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Phase transitions in spinor quantum gravity on a lattice

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We construct a well-defined lattice-regularized quantum theory formulated in terms of fundamental fermion and gauge fields, the same type of degrees of freedom as in the Standard Model. The theory is explicitly invariant under local Lorentz transformations and, in the continuum limit, under diffeomorphisms. It is suitable for describing large nonperturbative and fast-varying fluctuations of metrics. Although the quantum curved space turns out to be, on the average, flat and smooth owing to the noncompressibility of the fundamental fermions, the low-energy Einstein limit is not automatic: one needs to ensure that composite metrics fluctuations propagate to long distances as compared to the lattice spacing. One way to guarantee this is to stay at a phase transition. We develop a lattice mean-field method and find that the theory typically has several phases in the space of the dimensionless coupling constants, separated by the 2nd order phase transition surface. For example, there is a phase with a spontaneous breaking of chiral symmetry. The effective low-energy Lagrangian for the ensuing Goldstone field is explicitly diffeomorphism invariant. We expect that the Einstein gravitation is achieved at the phase transition. A bonus is that the cosmological constant is probably automatically zero.

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I. INTRODUCTION

We live in a world with fermions, and they must be included into general relativity. The standard way one couples Dirac fermions to gravity is via the Fock-Weyl action [1,2]: Fermions interact with the frame field e^A_μ (also known as *vierbein, repère,* or *tetrad*) and with the spin connection ω^{AB}_μ being the gauge field of the local Lorentz group. The frame and the spin connection are *a priori* independent field variables. The bosonic part of the action has to be written through e_μ and ω_μ accordingly. This is known for the last 90 years as Cartan's formulation of general relativity [3]. Speaking generally, it is distinct from the classic Einstein-Hilbert formulation based on the Riemann geometry, since it allows for a nonzero torsion. We stress that the presence of fermions in nature forces us to make a definite choice in favor of the Cartan, as contrasted to the Riemann geometry.

In practice, however, it is hardly possible to detect the difference. In the leading order in the gradient expansion of the gravitational action written down in terms of e_μ and ω_μ, the saddle-point equation for ω_μ says that torsion is on the average zero. Therefore, Cartan's theory reduces to that of Einstein.

In the next order in p^2/M_P^2 where M_P is the Planck mass and p is the characteristic momentum, a four-fermion contact interaction appears from integrating out torsion. Its strength is many orders of magnitude less than that of weak interactions [4] therefore this correction will hardly be detected any time soon in the laboratory. In principle, it modifies, e.g., the Friedman cosmological evolution equation that follows from the purely Riemannian approach. However, the correction remains tiny as long as fermions in the Universe have Fermi momentum or temperature that are much less than M_P [5]. If they reach that scale such that the four-fermion correction becomes of the order of the leading stress-energy term, the theory itself fails since the gradient expansion [5,6] from where it has been derived, becomes inapplicable. There is no agreed upon idea what the theory looks like at the Planck scale; in particular, quantum gravity effects are supposed to set up there.

Being indistinguishable from Einstein's equation in the range where observations are performed, Cartan's theory, however, has a critical feature when one attempts to quantize it. The bosonic part of the action is written in terms of e_μ and ω_μ. To preserve the required general covariance or invariance under the change of the coordinate system, called diffeomorphism, any action term is necessarily odd in the antisymmetric Levi-Civita tensor $\epsilon^{\kappa\lambda\mu\nu}$. That makes all possible diffeomorphism-invariant action terms not sign definite [7].

The simplest example is the invariant volume itself or the cosmological term, $\int d^4x \det(e)$. If the frame field is allowed to fluctuate, as supposed in quantum gravity, the sign of $\det(e)$ can continuously change from positive to negative or *vice versa*. Of course, $\det(e) = 0$ is a singularity where the curved space effectively loses one dimension but it is not possible to forbid such local happenings in the world with a fluctuating metric; see the illustration in Fig. 1, left. Moreover, if $\det(e)$ goes to zero linearly in some parameter t, it *has* to change sign by continuity; see Fig. 1, right. The same is true for any diffeomorphism-invariant action term.

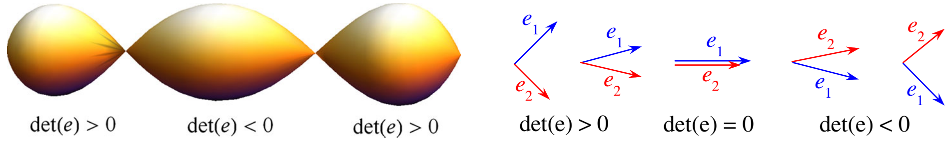

FIG. 1 (color online). Left: An example of a space with alternating sign of det(e); Right: det(e) changes sign by continuity of the frame field.

In the standard Riemannian formulation, one writes the invariant integration measure with the help of $\sqrt{\det(g)}$ where $g_{\mu\nu} = e^A_\mu e^A_\nu$ is the metric tensor, hence $\det(g) = (\det(e))^2$ is sign definite. Its square root, however, should be understood as $\sqrt{\det(g)} = \det(e)$ and can have any sign. If it passes through zero it changes sign by continuity [8].

This fundamental pathology of any diffeomorphism-invariant quantum theory has not been stressed before, probably for two reasons. First, one commonly deals with the perturbative quantization about a flat or, e.g., a de Sitter metric such that the main concern is the absence of runaway fluctuations from that point only. However, when quantizing gravity, one has to be concerned with large nonperturbative fluctuations as well. Second, usually Minkowski space-times are considered where the integration measure $\exp(i\mathrm{Action})$ is oscillating anyway independently of the action sign. However, a theory with a sign-indefinite action in Euclidian space where the weight is $\exp(-\mathrm{Action})$ is usually fundamentally sick also in Minkowski space. An illustration is provided by the scalar ϕ^3 theory; see Fig. 2. Perturbation theory exists there in the usual sense near $\phi = 0$. However, if in Euclidian space the theory does not exist, in Minkowski space one cannot define properly the nonperturbative Feynman propagator. There will be also other pathologies related to the possibility of tunneling to a bottomless state.

In gravity theory, the Euclidian formulation has its own right, for example, in problems related to thermodynamics and to tunneling, like in the Hawking radiation problem where paradoxes are encountered just because we do not know how to quantize Euclidian gravity. If a theory is well defined for a Euclidian signature, it is usually possible to Wick rotate it to the Minkowski world. Therefore, for clearness we shall discuss here Euclidian gravity.

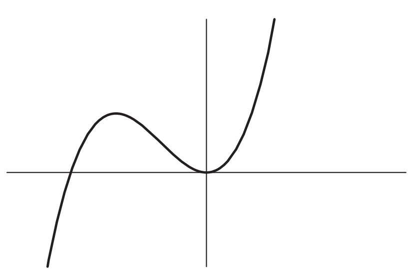

FIG. 2. The ϕ^3 theory is fundamentally sick both in Euclidean space where it is unbounded, and in Minkowski space where it can tunnel to a bottomless state.

Any diffeomorphism-invariant action, with any number of derivatives, is not sign definite in Euclidian space and hence cannot serve to define quantum gravity nonperturbatively.

At this time, we see only one way to overcome the sign problem, and that is to use in part fermionic variables in formulating quantum gravity microscopically, rather than only bosonic ones. Integrals over anticommuting Grassmann variables are well defined irrespectively of the overall sign in the exponent of a fermionic action. The reason is that in fermionic integrals introduced by Berezin [9] one actually picks up only certain finite order in the Taylor expansion of the exponent of the action, such that the overall sign does not matter. One calls it *spinor quantum gravity*. It has been advocated by Akama [10], Volovik [11], and recently in a series of papers by Wetterich [12–15] on other grounds.

More specifically, we suggest [7] (see also Ref. [16]) that at the fundamental, microscopic level gravity theory is defined as a theory of certain fundamental anticommuting spinor fields $\psi^\dagger$, ψ. We wish to preserve local gauge Lorentz symmetry exactly at all stages, and for that we need the explicit connection field ω_μ. The frame field e_μ and the metric tensor $g_{\mu\nu}$ will be composite fields making sense only at low energies. The basic independent variables will be $\psi^\dagger$, ψ and the gauge field ω_μ, the same type of degrees of freedom as in the Standard Model. We believe that using the same type of variables as in the Standard Model will help to unify all interactions [7]. As far as only gravity is concerned, the fundamental spinor fields $\psi^\dagger$, ψ may or may not be related to the fundamental matter fields. We introduce the main building blocks of the theory in Sec. II.

A quantum field theory is well defined if it is regularized in the ultraviolet. We shall regularize spinor quantum gravity by introducing simplicial lattice (made of triangles in 2 dimensions, tetrahedra in 3 dimensions, 5-cells or pentachorons in 4 dimensions, etc.) covering an abstract space, such that the simplex vertices are characterized and counted by integers i. Only the topology of this abstract number space matters, e.g., the number of nearest neighbors, etc.

Each vertex i in the number space corresponds to the real world coordinate by a certain map $x^\mu(i)$. Diffeomorphism invariance means that the theory should not depend on the coordinates $x^\mu(i)$ we ascribe to the vertices. Also, in the continuum limit (implying slowly varying fields) the action should be of the form $\int d^d x \mathcal{L}(x)$ and invariant under diffeomorphisms, $x^\mu \to x'^\mu(x)$. The integration measure over the fields in the path integral formulation should be also diffeomorphism invariant. In addition, we require exact gauge invariance under local Lorentz transformations. We build fermionic actions satisfying these conditions in Sec. III, and regularize them by putting on a lattice in Sec. IV.

After constructing a completely well-defined lattice-regularized quantum theory, the next question to address is whether the continuum limit can be achieved and whether it reduces to the Einstein-Cartan theory in the low-energy limit. The continuum limit is obtained when and if field correlations spread over a large distance in lattice units. The trouble is that the quantum theory one deals with in this approach is a typical strong-coupling theory where most of the correlations die out over a few lattice cells. Contrary to the standard lattice gauge theory where long-range correlations are ensured by simply taking the weak-coupling limit $\beta \to \infty$, in spinor quantum gravity there is no obvious handle to make the correlations long ranged.

The main trick and the invention of this paper is to ensure long-range correlations by adjusting the bare dimensionless coupling constants to the point (or line, or surface) where the theory undergoes a phase transition of the 2nd kind. At such point, all correlation functions become long ranged, and the Einstein theory will be guaranteed in the low-momenta limit by the inherent diffeomorphism invariance.

Second-order phase transitions occur in theories where there is an order parameter, usually related to the spontaneous breaking of a continuous symmetry. Our primary goal in this project is to demonstrate that second-order phase transitions are typical in the kind of diffeomorphism-invariant theories we consider. We develop in Sec. V an original mean-field method well suited for the search of the phase transitions, and check its accuracy in the Appendix where it is probed in an exactly solvable model, with very satisfactory results. Using this method, we unveil the phase diagram of a generic 2-dimensional lattice spinor gravity in the space of the bare coupling constants, Sec. VI. The model has two continuous symmetries: the $U(1)$ chiral symmetry and the $U(1)$ symmetry related to the fermion number conservation; both can be, in principle, spontaneously broken.

It turns out that there is a range of bare couplings where the fermionic lattice system experiences spontaneous breaking of chiral symmetry. In the particular model we studied we did not observe spontaneous breaking of the fermion number; however, it can happen in other models. This is an interesting finding *per se* but it may be also of use in the attempts to unify quantum gravity with the Standard Model. On this route, one expects one or several spontaneous breakups of continuous symmetries.

The 2-dimensional model we consider in some detail has certain nice features. First, the physical (invariant) volume $\langle V\rangle$ is extensive, i.e., proportional to the number of lattice points taken. This is not altogether trivial since nonperturbative metric fluctuations allow, in principle, "crumpling" of the space, and that is what some researchers indeed typically observe in alternative nonperturbative approaches to gravity. In spinor gravity, it is a natural result following from the noncompressibility of fermions. Second, the quantum average of the curvature turns out to be zero such that the empty space without sources is effectively flat. This is also a welcome feature since the natural result in nonperturbative gravity is that the curvature is of the order of the cutoff, that is, of the Planck mass, which is unacceptable. Third, despite flatness, the theory definitely describes a fluctuating *quantum* vacuum, as exemplified by the fact that the physical volume variance or susceptibility $\langle V^2\rangle - \langle V\rangle^2$ is nonzero.

As a result of the spontaneous breaking of continuous symmetry (here: chiral symmetry), a Goldstone field appears. We check by an explicit calculation in Sec. VII that the low-momentum effective ("chiral") Lagrangian for the Goldstone field is diffeomorphism invariant as expected. This invariance is rooted in the way we construct the original lattice action for spinors. The appearance of a Goldstone particle means that a definite bilinear combination of fermions is capable of propagating to large distances. However, this is not enough: in order for the system to totally lose memory about the original lattice, *all* degrees of freedom have to propagate to long distances in lattice units. This happens only at a phase transition where we expect that the Einstein-Hilbert action emerges as a low-energy effective action for the classical metric, with the cosmological constant being automatically zero; see Sec. VIII. In Sec. IX we discuss the dimensions of various quantities and fields used throughout the paper. We summarize in Sec. X.

