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Cosmological inflation and the quantum measurement problem

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According to cosmological inflation, the inhomogeneities in our Universe are of quantum-mechanical origin. This scenario is phenomenologically very appealing as it solves the puzzles of the standard hot big bang model and naturally explains why the spectrum of cosmological perturbations is almost scale invariant. It is also an ideal playground to discuss deep questions among which is the quantum measurement problem in a cosmological context. Although the large squeezing of the quantum state of the perturbations and the phenomenon of decoherence explain many aspects of the quantum-to-classical transition, it remains to understand how a specific outcome can be produced in the early Universe, in the absence of any observer. The continuous spontaneous localization (CSL) approach to quantum mechanics attempts to solve the quantum measurement question in a general context. In this framework, the wave function collapse is caused by adding new nonlinear and stochastic terms to the Schrödinger equation. In this paper, we apply this theory to inflation, which amounts to solving the CSL parametric oscillator case. We choose the wave function collapse to occur on an eigenstate of the Mukhanov-Sasaki variable and discuss the corresponding modified Schrödinger equation. Then, we compute the power spectrum of the perturbations and show that it acquires a universal shape with two branches, one which remains scale invariant and one with $n_S = 4$, a spectral index in obvious contradiction with the cosmic microwave background anisotropy observations. The requirement that the non-scale-invariant part be outside the observational window puts stringent constraints on the parameter controlling the deviations from ordinary quantum mechanics. Due to the absence of a CSL amplification mechanism in field theory, this also has the consequence that the collapse mechanism of the inflationary fluctuations is not efficient. Then, we determine the collapse time. On small scales the collapse is almost instantaneous, and we recover exactly the behavior of the CSL harmonic oscillator (a case for which we present new results), whereas, on large scales, we find that the collapse is delayed and can take several e-folds to happen. We conclude that recovering the observational successes of inflation and, at the same time, reaching a satisfactory resolution of the inflationary "macro-objectification" issue seems problematic in the framework considered here. This work also provides a complete solution to the CSL parametric oscillator system, a topic we suggest could play a very important role to further constrain the CSL parameters. Our results illustrate the remarkable power of inflation and cosmology to constrain new physics.

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I. INTRODUCTION

Inflation is currently the leading paradigm for explaining the physical conditions that prevailed in the very early Universe [1–5]. It solves the puzzles of the standard hot big bang phase and it explains the origin of the inhomogeneities in our Universe [6–11] (for reviews, see Refs. [12–18]). According to the inflationary scenario, these inhomogeneities result from the amplification of the unavoidable vacuum quantum fluctuations of the gravitational and inflaton fields during a phase of accelerated expansion. In particular, inflation predicts an almost scale invariant power spectrum for the cosmological fluctuations [19], a prediction which fits very well the high accuracy astrophysical data now at our disposal [20–26].

Often less emphasized is the fact that inflation is also particularly remarkable from the theoretical point of view. Indeed, the inflationary mechanism for the production of cosmological perturbations makes use of general relativity and quantum mechanics, two theories that are notoriously difficult to combine. Moreover, this mechanism leads to theoretical predictions that are possible to study observationally with great accuracy. In fact, inflation is probably the only case in physics where an effect based on general relativity and quantum mechanics leads to predictions that, given our present day technological capabilities, can be tested experimentally.

The situation described above can be used to investigate deep questions. Among these deep questions is how the quantum measurement problem looks in a cosmological context. According to inflation, the cosmic microwave background (CMB) radiation anisotropy [27] is an observable and is therefore described by a quantum operator. As a consequence, when one looks at a CMB map, one observes

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the result of a measurement of that observable. According to the postulates of quantum mechanics in the Copenhagen interpretation, this means that the wave function of the inflationary perturbations has collapsed to an eigenvector of this operator and that the CMB map corresponds to one of its eigenvalues. The problem with this approach is that the collapse is supposed to occur only when an observer performs a measurement on the system. Clearly, there was no observer before or when the CMB was emitted. This seems to contradict the phenomenological fact that large-scale structure formation started early in the history of the Universe since these structures are seeded by the same early physics which led to CMB fluctuations. As a matter of fact, CMB fluctuations can also be understood as the earliest hint that primordial inhomogeneities had already started to grow at that time. Furthermore, in some sense, the observers are actually the end product of the structure formation process. Of course, this measurement problem is already present in conventional laboratory situations but it seems to be exacerbated (to use the words of Ref. [28]) in a cosmological context.

Important steps towards a better understanding of these issues have already been accomplished. In particular, it was shown that the inflationary accelerated expansion transforms a coherent vacuum state into a strongly squeezed state [29], the corresponding squeezing being much more important than whatever can be realized in the laboratory [30]. In this limit, the predictions of the quantum formalism are indistinguishable from that of a theory where the fluctuations are just assumed to be realizations of a classical stochastic process [31–33]. The classical limit is a subtle concept in quantum mechanics but, in this sense (and in this sense only), the system can be characterized as being classical [34]. Moreover, the large-scale cosmological perturbations are not isolated and, as a consequence, the phenomenon of decoherence [35–37] is relevant for them. This has the consequence that their density matrix becomes diagonal before recombination, a criterion which is also considered as necessary in order to understand the quantum-to-classical transition [31,32,38–43]. However, it is known that decoherence *per se* does not solve the measurement problem [44,45]. Indeed, it remains to understand how a single outcome can be produced. This point is particularly important given that we only have one CMB map, that is to say only one measurement of the corresponding observable. In other words, even if the cosmological fluctuations can be viewed as a classical stochastic problem, this does not explain how a given realization of this process becomes an actual perception. This "macro-objectivation" problem is already present in a conventional situation but, as already mentioned before, it becomes particularly embarrassing in the context of inflation where the collapse of the wave function cannot be due to the presence of a conscious observer. Facing this situation, the common attitude is to postulate that decoherence should be combined with a new interpretational scheme, different from the Copenhagen interpretation [46,47]. Typically, in cosmology, the many world approach is often implicitly assumed [34,46–50]. Another frequently mentioned possibility, which seems to be particularly well suited to the cosmological context, is to consider that the wave function only represents the information that we have on the system [51]. In this case, the issue of the wave function collapse becomes irrelevant since it just corresponds to a situation where the observer updates their knowledge (in the Bayesian sense) about the physical properties of the system. Other attempts, such as the nonlocal hidden variable theories, have also been tried [52–57]. In all of these cases, the cosmological situation does not differ much from a conventional laboratory situation and, moreover, does not lead to new, falsifiable, predictions.[1] Then, it becomes a question of taste which approach best fits one's own prejudices.

