)\delta x^{b'}\delta x^{c'}\delta x^{d'} + \cdots. \tag{A24}$$

Substituting Eq. (A24) into the recursion relation (3.6a), which for the electromagnetic case becomes

$$2n\sigma^{;b'}V_n^{aa'}{}_{;b'} - 2nV_n^{aa'}\Delta^{-\frac{1}{2}}\sigma^{;b'}\Delta^{\frac{1}{2}}{}_{;b'} + 2n(n+1)V_n^{aa'} + (\delta^{a'}{}_{b'}\Box' - R^{a'}{}_{b'})V_{n-1}^{ab'} = 0, \tag{A25}$$

we can recursively solve for the coefficients of Eq. (A24), $v_n^{aa'}{}_{b}\ldots$. Once we have $V_n^{aa'}$ to the required n, we carry out the same remaining steps as in the scalar case and use Eq. (A20) to calculate the electromagnetic singular field.

8. Gravitational singular field

In the gravitational case, Eqs. (2.23) and (3.1) give

$$\bar{h}^{(S)}_{ab} = 2\mu\left[\frac{\Delta^{\frac{1}{2}}(x,x')g_{a'(a}g_{b)b'}(x,x')u^{a'}(x')u^{b'}(x')}{\sigma_{c'}(x,x')u^{c'}(x')}\right]^{x'=x_{(\mathrm{adv})}}_{x'=x_{(\mathrm{ret})}} + 2\mu\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V_{aba'b'}(x,z(\tau'))u^{a'}(x')u^{b'}(x')d\tau, \tag{A26}$$

where $V_{aba'b'}(x,z(\tau'))$ is given by Eq. (3.5). For the gravitational case, this is

$$V^{aba'b'}(x,x') = \sum_{n=0}^{\infty} V_n^{aba'b'}(x,x')\sigma^n(x,x'), \tag{A27}$$

where the appropriate metric at x or x' can be used to lower indices. The coordinate expansion for $V_0^{aba'b'}(x,x')$ is of the form

$$V_0^{aba'b'}(x,x') = v_0^{aba'b'}(x) + v_0^{aba'b'}{}_{c}(x)\delta x^{c'} + v_0^{aba'b'}{}_{cd}(x)\delta x^{c'}\delta x^{d'} + v_0^{aba'b'}{}_{cde}(x)\delta x^{c'}\delta x^{d'}\delta x^{e'} + \cdots. \tag{A28}$$

We replace $V_0^{aba'b'}(x,x')$ with Eq (A28) in the initial condition described by Eq. (3.6b), which for the gravitational case is

$$2\sigma^{;c'}V_0^{aba'b'}{}_{;c'} - 2V_0^{aba'b'}\Delta^{-\frac{1}{2}}\sigma^{;c'}\Delta^{\frac{1}{2}}{}_{;c'} + 2V_0^{aba'b'} + (\delta^{a'}{}_{c'}\delta^{b'}{}_{d'}\Box' + 2C^{a'}{}_{c'}{}^{b'}{}_{d'})(\Delta^{\frac{1}{2}}g^{c'(a}g^{b)d'}) = 0. \tag{A29}$$

This equation may be used to recursively solve for the coefficients of Eq. (A28), $v_0^{aba'b'}{}_{c}\ldots$. Next, the coordinate expansion of $V_n^{aba'b'}(x,x')$ for $n>0$ has the form

$$V_n^{aba'b'}(x,x') = v_n^{aba'b'}{}_{0}(x) + v_n^{aba'b'}{}_{c}(x)\delta x^{c'} + v_n^{aba'b'}{}_{cd}(x)\delta x^{c'}\delta x^{d'} + v_n^{aba'b'}{}_{cde}(x)\delta x^{c'}\delta x^{d'}\delta x^{e'} + \cdots. \tag{A30}$$

Substituting this into the recursion relation of Eq. (3.6a), which for the gravitational case has the form

$$2n\sigma^{;c'}V_n^{aba'b'}{}_{;c'} - 2nV_n^{aba'b'}\Delta^{-\frac{1}{2}}\sigma^{;c'}\Delta^{\frac{1}{2}}{}_{;c'} + 2n(n+1)V_n^{aba'b'} + (\delta^{a'}{}_{c'}\delta^{b'}{}_{d'}\Box' + 2C^{a'}{}_{c'}{}^{b'}{}_{d'})V_{n-1}^{abc'd'} = 0, \tag{A31}$$

we can recursively solve for the coefficients of Eq. (A30), $v_n^{aba'b'}{}_{c}\ldots$. As in the previous two cases, once we have $V_n^{aba'b'}(x,x')$ for the required n, it is straightforward to calculate the singular field using Eq. (A26)

APPENDIX B: COVARIANT BITENSOR EXPANSIONS

In this appendix, we give covariant expansions for the bitensors appearing in the formal expression for the singular field, Eq. (2.6). These are given in terms of the biscalars $\bar{s} \equiv (g^{\bar{a}\bar{b}} + u^{\bar{a}}u^{\bar{b}})\sigma_{\bar{a}}\sigma_{\bar{b}}$, [the projection of $\sigma_{\bar{a}}(x,\bar{x})$ orthogonal to the world line], and $\bar{r} = \sigma_{\bar{a}}u^{\bar{a}}$ [the projection of $\sigma_{\bar{a}}(x,\bar{x})$ along the world line]. In writing the coefficients, we use the notation $[T_{a_1\cdots a_n}]_{(k)}$ to denote the term of order ϵ^k in the expansion of the tensor $T_{a_1\cdots a_n}$, so that

$$T_{a_1\cdots a_n} = \sum_{k=0}^{\infty}[T_{a_1\cdots a_n}]_{(k)}\epsilon^k. \tag{B1}$$