II. COMPOSITE FRAME FIELDS

Following Ref. [7] we introduce a composite frame field e^A_μ built as a bilinear fermion "current." In d dimensions the frame field transforms as a vector of the $SO(d)$ Lorentz gauge group:

$$e^A_\mu(x) \xrightarrow{\text{Lorentz}} O^{AB}(x) e^B_\mu(x). \tag{1}$$

Since $A, B, \ldots = 1, \ldots, d$ are flat Euclidean indices, we can equivalently write them either as subscripts or superscripts. The frame field transforms also as a rank-one tensor (world vector) with respect to diffeomorphisms $x^\mu \to x'^\mu(x)$:

$$e^A_\mu(x) \xrightarrow{\text{diffeomorphism}} e^A_{\mu'}(x') \frac{\partial x'^{\mu'}}{\partial x^\mu}. \tag{2}$$

Our basic objects are fermion fields $\psi(x)$, $\psi^\dagger(x)$ assumed to be world scalars under diffeomorphisms, and transforming according to the spinor representation of the Lorentz group,

$$\psi(x) \to V(x)\psi(x), \quad \psi^\dagger(x) \to \psi^\dagger(x)V^\dagger(x), \quad V \in SO(d),$$
$$\psi, \psi^\dagger(x) \to \psi, \psi^\dagger(x'(x)). \tag{3}$$

The dimension of the spinor representation is $d_f = 2^{[d/2]}$, see, e.g., Ref. [17].

We introduce the covariant derivative in the spinor representation,

$$\nabla_\mu = \partial_\mu - \frac{i}{2}\omega_\mu^{AB}\Sigma_{AB}, \qquad \overleftarrow{\nabla}_\mu = \overleftarrow{\partial}_\mu + \frac{i}{2}\omega_\mu^{AB}\Sigma_{AB}, \tag{4}$$

where ω_μ^{AB} is the spin connection in the adjoint representation of the $SO(d)$ group, and Σ_{AB} are its $d_f \times d_f$ generators,

$$\Sigma_{AB} = \frac{i}{4}[\gamma_A\gamma_B], \tag{5}$$

built from Dirac matrices γ_A satisfying the Clifford algebra,

$$\{\gamma_A\gamma_B\} = 2\delta_{AB}\mathbf{1}_{d_f\times d_f}. \tag{6}$$

In the adjoint (antisymmetric tensor) representation the corresponding covariant derivative is

$$D_\mu^{AB} = \partial_\mu\delta^{AB} + \omega_\mu^{AB}. \tag{7}$$

Its commutator defines the curvature

$$[D_\mu, D_\nu]^{AB} = \mathcal{F}_{\mu\nu}^{AB}, \qquad [\nabla_\mu, \nabla_\nu] = -\frac{i}{2}\mathcal{F}_{\mu\nu}^{AB}\Sigma_{AB}, \tag{8}$$

where

$$\mathcal{F}_{\mu\nu}^{AB} = \partial_\mu\omega_\nu^{AB} + \omega_\mu^{AC}\omega_\nu^{CB} - (\mu \leftrightarrow \nu). \tag{9}$$

One can built two distinct bilinear combinations of the fermion fields, transforming as the frame field (1) and (2):

$$e_\mu^A = i(\psi^\dagger\gamma^A\nabla_\mu\psi + \psi^\dagger\overleftarrow{\nabla}_\mu\gamma^A\psi), \tag{10}$$

$$f_\mu^A = \psi^\dagger\gamma^A\nabla_\mu\psi - \psi^\dagger\overleftarrow{\nabla}_\mu\gamma^A\psi. \tag{11}$$

To check that e_μ^A and f_μ^A transform as a vector (1) one needs the relation between the matrix V rotating spinors (3) and the matrix O rotating vectors,

$$O^{AB} = \frac{1}{d_f}\,\mathrm{Tr}(V^\dagger\gamma^A V\gamma^B).$$

Given that ψ, $\psi^\dagger$ anticommute, the above bilinear operators are Hermitian.

We can define the bilinear fermion operator that plays the role of the torsion field, for example,

$$T_{\mu\nu}^A(e) \stackrel{d}{=} \frac{1}{2}(D_\mu^{AB}e_\nu^B - D_\nu^{AB}e_\mu^B) = \frac{i}{4}\mathcal{F}_{\mu\nu}^{AB}(\psi^\dagger\gamma_B\psi), \tag{12}$$

and similarly for the other composite frame field f_μ (11).

III. DIFFEOMORPHISM-INVARIANT ACTION TERMS

One can now construct a sequence of many-fermion actions that are invariant under local Lorentz transformations and also diffeomorphism invariant, using either e_μ^A or f_μ^A (or both) as building blocks:

$$S_k = \int d^dx \frac{1}{d!}\epsilon^{\mu_1\mu_2\ldots\mu_d}\epsilon^{A_1A_2\ldots A_d}(\mathcal{F}_{\mu_1\mu_2}^{A_1A_2}\cdots\mathcal{F}_{\mu_{2k-1}\mu_{2k}}^{A_{2k-1}A_{2k}})$$
$$\times (e_{\mu_{2k+1}}^{A_{2k+1}}\cdots e_{\mu_d}^{A_d}), \qquad k = 0, 1, \ldots, [d/2], \tag{13}$$

where $\epsilon^{\mu_1\mu_2\ldots\mu_d}$ is the totally antisymmetric (Levi-Civita) tensor. Notice that S_0 is the analog of the cosmological term but there are many of them since one can replace any number of e_μ^A's by f_μ^A's, S_1 is the analog of the Einstein-Hilbert-Cartan action linear in curvature, and the last action term $S_{[d/2]}$ for even d is a full derivative. Apart from full derivatives, there are 3 possible action terms in $2d$, 6 terms in $3d$, 8 terms in $4d$, 12 terms in $5d$, etc.

The use of $\epsilon^{\mu_1\mu_2\ldots\mu_d}$ is obligatory to support diffeomorphism invariance. In principle, one can construct Lorentz-invariant action terms by contracting the flat indices with Kronecker deltas instead of $\epsilon^{A_1\ldots A_d}$; however, that will make the action term P and T odd. For example, there is a well-known P, T-odd term in four dimensions, called sometimes the Holst action, $\epsilon^{\kappa\lambda\mu\nu}\mathcal{F}_{\kappa\lambda}^{AB}e_\mu^A e_\nu^B$ [18,19], but we do not consider such terms here.

One can add to the list of admissible action terms any of the actions (13) multiplied by any power of the world and Lorentz-group scalar $(\psi^\dagger\psi)$; we shall consider such kinds of terms later on in relation to the spontaneous breaking of chiral symmetry.

All action terms (13) are apparently invariant under two global $U(1)$ rotations:

(i) phase rotation related to the fermion number conservation, $\psi \to e^{i\alpha}\psi$, $\psi^\dagger \to \psi^\dagger e^{-i\alpha}$,

(ii) chiral rotation for even dimensions d, $\psi \to e^{i\beta\gamma_{d+1}}\psi$, $\psi^\dagger \to \psi^\dagger e^{i\beta\gamma_{d+1}}$, where $\gamma_{d+1} = i^{d/2}\gamma_1\gamma_2\ldots\gamma_d$, $\{\gamma_{d+1}\gamma_d\} = 0$, $\gamma_{d+1}^2 = \mathbf{1}$,

since both e_μ^A and f_μ^A are invariant under these transformations. The two corresponding Nöther currents are conserved. However, both symmetries can be spontaneously broken by interactions, and we shall see that this is what indeed typically happens.

IV. SPINOR GRAVITY ON THE LATTICE

In order to formulate quantum theory properly one has to regularize it at short distances. The most clear-cut regularization is by (lattice) discretization, however, diffeomorphism invariance imposes severe restrictions on it; see recent discussion by Wetterich [13–15]. We impose two basic requirements:

(i) explicit invariance under local gauge transformations of the Lorentz group, small or large (as in lattice gauge theory),

(ii) if the fields vary slowly in lattice units, i.e., in the continuum limit, the lattice action reduces to one of the diffeomorphism-invariant action terms (13) and the like.

A. Triangulation by simplices

To that end, we introduce an abstract discretized space where only the topology of vertices and edges connecting neighbor vertices is chosen beforehand and fixed. We find that the simplest hypercubic topology does not work. Only in two dimensions it is possible, for accidental reasons, to fulfill item 2 above by introducing a square lattice. In higher dimensions, the simplest but sufficient construction is to use a simplicial lattice. For uniformity, in two dimensions we also consider a triangle lattice made of three-vertex cells. In $3d$ simplices are tetrahedra or 4-cells, in $4d$ these are pentachorons or 5-cells, and so on.

It is always possible to cover the whole d-dimensional space by $(d+1)$-cells or simplices, although the number of edges entering one vertex may not be the same for all vertices. Alternatively, the number of edges coming from all vertices is the same but then the edges lengths may vary, if one attempts to force the lattice into flat space. Since only the topology of the nearest neighbors matters and the abstract "number" space does not need to be flat, this is also acceptable. The important thing is that the chosen set of cells should fill in the space without holes and without overlapping [20].

All vertices in a simplicial lattice can be characterized by a set of d integers. For brevity we label these d numbers by a single integer i. Each vertex has its unique integer label i, supplemented with a rule regarding which labels are ascribed to the neighbor vertices forming elementary cells. We shall denote the $d+1$ labels belonging to one cell by $i = 0, 1, \ldots, d$. In this section, we write down the full lattice action as a sum over actions for individual simplicial cells; therefore, we shall not be concerned with the precise geometric arrangement of the cells.

Each vertex in the abstract number space corresponds to the real world coordinate by a certain map $x^\mu(i)$. The goal is to write possible action terms in such a way that if the fields vary slowly from one vertex (or link) to the topological neighbor one, the action reduces to one of the possible diffeomorphism- and local Lorentz-invariant action term in Eq. (13).

We start by writing the volume of an elementary cell (simplex) in a given coordinate system in d dimensions. It can be presented as a determinant of a $d \times d$ matrix,

$$V_{\text{simplex}} = \frac{1}{d!} \det{}_{(\mu,i)}(x_i^\mu - x_0^\mu), \tag{14}$$

where x_0^μ is the coordinate ascribed to one of the vertices, and x_i^μ, $i = 1, \ldots, d$ are the coordinates ascribed to all the other vertices. We introduce the notion of a "positive order" of vertices i in the cell: it is such that for smooth functions x_i^μ the volume (14) is positive. An odd permutation of vertices in this set makes a "negative order."

It will be convenient to use the antisymmetric symbol

$$\epsilon^{i_0 i_1 \ldots i_d} = \begin{cases} 0 & \text{if } i_k \text{ does not belong to a given cell,} \\ 1 & \text{if the set } i_0, i_1 \ldots i_d \text{ is in the positive order,} \\ -1 & \text{if the set } i_0, i_1 \ldots i_d \text{ is in the negative order.} \end{cases} \tag{15}$$

With the help of this symbol the cell volume (14) can be written as

$$V_{\text{simplex}} = \frac{\epsilon^{i_0 i_1 i_2 \ldots i_d}}{(d+1)!} \frac{\epsilon_{\mu_1 \mu_2 \ldots \mu_d}}{d!} (x_{i_1}^{\mu_1} - x_{i_0}^{\mu_1})(x_{i_2}^{\mu_2} - x_{i_0}^{\mu_2}) \ldots \times (x_{i_d}^{\mu_d} - x_{i_0}^{\mu_d}). \tag{16}$$

B. Lattice action

The building blocks of our construction are anticommuting spinor fields ψ_i, $\psi_i^\dagger$ that are world scalars and "live" on lattice vertices i, and the parallel transporter U_{ij}. As in any lattice gauge theory, we replace the connection ω_μ by a unitary matrix "living" on lattice links [7],

$$U_{ij} = P\exp\left(-\frac{i}{2}\int_{x_i}^{x_j} \omega_\mu^{AB} \Sigma^{AB} dx^\mu\right), \quad U_{ji} = U_{ij}^\dagger. \tag{17}$$

In terms of these lattice variables, the discretized versions of the composite frame fields (10) and (11) are

$$\tilde{e}^{\,A}_{i,j} = i(\psi_j^\dagger U_{ji} \gamma^A U_{ij} \psi_j - \psi_i^\dagger \gamma^A \psi_i), \tag{18}$$

$$\tilde{f}^{\,A}_{i,j} = \psi_i^\dagger \gamma^A U_{ij} \psi_j - \psi_j^\dagger U_{ji} \gamma^A \psi_i. \tag{19}$$

The difference between $\tilde{e}$ and $\tilde{f}$ is that the first has both fermions in the same vertex whereas in the second, fermions are residing in the neighbor vertices.