However, there exists an exception to the conclusion of the previous discussion, namely the case of the collapse models [61–66] (for reviews, see Refs. [67,68]). In this approach, the Schrödinger equation is modified by adding nonlinear and stochastic terms which render dynamical the collapse of the wave function. The model has nice features: first, the approach seems to follow a conservative strategy since, in physics, it is standard to first consider a linear theory and then, in order to have a more accurate description, to consider nonlinear corrections; in some sense, the collapse theories follow this line of argument. Second, there is now a single law of evolution for the state vector and, third, the Born laws can be derived instead of postulated. There are also disadvantages such as the property that energy is not conserved or the fact that the relativistic formulation of the theory appears to be technically and conceptually difficult to develop (however, see Ref. [69]). But, clearly, the main advantage in comparison to the possibilities discussed above is that this approach is falsifiable since it leads to predictions different from that of conventional quantum mechanics. This fact has been widely used in order to constrain collapse theories in the laboratory [68,70–73] but, clearly, it is also important to see whether this could be done in a cosmological context [74–76]. It is therefore interesting to investigate what the collapse theories have to say about the inflationary mechanism. Notice that, regardless of one's opinion about collapse theories, the subject is worth studying: a supporter would argue that the cosmological measurement problem can possibly find a natural solution within this theory and an opponent would hope that the constraints obtained in a cosmological context can rule out the theory. In fact, this last question turns out to be very important. Indeed, as

[1] In the case of the Bohm-de Broglie approach, there could be a transitory regime, before "quantum equilibrium" is reached, where the predictions differ from conventional quantum mechanics [58]. Cosmology is also precisely considered as a situation where this regime could be relevant [59,60].

already mentioned, the constraints that exist on collapse theories are usually obtained from physical phenomena that can be observed in the laboratory. Therefore, by studying collapse theories in the context of cosmology and inflation, one can hope to derive very relevant new constraints since one now deals with characteristic scales (energy, length, etc.) which typically differ by many orders of magnitude from those used in a down-to-earth context. This illustrates again the conceptual relevance of inflation when it comes to very fundamental questions and its power to constrain alternatives to gravity but also to quantum mechanics. In some sense, inflation represents an ideal playground to test new theories. Notice in passing that the very same strategy was used in the case of the so-called trans-Planckian problem of inflation [77–79] where it was shown that the inflationary observables could possibly contain an imprint (although probably small) of string theory.

We are using a (modified) Schrödinger-type of equation to describe the behavior of cosmological perturbations. This is justified because each Fourier mode of those effectively evolves in an independent way and cosmological expansion permits one to define a privileged time. This allows for a sensible treatment of cosmological perturbations even though a fully relativistic continuous spontaneous localization (CSL) model, which could be naively expected to be required, is still lacking. At this moment, surprisingly, it is easier to treat inflationary perturbations than ordinary particle physics.

It should also be emphasized that the idea of applying collapse theories to inflationary perturbations of quantum-mechanical origin was first considered in Refs. [80–82]. In these articles, a phenomenological model for the collapse process was assumed and the corresponding physical properties were derived. In particular, the power spectrum of the perturbations was calculated and was shown to deviate from the standard predictions. Therefore, Refs. [80–82] have demonstrated that, in principle, it is possible to observationally test collapse theories in a cosmological context. Our approach differs from that of Refs. [80–82] in the fact that we use the CSL model to implement the collapse dynamics. This has the advantage that our calculations can be directly confronted and compared to other results obtained in other branches of physics.

This paper is organized as follows. In the next section, Sec. II, we present a brief review of the theory of inflationary cosmological perturbations of quantum-mechanical origin. We especially focus on the calculation of the power spectrum since this quantity is the tool that allows us to relate the inflationary theory with the CMB observations. Then, in Sec. III, we discuss the cosmological measurement problem and we explain how high accuracy CMB measurements can constrain inflation. In Sec. IV, we consider collapse theories, in particular, its CSL version, which is, as already mentioned, the case we use in this article. These sections aim at rendering the present work self-contained for readers with different expertise. Then, we show how the harmonic oscillator can be treated in this context. This case is particularly relevant for cosmological fluctuations since it corresponds to the small-scale limit (in comparison to the Hubble radius) of the theory of cosmological perturbations. In Sec. V, we apply the CSL theory to inflation and to the calculation of the power spectrum. We use this result to constrain the parameter that controls the deviations from ordinary quantum mechanics. In Sec. VI, we study in more details the collapse phenomenon and explicitly compute the collapse time on small and large scales. In Sec. VII, we summarize our results and present our conclusions. We end the paper with an Appendix where it is shown that changing the "temporal gauge" in which the modified Schrödinger equation is written does not affect the shape of the power spectrum. This calculation reinforces the generic character of the results obtained in this work.