1. Advanced and retarded points

Equation (2.6) for the singular field includes bitensors at points x' on the world line between the advanced and retarded points of x. We consolidate this dependence to a single arbitrary point, $\bar{x}$, on the world line by expanding the dependence on x' about $\bar{x}$. Denoting the proper distance along the world line between $x_{(\mathrm{adv})}/x_{(\mathrm{ret})}$ and $\bar{x}$ by $\Delta\tau$, we may write the expansion of this distance in powers of ϵ as

$$\Delta\tau_{(1)} = \bar{r}\pm\bar{s}, \qquad \Delta\tau_{(2)} = 0, \qquad \Delta\tau_{(3)} = \mp\frac{(\bar{r}\pm\bar{s})^2R_{u\sigma u\sigma}}{6\bar{s}}, \qquad \Delta\tau_{(4)} = \mp\frac{(\bar{r}\pm\bar{s})^2((\bar{r}\pm\bar{s})R_{u\sigma u\sigma;u} - R_{u\sigma u\sigma;\sigma})}{24\bar{s}},$$

$$\Delta\tau_{(5)} = \mp\frac{(\bar{r}\pm\bar{s})^2}{360\bar{s}^3}\{5R_{u\sigma u\sigma}R_{u\sigma u\sigma}(\bar{r}\mp3\bar{s})(\bar{r}\pm\bar{s}) + \bar{s}^2[(\bar{r}\pm\bar{s})^2(3R_{u\sigma u\sigma;uu} + 4R_{u\sigma u\bar{a}}R_{u\sigma u}{}^{\bar{a}}) - (\bar{r}\pm\bar{s})(3R_{u\sigma u\sigma;u\sigma} - 16R_{u\sigma u\bar{a}}R_{u\sigma\sigma}{}^{\bar{a}}) + 3R_{u\sigma u\sigma;\sigma\sigma} + 4R_{u\sigma\sigma\bar{a}}R_{u\sigma\sigma}{}^{\bar{a}}]\},$$

$$\begin{aligned}\Delta\tau_{(6)} = \pm\frac{(\bar{r}\pm\bar{s})^2}{4320\bar{s}^3}(&30R_{u\sigma u\sigma}[R_{u\sigma u\sigma;\sigma}(\bar{r}\mp3\bar{s})(\bar{r}\pm\bar{s}) - R_{u\sigma u\sigma;u}(\bar{r}\mp4\bar{s})(\bar{r}\pm\bar{s})^2] + \bar{s}^2\{6R_{u\sigma u\sigma;\sigma\sigma\sigma} + 36R_{u\sigma\sigma\bar{a};\sigma}R_{u\sigma\sigma}{}^{\bar{a}}\\ &- 2(\bar{r}\pm\bar{s})[3R_{u\sigma u\sigma;u\sigma\sigma} - 36R_{u\sigma\sigma\bar{a};\sigma}R_{u\sigma u}{}^{\bar{a}} - R_{u\sigma\sigma}{}^{\bar{a}}(16R_{u\sigma u\bar{a};\sigma} + 5R_{u\sigma u\sigma;\bar{a}} - 10R_{u\sigma\sigma\bar{a};u})]\\ &+ 2(\bar{r}\pm\bar{s})^2[3R_{u\sigma u\sigma;uu\sigma} - 30R_{u\sigma u\bar{a};u}R_{u\sigma\sigma}{}^{\bar{a}} + R_{u\sigma u}{}^{\bar{a}}(13R_{u\sigma u\bar{a};\sigma} + 5R_{u\sigma u\sigma;\bar{a}} - 25R_{u\sigma\sigma\bar{a};u})]\\ &- 6(\bar{r}\pm\bar{s})^3(R_{u\sigma u\sigma;uuu} + 6R_{u\sigma u\bar{a};u}R_{u\sigma u}{}^{\bar{a}})\}).\end{aligned}$$

2. Advanced and retarded distances

Taking two derivatives of the world function, we obtain a bitensor which has the covariant expansion

$$\begin{aligned}\sigma_{ab} &= g_{ab} - \frac{1}{3}R_{acbd}\sigma^c\sigma^d + \frac{1}{12}R_{acbd;e}\sigma^c\sigma^d\sigma^e + \left(\frac{1}{45}R_{acpd}R^p{}_{ebf} + \frac{1}{60}R_{acbd;ef}\right)\sigma^c\sigma^d\sigma^e\sigma^f + \bigg(\frac{1}{120}R_{acpd;e}R^p{}_{fbg}\\ &\quad + \frac{1}{120}R_{acpd}R^p{}_{ebf;g} + \frac{1}{360}R_{acbd;efg}\bigg)\sigma^c\sigma^d\sigma^e\sigma^f\sigma^g - \bigg(\frac{2}{945}R_{acpd}R^p{}_{eqf}R^q{}_{gbh} + \frac{1}{504}R_{acpd;ef}R^p{}_{gbh}\\ &\quad + \frac{17}{5040}R_{acpd;e}R^p{}_{fbg;h} + \frac{1}{504}R_{acpd}R^p{}_{ebf;gh} + \frac{1}{2520}R_{acbd;efgh}\bigg)\sigma^c\sigma^d\sigma^e\sigma^f\sigma^g\sigma^h + \bigg(\frac{17}{20160}R_{acpd;e}R^p{}_{fqg}R^q{}_{hbi}\\ &\quad + \frac{29}{30240}R_{acpd}R^p{}_{eqf;g}R^q{}_{hbi} + \frac{11}{30240}R_{acpd;efg}R^p{}_{hbi} + \frac{17}{20160}R_{acpd;ef}R^p{}_{gbh;i} + \frac{17}{20160}R_{acpd}R^p{}_{eqf}R^q{}_{gbh;i}\\ &\quad + \frac{17}{20160}R_{acpd;e}R^p{}_{fbg;hi} + \frac{11}{30240}R_{acpd}R^p{}_{ebf;ghi} + \frac{1}{20160}R_{acbd;efghi}\bigg)\sigma^c\sigma^d\sigma^e\sigma^f\sigma^g\sigma^h\sigma^i. \qquad \text{(B3)}\end{aligned}$$