Expanding all fields in Eqs. (18) and (19) around the center of a cell $x = \frac{1}{d+1}\sum_{i=0}^d x_i$ we obtain

$$\tilde{e}^{\,A}_{i,j} = (x_j^\mu - x_i^\mu) e_\mu^A(x) + \mathcal{O}(\Delta x^2), \tag{20}$$

$$\tilde{f}^{\,A}_{i,j} = (x_j^\mu - x_i^\mu) f_\mu^A(x) + \mathcal{O}(\Delta x^2), \tag{21}$$

where e_μ^A, f_μ^A are given by their continuum expressions (10) and (11), and the correction term is proportional to the derivatives of the fields and to the squares of the lengths of the cell edges. If the fields are slowly varying, meaning that the derivatives are small, the correction term can be neglected. This is what we mean by the continuum limit.

We also need the discretized version of the curvature tensor $\mathcal{F}^{AB}_{\mu\nu}$: it is a plaquette. In our case the plaquettes are triangles, and we define the parallel transporter along a closed triangle spanning the i, j, k vertices:

$$P_{ijk} = U_{ij}U_{jk}U_{ki}, \qquad P^{AB}_{ijk} = \frac{1}{d_f}\,\mathrm{Tr}(\Sigma^{AB}P_{ijk}). \tag{22}$$

Expanding P_{ijk} around the center of the cell x we obtain

$$P_{ijk} = 1 - \frac{i}{4}(x^\mu_j - x^\mu_i)(x^\nu_k - x^\nu_i)\mathcal{F}_{\mu\nu}(x) + \mathcal{O}(\Delta x^3),$$

and

$$P^{AB}_{ijk} = -\frac{i}{4}(x^\mu_j - x^\mu_i)(x^\nu_k - x^\nu_i)\mathcal{F}^{AB}_{\mu\nu}(x) + \mathcal{O}(\Delta x^3).$$

Using the above ingredients one can easily construct the lattice-regularized version of the action terms (13). For example, the discretized cosmological term S_0 has the form:

$$\tilde{S}_0 = \sum_{\text{all cells}} \frac{\epsilon^{i_0 i_1 \ldots i_d}}{(d+1)!}\frac{\epsilon^{A_1A_2\ldots A_d}}{d!}\tilde{e}^{A_1}_{i_0i_1}\tilde{e}^{A_2}_{i_0i_2}\ldots\tilde{e}^{A_d}_{i_0i_d}, \tag{23}$$

where any number of $\tilde{e}$'s can be replaced by $\tilde{f}$'s. In the continuum limit one uses Eqs. (20) and (21) and obtains

$$\tilde{S}_0 = \sum_{\text{all cells}} \frac{\epsilon^{i_0 i_1 \ldots i_d}}{(d+1)!}\frac{\epsilon^{\mu_1\mu_2\ldots\mu_d}}{d!}(x^{\mu_1}_{i_1} - x^{\mu_1}_{i_0})\ldots(x^{\mu_d}_{i_d} - x^{\mu_d}_{i_1}) \times \det(e)[1 + \mathcal{O}(\Delta x)]. \tag{24}$$

The coordinate factors combine into the volume of the cell (16) and one gets

$$\tilde{S}_0 = \sum_{\text{cells}} V(\text{cell})\det(e)[1 + \mathcal{O}(\Delta x)] \to \int d^dx\,\det(e) = S_0, \tag{25}$$

where $\det(e)$ is composed from the continuum tetrad (10) and is attributed to the center of a cell. Eq. (25) proves that the lattice action (23) becomes the needed continuum action (13) if the fields involved are slowly varying from one lattice vertex to the neighbor ones.

Similarly, one finds the lattice version of all other action terms S_k of Eq. (13):

$$\tilde{S}_k = (4i)^k \sum_{\text{cells}} \frac{\epsilon^{i_0 i_1 \ldots i_d}}{(d+1)!}\frac{\epsilon^{A_1A_2\ldots A_d}}{d!}(P^{A_1A_2}_{i_0i_1i_2}P^{A_3A_4}_{i_0i_3i_4}\ldots P^{A_{2k-1}A_{2k}}_{i_0i_{2k-1}i_{2k}}) \times (\tilde{e}^{A_{2k+1}}_{i_0i_{2k+1}}\ldots\tilde{e}^{A_d}_{i_0i_d}) \to S_k, \tag{26}$$

where the total number of plaquette factors P (22) is k, $k = 0, 1, \ldots, [d/2]$. In fact one can write a variety of such action terms replacing any number of composite frame fields $\tilde{e}$ (18) by the composite frame fields $\tilde{f}$ (19).

C. Lattice partition function

The lattice-regularized partition function for the spinor quantum gravity is quite similar to that of the common lattice gauge theory. One integrates with the Haar measure over link variables U_{ij} living on lattice edges, and over anticommuting fermion variables ψ_i, $\psi^\dagger_i$ living on lattice sites. The lattice, though, must be simplicial, otherwise the trick used, e.g., in Eq. (24) to get the diffeomorphism-invariant action in the continuum limit, would not work.

Because of the requirement of diffeomorphism invariance, the lattice action is quite different from those used in common lattice gauge theory. Typically one has many-fermion terms in the action. There are no action terms without fermions. One can write 3 action terms in $2d$ (all of them are four-fermion), 6 terms in $3d$ (four are six-fermion and two are two-fermion), 8 terms in $4d$ (five are eight-fermion and three are four-fermion), etc. We assume that spinor fields are dimensionless since we normalize the basic Berezin integrals as

$$\int d\psi\,\psi = 1, \qquad \int d\psi^\dagger\psi^\dagger = 1,$$
$$\int d\psi = 0, \qquad \int d\psi^\dagger = 0, \tag{27}$$

hence, all quantities in Eq. (26) are dimensionless. Therefore, the "coupling constants" λ_k one puts as arbitrary coefficients in front of the action terms $\tilde{S}_k$ (26) are all dimensionless.

The partition function is

$$Z = \prod_{\text{vertices } i}\int d\psi^\dagger_i d\psi_i \prod_{\text{links } ij}\int dU_{ij} \times \exp\Bigl(\sum_{\text{cells}} \lambda^{(m)}_k \tilde{S}^{(m)}_k(\psi^\dagger, \psi, U)\Bigr), \tag{28}$$

where $\tilde{S}^{(m)}_k$ are lattice actions of the type (26) with any number of composite frame fields $\tilde{e}$ (18) replaced by the other composite frame fields $\tilde{f}$ (19).

V. MEAN-FIELD APPROXIMATION

The partition function (28) defines a new type of a theory, and new methods—exact, numerical, and approximate—have to be developed.

In principle, in order to compute the partition function (28) as well as correlation functions, etc., one has to Taylor expand the exponent in Eq. (28) to certain powers of the fermionic action terms S_k such that at all lattice sites there is precisely the same number of fermion operators $\psi^\dagger$ and ψ as there are integrations, since all other contributions are identically zero by the Berezin integration rule (27) for anticommuting variables. The subsequent integration over link variables with the Haar measure is simple [7] since link matrices U_{ij} never appear in a large power. Moreover, the majority of potentially possible contributions are killed by link integration.

In practice, however, the arising combinatorial problem is tremendous, and we did not manage yet to find a computational algorithm that would be faster than the exponent of the lattice volume. So far we have done a toy model in $1d$ exactly (see the Appendix) and succeeded in computing numerically correlation functions in $2d$ for limited volumes. There is a hope that the $2d$ model may be solved exactly but the method can hardly be extended to higher dimensions.

Therefore, for this pilot study, we have developed an approximate mean-field method to get the first glance on the dynamics of the new interesting theory at hand. Comparing the results with an exactly solvable model we see that the mean-field accuracy is within a few percent. In the $2d$ model there are a few exact functional relations that are satisfied with the accuracy better than 15%, and this can be systematically improved. More important, the mean-field approximation reveals a nontrivial phase structure of the theory in the space of the coupling constants λ_k. This is the main finding of this study that may have important physical implications; see the Introduction.

The mean-field approximation we use is an extension of methods developed in condensed matter physics that go under the names "dynamical mean-field approach" or "local impurity self-consistent approximation" or "cavity method"; see Ref. [21] for a review. Roughly speaking, the idea of the method is the following: One first picks up a simple element of the lattice (e.g., one simplex or a group of simplices with or without the boundary, let us call this fixed element "the cavity"), and calculates the effective action for the fields inside the cavity in the collective background of the fields outside it, replacing the background by the supposed mean field. At the second stage, one makes the method self-consistent, namely, one calculates the mean field by integrating over the "live" variables inside the chosen cavity using the effective action found and expressed through the mean field at the first stage. As a result, one gets a system of highly nonlinear self-consistent equations for a set of mean values of the field operators. Solving those equations one obtains the mean-field values as function of the coupling constants $\lambda_k^{(m)}$. This gives the phase diagram of the theory in the space of the coupling constants.

The method has the advantage that it can be systematically improved by enlarging the chosen cavity. In the limit when the cavity covers the whole lattice, it is an exact calculation. Also, it is known that the accuracy of the mean-field method is better the more nearest neighbors there are [21]. In simplicial lattices, the number of neighbor cells is large, and the mean-field method becomes exact in the limit $d \to \infty$.

Let us formulate the method more mathematically. We choose the cavity, for example, the elementary simplex with the boundary. We label the "live" fields belonging to the cavity by $m, n, \ldots$, and the fields outside the cavity (that will be replaced by mean fields) by $i, j, \ldots$. The full partition function can be written symbolically as

$$Z = \int d\psi_m^\dagger d\psi_m dU_{mn} e^{S_{mn}} \int dU_{mi} e^{S_{mi}} \int d\psi_i^\dagger d\psi_i dU_{ij} e^{S_{ij}}, \tag{29}$$

where S_{mn} is the part of the action that contains only fields from the cavity, S_{ij} contains only fields from outside the cavity, and S_{mi} contains both. The link elements U_{mi} are connecting vertices from the cavity with their nearest neighbors outside.

The last integral in Eq. (29) is the full partition function with the cavity cut out. When the lattice volume goes to infinity, cutting out a finite cell does not change the averages of operators as compared to the averages computed on a full lattice; we denote them as

$$\langle\langle O \rangle\rangle = \frac{\int d\psi_i^\dagger d\psi_i dU_{ij} e^{S_{ij}} O(\psi_i^\dagger, \psi_i, U_{ij})}{\int d\psi_i^\dagger d\psi_i dU_{ij} e^{S_{ij}}}. \tag{30}$$

The integration over the links U_{mi} connecting the cavity with the outside neighborhood must be performed explicitly in Eq. (29). We expand $e^{S_{mi}}$ in powers of the mixed action; since S_{mi} is a fermion operator, the power series is finite. Integrating over U_{mi} splits all terms involved into a sum of products of operators composed of the cavity fields $O(\psi_m^\dagger, \psi_m, U_{mn})$ and those living outside the cavity $O'(\psi_i^\dagger, \psi_i, U_{ij})$:

$$\int dU_{mi} e^{S_{mi}} = 1 + \sum_p O_p(\psi_m^\dagger, \psi_m, U_{mn}) O'_p(\psi_i^\dagger, \psi_i, U_{ij}), \tag{31}$$

where the sum goes over various fermion operators labeled by p. Operators built from the cavity fields are left intact whereas the outside operators are replaced by the averages according to Eq. (30). We, thus, obtain the effective action for the fields inside the cavity:

$$e^{S_{\mathrm{eff},mn}} = e^{S_{mn}} \Big(1 + \sum_p O_p(\psi_m^\dagger, \psi_m, U_{mn}) \langle\langle O'_p \rangle\rangle \Big). \tag{32}$$

Finally, we make the calculation self-consistent by requesting that the operator averages $\langle O_p \rangle$ computed from the cavity fields alone with the effective action (32) coincide with the full ones $\langle\langle O_p \rangle\rangle$:

$$\langle O_p \rangle = \frac{\int d\psi_m^\dagger d\psi_m dU_{mn} e^{S_{\mathrm{eff},mn}} O_p(\psi_m^\dagger, \psi_m, U_{mn})}{\int d\psi_m^\dagger d\psi_m dU_{mn} e^{S_{\mathrm{eff},mn}}} = \langle\langle O_p \rangle\rangle. \tag{33}$$

Since S_{eff} depends on the averages $\langle\langle O_p \rangle\rangle$ the self-consistency equation, (33), is in fact a set of nonlinear equations on the mean values of the operators introduced in this derivation. Solving those equations, one finds the values of the average operators as function of the coupling constants of the theory.