II. INFLATIONARY COSMOLOGICAL PERTURBATIONS

A. Basic formalism

By definition, inflation is a phase of accelerated expansion that took place in the very early Universe, prior to the standard hot big bang phase [1–5] (for reviews, see Refs. [13–15]). As is well known, postulating such a phase of evolution allows us to solve the standard problems of the hot big bang model. Given that at very high energies, field theory is the relevant framework to describe matter, a natural way to realize inflation is to consider that a real scalar field (the "inflaton" field) dominated the energy density budget of matter in the early Universe. Moreover, this assumption is compatible with the observed homogeneity, isotropy and flatness of the early Universe. Technically, the above-mentioned situation can be described by the metric tensor $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$, where $a(t)$ is the Friedman-Lemaître-Robertson-Walker (FLRW) scale factor and t the cosmic time.[2] The Einstein equations imply that $\ddot{a}/a = -(\rho + 3p)/(6M_{\mathrm{Pl}}^2)$, ρ and p being the energy density and pressure of the matter sourcing the gravitational field and M_{Pl} the Planck mass (a dot denotes a derivative with respect to the cosmic time t). For a scalar field, this reduces to $\ddot{a}/a = V(\varphi) \times (1 - \dot{\varphi}^2/V)/(3M_{\mathrm{Pl}}^2)$, where $V(\varphi)$ is the scalar field potential. This means that inflation (i.e., $\ddot{a} > 0$) can be obtained provided the inflaton slowly rolls down its potential so that its potential energy dominates over its kinetic energy. This also shows that the inflaton potential must be sufficiently flat, a requirement which is not always easy to obtain in

[2]Unless explicit mention of the contrary, we shall in what follows assume natural units in which $\hbar = c = 1$ so that the Newton constant G_{N} is related with the Planck mass M_{Pl} through $8\pi G_{\mathrm{N}} = M_{\mathrm{Pl}}^{-2}$.

realistic situations and makes the inflationary model building problem a difficult issue [83]. The physical nature of the inflaton field has not been identified (there are many candidates) and, as a consequence, the shape of $V(\varphi)$ is not known. Of course, different $V(\varphi)$ lead to different inflationary expansions but, since these different potentials must all be sufficiently flat, the corresponding scale factors are all approximately given by the de Sitter solution. This solution is described by the scale factor $a(t) \simeq e^{Ht}$, where $H \equiv \dot{a}/a$ is the Hubble parameter, a slowly-varying quantity directly related to the energy scale of inflation. Observationally, this last quantity is not known but is constrained [22] to be between the grand unified theory (GUT) scale, that is to say $\sim 10^{15}$ GeV, and ~ 1 TeV. The previous considerations show that inflation can also be viewed as a phase of quasiexponential expansion.

A concrete illustration of the above discussion consists in considering power-law inflation [84]. Although it is based on a specific model with potential $V(\varphi) = M^4 e^{-\alpha\varphi/M_{\rm Pl}}$ (with α constant), it captures, in a simple way, all the essential properties of inflation and, moreover, is the only scenario which permits an exact integration of the equations of motion (at the background level but also at the perturbative level, see below). The corresponding scale factor is given by

$$a(\eta) = \ell_0(-\eta)^{1+\beta}, \tag{1}$$

where ℓ_0 is a length the value of which is fixed once the energy scale of inflation is known and η in the conformal time defined by $dt = a d\eta$, see Eq. (2). The quantity β is a free parameter such that $\beta \le -2$ and is related to α through $\alpha^2/2 = (\beta + 2)/(\beta + 1)$. The case $\beta = -2$ represents the de Sitter solution since it implies $\alpha = 0$, i.e., a flat potential (and, of course, in cosmic time, the solution $a \propto 1/\eta$ is given by an exponential). Therefore, different β represents different inflationary solutions and β must always be close to -2 in order for the potential to be sufficiently flat. As announced, power-law inflation illustrates well the discussion of the previous paragraph.

The above arguments can be considered as strong hints in favor of inflation. However, soon after its advent, it was realized that inflation, combined with quantum mechanics, leads to an even more impressive result, namely it naturally explains the origin of the CMB anisotropies and of the large-scale structures. According to the inflationary paradigm, these deviations from homogeneity and isotropy originate from the unavoidable zero-point quantum fluctuations of the coupled inflaton and gravitational fields. Statistically, the fluctuations are characterized by their two-point correlation function or power spectrum. The observations [20–26] indicate that the corresponding power spectrum is close to the Harrison-Zel'dovich, scale invariant, power spectrum with equal power on all scales. That this power spectrum represents a good fit to the astrophysical data was in fact realized before the advent of inflation but no convincing fundamental theory was known to explain this result.

The main success of inflation is that it precisely predicts an almost scale invariant power spectrum, the small deviations from scale invariance being connected with the microphysics of inflation [6–11]. The fact that different types of inflationary scenarios lead to a power spectrum which is, at leading order, always close to scale invariance is connected with the fact that the inflationary scale factor is always close to the de Sitter solution (see above) or, equivalently, with the fact that the inflaton potential is always almost flat. The deviations from scale invariance are related to the deviations from a flat potential and, therefore, depend on the detailed shape of the potential. As a consequence, measuring them allows us to say something about $V(\varphi)$ and there is currently an important effort in this direction using the high accuracy CMB data that have been released in the past years.