For the singular field, we require the expansion of $[\sigma_{a'}u^{a'}](z_\pm, x)$. Writing $[\sigma_{a'}u^{a'}](\tau) = [\sigma_{a'}u^{a'}](z(\tau), x)$, expanding the dependence on τ about $\bar{x}$ (using the method of Sec. III A and making use of the above expansion of σ_{ab}) and evaluating at $\tau = \tau_\pm$, we obtain the coefficients of the expansion of $[\sigma_{a'}u^{a'}]_\pm \equiv [\sigma_{a'}u^{a'}](z_\pm, x)$ about $\bar{x}$. They are

$$\begin{aligned} r_{(1)} &= \bar{s}, \qquad r_{(3)} = -\frac{\bar{r}^2 - \bar{s}^2}{6\bar{s}}R_{u\sigma u\sigma}, \qquad r_{(4)} = \frac{\bar{r}\pm\bar{s}}{24\bar{s}}[R_{u\sigma u\sigma;\sigma}(\bar{r}\mp\bar{s}) - R_{u\sigma u\sigma;u}(\bar{r}\pm\bar{s})(\bar{r}\mp 2\bar{s})],\\ r_{(5)} &= -\frac{1}{360\bar{s}^3}\{\bar{s}^2[(\bar{r}^2-\bar{s}^2)(3R_{u\sigma u\sigma;\sigma\sigma} + 4R_{u\sigma\sigma}{}^{\bar{a}}R_{u\sigma\sigma\bar{a}}) - (\bar{r}\pm\bar{s})^2(\bar{r}\mp 2\bar{s})(3R_{u\sigma u\sigma;\sigma u} - 16R_{u\sigma\sigma}{}^{\bar{a}}R_{u\sigma u\bar{a}})\\ &\quad + (\bar{r}\pm\bar{s})^3(\bar{r}\mp 3\bar{s})(3R_{u\sigma u\sigma;uu} + 4R_{u\sigma u}{}^{\bar{a}}R_{u\sigma u\bar{a}})] + 5[(\bar{r}^2-\bar{s}^2)R_{u\sigma u\sigma}]^2\},\\ r_{(6)} &= \frac{1}{4320\bar{s}^3}(\bar{s}^2\{6(\bar{r}^2-\bar{s}^2)(R_{u\sigma u\sigma;\sigma\sigma\sigma} + 6R_{u\sigma\sigma}{}^{\bar{a}}{}_{;\sigma}R_{u\sigma\sigma\bar{a}}) - 6(\bar{r}\pm\bar{s})^4(\bar{r}\mp 4\bar{s})(R_{u\sigma u\sigma;uuu} + 6R_{u\sigma u\bar{a};u}R_{u\sigma u}{}^{\bar{a}})\\ &\quad - 2(\bar{r}\pm\bar{s})^2(\bar{r}\mp 2\bar{s})[3R_{u\sigma u\sigma;u\sigma\sigma} - 36R_{u\sigma\sigma\bar{a};\sigma}R_{u\sigma u}{}^{\bar{a}} - R_{u\sigma\sigma}{}^{\bar{a}}(16R_{u\sigma u\bar{a};\sigma} + 5R_{u\sigma u\sigma;\bar{a}} - 10R_{u\sigma\sigma\bar{a};u})]\\ &\quad + 2(\bar{r}\pm\bar{s})^3(\bar{r}\mp 3\bar{s})[3R_{u\sigma u\sigma;uu\sigma} - 30R_{u\sigma u\bar{a};u}R_{u\sigma\sigma}{}^{\bar{a}} + R_{u\sigma u}{}^{\bar{a}}(13R_{u\sigma u\bar{a};\sigma} + 5R_{u\sigma u\sigma;\bar{a}} - 25R_{u\sigma\sigma\bar{a};u})]\}\\ &\quad + 30R_{u\sigma u\sigma}[(\bar{r}^2-\bar{s}^2)^2R_{u\sigma u\sigma;\sigma} - (\bar{r}\pm\bar{s})^3(\bar{r}^2\mp 3\bar{r}\,\bar{s} + 4\bar{s}^2)R_{u\sigma u\sigma;u}]). \qquad \text{(B4)}\end{aligned}$$

3. Van Vleck–Morette determinant

The Van Vleck determinant has the covariant expansion

$$\begin{aligned}\Delta^{1/2}(x,x') &= 1 + \frac{1}{12}R_{ab}\sigma^a\sigma^b - \frac{1}{24}R_{ab;c}\sigma^a\sigma^b\sigma^c + \left(\frac{1}{360}R_{paqb}R^p{}_c{}^q{}_d + \frac{1}{288}R_{ab}R_{cd} + \frac{1}{80}R_{ab;cd}\right)\sigma^a\sigma^b\sigma^c\sigma^d\\ &\quad - \left(\frac{1}{360}R_{paqb}R^p{}_c{}^q{}_{d;e} + \frac{1}{288}R_{ab}R_{cd;e} + \frac{1}{360}R_{ab;cde}\right)\sigma^a\sigma^b\sigma^c\sigma^d\sigma^e + \bigg(\frac{1}{1260}R_{paqb}R^p{}_c{}^q{}_{d;ef}\\ &\quad + \frac{1}{1344}R_{paqb;c}R^p{}_d{}^q{}_{e;f} + \frac{1}{5670}R_{paqb}R_{rc}{}^p{}_dR^q{}_e{}^r{}_f + \frac{1}{4320}R_{paqb}R^p{}_c{}^q{}_dR_{ef} + \frac{1}{10368}R_{ab}R_{cd}R_{ef}\\ &\quad + \frac{1}{1152}R_{ab;c}R_{de;f} + \frac{1}{960}R_{ab}R_{cd;ef} + \frac{1}{2016}R_{ab;cdef}\bigg)\sigma^a\sigma^b\sigma^c\sigma^d\sigma^e\sigma^f - \bigg(\frac{1}{6048}R_{paqb}R^p{}_c{}^q{}_{d;efg}\\ &\quad + \frac{1}{2240}R_{paqb;c}R^p{}_d{}^q{}_{e;fg} + \frac{1}{3780}R_{paqb}R_{rc}{}^p{}_dR^q{}_e{}^r{}_{f;g} + \frac{1}{4320}R_{paqb}R^p{}_c{}^q{}_{d;e}R_{fg} + \frac{1}{8640}R_{paqb}R^p{}_c{}^q{}_dR_{ef;g}\\ &\quad + \frac{1}{6912}R_{ab}R_{cd}R_{ef;g} + \frac{1}{1920}R_{ab;c}R_{de;fg} + \frac{1}{4320}R_{ab}R_{cd;efg} + \frac{1}{13440}R_{ab;cdefg}\bigg)\sigma^a\sigma^b\sigma^c\sigma^d\sigma^e\sigma^f\sigma^g. \qquad \text{(B5)}\end{aligned}$$