Of special interest are the cases where certain operator averages (the "condensates") violate the continuous symmetries of the original theory. It signals the spontaneous breaking of symmetry and leads to a nontrivial phase diagram for the theory. In the next section we illustrate it in a general $2d$ model.

VI. TWO-DIMENSIONAL SPINOR GRAVITY

The partition function is defined by Eq. (28) where the action has, in general, three terms with three arbitrary coupling constants $\lambda_{1,2,3}$,

$$S = \int d^2x\left(\lambda_1 \det(e) + \lambda_2 \det(f) + \lambda_3 \frac{1}{2!}\epsilon^{AB}\epsilon^{\mu\nu}e^A_\mu f^B_\nu\right), \quad A, B = 1, 2. \tag{34}$$

The lattice-regularized version of it is, according to Eq. (23),

$$\tilde{S} = \sum_{\text{cells}} \frac{\epsilon^{ijk}}{3!}\frac{\epsilon^{AB}}{2!}(\lambda_1 \tilde{e}^A_{ij}\tilde{e}^B_{ik} + \lambda_2 \tilde{f}^A_{ij}\tilde{f}^B_{ik} + \lambda_3 \tilde{e}^A_{ij}\tilde{f}^B_{ik}), \tag{35}$$

where $i, j, k = 0, 1, 2$ label the vertices of a cell which in $2d$ is a triangle. Using integration by parts, the first term in (34) can be rewritten as $-\mathcal{F}^{12}_{12}(\psi^\dagger\psi)^2$. It gives an alternative discretization for the same continuum action:

$$\tilde{S} = \sum_{\text{cells}} \frac{\epsilon^{ijk}}{3!}\frac{\epsilon^{AB}}{2!} \times\left(-\frac{i}{3}\lambda_1 P^{AB}_{ijk}(\psi^\dagger_i\psi_i)^2 + \lambda_2 \tilde{f}^A_{ij}\tilde{f}^B_{ik} + \lambda_3 \tilde{e}^A_{ij}\tilde{f}^B_{ik}\right). \tag{36}$$

Although in the continuum limit the lattice actions (35) and (36) differ by a full derivative, the lattice mean-field approximation gives numerically slightly different results depending on whether we start from Eq. (35) or from Eq. (36). The deviation serves as one of the checks of the accuracy of the approximation, and we find it consistent with other accuracy checks.

In $d = 2$ the Lorentz group is the Abelian $SO(2) \simeq U(1)$ group. The spinors are two component, and the γ matrices are the Pauli matrices $\gamma^A = \sigma^A$, $A = 1, 2$. The Lorentz rotations generator is $\Sigma^{12} = -\sigma^3/2$; see Eq. (5). The analog of the "gamma-five" matrix in $2d$ is $\gamma^3 = i\gamma^1\gamma^2 = -\sigma^3$.

Both variants of the frame field e^A_μ and f^A_μ as well as their lattice extensions, $\tilde{e}^A_{ij}$ and $\tilde{f}^A_{ij}$, are invariant under two global $U(1)_V \times U(1)_A$ transformations:

$$\text{vector transformation}: \psi \to e^{i\frac{\beta}{2}}\psi, \quad \psi^\dagger \to \psi^\dagger e^{-i\frac{\beta}{2}}, \tag{37}$$

$$\text{axial transformation}: \psi \to e^{i\frac{\alpha}{2}\sigma^3}\psi, \quad \psi^\dagger \to \psi^\dagger e^{i\frac{\alpha}{2}\sigma^3}. \tag{38}$$

Therefore, both the continuum (34) and the lattice (35) actions possess these two global symmetries also; the corresponding Nöther currents are conserved. The vector symmetry means that the fermion number is conserved whereas the axial means that the difference between the numbers of "left-handed" and "right-handed" fermions (described by the upper and lower components of the spinors, respectively) is also conserved. It is also called the helicity conservation, or chiral symmetry.

A. Exact results

In the $2d$ partition function (28), there are four integrals per site over fermion variables ψ^1, ψ^2, $\psi^\dagger_1$, $\psi^\dagger_2$, and one integration per link over the Abelian matrix $U_{ij} = \exp(-i\frac{\omega_{ij}}{4}\sigma^3)$. Berezin's integrals over fermions (27) are nonzero only when every lattice site takes exactly four fermion operators from the action exponent. Meanwhile, each term in the action (35) or (36) is four-fermion. From counting the number of fermion fields coming from the action (which must be equal to the number of integrations) we conclude that the partition function Z is a homogenous polynomial of the coupling constants $\lambda_{1,2,3}$ of order N, where N is the total number of sites in the lattice,

$$Z = \lambda_1^N F\left(\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1}\right) \to \begin{cases} C_1\lambda_1^N & \lambda_{2,3} \to 0 \\ C_2\lambda_2^N & \lambda_{1,3} \to 0 \\ C_3\lambda_3^N & \lambda_{1,2} \to 0. \end{cases} \tag{39}$$

Since there are two types of frame fields, e and f, we can define three types of "physical" or invariant volumes of the generally curved space, averaged over quantum fluctuations of the composite frame fields,

$$\langle V_1\rangle \stackrel{d}{=} \left\langle \int d^2x \det(e)\right\rangle = \frac{1}{Z}\frac{\partial Z}{\partial\lambda_1} = \frac{\partial \log Z}{\partial\lambda_1}, \tag{40}$$

$$\langle V_2\rangle \stackrel{d}{=} \left\langle \int d^2x \det(f)\right\rangle = \frac{1}{Z}\frac{\partial Z}{\partial\lambda_2} = \frac{\partial \log Z}{\partial\lambda_2}, \tag{41}$$

$$\langle V_3\rangle \stackrel{d}{=} \left\langle \int d^2x \frac{1}{2!}\epsilon^{AB}\epsilon^{\mu\nu}e^A_\mu f^B_\nu\right\rangle = \frac{1}{Z}\frac{\partial Z}{\partial\lambda_3} = \frac{\partial \log Z}{\partial\lambda_3}. \tag{42}$$

The immediate conclusion from Eqs. (39)–(42) is that the average action is

$$\langle S\rangle = \lambda_1\langle V_1\rangle + \lambda_2\langle V_2\rangle + \lambda_3\langle V_3\rangle = N = \frac{M}{2}, \tag{43}$$

irrespective of the coupling constants, where M is the number of triangle cells, which is twice the number of vertices N for large simplicial lattices.

Further on, one can introduce "physical volume susceptibility" or variance

$$\langle \Delta V_1^2\rangle \stackrel{d}{=} \langle (V_1 - \langle V_1\rangle)^2\rangle = \langle V_1^2\rangle - \langle V_1\rangle^2 = \frac{\partial^2 \log Z}{\partial\lambda_1^2}, \tag{44}$$

$$\langle \Delta V_2^2 \rangle \stackrel{d}{=} \langle (V_2 - \langle V_2 \rangle)^2 \rangle = \langle V_2^2 \rangle - \langle V_2 \rangle^2 = \frac{\partial^2 \log Z}{\partial \lambda_2^2}, \quad (45)$$

$$\langle \Delta V_3^2 \rangle \stackrel{d}{=} \langle (V_3 - \langle V_3 \rangle)^2 \rangle = \langle V_3^2 \rangle - \langle V_3 \rangle^2 = \frac{\partial^2 \log Z}{\partial \lambda_3^2}. \quad (46)$$

Therefore, from Eq. (39) we know exactly the average physical volumes and volume susceptibilities at least at the edges of the parameter space $\Lambda =^d (\lambda_1, \lambda_2, \lambda_3)$:

$$\langle V_1 \rangle_{\lambda_{2,3}\to 0} = \frac{M}{2\lambda_1}, \quad \langle V_2 \rangle_{\lambda_{1,3}\to 0} = \frac{M}{2\lambda_2}, \quad \langle V_3 \rangle_{\lambda_{1,2}\to 0} = \frac{M}{2\lambda_3}, \quad (47)$$

$$\langle \Delta V_1^2 \rangle_{\lambda_{2,3}\to 0} = -\frac{M}{2\lambda_1^2}, \qquad \langle \Delta V_2^2 \rangle_{\lambda_{1,3}\to 0} = -\frac{M}{2\lambda_2^2},$$
$$\langle \Delta V_3^2 \rangle_{\lambda_{1,2}\to 0} = -\frac{M}{2\lambda_3^2}, \quad (48)$$

where $M = 2N$ is the total number of simplicial cells in the lattice. The proportionality of these quantities to M is a very general property (valid not only at the edges of the parameter space but everywhere) following from Eq. (39). It shows that the physical volume is an extensive quantity, as it should be. This is not altogether trivial since nonperturbative metric fluctuations allow, in principle, "crumpling" of the space, or the formation of "branched polymers," and that is what some researchers observe in alternative nonperturbative approaches to gravity. In spinor gravity, it is a natural result following physically from the noncompressibility of fermions and mathematically expressed by Eq. (39).

The susceptibilities (48) are also extensive, as should be expected. In the classical ground state there are no quantum fluctuations, so $\Delta V = 0$. The fact that (48) is nonzero means that we are dealing with a fluctuating quantum vacuum. At the same time for large volumes, the relative strength of the fluctuations die out: $\sqrt{\Delta V^2}/V \sim 1/\sqrt{M} \to 0$.

There are theorems for mixed derivatives, valid in the whole parameter space Λ, that can be used to check the accuracy of approximate calculations, for example,

$$\frac{\partial \langle \int d^2x \det(e) \rangle}{\partial \lambda_2} = \frac{\partial \log Z}{\partial \lambda_1 \partial \lambda_2} = \frac{\partial \langle \int d^2x \det(f) \rangle}{\partial \lambda_1}. \quad (49)$$

Finally, there is an exact statement about the average curvature. The number of link variables in all terms of the action (35) is even. That gives a nonzero result from integration over links for the partition function. However, if one attempts to compute the average of the curvature proportional to $\mathcal{F}$ that in the lattice formulation is given by a product of three links [see Eq. (22)], the number of link variables becomes odd, and link integration yields an identical zero. Therefore, we conclude that the average Cartan curvature proportional to the average scalar curvature is zero,

$$\langle \det(e) R \rangle = 2\langle \mathcal{F}_{12}^{12} \rangle = 0. \quad (50)$$

This result in $2d$ is, of course, in conformity with the zero Euler characteristic of a torus; no other result could be correct. It is illuminating, however, to see how "microscopically" the Euler theorem works for fluctuating spaces. In higher dimensions $\det(e)R$ is not a full derivative but it still may be possible to find its average in a similar way.

At the same time, owing to quantum fluctuations the average curvature squared is generally nonzero and extensive,

$$\left\langle \left(\int d^2x \det(e) R \right)^2 \right\rangle \sim M, \quad (51)$$

implying that the volume-independent combination dies out in the thermodynamic limit,

$$\frac{\sqrt{\langle (\int d^2x \det(e) R)^2 \rangle}}{\langle \int d^2x \det(e) \rangle} \sim \frac{1}{\sqrt{M}} \to 0. \quad (52)$$

Equations (47), (50), and (52) mean that although we apparently deal with a quantum fluctuating vacuum, the space is on the average large and flat in the absence of external sources. Therefore, one can say that the model describes a flat background metric $G_{\mu\nu}$ that is a unity matrix in a particular frame representing the flat space but transforms as a tensor under the change of coordinates. We shall use this notion in Sec. VII.

B. Mean-field approximation for one simplex cavity

In this subsection we apply the mean-field method formulated in Sec. V to the lattice action (36) where we first put for simplicity $\lambda_3 = 0$. At the end of this section we formulate the main results for $\lambda_3 \neq 0$.

In the first approximation to the mean-field method, we choose the elementary triangle cell (m, n, p) as the "cavity"; see Fig. 3. The fields inside the triangle cavity are considered as real quantum fields, whereas the fields outside the cavity are combined into certain gauge-invariant operators that are frozen to their mean-field

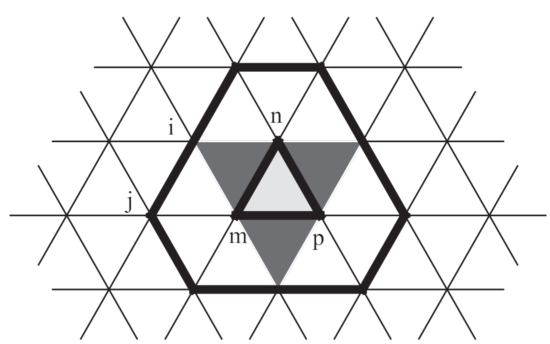

FIG. 3. The simplest triangle cavity (m, n, p) and its neighbors used in the mean-field calculation.

values. The triangle cavity is surrounded by three black triangles of the type (i, m, n) with a common edge, and by nine white triangles of the type (i, j, m) with a common vertex. The effective action for the fields inside the cavity gets contributions from both types of neighbors.