Let us now see how the results reviewed before can be derived. Clearly, in order to model the cosmological fluctuations, one needs to go beyond homogeneity and isotropy. The most general metric describing small fluctuations of the scalar type on top of a FLRW Universe can be written as [12]

$$ds^2 = a^2(\eta)\{-(1-2\phi)d\eta^2 + 2(\partial_i B)dx^i d\eta + [(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E]dx^i dx^j\}. \tag{2}$$

A similar approach could be used to take into account tensor perturbations (i.e., gravity waves). Here, we do not include them since they are subdominant in the CMB, representing less than $\sim 20\%$ at 2σ confidence level [22] and, in addition, doing it would not bring any new aspects to the question we want to investigate in this article. In Eq. (2), the four functions ϕ, B, ψ and E are of course functions of time and space since we consider an inhomogeneous and anisotropic situation. As is well known, the above approach is redundant because of gauge freedom [12,85,86]. A careful study of this question shows that the gravitational sector can in fact be described by a single, gauge-invariant, quantity, the Bardeen potential Φ_B defined by [85]

$$\Phi_B(\eta, \boldsymbol{x}) = \phi + \frac{1}{a}[a(B - E')]', \tag{3}$$

where a prime denotes a derivative with respect to the conformal time η. In the same manner, the matter sector can be modeled by the gauge invariant fluctuation of the scalar field

$$\delta\varphi^{(gi)}(\eta, \boldsymbol{x}) = \delta\varphi + \varphi'(B - E'). \tag{4}$$

The two quantities Φ_B and $\delta\varphi^{(gi)}$ are related by a perturbed Einstein constraint. This implies that the scalar sector can in fact be described by a single quantity. For this reason, we now introduce the so-called Mukhanov-Sasaki variable [6,87] which is a combination of the Bardeen potential and of the gauge invariant field

$$v(\eta, \boldsymbol{x}) = a\left[\delta\varphi^{(\mathrm{gi})} + \varphi' \frac{\Phi_{\mathrm{B}}}{\mathcal{H}}\right], \tag{5}$$

where $\mathcal{H} \equiv a'/a$. All the other relevant quantities can be expressed in terms of $v(\eta, \boldsymbol{x})$ which, therefore, fully characterizes the scalar sector.

The next step consists in deriving an equation of motion for $v(\eta, \boldsymbol{x})$. This can be done directly from the perturbed Einstein equations but, here, we first establish the action for the quantity $v(\eta, \boldsymbol{x})$. Expanding the action of the system (i.e., Einstein-Hilbert action plus the action of a scalar field) up to second order in the perturbations, one obtains [12]

$$^{(2)}\delta S = \frac{1}{2}\int \mathrm{d}^4x\left[(v')^2 - \delta^{ij}\partial_i v \partial_j v + \frac{(a\sqrt{\epsilon_1})''}{a\sqrt{\epsilon_1}} v^2\right], \tag{6}$$

where $\epsilon_1 = 1 - \mathcal{H}'/\mathcal{H}^2$ is the first slow-roll parameter [88,89]. As the formula $\ddot{a}/a = H^2(1 - \epsilon_1)$ shows, the condition $\epsilon_1 < 1$ is in fact sufficient to have inflation. Moreover, we have slow-roll inflation [19,88–91] if $\epsilon_1 \ll 1$. In this case, it is easy to show that $\epsilon_1 \simeq (M_{\mathrm{Pl}}^2/2V^2)(\mathrm{d}V/\mathrm{d}\varphi)^2$, i.e., ϵ_1 is in fact a measure of how much the inflaton potential deviates from a flat potential. Equivalently, according to the previous considerations, this is also a measure of how much the inflationary expansion deviates from a pure de Sitter solution. In the case of power-law inflation, one has $\epsilon_1 = (2+\beta)/(1+\beta)$ and, of course, $\epsilon_1 = 0$ when $\beta = -2$ (de Sitter solution). The scale factor can also be rewritten as $a(\eta) \simeq \ell_0(-\eta)^{-1-\epsilon_1}$ and this formula is in fact valid for any slow-roll model of inflation, i.e., for arbitrary shaped potentials, not necessarily of the exponential type. In this sense, power-law inflation with $\beta \lesssim -2$ is a simple representative of all the slow-roll scenarios. Therefore, the fact that, in this paper, we focus on this particular model for technical reasons (again, because this model allows an easy integration of the equations of motion at the background and perturbative level) does not restrict in any way the generality of our considerations.

Our next move consists in Fourier transforming the quantity $v(\eta, \boldsymbol{x})$. This is of course justified by the fact that we work with a linear theory and, hence, all the modes evolve independently. We have

$$v(\eta, \boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k}\, v_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{7}$$

with $v_{-\boldsymbol{k}} = v_{\boldsymbol{k}}^*$ because $v(\eta, \boldsymbol{x})$ is real. Then inserting this expansion into Eq. (6), one arrives at [12]

$$^{(2)}\delta S = \int \mathrm{d}\eta \int \mathrm{d}^3\boldsymbol{k}\left\{v_{\boldsymbol{k}}' v_{\boldsymbol{k}}^{*\prime} + v_{\boldsymbol{k}} v_{\boldsymbol{k}}^*\left[\frac{(a\sqrt{\epsilon_1})''}{a\sqrt{\epsilon_1}} - k^2\right]\right\}, \tag{8}$$

where the integral over $\boldsymbol{k}$ is taken over half the Fourier space only. Next, we define $p_{\boldsymbol{k}}$, the variable canonically conjugate to $v_{\boldsymbol{k}}$

$$p_{\boldsymbol{k}} = \frac{\delta \mathcal{L}}{\delta v_{\boldsymbol{k}}^{*\prime}} = v_{\boldsymbol{k}}', \tag{9}$$

where $\mathcal{L}$ is the Lagrangian density in Fourier space that can be derived from Eq. (8). This allows us to calculate the Hamiltonian which reads

$$H = \int \mathrm{d}^3\boldsymbol{k}\left\{p_{\boldsymbol{k}} p_{\boldsymbol{k}}^* + v_{\boldsymbol{k}} v_{\boldsymbol{k}}^*\left[k^2 - \frac{(a\sqrt{\epsilon_1})''}{a\sqrt{\epsilon_1}}\right]\right\}. \tag{10}$$