Writing $\Delta^{1/2}(\tau) = \Delta^{1/2}(z(\tau), x)$, expanding the dependence on τ about $\bar{x}$ (using the method of Sec. III A) and evaluating at $\tau = \tau_\pm$, we obtain the coefficients in the expansion of $\Delta^{1/2}_\pm \equiv \Delta^{1/2}(z_\pm, x)$ about $\bar{x}$. Specialized to the vacuum case, they are

$$
\begin{aligned}
&\Delta^{1/2}_{(0)} = 1, \qquad \Delta^{1/2}_{(1)} = 0, \qquad \Delta^{1/2}_{(2)} = 0, \qquad \Delta^{1/2}_{(3)} = 0,\\
&\Delta^{1/2}_{(4)} = \frac{1}{360}[C_{\sigma\bar{a}\sigma\bar{b}} + 2(\bar{r} \pm \bar{s})C_{u(\bar{a}|\sigma|\bar{b})} + (\bar{r} \pm \bar{s})^2 C_{u\bar{a}u\bar{b}}][C_{\sigma}{}^{\bar{a}}{}_{\sigma}{}^{\bar{b}} + 2(\bar{r} \pm \bar{s})C_{u}{}^{\bar{a}}{}_{\sigma}{}^{\bar{b}} + (\bar{r} \pm \bar{s})^2 C_{u}{}^{\bar{a}}{}_{u}{}^{\bar{b}}],\\
&\Delta^{1/2}_{(5)} = \frac{1}{360}[C_{\sigma\bar{a}\sigma\bar{b}} + 2(\bar{r} \pm \bar{s})C_{u(\bar{a}|\sigma|\bar{b})} + (\bar{r} \pm \bar{s})^2 C_{u\bar{a}u\bar{b}}][(\bar{r} \pm \bar{s})(C_{\sigma}{}^{\bar{a}}{}_{\sigma}{}^{\bar{b}}{}_{;u} - 2C_{u}{}^{\bar{a}}{}_{\sigma}{}^{\bar{b}}{}_{;\sigma}) - (\bar{r} \pm \bar{s})^2(C_{u}{}^{\bar{a}}{}_{u}{}^{\bar{b}}{}_{;\sigma} - 2C_{u}{}^{\bar{a}}{}_{\sigma}{}^{\bar{b}}{}_{;u})\\
&\qquad + (\bar{r} \pm \bar{s})^3 C_{u}{}^{\bar{a}}{}_{u}{}^{\bar{b}}{}_{;u} - C_{\sigma}{}^{\bar{a}}{}_{\sigma}{}^{\bar{b}}{}_{;\sigma}].
\end{aligned} \tag{B6}
$$

4. Bivector of parallel transport

The derivative of the bivector of parallel transport has the covariant expansion

$$
\begin{aligned}
g_{a}{}^{a'} g_{a'b;c}(x, x') = {}& -\frac{1}{2}R_{bacd}\sigma^d + \frac{1}{6}R_{bacd;e}\sigma^d\sigma^e - \frac{1}{24}(R_{bapd}R^p{}_{ecf} + R_{bacd;ef})\sigma^d\sigma^e\sigma^f + \bigg(\frac{1}{60}R_{bapd}R^p{}_{ecf;g}\\
&+ \frac{7}{360}R_{bapd;e}R^p{}_{fcg} + \frac{1}{120}R_{bacd;efg}\bigg)\sigma^d\sigma^e\sigma^f\sigma^g - \bigg(\frac{1}{240}R_{bapd}R^p{}_{ecf;gh}\frac{1}{120}R_{bapd;e}R^p{}_{fcg;h}\\
&+ \frac{1}{180}R_{bapd;ef}R^p{}_{gch} + \frac{1}{240}R_{bapd}R^p{}_{eqf}R^q{}_{gch} + \frac{1}{720}R_{bacd;efgh}\bigg)\sigma^d\sigma^e\sigma^f\sigma^g\sigma^h.
\end{aligned} \tag{B7}
$$