Following the method of Sec. V, we expand the action exponent for every border cell, and integrate over the link variables U_{mi} connecting the cavity with the outer lattice. As a result, we obtain the product of operators built of the fields inside the cavity and those built of the outside fields. The latter operators are replaced by the averages to be found later from the self-consistency condition. We stress that, after integrating over U_{im}, the operators on both sides can be only gauge invariant.

For the single cell cavity we obtain operators of two types: single-site operators (they arise from the white cells), and double-site operators built from fermions at adjacent vertices (they arise from black cells).

Here is the list of operators that appear in this calculation. First of all, there are operators that are invariant under the $U(1)_V \times U(1)_A$ transformations described in Eqs. (37) and (38)

$$\begin{aligned} O_1(i) &= (\psi_i^\dagger \psi_i)^2, \\ O_2(i,j) &= (\psi_i^\dagger U_{ij} \psi_j)^2 + (\psi_j^\dagger U_{ji} \psi_i)^2 \\ &\quad + (\psi_i^\dagger \psi_i)(\psi_j^\dagger \psi_j) - (\psi_i^\dagger \sigma^3 \psi_i)(\psi_j^\dagger \sigma^3 \psi_j), \\ O_3(i,j) &= (\psi_i^\dagger \psi_i)^2 (\psi_j^\dagger \psi_j)^2. \end{aligned} \tag{53}$$

O_1 is a single-site operator while O_2 and O_3 are double-site operators.

To be able to study the potential breaking of the $U(1)_V \times U(1)_A$ symmetries we introduce operators that transform under those rotations. The chiral noninvariant operators that transform under $U(1)_A$ (38) are

$$\begin{aligned} &C_1(i) = i(\psi_i^\dagger \psi_i), \quad \bar{C}_1(i) = i(\psi_i^\dagger \sigma^3 \psi_i,), \\ &C_2(i,j) = i[(\psi_i^\dagger \psi_i)^2 (\psi_j^\dagger \psi_j) + (\psi_j^\dagger \psi_j)^2 (\psi_i^\dagger \psi_i)], \\ &\bar{C}_2(i,j) = i[(\psi_i^\dagger \psi_i)^2 (\psi_j^\dagger \sigma^3 \psi_j) + (\psi_j^\dagger \psi_j)^2 (\psi_i^\dagger \sigma^3 \psi_i)]. \end{aligned} \tag{54}$$

Fermion number violating operators transforming under $U(1)_V$ (37) are

$$\begin{aligned} W_1(i) &= \psi_{i,1} \psi_{i,2}, \\ W_2(i,j) &= (\psi_i^\dagger \psi_i)^2 \psi_j^1 \psi_j^2 + (\psi_j^\dagger \psi_j)^2 \psi_i^1 \psi_i^2. \end{aligned} \tag{55}$$

All operators are Hermitian.

The effective action for the fields inside the triangle cavity is computed as described in Sec. V. From the black neighbors we obtain the double-site effective action

$$\begin{aligned} e^{S(m,n)} &= 1 + \frac{8}{9} \lambda_1^2 [(O_1(m) + O_1(n)) \langle O_1 \rangle + O_3] \\ &\quad + \frac{\lambda_2^2}{36} [O_2 \langle O_1 \rangle - 2(W_2^\dagger \langle W_1 \rangle + W_2 \langle W_1^\dagger \rangle) \\ &\quad - C_2 \langle C_1 \rangle + \bar{C}_2 \langle \bar{C}_1 \rangle], \end{aligned} \tag{56}$$

where all double-site operators refer to the cavity vertices m and n. From the white neighbors we obtain the one-site effective action

$$\begin{aligned} e^{S(m)} &= 1 + \frac{8}{9} \lambda_1^2 [2 O_1 \langle O_1 \rangle + \langle O_3 \rangle] + \frac{\lambda_2^2}{36} [O_1 \langle O_2 \rangle \\ &\quad - 2(W_1^\dagger \langle W_2 \rangle + W_1 \langle W_2^\dagger \rangle) - C_1 \langle C_2 \rangle + \bar{C}_1 \langle \bar{C}_2 \rangle]. \end{aligned} \tag{57}$$

Actually, the operator averages in Eqs. (56) and (57) imply averaging over the cyclic permutation of lattice sites in the cavity: e.g., $\langle O_1 \rangle = \frac{1}{3} \langle O_1(m) + O_1(n) + O_1(p) \rangle$, and similarly for the double-site operators. The full effective action for the cavity is a sum over all 12 neighbor cells,

$$\begin{aligned} e^{S_{\rm eff}} &= \exp[S(m,n,p) + (S(m,n) + S(n,p) \\ &\quad + S(p,m)) + 3(S(m) + S(n) + S(p))], \end{aligned} \tag{58}$$

where $S(m, n, p)$ is the original action for the cavity triangle (m, n, p), as given by Eq. (36).

We see that the effective action for the fields living in the cavity cell depend explicitly on the yet unknown operator averages $\langle O \rangle$, $\langle C \rangle$, $\langle W \rangle$. To find them, one equates the operator averages as defined by the effective action (58) to those introduced previously; see Eq. (33). As a result, one obtains a system of nonlinear self-consistency equations on the averages $\langle O \rangle$, $\langle C \rangle$, $\langle W \rangle$. Solving those equations, one finds the averages as function of the coupling constants $\lambda_{1,2}$.

This calculation is straightforward but the equations are rather lengthy. Therefore, we just comment here on its most important features.

First of all, we notice that $S_{\rm eff}$ is quadratic in the symmetry breaking operators $C_{1,2}$ and $W_{1,2}$, as it should be; therefore, one gets a system of linear homogeneous self-consistency equations on the averages $\langle C_{1,2} \rangle$, $\langle W_{1,2} \rangle$ that always have a zero solution, unless the determinant of this set of linear equations is zero. If the determinant is nonzero in the whole range of the parameter space Λ, there is no spontaneous symmetry breaking. If the determinant passes through zero at some surface in the Λ space, it is where the second order phase transition takes place. Inside the domain where one of the $U(1)$ symmetries is spontaneously broken, the condensates $\langle C_{1,2} \rangle$ or $\langle W_{1,2} \rangle$ are nonzero and are found as anomalous solutions of the nonlinear equations, together with the symmetry-preserving averages $\langle O_{1,2,3} \rangle$.

In the absence of symmetry breaking one puts $\langle C_{1,2}\rangle = \langle W_{1,2}\rangle = 0$ and solves the system of three nonlinear equations on the averages $\langle O_{1,2,3}\rangle$. There are, in general, several solutions but none are real in the whole Λ space. We pick up the solution that is real near the line $\lambda_2 = 0$. However, it develops a cut and becomes complex at the lines $|\lambda_2| = 8.69|\lambda_1|$ signalling that there can be a phase transition along these lines. A careful study in the next subsection shows that, indeed, these are the border lines separating the phase with spontaneous chiral symmetry breaking; see Fig. 4.

Outside this domain, i.e., at $|\lambda_2| < 8.69|\lambda_1|$, chiral symmetry is not broken, the solution for the normal, symmetry-preserving operators is real, and one can approach the line $\lambda_2 = 0$ where we can check the accuracy of the mean-field method by comparing the average physical volume $\langle V_1\rangle \approx \sum_{\rm cells}\langle \det(\tilde{e})\rangle$ where the average is computed over one (cavity) cell with the effective action (58), with the exact result (47). We find numerically

$$\langle V_1\rangle_{\lambda_{2,3}\to 0} = 0.572\frac{M}{\lambda_1}\ \text{(mean field)}\ \text{vs}\ 0.5\frac{M}{\lambda_1}\ \text{(exact)}, \tag{59}$$

where M is the total number of lattice cells. We note that the functional dependence on λ_1 is correct whereas the numerical coefficient deviates from the exact one by 15%. A more powerful check comes from computing the average action $\langle S\rangle$ which turns out to be a constant up to the third digit in the whole range of analyticity in $\lambda_{1,2,3}$, equal to 0.57, instead of the exact result (43) being 0.5. This is the typical accuracy with which other checks with exact results are fulfilled.

We have also tested a more primitive mean-field approximation where the cavity is taken in the form of two neighbor vertices connected by a link. It is also capable of detecting the spontaneous breaking of chiral symmetry but the accuracy is, of course, worse: it is at a level of 40%.

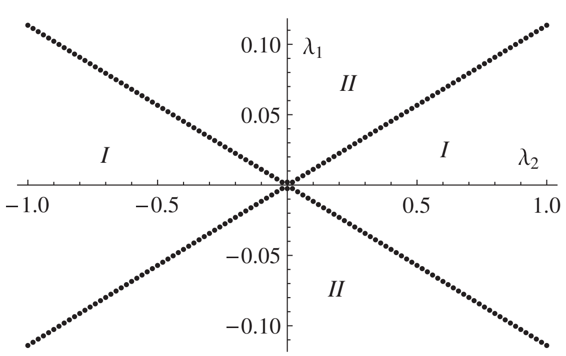

FIG. 4. The phase diagram of the $2d$ spinor gravity in the (λ_1, λ_2) plane at $\lambda_3 = 0$. Region I corresponds to the chiral-symmetry broken phase; region II is a regular phase. The dots show the lines of the 2nd phase transition: $|\lambda_2| \simeq 8.69|\lambda_1|$.

C. Spontaneous chiral-symmetry breaking

An accurate way to study spontaneous breaking of a continuous symmetry is to introduce a small term in the action that violates the symmetry in question explicitly. Since we are interested in the spontaneous breaking of the $U(1)_A$ or chiral symmetry we introduce the simplest diffeomorphism-invariant "mass term"

$$S_{\chi\text{-odd}} = \int d^2x \det(\tilde{e}) im\psi^\dagger\psi, \tag{60}$$

that is not invariant under chiral rotations (38). Its discretized lattice version is obvious; see the first term in Eq. (35).

Adding this term we repeat the same mean-field derivation of the effective action for the triangle cavity as in Eqs. (56) and (57) which now obtain an additional

$$S_{\chi\text{-odd}}(m,n) = \frac{\lambda_2 m}{27}(O_3\langle C_1\rangle + C_2\langle O_1\rangle) - \frac{m^2}{54}O_3\langle O_1\rangle, \tag{61}$$

$$S_{\chi\text{-odd}}(m) = \frac{\lambda_2 m}{27}(C_1\langle O_3\rangle + O_1\langle C_2\rangle) - \frac{m^2}{54}O_1\langle O_3\rangle. \tag{62}$$

Let us note that the terms linear in the mass parameter m are also linear in the chirality-odd operators $C_{1,2}$.

With this addition to the previous effective action (58), we now turn to solving the self-consistency equations for the operator averages $\langle O_{1,2,3}\rangle$ and $\langle C_{1,2}\rangle$, $\langle \bar{C}_{1,2}\rangle$. At $m \neq 0$ there is a solution for the chiral condensates $\langle C_{1,2}\rangle$ in the whole (λ_1, λ_2) plane (we still keep for simplicity $\lambda_3 = 0$). However, the dependence of the chiral condensates on the mass parameter m is totally different depending on whether we are in region I where chiral symmetry is broken, or in region II where it is preserved.

In region II the dependence of the chiral condensates on the mass is linear at small m; if m goes to zero the chiral condensates vanish. In region I the dependence of the chiral condensates on the small parameter m that breaks

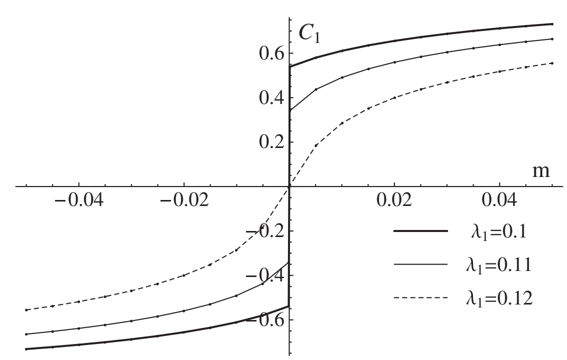

FIG. 5. The dependence of the chiral condensate $\langle C_1\rangle$ on the mass parameter m with the varying value of λ_1 at fixed value of λ_2. The steplike behavior $\sim$sign(m) signals the spontaneous breaking of symmetry. In this example, $\lambda_1 = 0.12$ is the 2nd order phase transition point where the chiral condensate abruptly vanishes when $m = 0$.