This Hamiltonian represents a collection of parametric oscillators (i.e., one oscillator per mode), the time-dependent frequency of which can be expressed as

$$\omega^2(\eta, \boldsymbol{k}) = k^2 - \frac{(a\sqrt{\epsilon_1})''}{a\sqrt{\epsilon_1}}. \tag{11}$$

We see that the frequency depends on the scale factors and its derivatives (up to the fourth). This means that different inflationary backgrounds (i.e., different inflaton potentials) lead to different $\omega(\eta, \boldsymbol{k})$ and, therefore, to different behaviors for $v_{\boldsymbol{k}}(\eta)$. From Eq. (10) or Eq. (8), it is easy to derive the equation of motion for the Mukhanov-Sasaki variable. One obtains

$$v_{\boldsymbol{k}}'' + \omega^2(\eta, \boldsymbol{k}) v_{\boldsymbol{k}} = 0, \tag{12}$$

which confirms that each mode behaves as a parametric oscillator. Once a model of inflation has been chosen, the potential $V(\varphi)$ is known and, hence, the corresponding scale factor can be calculated. This, in turn, allows us to determine $\omega^2(\eta, \boldsymbol{k})$ and, then, one can solve the equation of motion (12). However, in order to find the solution for the Fourier component of the Mukhanov-Sasaki variable, one also needs to specify the initial conditions. Classically, there does not seem to exist a natural criterion to choose them. However, when quantization has been performed, the requirement that it be initially in the vacuum state of the theory leads to well-defined initial conditions. We now turn to these questions.

B. Quantization in the Schrödinger picture

In this section, we review how the cosmological perturbations are quantized. Very often in the literature, this is done in the Heisenberg picture. Here, we carry out the quantization in the Schrödinger picture [15] because this is more convenient for the problem we want to investigate in this article. In order to quantize the system, it is also more convenient to work with real variables. Therefore, we introduce the following definitions:

$$v_{\boldsymbol{k}} \equiv \frac{1}{\sqrt{2}}(v_{\boldsymbol{k}}^{\mathrm{R}} + i v_{\boldsymbol{k}}^{\mathrm{I}}), \qquad p_{\boldsymbol{k}} \equiv \frac{1}{\sqrt{2}}(p_{\boldsymbol{k}}^{\mathrm{R}} + i p_{\boldsymbol{k}}^{\mathrm{I}}). \tag{13}$$

In the Schrödinger approach, the quantum state of the system is described by a wave functional, $\Psi[v(\eta, \boldsymbol{x})]$. Since we work in Fourier space (and since the theory is still free in the sense that it does not contain terms with power higher than 2 in the Lagrangian), the wave functional can also be factorized into mode components as

$$\Psi[v(\eta, \boldsymbol{x})] = \prod_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}(v_{\boldsymbol{k}}^{\mathrm{R}}, v_{\boldsymbol{k}}^{\mathrm{I}}) = \prod_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}^{\mathrm{R}}(v_{\boldsymbol{k}}^{\mathrm{R}}) \Psi_{\boldsymbol{k}}^{\mathrm{I}}(v_{\boldsymbol{k}}^{\mathrm{I}}). \quad (14)$$

Quantization is achieved by promoting $v_{\boldsymbol{k}}$ and $p_{\boldsymbol{k}}$ to quantum operators, $\hat{v}_{\boldsymbol{k}}$ and $\hat{p}_{\boldsymbol{k}}$, and by requiring the canonical commutation relations

$$[\hat{v}_{\boldsymbol{k}}^{\mathrm{R}}, \hat{p}_{\boldsymbol{q}}^{\mathrm{R}}] = i\delta(\boldsymbol{k} - \boldsymbol{q}), \qquad [\hat{v}_{\boldsymbol{k}}^{\mathrm{I}}, \hat{p}_{\boldsymbol{q}}^{\mathrm{I}}] = i\delta(\boldsymbol{k} - \boldsymbol{q}). \quad (15)$$

These relations admit the following representation:

$$\hat{v}_{\boldsymbol{k}}^{\mathrm{R,I}}\Psi = v_{\boldsymbol{k}}^{\mathrm{R,I}}\Psi, \qquad \hat{p}_{\boldsymbol{k}}^{\mathrm{R,I}}\Psi = -i\frac{\partial \Psi}{\partial v_{\boldsymbol{k}}^{\mathrm{R,I}}}. \quad (16)$$

The wave functional $\Psi[v(\eta, \boldsymbol{x})]$ obeys the Schrödinger equation which, in this context, is a functional differential equation. However, since each mode evolves independently, this functional differential equation can be reduced to an infinite number of differential equations for each $\Psi_{\boldsymbol{k}}$. Concretely, we have

$$i\frac{\Psi_{\boldsymbol{k}}^{\mathrm{R,I}}}{\partial \eta} = \hat{\mathcal{H}}_{\boldsymbol{k}}^{\mathrm{R,I}} \Psi_{\boldsymbol{k}}^{\mathrm{R,I}}, \quad (17)$$

where the Hamiltonian densities $\hat{\mathcal{H}}_{\boldsymbol{k}}^{\mathrm{R,I}}$ are related to the Hamiltonian by $\hat{H} = \int \mathrm{d}^3\boldsymbol{k}(\hat{\mathcal{H}}_{\boldsymbol{k}}^{\mathrm{R}} + \hat{\mathcal{H}}_{\boldsymbol{k}}^{\mathrm{I}})$. They can be expressed as

$$\hat{\mathcal{H}}_{\boldsymbol{k}}^{\mathrm{R,I}} = -\frac{1}{2}\frac{\partial^2}{\partial (v_{\boldsymbol{k}}^{\mathrm{R,I}})^2} + \frac{1}{2}\omega^2(\eta, \boldsymbol{k})(\hat{v}_{\boldsymbol{k}}^{\mathrm{R,I}})^2, \quad (18)$$

where we have made use of the representations (16).