For the singular field, we require the expansion of $g_{aa'}u^{a'}(z_\pm, x)$. Writing $[g_{aa'}u^{a'}](\tau) = [g_{aa'}u^{a'}](z(\tau), x)$, expanding the dependence on τ about $\bar{x}$ (using the method of Sec. III A and making use of the above expansion of the bivector of parallel transport) and evaluating at $\tau = \tau_\pm$, we obtain the coefficients of the expansion of $[g_{aa'}u^{a'}]_\pm \equiv g_{aa'}u^{a'}(z_\pm, x)$ about $\bar{x}$. They are

$$
\begin{aligned}
&[g_{aa'}u^{a'}]_{(0)} = g_{a\bar{a}}u^{\bar{a}}, \qquad [g_{aa'}u^{a'}]_{(1)} = 0, \qquad [g_{aa'}u^{a'}]_{(2)} = -\frac{1}{2}(\bar{r} \pm \bar{s})g_{a}{}^{\bar{a}}R_{u\sigma u\bar{a}},\\
&[g_{aa'}u^{a'}]_{(3)} = \frac{1}{6}(\bar{r} \pm \bar{s})g_{a}{}^{\bar{a}}(R_{u\sigma u\bar{a};\sigma} - (\bar{r} \pm \bar{s})R_{u\sigma u\bar{a};u}),\\
&[g_{aa'}u^{a'}]_{(4)} = \pm\frac{1}{24\bar{s}}g_{a}{}^{\bar{a}}(\bar{r} \pm \bar{s})\{2(\bar{r} \pm \bar{s})R_{u\sigma u\bar{a}}R_{u\sigma u\sigma} - \bar{s}[R_{u\sigma u\bar{a};\sigma\sigma} + R_{u\bar{a}\sigma\bar{b}}R_{u\sigma\sigma}{}^{\bar{b}} + (\bar{r} \pm \bar{s})^2 R_{u\sigma ua;uu}\\
&\qquad + (\bar{r} \pm \bar{s})(R_{u\sigma u}{}^{\bar{b}}(2R_{u\bar{a}\sigma\bar{b}} + 3R_{u\sigma\bar{a}\bar{b}}) - R_{u\sigma u\bar{a};u\sigma} + R_{u\bar{a}u}{}^{\bar{b}}(R_{u\sigma\sigma\bar{b}} + (\bar{r} \pm \bar{s})R_{u\sigma u\bar{b}}))]\},\\
&[g_{aa'}u^{a'}]_{(5)} = \pm\frac{1}{2160\bar{s}}g_{a}{}^{\bar{a}}(\bar{r} \pm \bar{s})(\bar{s}\{18[R_{u\sigma ua;\sigma\sigma\sigma} - R_{u\sigma ua;u\sigma\sigma}(\bar{r} \pm \bar{s}) + R_{u\sigma ua;uu\sigma}(\bar{r} \pm \bar{s})^2 - R_{u\sigma ua;uuu}(\bar{r} \pm \bar{s})^3]\\
&\qquad + 6R_{u\sigma\sigma b}[7R_{ua\sigma}{}^{b}{}_{;\sigma} + (\bar{r} \pm \bar{s})(3R_{uau}{}^{b}{}_{;\sigma} - 4R_{ua\sigma}{}^{b}{}_{;u} + 4R_{u\sigma ua}{}^{;b}) - 8R_{uau}{}^{b}{}_{;u}(\bar{r} \pm \bar{s})^2]\\
&\qquad + 9R_{ua\sigma b}[4R_{u\sigma\sigma}{}^{b}{}_{;\sigma} + (\bar{r} \pm \bar{s})(5R_{u\sigma u}{}^{b}{}_{;\sigma} - 3R_{u\sigma\sigma}{}^{b}{}_{;u}) - 6R_{u\sigma u}{}^{b}{}_{;u}(\bar{r} \pm \bar{s})^2] + 2R_{u\sigma ub}[21(\bar{r} \pm \bar{s})(2R_{ua\sigma}{}^{b}{}_{;\sigma}\\
&\qquad + 3R_{u\sigma a}{}^{b}{}_{;\sigma}) + (\bar{r} \pm \bar{s})^2(17R_{uau}{}^{b}{}_{;\sigma} - 32R_{ua\sigma}{}^{b}{}_{;u} - 36R_{u\sigma a}{}^{b}{}_{;u} + 16R_{u\sigma ua}{}^{;b}) - 21R_{uau}{}^{b}{}_{;u}(\bar{r} \pm \bar{s})^3]\\
&\qquad + 9R_{uaub}[4R_{u\sigma\sigma}{}^{b}{}_{;\sigma}(\bar{r} \pm \bar{s}) + 3(\bar{r} \pm \bar{s})^2(R_{u\sigma u}{}^{b}{}_{;\sigma} - R_{u\sigma\sigma}{}^{b}{}_{;u}) - 4R_{u\sigma u}{}^{b}{}_{;u}(\bar{r} \pm \bar{s})^3] + 54(\bar{r} \pm \bar{s})R_{u\sigma ab}[R_{u\sigma u}{}^{b}{}_{;\sigma}\\
&\qquad - 2R_{u\sigma u}{}^{b}{}_{;u}(\bar{r} \pm \bar{s})]\} + 15(\bar{r} \pm \bar{s})\{4R_{u\sigma u\sigma}[2(\bar{r} \pm \bar{s})R_{u\sigma ua;u} - R_{u\sigma ua;\sigma}] + 3R_{u\sigma ua}[R_{u\sigma u\sigma;u}(\bar{r} \pm \bar{s}) - R_{u\sigma u\sigma;\sigma}]\}).
\end{aligned} \tag{B8}
$$

5. Scalar tail

The scalar tail bitensor, $V(x, x')$, may be expanded in a covariant series by writing it in the form of a Hadamard series,

$$
V(x, x') = V_0(x, x') + V_1(x, x')\sigma(x, x') + \cdots, \tag{B9}
$$

and expanding each of the Hadamard coefficients $V_0(x, x')$, $V_1(x, x')$, $\cdots$ in a covariant Taylor series,

$$
\begin{aligned}
V_0 &= v_0 - \frac{1}{2}v_{0;c}\sigma^c + \frac{1}{2}v_{0cd}\sigma^c\sigma^d + \frac{1}{6}\bigg(-\frac{3}{2}v_{0(cd;e)} + \frac{1}{4}v_{0;(cde)}\bigg)\sigma^c\sigma^d\sigma^e + \cdots,\\
V_1 &= v_1 - \frac{1}{2}v_{1;c}\sigma^c + \cdots.
\end{aligned}
$$