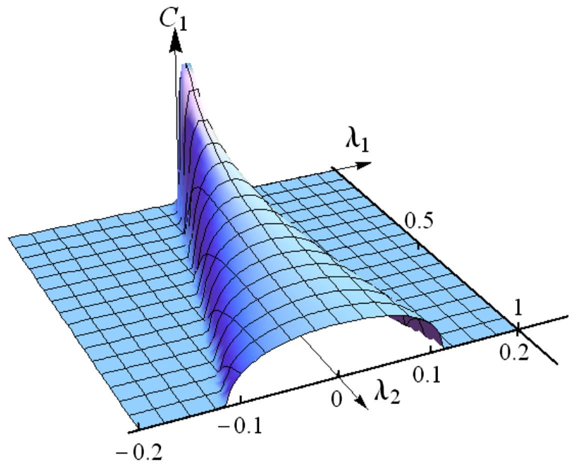

FIG. 6 (color online). The value of the chiral condensate $\langle C_1\rangle$ in the (λ_1, λ_2) plane. At the phase transition line $|\lambda_2| = 8.69|\lambda_1|$ the condensate vanishes with an infinite first derivative.

the symmetry explicitly is *nonanalytic*. Actually the chiral condensates are proportional to the sign functions of m, $\langle C_{1,2}\rangle \sim \mathrm{sign}(m)$; see Fig. 5. The behavior of the chiral condensate $\langle C_{1,2}\rangle$ in the whole (λ_1, λ_2) plane (at $\lambda_3 = 0$) is shown in Fig. 6.

Figures 5 and 6 clearly demonstrate that there is a range of coupling constants where the theory undergoes spontaneous breaking of the continuous $U(1)_A$ chiral symmetry, with a line of the 2nd order phase transition separating the phases.

D. No fermion number violation

The effective action (58) contain operators $W_{1,2}$ violating the $U(1)_V$ symmetry (37) related to the fermion number conservation. If $\langle W_{1,2}\rangle \neq 0$, it signals the spontaneous violation of this symmetry. Fermion number conservation is spontaneously broken, e.g., in ordinary superconductors and "color" superconductors in QCD. However, in contrast to chiral symmetry that is also broken in QCD, spontaneous fermion condensation usually happens not in the vacuum but at nonzero chemical potential for fermions, since interactions are effectively amplified near the Fermi surface. In this subsection we look for the spontaneous fermion number nonconservation in the same $2d$ model where we observe spontaneous breaking of the $U(1)_A$ symmetry in the mean-field approximation.

Following the same logic as in the previous subsection, we introduce a small action term that violates the $U(1)_V$ symmetry explicitly,

$$S_{\text{B-odd}} = \int d^2x \det(\tilde{e}) b\psi^1\psi^2.$$

This operator preserves the chiral $U(1)_A$ symmetry. The correction to the effective one-triangle action is

$$S_{\text{B-odd}}(m, n) = \frac{\lambda_2 b}{27}(W_2\langle O_1\rangle + O_3\langle W_1\rangle),$$
$$S_{\text{B-odd}}(m) = \frac{\lambda_2 b}{27}(O_1\langle W_2\rangle + W_1\langle O_3\rangle).$$

We solve again the self-consistency equations on the operator averages but now with this addition, and look for nonanalytic dependence on the small parameter b. In contrast to the case of spontaneous chiral symmetry breaking, we do not find such solutions in the whole parameter space $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$.

We conclude that the fermion number conservation is not broken spontaneously in the model, except maybe along the line of the chiral phase transition. We did not study the inclusion of a chemical potential for fundamental fermions—that would explicitly violate Lorentz symmetry but presumably make the phase diagram of the model more rich.

There are no reasons why fermion number conservation would not break spontaneously, say, in $4d$, and the mean-field method suggested here is a simple way to detect it.

E. Full phase diagram

The full action compatible with the principles proclaimed has, in $2d$, three terms and consequently three coupling constants. In the previous subsections we have restricted our study to the case of $\lambda_3 = 0$.

Actually, we repeat all the steps described above also for $\lambda_3 \neq 0$. The algebra becomes more cumbersome but still doable. We find that the chiral symmetry breaking phase I occupies the cone

$$\lambda_2^2 < 77.23\lambda_1^2 + 5.36\lambda_3^2, \tag{63}$$

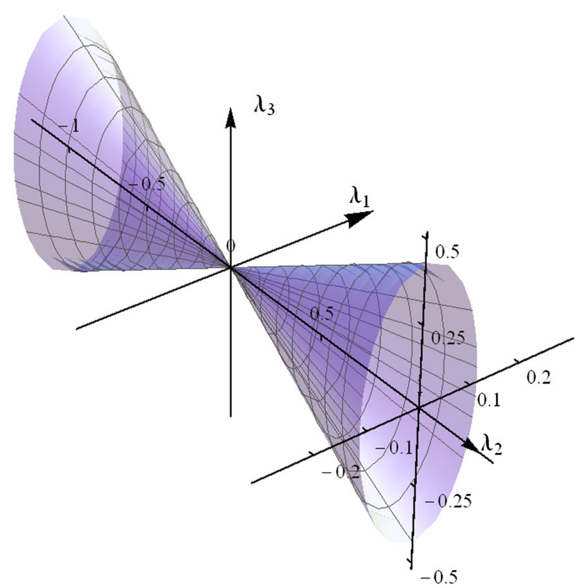

FIG. 7 (color online). Spontaneous chiral symmetry breaking takes place inside a cone in the full parameter space $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$.

shown in Fig. 7; Fig. 4 is its section at $\lambda_3 = 0$. We remark that the accuracy of the mean-field approximation for some reason deteriorates as λ_3 grows. Still the exact relation (43) holds even at $\lambda_3 \to \infty$ up to a factor of 1.6.

VII. LOW-ENERGY ACTION FOR PROPAGATING FIELDS

The theory defined by the partition function (28) is in fact ultralocal: all correlation functions of gauge-invariant operators generally decay exponentially at the separation of a few lattice cells. This is clear on general grounds but we have also checked it by numerical simulations on a $2d$ lattice of limited volumes. Special measures should be taken to ensure that certain degrees of freedom propagate to distances that are large in lattice units. The situation here is different from the common lattice gauge theory where it is sufficient to take the limit $\beta \to \infty$ where β is the inverse gauge coupling to guarantee long-range correlations in lattice units. In our theory, there is no such obvious handle.

However, there are ways to *guarantee* that long-range correlations appear; moreover, that can be checked in the mean-field approximation. An example which we consider here is provided by the Goldstone theorem: If a global continuous symmetry is broken spontaneously, the associated Goldstone bosons are exactly massless and hence propagate to large distances.

In the previous section we have shown that the continuous $U(1)_A$ or chiral symmetry is spontaneously broken in a broad range of the space of the coupling constants. Supposing the coupling constants are chosen inside that range (inside the cone in Fig. 7), there is a massless Goldstone excitation $\alpha(x)$ which is the phase of the $U(1)_A$ rotation (38).

Under this rotation, the chirality-violating operators $C_{1,2}$ and $\bar{C}_{1,2}$ transform as

$$\begin{aligned} C_1^\pm &= C_1 \pm \bar{C}_1 \to e^{\pm i\alpha} C_1^\pm, \\ C_2^\pm &= C_2 \pm \bar{C}_2 \to e^{\pm i\alpha} C_2^\pm. \end{aligned} \tag{64}$$

To derive the low-energy action for the Goldstone field we allow the phase α to vary slowly from cell to cell:

$$\langle C_{1,2}^\pm \rangle = \rho_{1,2} e^{\pm i\alpha(\text{cell})}. \tag{65}$$

We parametrize the operator averages $\langle C_{1,2}^\pm \rangle$ in the same way and rederive the effective action (58) for the fields inside the triangle cavity, taking now into account that the operator averages have slightly different phases in the cells surrounding the cavity. Then, integrating over the fields inside the cavity we find the effective one-cell partition function Z_1 modified by the varying nearest neighborhood. If α is the same for all neighboring cells, it is the same expression as in Sec. V; let us call it Z_{10}. However, there will be further terms depending on the gradients of $\alpha(x)$, we are now after.

The full partition function is, in the mean-field approximation, a product of Z_1's over all cells whose number is M. Therefore, the action for the Goldstone field $\alpha(x)$ is

$$\begin{aligned} S_G &= -M \ln Z_1 \\ &= -M \ln Z_{10} - M\bigg[\frac{1}{Z_{10}} \frac{\partial Z_1}{\partial \alpha_i} \Delta\alpha_i + \frac{1}{2Z_{10}} \bigg(\frac{\partial^2 Z_1}{\partial\alpha_i \partial\alpha_j} \\ &\quad - \frac{1}{Z_{10}} \frac{\partial Z_1}{\partial\alpha_i} \frac{\partial Z_1}{\partial\alpha_j}\bigg) \Delta\alpha_i \Delta\alpha_j + \mathcal{O}(\Delta\alpha^3)\bigg], \end{aligned} \tag{66}$$

where α_i is the value of the phase attributed to one of the 12 neighbor cells i, and $\Delta\alpha_i$ is the difference between α_i and α_0 attributed to the central cavity cell; the summation goes over all neighbor cells. It is important that the dependence of Z_1 on α_i starts from quadratic terms, which is the consequence of chiral symmetry; hence, $\partial Z_1/\partial\alpha_i = 0$, and we are left with second derivatives.

Ignoring the first α-independent term in Eq. (66), we find that the action is quadratic in the jumps $\Delta\alpha$ from one cell to the neighbor ones,

$$S_G = -M \frac{1}{2Z_{10}} \frac{\partial^2 Z_1}{\partial\alpha_i \partial\alpha_j} \Delta\alpha_i \Delta\alpha_j + \mathcal{O}(\Delta\alpha^3). \tag{67}$$

We now introduce a coordinate system by mapping the centers of the cells to coordinates $x^\mu(i)$ (Sec. IV). If the changes of α from a cell to neighbor cells are small we can expand

$$\Delta\alpha_i = \partial_\mu \alpha \Delta x_i^\mu + \frac{1}{2} \partial_\mu \partial_\nu \alpha \Delta x_i^\mu \Delta x_i^\nu + \ldots, \tag{68}$$

where $\Delta x_i^\mu = x^\mu(i) - x^\mu(0)$ is the distance between the coordinate attributed to the cell i and that attributed to the cavity cell, in a given coordinate frame $x^\mu(i)$. Putting this expansion into Eq. (67) we obtain

$$S_G = -M \frac{1}{2Z_{10}} \frac{\partial^2 Z_1}{\partial\alpha_i \partial\alpha_j} \Delta x_i^\mu \Delta x_j^\nu \partial_\mu \alpha \partial_\nu \alpha + \mathcal{O}(\Delta x^3). \tag{69}$$

The first factor M, the full number of cells on the lattice can be written as

$$M = \sum_{\text{cells}} = \int \frac{d^2x}{V(\text{cell})}, \tag{70}$$

where $V(\text{cell})$ is the cell volume in a given frame; see Eq. (14). The combination

$$\lim_{\Delta x \to 0} \frac{1}{V(\text{cell})} \frac{1}{Z_{10}} \frac{\partial^2 Z_1}{\partial\alpha_i \partial\alpha_j} \Delta x_i^\mu \Delta x_j^\nu \overset{d}{=} -\sqrt{G} G^{\mu\nu}, \tag{71}$$

transforms under the change of the map $x^\mu \to x'^\mu(x)$ as a product of the contravariant tensor times the square root of the determinant of a covariant tensor, hence the notations in the right-hand side or Eq. (71). Its particular form depends, of course, on the coordinate system chosen. For a concrete map to the Cartesian coordinates of the lattice drawn in Fig. 3 we find that it is proportional to a unity tensor,

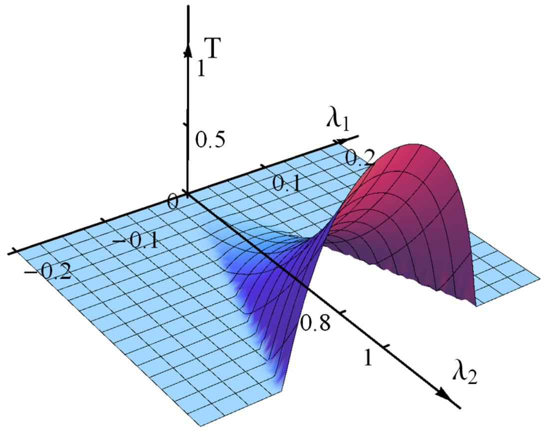

FIG. 8 (color online). The normalization factor $T(\lambda_1, \lambda_2, 0)$ in the low-energy effective chiral Lagrangian, Eq. (72).

$$\sqrt{G}G^{\mu\nu}|_{\text{regular lattice}} = T(\lambda_{1,2,3})\delta^{\mu\nu}, \tag{72}$$

where the proportionality coefficient $T(\lambda_{1,2,3})$ is shown in Fig. 8; it is proportional to a combination of the moduli of the chiral condensates $\rho_{1,2}$; see Eq. (65). This result is in conformity with the average flatness of the space found in Sec. VI A. If one chooses another coordinate map $\sqrt{G}G^{\mu\nu}$ changes accordingly.