We are now in a position where we can solve the Schrödinger equation. Let us consider the following Gaussian state

$$\Psi_{\boldsymbol{k}}^{\mathrm{R,I}}(\eta, v_{\boldsymbol{k}}^{\mathrm{R,I}}) = N_{\boldsymbol{k}}(\eta)\mathrm{e}^{-\Omega_{\boldsymbol{k}}(\eta)(v_{\boldsymbol{k}}^{\mathrm{R,I}})^2}. \quad (19)$$

The functions $N_{\boldsymbol{k}}(\eta)$ and $\Omega_{\boldsymbol{k}}(\eta)$ are time dependent and do not carry the subscripts "R" and/or "I" because they are the same for the wave functions of the real and imaginary parts of the Mukhanov-Sasaki variable (see below). Then, inserting $\Psi_{\boldsymbol{k}}$ given by Eq. (19) into the Schrödinger equation (17) implies that $N_{\boldsymbol{k}}$ and $\Omega_{\boldsymbol{k}}$ obey the differential equations

$$i\frac{N_{\boldsymbol{k}}'}{N_{\boldsymbol{k}}} = \Omega_{\boldsymbol{k}}, \qquad \Omega_{\boldsymbol{k}}' = -2i\Omega_{\boldsymbol{k}}^2 + \frac{i}{2}\omega^2(\eta, \boldsymbol{k}). \quad (20)$$

The solutions can be easily found and read

$$|N_{\boldsymbol{k}}| = \left(\frac{2\Re\mathrm{e}\Omega_{\boldsymbol{k}}}{\pi}\right)^{1/4}, \qquad \Omega_{\boldsymbol{k}} = -\frac{i}{2}\frac{f_{\boldsymbol{k}}'}{f_{\boldsymbol{k}}}, \quad (21)$$

where $f_{\boldsymbol{k}}$ is a function obeying the equation $f_{\boldsymbol{k}}'' + \omega^2 f_{\boldsymbol{k}} = 0$, that is to say exactly Eq. (12). The first equation (21) guarantees that the wave function is properly normalized, i.e.,

$$\int \Psi_{\boldsymbol{k}}^{\mathrm{R,I}}\Psi_{\boldsymbol{k}}^{\mathrm{R,I}*}\mathrm{d}v_{\boldsymbol{k}}^{\mathrm{R,I}} = 1. \quad (22)$$

Let us now discuss the initial conditions. The fundamental assumption of inflation is that the perturbations are initially in their ground state. At the beginning of inflation, all the modes of astrophysical interest today have a physical wavelength smaller than the Hubble radius, i.e., $k/(aH) \to \infty$. In this regime, one has $\omega^2(\eta, \boldsymbol{k}) \to k^2$ and each mode now behaves as an harmonic oscillator (as opposed to a parametric oscillator in the generic case) with frequency $\omega = k$. As a consequence, the differential equation for $f_{\boldsymbol{k}}(\eta)$ can easily be solved and the solution reads $f_{\boldsymbol{k}} = A_{\boldsymbol{k}}\mathrm{e}^{ik\eta} + B_{\boldsymbol{k}}\mathrm{e}^{-ik\eta}$, $A_{\boldsymbol{k}}$ and $B_{\boldsymbol{k}}$ being integration constants. Upon using the second equation (21), one has

$$\Omega_{\boldsymbol{k}} \to \frac{k}{2}\frac{A_{\boldsymbol{k}}\mathrm{e}^{ik\eta} - B_{\boldsymbol{k}}\mathrm{e}^{-ik\eta}}{A_{\boldsymbol{k}}\mathrm{e}^{ik\eta} + B_{\boldsymbol{k}}\mathrm{e}^{-ik\eta}}. \quad (23)$$

The wave function (19) represents the ground state wave function of an harmonic oscillator if $\Omega_{\boldsymbol{k}} = k/2$. Therefore, one must choose the initial conditions such that $B_{\boldsymbol{k}} = 0$. Moreover, it is easy to check that the Wronskian $W \equiv f_{\boldsymbol{k}}' f_{\boldsymbol{k}}^* - f_{\boldsymbol{k}}'^* f_{\boldsymbol{k}}$ is a conserved quantity, $\mathrm{d}W/\mathrm{d}\eta = 0$, thanks to the equation of motion of $f_{\boldsymbol{k}}$. Straightforward calculation leads to $W = 2ik|A_{\boldsymbol{k}}|^2$. In the Heisenberg picture the canonical commutation relations require that $W = i$. Even if in the Schrödinger picture presently used, the specific value of W is irrelevant since it cancels out on all calculable physical quantities, this value is conventionally adopted, which amounts to setting $A_{\boldsymbol{k}} = 1/\sqrt{2k}$. As announced, requiring the initial state to be the ground state has completely fixed the initial conditions. We see that Eq. (12) (or, equivalently, the equation for $f_{\boldsymbol{k}}$) should thus be solved with the boundary condition

$$\lim_{k/(aH)\to+\infty} f_{\boldsymbol{k}} = \frac{1}{\sqrt{2k}}\mathrm{e}^{ik\eta}. \quad (24)$$

This choice of initial conditions is referred to as the Bunch-Davies vacuum.