The series coefficients required to obtain the expansion of the singular field to $\mathcal{O}(\epsilon^4)$ [$V(x, x')$ to $\mathcal{O}(\epsilon^3)$] are given by

$$v_0 = \frac{1}{2}\Big(\Big(\xi - \frac{1}{6}\Big)R + m^2\Big),$$
$$v_0{}^{cd} = -\frac{1}{180}R_{pqr}{}^{c}R^{pqrd} - \frac{1}{180}R^{c}{}_{p}{}^{d}{}_{q}R^{pq} + \frac{1}{90}R^{c}{}_{p}R^{dp} - \frac{1}{120}\Box R^{cd} + \frac{1}{12}\Big(\xi - \frac{1}{6}\Big)RR^{cd} + \Big(\frac{1}{6}\xi - \frac{1}{40}\Big)R^{;cd} + \frac{1}{12}m^2R^{cd},$$
$$v_1 = \frac{1}{720}R_{pqrs}R^{pqrs} - \frac{1}{720}R_{pq}R^{pq} + \frac{1}{8}\Big(\xi - \frac{1}{6}\Big)^2R^2 - \frac{1}{24}\Big(\xi - \frac{1}{5}\Big)\Box R + \frac{1}{4}m^2\Big(\xi - \frac{1}{6}\Big)R + \frac{1}{8}m^4.$$

For the singular field, we require the expansion of $\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V d\tau'$. Writing $V(\tau) = V(z(\tau), x)$ and expanding the dependence on τ about $\bar{x}$ (using the method of Sec. III A and making use of the above expansion of V), we obtain an expansion in powers of $\Delta\tau$ which can be trivially integrated between $\tau = \tau_-$ and $\tau = \tau_+$. Specialized to the vacuum case, the required expansion coefficients are then

$$\Big[\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V d\tau'\Big]_{(1)} = 0, \qquad \Big[\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V d\tau'\Big]_{(2)} = 0, \qquad \Big[\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V d\tau'\Big]_{(3)} = 0, \qquad \Big[\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V d\tau'\Big]_{(4)} = 0.$$

6. Electromagnetic tail

The electromagnetic tail bitensor, $V_{ab'}(x, x')$, may be expanded in a covariant series by writing it in the form of a Hadamard series,

$$V_{ab'}(x, x') = g_{b'}{}^{b}[V_{0ab}(x, x') + V_{1ab}(x, x')\sigma(x, x') + \cdots], \tag{B10}$$

and expanding each of the Hadamard coefficients, $V_{0ab}(x, x'), V_{1ab}(x, x'), \cdots$, in a covariant Taylor series,

$$V_{0ab} = v_{0(ab)} + \Big(-\frac{1}{2}v_{0(ab);c} + v_{0[ab]c}\Big)\sigma^c + \frac{1}{2}(v_{0(ab)cd} - v_{0[ab](c;d)})\sigma^c\sigma^d$$
$$+ \frac{1}{6}\Big(-\frac{3}{2}v_{0(ab)(cd;e)} + \frac{1}{4}v_{0(ab);(cde)} + v_{0[ab]cde}\Big)\sigma^c\sigma^d\sigma^e + \cdots,$$
$$V_{1ab} = v_{1(ab)} + \Big(-\frac{1}{2}v_{1(ab);c} + v_{1[ab]c}\Big)\sigma^c + \cdots.$$

The series coefficients required to obtain the expansion of the singular field to $\mathcal{O}(\epsilon^4)$ [$V_{ab'}(x, x')$ to $\mathcal{O}(\epsilon^3)$] are given by

$$v_{0(ab)} = \frac{1}{2}R_{ab} - \frac{1}{12}Rg_{ab}, \qquad v_{0[ab]}{}^{c} = \frac{1}{6}R^{c}{}_{[b;a]},$$
$$v_{0(ab)}{}^{cd} = \frac{1}{6}R_{ab}{}^{;(cd)} + \frac{1}{12}R_{ab}R^{cd} + \frac{1}{12}R_{(a}{}^{pqc}R_{b)pq}{}^{d} + g_{ab}\Big(-\frac{1}{180}R_{pqr}{}^{c}R^{pqrd} - \frac{1}{180}R^{c}{}_{p}{}^{d}{}_{q}R^{pq}$$
$$+ \frac{1}{90}R^{c}{}_{p}R^{dp} - \frac{1}{72}RR^{cd} - \frac{1}{40}R^{;cd} - \frac{1}{120}\Box R^{cd}\Big),$$
$$v_{0[ab]}{}^{cde} = -\frac{3}{20}R^{(c}{}_{[a;b]}{}^{de)} - \frac{1}{12}R^{(c}{}_{[a;b]}R^{de)} - \frac{1}{20}R_{[a}{}^{pq(c}R_{b]pq}{}^{d;e)} - \frac{1}{30}R_{ab}{}^{p(c}{}_{;q}R_{p}{}^{de)q}$$
$$+ \frac{1}{60}R_{abp}{}^{(c;d}R^{e)p} + \frac{1}{20}R_{abp}{}^{(c}R^{de);p} - \frac{1}{20}R_{ab}{}^{p(c}R_{p}{}^{d;e)}$$