We thus arrive at a diffeomorphism-invariant low-energy action for the massless Goldstone field:

$$S_G = \frac{1}{2}\int d^2x\sqrt{G}G^{\mu\nu}\partial_\mu\alpha\partial_\nu\alpha. \tag{73}$$

This field can propagate infinitely far in lattice units since its masslessness is guaranteed by the Goldstone theorem.

To complete our study of the spontaneous chiral symmetry breaking, we derive the analog of the Gell-Mann–Oakes–Renner relation for the pion mass in QCD, expressed through the quark masses. If chiral symmetry is broken explicitly by a small fermion mass term (60) the phase of the chiral condensate becomes a pseudo-Goldstone field with a mass proportional to the *square root* of the fermion mass m.

Indeed, the addition of the mass term (60) changes the effective one-cavity partition function:

$$Z_1 \to Z_1 + mZ_m + \mathcal{O}(m^2). \tag{74}$$

The Z_m piece depends explicitly on the chiral condensate phase α introduced in Eq. (65), and from symmetry considerations it is clear that the expansion starts from the α^2 term; direct calculation confirms it.

Summing up the mean-field action over the whole lattice, one uses the relation (70) where the right-hand side behaves as $\sim\int d^2x\sqrt{G}$ according to the transformation properties under the change of the coordinate system $x^\mu(i)$ attributed to the lattice. We obtain thus the action for the pseudo-Goldstone mode in the continuum limit

$$S_G = \frac{1}{2}\int d^2x\sqrt{G}(G^{\mu\nu}\partial_\mu\alpha\partial_\nu\alpha + \mu^2\alpha^2) + \mathcal{O}(\partial^2\alpha\partial^2\alpha) + \mathcal{O}(\alpha^4), \qquad \mu^2 \sim m, \tag{75}$$

where μ is proportional to the pseudo-Goldstone boson mass. We see that it is proportional to the *square root* of the mass parameter m that breaks chiral symmetry explicitly. In QCD, this is known as the Gell–Mann–Oakes–Renner relation for the pion mass. The coefficient in this relation depends on the coupling constants $\lambda_{1,2,3}$. At the 2nd order phase transition surface of the cone in Fig. 7 the pseudo-Goldstone mass goes to zero at fixed m.

There is a famous Mermin-Wagner theorem stating that a continuous symmetry cannot be spontaneously broken in $2d$ as the resulting Goldstone bosons would have an unacceptably large, actually divergent free energy. Since the mean-field approximation misses the Goldstone physics, one can argue that the spontaneous chiral symmetry breaking we observe is an artifact of the approximation. If, however, the Goldstone field $\alpha(x)$ is Abelian as here, the actual phase is, most likely, that of Berezinsky-Kosterlitz-Thouless where the chiral condensate $\rho e^{i\alpha}$ indeed vanishes owing to the violent fluctuations of $\alpha(x)$ defined on a circle $(0, 2\pi)$, but the correlation functions of the type $\langle e^{i\alpha(x)} e^{-i\alpha(y)}\rangle$ have a powerlike behavior, and there is a phase transition depending on the original couplings of the theory.

In any case, our primary goal here is to learn how to deal with the lattice-regularized spinor quantum gravity which is a new type of a theory. The mean-field approximation is one possible approach that is expected to work even better in higher dimensions where, as a matter of fact, the Mermin-Wagner theorem does not apply.

VIII. HOW TO OBTAIN EINSTEIN'S LIMIT?

The apparent diffeomorphism-invariance of Eq. (75) is built in by our construction of the lattice and lattice action in Sec. IV. As soon as there are degrees of freedom that can propagate to long distances, their low-energy effective action is diffeomorphism-invariant in the continuum limit.

In the previous Section the appearance of long-propagating mode has been guaranteed by the Goldstone theorem. However, it concerns only the specific Goldstone modes associated with the spontaneous breaking of continuous symmetry. Other degrees of freedom remain heavy: their correlation functions decay exponentially after a few lattice cells. If one attempts to write an effective low-energy action for the classical metric tensor $g^{\text{cl}}_{\mu\nu}$ (see below its exact definition) it will have the diffeomorphism-invariant form,

$$S_{\text{low}} = \int dx\sqrt{g^{\text{cl}}}(-c_1 + c_2 R(g^{\text{cl}}) + \ldots), \tag{76}$$

with the constants $c_{1,2}$ computable, in principle, from the original coupling constants of the lattice-regularized theory. However, if one does not take special measures, the ratio $\sqrt{c_1/c_2}$, playing the role of the graviton mass, will be on the order of the inverse lattice spacing. In such a situation, it is senseless to introduce the metrics in the first place. It makes sense only if $\sqrt{c_1/c_2}$ happens to be zero or very small, such that the graviton and the Newton force propagates to large distances.

To ensure it, it is sufficient to stay, e.g., at the phase transition of the second order, where all degrees of freedom become massless. The classical metric tensor $g^{\rm cl}_{\mu\nu}$ and the effective action functional $\Gamma[g^{\rm cl}_{\mu\nu}]$ can be introduced by means of the Legendre transform (proposed in this context also by Wetterich [14]). One introduces first the generating functional for the stress-energy tensor $\Theta^{\mu\nu}$ as an external source,

$$e^{W[\Theta]} = \int d\psi^\dagger d\psi d\omega_\mu \exp\left(S + \int \hat{g}_{\mu\nu}\Theta^{\mu\nu}\right), \tag{77}$$

where S the fermionic action and $\hat{g}_{\mu\nu}$ is a four-fermion operator built from the frame fields (10) and (11) or, after discretization, from their lattice versions (18) and (19). The classic metric field is by definition

$$g^{\rm cl}_{\mu\nu} \stackrel{d}{=} \langle \hat{g}_{\mu\nu}\rangle = \frac{\delta W[\Theta]}{\delta \Theta^{\mu\nu}}. \tag{78}$$

This equation can be solved to give the functional $\Theta^{\mu\nu}[g^{\rm cl}]$. Using it one can construct the effective action as the Legendre transform:

$$\Gamma[g^{\rm cl}] = W[\Theta] - g^{\rm cl}_{\mu\nu}\Theta^{\mu\nu}. \tag{79}$$

At the phase transition fluctuations are long ranged. For long-range fluctuations it is legal to take the continuum limit of the lattice, which is diffeomorphism invariant. The low-energy limit of diffeomorphism-invariant actions for a quantity transforming as a metric tensor is uniquely given by Eq. (76). Moreover, the cosmological term necessarily has a zero coefficient, $c_1 = 0$, since otherwise the graviton would propagate to a finite distance $\sqrt{c_2/c_1}$, which contradicts the masslessness of the fluctuations at the phase transition. This is how one can recover Einstein's gravity from the lattice-regularized spinor theory.

In principle, the effective Einstein-Hilbert action from spinor quantum gravity can be derived in the mean-field approximation similarly to our derivation of the low-energy effective chiral Lagrangian in Sec. VII. However, in $2d$, where we have so far succeeded in developing the mean-field method, the Einstein-Hilbert action is a full derivative and there are no gravitons or the Newton force. Therefore, the derivation of Eq. (76) has to be postponed until higher dimensions are studied along the lines of the present paper.

IX. DIMENSIONS

In this paper, we use unconventional dimensions of the fields, which, however, we believe are natural and adequate for a microscopic theory of quantum gravity. The fermion fields are normalized by the Berezin integral (27) and are dimensionless; hence, the composite frame field (10) has the dimension 1/length and the metric tensor has the dimension $1/\text{length}^2$, in contrast to the conventional dimensionless metric tensor. On the other hand, all diffeomorphism-invariant quantities are dimensionless in our approach. In this section we explain why it is convenient, and what is the relation to the usual approach.

The historic tradition in general relativity is that the space-time at infinity is flat, and therefore one can safely choose the coordinate system such that $g_{\mu\nu}$ is a unity matrix there. This sets the traditional dimensions of the fields. In particular, the scalar curvature has the dimension $1/\text{length}^2$, the fermion fields have the dimension $1/\text{length}^{\frac{3}{2}}$, etc. However, in a diffeomorphism-invariant quantum theory where one can perform arbitrary change of coordinates, $x^\mu \to x'^\mu(x)$ not necessarily identical at infinity, for example, a dilatation $x^\mu \to x^\mu/\rho$, and where $g_{\mu\nu}$ can *a priori* strongly fluctuate at infinity, this convention is not convenient.

The natural dimensions of the fields are those that are in accordance with their transformation properties: any contravariant vector transforms as x^μ and has the dimension of length, a covariant vector, in particular, the frame field e_μ transforms as a derivative and has the dimension 1/length, $g_{\mu\nu}$ has the dimension $1/\text{length}^2$, etc. World scalars like the scalar curvature and the fermion fields are, naturally, dimensionless. In fact it is a tautology: a quantity invariant under diffeomorphisms is, in particular, invariant under dilatations and hence has to be dimensionless.

In this convention, any diffeomorphism-invariant action term is, by construction, dimensionless and is accompanied by a dimensionless coupling constant, as in Eq. (28).

Let us suppose that we have a microscopic quantum gravity theory at hand that successfully generates the first terms in the derivative expansion of the effective action,

$$\Gamma = -c_1 \int d^4x\sqrt{g} + c_2 \int d^4x\sqrt{g}R + \ldots, \tag{80}$$

where $c_{1,2}$ are certain dimensionless constants expressed through the dimensionless couplings $\lambda_{1,2,\ldots}$ of the original microscopic theory. The ground state of the action (1) is the space with constant curvature $R = 2c_1/c_2$, represented, e.g., by a conformal-flat metric

$$g_{\mu\nu} = \frac{6c_2}{c_1}\left(\frac{2\rho}{((x-x_0)^2+\rho^2)}\right)^2 \delta_{\mu\nu}, \tag{81}$$

where x_0 and ρ are arbitrary. At the vicinity of some observation point x_0, it can be made a unity matrix by rescaling the metric tensor,

$$g_{\mu\nu} = m^2 \bar{g}_{\mu\nu}, \qquad \bar{g}_{\mu\nu} = \delta_{\mu\nu}, \qquad m = \sqrt{\frac{6c_2}{c_1}}\frac{2}{\rho}, \tag{82}$$

where the rescaling factor m has the dimension of mass, and $\bar{g}_{\mu\nu}$ has the conventional zero dimension. At this point one can rescale other fields to conventional dimensions, in particular, and introduce the new fermion field $\bar{\psi}$ of conventional dimension $m^{3/2}$:

$$\psi = m^{-3/2}\bar{\psi}, \qquad \psi^\dagger = m^{-3/2}\bar{\psi}^\dagger. \tag{83}$$

The new composite dimensionless tetrad field compatible both with Eqs. (82) and (83) is

$$\bar{e}^A_\mu = \frac{1}{m} e^A_\mu = \frac{1}{m^4} i(\bar{\psi}^\dagger \gamma^A \nabla_\mu \bar{\psi} + \bar{\psi}^\dagger \overleftarrow{\nabla}_\mu \gamma^A \bar{\psi}). \tag{84}$$

One can now rewrite the action (80) together with the fermionic matter in terms of the new rescaled fields denoted by a bar,

$$S = -\underbrace{c_1 m^4}_{2\Lambda=\lambda^4} \int d^4x\sqrt{\bar{g}} + \underbrace{c_2 m^2}_{M_P^2 = 1/\sqrt{16\pi G_N}} \int d^4x\sqrt{\bar{g}}\bar{R} + m^0 \int d^4x\sqrt{\bar{g}}\bar{e}^{A\mu}(\bar{\psi}^\dagger\gamma^A\nabla_\mu\bar{\psi} + \text{H.c.}). \tag{85}$$

Underbraced are the cosmological constant and the Plank mass squared, respectively; numerically, $\lambda = 2.39\times 10^{-3}$ eV, $M_P = 1.72\times 10^{18}$ GeV. The dimensionless ratio of these values,

$$\frac{\lambda}{M_P} = \left(\frac{c_1 m^4}{c_2^2 m^4}\right)^{\frac{1}{4}} = \left(\frac{c_1}{c_2^2}\right)^{\frac{1}{4}} = 1.39\times 10^{-30}, \tag{86}$$

is the only meaningful quantity in pure gravity theory, independent of the arbitrary scale parameter m. If a fermion obtains an effective mass, e.g., as a result of the spontaneous chiral symmetry breaking, leading to an additional term in the effective low-energy action,

$$S_m = \int d^4x\sqrt{g}\psi^\dagger\mathcal{M}\psi = \underbrace{m\mathcal{M}}_{\text{fermion mass } m_f} \int d^4x\sqrt{\bar{g}}\bar{\psi}^\dagger\bar{\psi}, \tag{87}$$

then the "theory of everything" has to predict also other dimensionless ratios. For example, taking the top quark mass $m_t = 172$ GeV one has to be able to explain the ratio

$$\frac{m_t}{\sqrt{\lambda M_P}} = \frac{\mathcal{M}}{c_1^{\frac{1}{4}} c_2^{\frac{1}{2}}} = 0.0848. \tag{88}$$

In other words, one can measure the Newton constant (or the Planck mass) or the cosmological constant in units of the quark or lepton masses or the Bohr radius. Only dimensionless ratios make sense and can be, as a matter of principle, calculated from a microscopic theory. To that end it is convenient and legitimate to use natural dimensions when $g_{\mu\nu}$ has the dimension $1/\text{length}^2$ whereas all world scalars are dimensionless, be it the scalar curvature R, the interval ds, the fermion field ψ, or any diffeomorphism-invariant action term.