C. The power spectrum

Let us now turn to the calculation of the power spectrum and first introduce the two-point correlation function, defined by

$$\langle\Psi|\hat{v}(\eta, \boldsymbol{x})\hat{v}(\eta, \boldsymbol{x}+\boldsymbol{r})|\Psi\rangle = \int \prod_{\boldsymbol{k}} \mathrm{d}v_{\boldsymbol{k}}^{\mathrm{R}}\mathrm{d}v_{\boldsymbol{k}}^{\mathrm{I}}\Psi_{\boldsymbol{k}}^*(v_{\boldsymbol{k}}^{\mathrm{R}}, v_{\boldsymbol{k}}^{\mathrm{I}})v(\eta, \boldsymbol{x})v(\eta, \boldsymbol{x}+\boldsymbol{r})\Psi_{\boldsymbol{k}}(v_{\boldsymbol{k}}^{\mathrm{R}}, v_{\boldsymbol{k}}^{\mathrm{I}}). \quad (25)$$

The next step consists in using the Fourier transform of the Mukhanov-Sasaki variable, see Eq. (7) and the explicit form of the wave function of Eq. (19). One arrives at

$$\langle\Psi|\hat{v}(\eta,\boldsymbol{x})\hat{v}(\eta,\boldsymbol{x}+\boldsymbol{r})|\Psi\rangle = \frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{p}\mathrm{d}\boldsymbol{q}\,\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}\,\mathrm{e}^{i\boldsymbol{q}\cdot(\boldsymbol{x}+\boldsymbol{r})}\prod_{\boldsymbol{k}}\left(\frac{2\Re\mathrm{e}\Omega_{\boldsymbol{k}}}{\pi}\right) \times \int\prod_{\boldsymbol{k}}\mathrm{d}v_{\boldsymbol{k}}^{\mathrm{R}}\mathrm{d}v_{\boldsymbol{k}}^{\mathrm{I}}\,\mathrm{e}^{-2\sum_{\boldsymbol{k}}\Re\mathrm{e}\Omega_{\boldsymbol{k}}[(v_{\boldsymbol{k}}^{\mathrm{R}})^2+(v_{\boldsymbol{k}}^{\mathrm{I}})^2]}v_{\boldsymbol{p}}v_{\boldsymbol{q}}. \quad (26)$$

If $\boldsymbol{p}\neq\pm\boldsymbol{q}$, the result of the integration is zero since the integrand (up to the Gaussian weight) becomes linear in $v_{\boldsymbol{p}}^{\mathrm{R,I}}$ or $v_{\boldsymbol{q}}^{\mathrm{R,I}}$. If $\boldsymbol{p}=\boldsymbol{q}$, then the only nonlinear term in the integrand is given by $[(v_{\boldsymbol{p}}^{\mathrm{R}})^2-(v_{\boldsymbol{p}}^{\mathrm{I}})^2]/2$. Each term contributes the same amount, so the difference vanishes. The only possibility left is therefore $\boldsymbol{p}=-\boldsymbol{q}$, such that $v_{\boldsymbol{p}}v_{\boldsymbol{q}}=[(v_{\boldsymbol{p}}^{\mathrm{R}})^2+(v_{\boldsymbol{p}}^{\mathrm{I}})^2]/2$, the factor $1/2$ coming from the definition of $v_{\boldsymbol{k}}^{\mathrm{R,I}}$, see Eqs. (13). This leads to

$$\langle\Psi|\hat{v}(\eta,\boldsymbol{x})\hat{v}(\eta,\boldsymbol{x}+\boldsymbol{r})|\Psi\rangle = \frac{2}{(2\pi)^3}\frac{1}{2}\int \mathrm{d}\boldsymbol{p}\,\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{r}}\prod_{\boldsymbol{k}}^{N}\left(\frac{2\Re\mathrm{e}\Omega_{\boldsymbol{k}}}{\pi}\right) \times\int\prod_{\boldsymbol{k}}^{N}\mathrm{d}v_{\boldsymbol{k}}^{\mathrm{R}}\mathrm{d}v_{\boldsymbol{k}}^{\mathrm{I}}\,\mathrm{e}^{-2\sum_{\boldsymbol{k}}\Re\mathrm{e}\Omega_{\boldsymbol{k}}[(v_{\boldsymbol{k}}^{\mathrm{R}})^2+(v_{\boldsymbol{k}}^{\mathrm{I}})^2]}(v_{\boldsymbol{p}}^{\mathrm{R}})^2, \quad (27)$$

the factor of 2 originating from the fact that we have two contributions, one given by the term $(v_{\boldsymbol{p}}^{\mathrm{R}})^2$ and the other by $(v_{\boldsymbol{p}}^{\mathrm{I}})^2$. The Gaussian integrals can easily be carried out. They are all of the form "$\int\mathrm{d}x\mathrm{e}^{-\alpha x^2}$," except of course the one over $v_{\boldsymbol{p}}^{\mathrm{R}}$ which is of the form "$\int\mathrm{d}x x^2\mathrm{e}^{-\alpha x^2}$." As a consequence, one obtains

$$\langle\Psi|\hat{v}(\eta,\boldsymbol{x})\hat{v}(\eta,\boldsymbol{x}+\boldsymbol{r})|\Psi\rangle = \frac{1}{(2\pi)^3}\int\mathrm{d}\boldsymbol{p}\,\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{r}}\prod_{\boldsymbol{k}}^{N}\left(\frac{2\Re\mathrm{e}\Omega_{\boldsymbol{k}}}{\pi}\right)\frac{1}{2}\left[\frac{\sqrt{\pi}}{\left(\sqrt{2\Re\mathrm{e}\Omega_{\boldsymbol{p}}}\right)^3}\right] \times\prod_{\boldsymbol{k}}^{N}\left(\frac{\sqrt{\pi}}{\sqrt{2\Re\mathrm{e}\Omega_{\boldsymbol{k}}}}\right)\prod_{\boldsymbol{k}}^{N-1}\left(\frac{\sqrt{\pi}}{\sqrt{2\Re\mathrm{e}\Omega_{\boldsymbol{k}}}}\right). \quad (28)$$