and

$$v_{1(ab)} = -\frac{1}{48}R_{apqr}R_{b}{}^{pqr} + \frac{1}{8}R_{ap}R_{b}{}^{p} - \frac{1}{24}RR_{ab} - \frac{1}{24}\Box R_{ab} + g_{ab}\Big(\frac{1}{720}R_{pqrs}R^{pqrs} - \frac{1}{720}R_{pq}R^{pq} + \frac{1}{288}R^2 + \frac{1}{120}\Box R\Big),$$
$$v_{1[ab]}{}^{c} = \frac{1}{240}R_{[a}{}^{pqr}R_{b]pqr}{}^{;c} + \frac{1}{24}R_{[a}{}^{pqc}R_{b]p;q} + \frac{1}{120}R_{[a}{}^{pqc}R_{b]q;p} - \frac{1}{120}R^{c}{}_{pq[a}R^{pq}{}_{;b]} + \frac{1}{24}R^{pc}{}_{;[a}R_{b]p} + \frac{1}{24}R^{p}{}_{[a}R_{b]}{}^{c}{}_{;p}$$
$$- \frac{1}{360}R^{pc}R_{p[a;b]} - \frac{1}{24}R^{p}{}_{[a}R_{b]p}{}^{;c} + \frac{1}{72}RR^{c}{}_{[a;b]} + \frac{1}{120}R^{c}{}_{[a;b]p}{}^{p} - \frac{1}{540}R_{abpq;r}R^{pqrc} - \frac{1}{540}R_{abpq;r}R^{rqpc}$$
$$- \frac{1}{360}R_{abp}{}^{c}{}_{;q}R^{pq} + \frac{1}{120}R_{abpq}R^{cp;q} - \frac{1}{120}R_{ab}{}^{pc}R_{;p}.$$

These are the same as those given by Brown and Ottewill [88] with the exception of $v_{1[ab]}{}^{c}$, where we have corrected a sign error in one of their terms and combined another two terms into a single term.

For the singular field, we require the expansion of $\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V_{ab'}u^{b'}d\tau'$. Writing $[V_{ab'}u^{b'}](\tau) = [V_{ab'}u^{b'}](z(\tau), x)$ and expanding the dependence on τ about $\bar{x}$ (using the method of Sec. III A and making use of the above expansion of $V_{ab'}$), we obtain an expansion in powers of $\Delta\tau$ which can be trivially integrated between $\tau = \tau_-$ and $\tau = \tau_+$. Specialized to the vacuum case, the required expansion coefficients are then

$$\left[\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V_{ab'}u^{b'}d\tau'\right]_{(1)} = 0, \qquad \left[\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V_{ab'}u^{b'}d\tau'\right]_{(2)} = 0,$$

$$\begin{aligned}\left[\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V_{ab'}u^{b'}d\tau'\right]_{(3)} &= \frac{1}{1152}(\bar{r}\pm\bar{s})g_a{}^{\bar{a}}[16R_{ubac}R_u{}^b{}_u{}^c(\bar{r}\pm\bar{s})^2 + 48R_u{}^b{}_\sigma{}^c R_{\sigma cab} + 24(\bar{r}\pm\bar{s})(R_u{}^b{}_\sigma{}^c R_{ucab} \\ &\quad + R_u{}^b{}_u{}^c R_{\sigma bac} + (\bar{r}\mp 2\bar{s})R_{bcde}R^{bcde}u_a)],\end{aligned}$$

$$\begin{aligned}\left[\int_{\tau_{(\mathrm{ret})}}^{\tau_{(\mathrm{adv})}} V_{ab'}u^{b'}d\tau'\right]_{(4)} &= \frac{1}{51840}(\bar{r}\pm\bar{s})g_a{}^{\bar{a}}(144(2R_{ua\sigma}{}^{b;c}R_{\sigma b\sigma c} - 6R_\sigma{}^b{}_a{}^c{}_{;\sigma}R_{uc\sigma b} - 9R_u{}^b{}_\sigma{}^c{}_{;\sigma}R_{\sigma cab} + R_\sigma{}^{bcd}{}_{;\sigma}R_{\sigma bcd}u_a) \\ &\quad - 4(\bar{r}\pm\bar{s})\{18[6R_\sigma{}^b{}_a{}^c{}_{;\sigma}R_{ubuc} - 2R_{ua\sigma}{}^{b;c}R_{ub\sigma c} + 9R_u{}^b{}_\sigma{}^c{}_{;\sigma}R_{ucab} + R_{uc\sigma b}(6R_u{}^b{}_a{}^c{}_{;\sigma} - 2R_{ua\sigma}{}^{b;c} \\ &\quad - 9R_\sigma{}^b{}_a{}^c{}_{;u}) + 9R_u{}^b{}_u{}^c{}_{;\sigma}R_{\sigma bac} - 2R_{uau}{}^{b;c}R_{\sigma b\sigma c} - 6R_u{}^b{}_\sigma{}^c{}_{;u}R_{\sigma cab}] + (\bar{r}\mp 2\bar{s})[27R_u{}^{bcd}{}_{;\sigma}R_{abcd} \\ &\quad + 18R_a{}^{bcd}{}_{;\sigma}R_{ubcd} + 4R_{ua}{}^{bc;d}(R_{\sigma bcd} - R_{\sigma dbc})] - 3[6R_\sigma{}^{bcd}{}_{;\sigma}R_{ubcd} + 6(R_u{}^{bcd}{}_{;\sigma} - R_\sigma{}^{bcd}{}_{;u})R_{\sigma bcd} \\ &\quad + R^{bcde}{}_\sigma R_{bcde}(\bar{r}\mp 2\bar{s})]u_a\} + (\bar{r}\pm\bar{s})^2\{48[R_{ubuc}(2R_{ua\sigma}{}^{b;c} - 6R_u{}^b{}_a{}^c{}_{;\sigma} + 9R_\sigma{}^b{}_a{}^c{}_{;u}) \\ &\quad + 6R_u{}^b{}_\sigma{}^c{}_{;u}R_{ucab} + 9R_u{}^b{}_a{}^c{}_{;u}R_{uc\sigma b} + 2R_{uau}{}^{b;c}(R_{ub\sigma c} + R_{uc\sigma b}) + 6R_u{}^b{}_u{}^c{}_{;u}R_{\sigma bac}] \\ &\quad + (\bar{r}\mp 3\bar{s})[R_{ubcd}(27R_a{}^{bcd}{}_{;u} - 4R_{ua}{}^{bc;d}) + 4R_{ua}{}^{bc;d}R_{udbc}] + 3[16(R_u{}^{bcd}{}_{;\sigma} - R_\sigma{}^{bcd}{}_{;u})R_{ubcd} \\ &\quad - R^{bcde}{}_{;u}R_{bcde}(\bar{r}\mp 3\bar{s})]u_a + 6R_u{}^{bcd}{}_{;u}[3R_{abcd}(\bar{r}\mp 3\bar{s}) - 8R_{\sigma bcd}u_a] - 432R_u{}^b{}_u{}^c{}_{;\sigma}R_{ubac}\} \\ &\quad + 36(\bar{r}\pm\bar{s})^3[6R_u{}^b{}_u{}^c{}_{;u}R_{ubac} + R_{ubuc}(2R_{uau}{}^{b;c} + 9R_u{}^b{}_a{}^c{}_{;u}) - R_u{}^{bcd}{}_{;u}R_{ubcd}u_a]).\end{aligned} \tag{B11}$$