X. CONCLUSIONS

We have formulated a lattice-regularized spinor quantum gravity that is well defined and well behaved both for large-amplitude and high-frequency fluctuations. In any number of dimensions one can construct a variety of fermionic actions that are invariant (i) under local Lorentz transformations and (ii) under diffeomorphisms in the continuum limit. We have built quite a few action terms satisfying (i) and (ii) for any number of dimensions. In fact our list of possible fermionic action terms can be expanded further if some of the additional requirements are relaxed. Therefore, we actually formulate a whole class of new kinds of theories in any number of dimensions, characterized by a set of dimensionless coupling constants $\lambda_{1,2,\ldots}$.

The continuum limit shows up if all degrees of freedom, or at least some of them, are slowly varying fields from one lattice cell to another. This is, generally, not fulfilled, generically, all correlation functions decay exponentially over a few lattice cells. For such "massive" degrees of freedom the theory is at the "strong coupling" regime where the continuum limit is not achieved and remains dormant.

There must be special physical reasons for massless excitations in the theory, for which the continuum limit makes sense and diffeomorphism-invariance becomes manifest. One such reason is spontaneous breaking of continuous symmetry where the existence of massless fields is guaranteed by the Goldstone theorem. To show that spontaneous breaking may be typical in such kinds of theories, we have developed a new mean-field approximation. We have checked its accuracy in a $1d$ exactly solvable model, and in a full $2d$ theory where certain exact relations can be derived. The exact relations tell us nice things: the physical or invariant volume occupied by the system is extensive as due to the noncompressibility of fermions, the volume variance (or susceptibility) is also extensive showing that it is a true quantum vacuum, and the average curvature is, at least in $2d$, zero meaning that the quantum space is on the average flat. They also tell us that our mean-field approximation is rather accurate, and the accuracy can be systematically improved.

We show, within the mean-field method, that the spontaneous breaking of chiral symmetry happens in a broad range of the coupling constants and that in this range the low-energy action for the Goldstone field (or pseudo-Goldstone if we add a term explicitly breaking symmetry) is diffeomorphism invariant, as expected.

To obtain the low-energy Einstein limit, one has to stay at the second-order phase transition surface in the space of the coupling constants. There the masslessness of excitations, and not only of the Goldstone ones, is guaranteed. Hence, one can go to the continuum limit where the diffeomorphism invariance is also guaranteed by construction. Therefore, we expect that the effective low-energy action for the classical metric tensor, derived through the Legendre transform, is just the Einstein-Hilbert action, with the zero cosmological term. This can be probably seen already in the mean-field approach for dimensions higher than two. This work is in progress.

ACKNOWLEDGMENTS

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APPENDIX A: MEAN-FIELD METHOD IN A 1*d* MODEL

We consider here a $1d$ toy model with fermions and $U(1)$ gauge symmetry, that can be solved exactly. We then apply the mean-field method to this model to check how accurately it reproduces the exact solution. We obtain quite satisfactory results.

We take the same fields as in the full $2d$ model, namely the doublet of fermion fields ψ^α, $\psi^\dagger_\alpha$, $\alpha = 1, 2$, that transform under the $U(1)$ gauge transformation as $\psi \to V\psi$, $\psi^\dagger \to \psi^\dagger V^\dagger$, $V = \exp(i\alpha\sigma_3)$. The gauge field is represented by link variables $U_{ij} = \exp(-i\omega_{ij}\sigma^3/4)$ that transform as $U_{ij} \to V_i U_{ij} V_j^\dagger$.

We construct the lattice version of the two "frame" fields, as in Eqs. (18) and (19),

$$\tilde{e}^A_{i,j} = i(\psi^\dagger_j U_{ji}\sigma^A U_{ij}\psi_j - \psi^\dagger_i \sigma^A \psi_i), \quad A = 1, 2,$$
$$\tilde{f}^A_{i,j} = \psi^\dagger_i \sigma^A U_{ij}\psi_j - \psi^\dagger_j U_{ji}\sigma^A\psi_i, \tag{A1}$$

(which, however, do not have the meaning of frame fields in $1d$), and form the action that is quite similar to the full $2d$ action (36):

$$S = \sum_{i=1}^N \Big[\frac{\lambda_1}{8}(\tilde{e}^A_{i,i+1}\tilde{e}^A_{i,i+1} + \tilde{e}^A_{i+1,i}\tilde{e}^A_{i+1,i}) + \frac{\lambda_2}{4}\tilde{f}^A_{i,i+1}\tilde{f}^A_{i,i+1} + 2\mu(\psi^\dagger_i\psi_i)^2(\psi^\dagger_{i+1}\psi_{i+1})^2\Big], \tag{A2}$$

where $\lambda_{1,2}$ and μ are the coupling constants. The partition function is defined as a product of Berezin integrals on the $1d$ lattice with N points:

$$Z = \prod_{i=1}^N \int d\psi^1_i d\psi^2_i d\psi^\dagger_{i1} d\psi^\dagger_{i2} dU_{i,i+1} e^S. \tag{A3}$$

We imply antiperiodic boundary conditions for fermion fields and periodic boundary conditions for link variables.

The partition function (A3) is *exactly computable* by a kind of transfer-matrix method. Diagonalizing the transfer matrix we obtain a nontrivial result:

$$Z = 2(1 + (-1)^N)\lambda_2^N + (\lambda_1 - \sqrt{2}\sqrt{\lambda_1^2 + \lambda_2^2 + \mu})^N + (\lambda_1 + \sqrt{2}\sqrt{\lambda_1^2 + \lambda_2^2 + \mu})^N. \tag{A4}$$

The fact that the partition function has the form of a sum of extensive exponents means that actually it describes simultaneously four independent phases or states of the system that do not compete and hence do not mix up in the thermodynamic limit $N \to \infty$. The stable phase is the one with the lowest free energy, that is, with the largest partition function.

Depending on the relation between the coupling constants, one of the terms in Eq. (A4) prevails at $N \to \infty$:

(i) Phase 0: $|\lambda_2| > |\lambda_1|$ and $-(\lambda_1^2 + \lambda_2^2) < \mu < -\frac{1}{2}\times(\lambda_1 \pm \lambda_2)^2$; the partition function is given by the first term,

(ii) Phase 1: $\mu > -\frac{1}{2}(\lambda_1 + \lambda_2)^2$, $\lambda_1 < 0$; the partition function is given by the second term,

(iii) Phase 2: $\mu > -\frac{1}{2}(\lambda_1 + \lambda_2)^2$, $\lambda_1 > 0$; the partition function is given by the third term,

(iv) Phase 3: $\mu < -(\lambda_1^2 + \lambda_2^2)$, $|\lambda_1| > |\lambda_2|$; the partition function is complex and has no smooth thermodynamic limit.

In phase 0 fermions in the neighbor lattice sites form an ordered state of the type "pair-gap-pair-gap..." all over the lattice, where the "pair" means that there are four link matrices in the integration over link variables, and "gap" means zero matrices. It can be realized only on even-N lattices; hence, it is a lattice artifact, and we do not consider it further. Phases 1 and 2 are states where two link matrices appear in all link integrations. We concentrate of phases 1 and 2 only in what follows.

We also calculate exactly average values of the following four-fermion operators:

$$\langle\tilde{e}^A_{ij}\tilde{e}^A_{ij}\rangle = \begin{cases} 4\dfrac{\lambda_1\sqrt{\lambda_1^2+\lambda_2^2+\mu}-\sqrt{2}(\lambda_2^2+\mu)}{\sqrt{\lambda_1^2\lambda_2^2+\mu(2\lambda_2^2+\lambda_1^2+2\mu)}} & \text{in phase 1} \\ 4\dfrac{\lambda_1\sqrt{\lambda_1^2+\lambda_2^2+\mu}+\sqrt{2}(\lambda_2^2+\mu)}{\sqrt{\lambda_1^2\lambda_2^2+\mu(2\lambda_2^2+\lambda_1^2+2\mu)}} & \text{in phase 2,} \end{cases} \tag{A5}$$

$$\langle f^A_{ij}\tilde{f}^A_{ij}\rangle = \begin{cases} 4\dfrac{\sqrt{2}\lambda_2(\lambda_1+\sqrt{2}\sqrt{\lambda_1^2+\lambda_2^2+\mu})}{\sqrt{\lambda_1^2\lambda_2^2+\mu(2\lambda_2^2+\lambda_1^2+2\mu)}} & \text{in phase 1} \\ 4\dfrac{\sqrt{2}\lambda_2(-\lambda_1+\sqrt{2}\sqrt{\lambda_1^2+\lambda_2^2+\mu})}{\sqrt{\lambda_1^2\lambda_2^2+\mu(2\lambda_2^2+\lambda_1^2+2\mu)}} & \text{in phase 2,} \end{cases} \tag{A6}$$

$$\langle O\rangle \overset{d}{=} \langle(\psi^\dagger\psi)^2\rangle = \begin{cases} \dfrac{-1}{2\sqrt{2}\sqrt{\lambda_1^2+\lambda_2^2+\mu}} & \text{in phase 1} \\ \dfrac{1}{2\sqrt{2}\sqrt{\lambda_1^2+\lambda_2^2+\mu}} & \text{in phase 2.} \end{cases} \tag{A7}$$

We now turn to constructing the mean-field approximation to the model, to check its accuracy against the exact calculation. We apply the general method of Sec. V, which

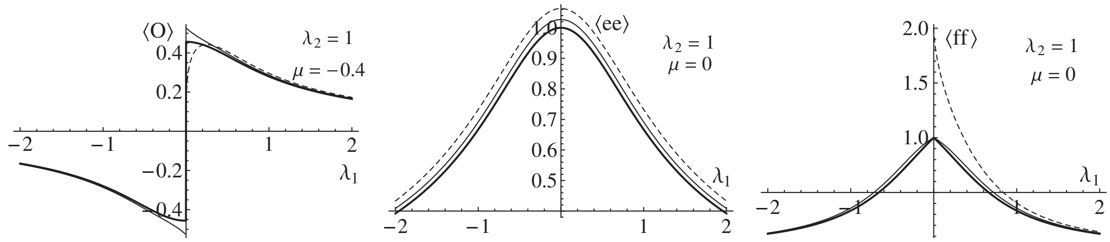

FIG. 9. Averages of three operators in the $1d$ model. The bold line is the exact result, the dashed line is the result of the 1st mean-field approximation using one lattice segment, and the solid line is the result of the 2nd (two-segment) mean-field approximation.

is rather straightforward in this simple case. We first take the "cavity" in the form of two neighbor lattice sites connected by a link (1st approximation), and then three adjacent sites connected by two links (2nd approximation). Both mean-field approximations give satisfactory accuracy when compared to the exact results but the second is, of course, better.

In both cases, the cavity boundary is just the neighbor sites connected to the cavity by link variables U_{mi}. Expanding $e^{S_{mi}}$ up to the second power (higher powers are zero because of too many fermion operators) and integrating over U_{mi}, we obtain several operator structures. Splitting them into the product of operators composed of fields inside the chosen cavity, and the operator built of the outside fields, we replace the latter by the averages, to be found self-consistently. Most of the operators break the σ^3 symmetry of the original action, and we ignore them. The only operator with proper symmetries left is $O = (\psi_i^\dagger \psi_i)^2$. We find the effective action for the two-site cavity

$$e^{S(m)} = 1 + \lambda_1(\langle O\rangle + (\psi_m^\dagger \psi_m)^2) + 2(\lambda_1^2 + \lambda_2^2 + \mu)\langle O\rangle \times (\psi_m^\dagger \psi_m)^2. \tag{A8}$$

To obtain the self-consistency equation we equate the average of O found from the effective action (A8), to $\langle O\rangle$. The resulting nonlinear equation on $\langle O\rangle$ has three solutions. We choose the solutions $\langle O\rangle(\lambda_1, \lambda_2, \mu)$ that are real in the ranges 1 and 2 above.

The results for the averages of the three operators in the first and second approximations as well as their exact values (A5)–(A7) are presented in Fig. 9. There is a "phase transition" between phases 1 and 2 at $\lambda_1 = 0$. We see that the second mean-field approximation corresponding to a three-site, two-segment cavity gives a very satisfactory accuracy.

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