7. Gravitational tail

The gravitational tail bitensor, $V_{aa'bb'}(x, x')$, may be expanded in a covariant series by writing it in the form of a Hadamard series,

$$V_{aa'bb'}(x, x') = g_{a'}{}^c g_{b'}{}^d[V_{0acbd}(x, x') + V_{1acbd}(x, x')\sigma(x, x') + \cdots], \tag{B12}$$

and expanding each of the Hadamard coefficients, $V_{0acbd}(x, x'), V_{1acbd}(x, x'), \cdots$, in a covariant Taylor series,

$$\begin{aligned}V_{0AB} &= v_{0(AB)} + \left(-\frac{1}{2}v_{0(AB);e} + v_{0[AB]e}\right)\sigma^e + \frac{1}{2}(v_{0(AB)ef} - v_{0[AB](e;f)})\sigma^e\sigma^f \\ &\quad + \frac{1}{6}\left(-\frac{3}{2}v_{0(AB)(ef;g)} + \frac{1}{4}v_{0(AB);(efg)} + v_{0[AB]efg}\right)\sigma^e\sigma^f\sigma^g + \cdots, \\ V_{1AB} &= v_{1(AB)} + \left(-\frac{1}{2}v_{1(AB);c} + v_{1[AB]e}\right)\sigma^e + \cdots.\end{aligned}$$

The required coefficients for V_{0AB} are

$$v_{0(AB)} = v_{0(\overline{ab}\,\overline{cd})} = -C_{acbd}, \tag{B13}$$

$$v_{0[AB]}{}^e = 0, \tag{B14}$$

$$v_{0(AB)}{}^{ef} = v_{0(\overline{ab}\,\overline{cd})}{}^{ef} = -\frac{1}{3}C_{acbd}{}^{;(ef)} - \frac{1}{6}C_{ac}{}^{p(e}C_{bdp}{}^{f)} + \frac{1}{6}g_{ac}C_b{}^{pq(e}C_{dpq}{}^{f)} - \frac{1}{720}\Pi_{abcd}g^{ef}C^{pqrs}C_{pqrs}, \tag{B15}$$

and

$$v_{0[AB]}{}^{efg} = v_{0[\overline{ab}\,\overline{cd}]}{}^{efg} = \frac{1}{10}g_{ac}C_{[b}{}^{pq(e;f}C_{d]pq}{}^{g)} + \frac{1}{15}g_{ac}C_{bdpe;q}C^p{}_f{}^q{}_g, \tag{B16}$$

where

$$\Pi_{abcd} = \frac{1}{2} g_{ac} g_{bd} + \frac{1}{2} g_{ad} g_{bc} + \kappa g_{ab} g_{cd}, \tag{B17}$$

and κ is the real parameter introduced in Eq. (C1) of [90] for comparison purposes. In Eqs. (B13)–(B16), the right-hand sides are understood to be symmetrized on the index pairs (ab) and (cd). The required coefficients for V_{1AB} are

$$v_1(AB) = v_1(\overline{ab}\,\overline{cd}) = \frac{1}{12} \Box C_{acbd} + \frac{1}{2} C_a{}^p{}_b{}^q C_{cpdq} + \frac{1}{24} C_{ac}{}^{pq} C_{bdpq} - \frac{1}{96} g_{ac} g_{bd} C^{pqrs} C_{pqrs} + \frac{1}{720} \Pi_{abcd} C^{pqrs} C_{pqrs} \tag{B18}$$

and

$$v_{1[AB]}{}^e = v_{1[\overline{ab}\,\overline{cd}]}{}^e = \frac{1}{12} (C_a{}^p{}_b{}^q C_{cpdq}{}^{;e} - C_c{}^p{}_d{}^q C_{apbq}{}^{;e}) + \frac{1}{12} (C_a{}^{peq} C_{bcdp;q} - C_c{}^{peq} C_{dabp;q}) + \frac{1}{90} g_{ac} C^{epqr} C_{bdpq;r}, \tag{B19}$$

where again there is implicit symmetrization on the index pairs (ab) and (cd). When $\kappa = -1/2$, Eq. (B18) agrees with Eq. (A23) of Allen, Folacci and Ottewill [89] specialized to the vacuum case. Our expressions also agree with Anderson, Flanagan, and Ottewill [90], but we write them here in a slightly more compact form. Note that the expressions (B13)–(B16), (B18), and (B19) are all traceless on the index pair (cd), aside from the terms involving the tensor Π_{abcd}. This means that performing a trace reversal on the index pair (cd) is equivalent to changing the value of κ from 0 to $-1/2$. For the calculation of the gravitational singular field, we require an expansion of the trace reversed singular field, and so we choose $\kappa = 0$.

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