The infinite product "$\prod_{\boldsymbol{k}}^{N-1}$" means a product over all the wave vectors but $\boldsymbol{p}$. One can always write this product as "$\sqrt{2\Re\mathrm{e}\Omega_{\boldsymbol{p}}/\pi}\prod_{\boldsymbol{k}}^{N}$," then the two last infinite products in the above expression exactly cancel the first one. Therefore, we are left with

$$\langle\Psi|\hat{v}(\eta,\boldsymbol{x})\hat{v}(\eta,\boldsymbol{x}+\boldsymbol{r})|\Psi\rangle = \frac{1}{(2\pi)^3}\int\mathrm{d}\boldsymbol{p}\,\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{r}}\frac{1}{4\Re\mathrm{e}\Omega_{\boldsymbol{p}}}. \quad (29)$$

We now need to express $\Re\mathrm{e}\Omega_{\boldsymbol{p}}$ in terms of the function $f_{\boldsymbol{p}}$. From the second Eq. (21), one easily shows that

$$\Re\mathrm{e}\Omega_{\boldsymbol{p}} = -\frac{i}{4}\frac{W}{|f_{\boldsymbol{p}}|^2}, \quad (30)$$

and we obtain our final expression for the two-point correlation function

$$\langle\Psi|\hat{v}(\eta,\boldsymbol{x})\hat{v}(\eta,\boldsymbol{x}+\boldsymbol{r})|\Psi\rangle = \frac{1}{(2\pi)^3}\int\mathrm{d}\boldsymbol{p}\,\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{r}}\frac{i}{W}|f_{\boldsymbol{p}}|^2 = \frac{1}{2\pi^2}\int_0^{+\infty}\frac{\mathrm{d}p}{p}\frac{\sin pr}{pr}p^3|f_{\boldsymbol{p}}|^2, \quad (31)$$

where, in the last expression, we have used our choice $W=i$. The power spectrum is just defined as the square of the Fourier amplitude per logarithmic interval at a given scale, i.e.,

$$\mathcal{P}_v(k) = \frac{k^3}{2\pi^2}|f_{\boldsymbol{k}}|^2. \quad (32)$$

The same manipulations allow us to express the two-point correlation of two Fourier amplitudes. It can be written as

$$\langle\Psi|\hat{v}_{\boldsymbol{k}}\hat{v}_{\boldsymbol{p}}^*|\Psi\rangle = \int\prod_{\boldsymbol{q}}\mathrm{d}v_{\boldsymbol{q}}^{\mathrm{R}}\mathrm{d}v_{\boldsymbol{q}}^{\mathrm{I}}\Psi_{\boldsymbol{q}}^*\hat{v}_{\boldsymbol{k}}\hat{v}_{\boldsymbol{p}}^*\Psi_{\boldsymbol{q}}. \quad (33)$$

This integral is nonvanishing only if $\boldsymbol{k}=\boldsymbol{p}$ (otherwise one has to integrate an odd function) and receives two contributions, one from $(v_{\boldsymbol{k}}^{\mathrm{R}})^2$ and the other from $(v_{\boldsymbol{k}}^{\mathrm{I}})^2$. Repeating calculations already performed before, one finally arrives at

$$\langle\Psi|\hat{v}_{\boldsymbol{k}}\hat{v}_{\boldsymbol{p}}^*|\Psi\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_v(k)\delta(\boldsymbol{k}-\boldsymbol{p}). \quad (34)$$

We now need to explain how the cosmological perturbations of quantum-mechanical origin studied above are related to observables in cosmology. This is the goal of the next section.

D. From quantum fluctuations to CMB anisotropies

The presence of quantum fluctuations in the inflaton and gravitational fields has many observational implications. Here, we focus on one of them, namely the existence of CMB temperature anisotropies. The importance of this observable is that we now have at our disposal very high accuracy measurements of those anisotropies [20,21]. Moreover, even more accurate data will be released soon [92]. The relation between the temperature fluctuations along a given direction $\boldsymbol{e}$ and the cosmological perturbations is expressed by the so-called Sachs-Wolfe effect [93,94]. A simplified version of this result, valid on large angular scales only, can be written as [94]

$$\frac{\delta T}{T}(\boldsymbol{e}) = \frac{1}{5}\zeta[\eta_{\ell\mathrm{ss}}, -\boldsymbol{e}(\eta_{\ell\mathrm{ss}}-\eta_0)+\boldsymbol{x}_0], \quad (35)$$

where T represents the averaged background temperature, i.e., $T\simeq 2.7$ K, $\eta_{\ell\mathrm{ss}}$ is the conformal time at emission (that is to say at the surface of last scattering) and η_0 is the present conformal time. The vector $\boldsymbol{x}_0$ landmarks the place of reception, in the present case Earth (or a satellite orbiting the Earth). The quantity ζ denotes the curvature perturbation. It is related to the Bardeen potential defined in Eq. (3) through the following expression [12,86,95]:

$$\zeta = \frac{2}{3}\frac{\mathcal{H}^{-1}\Phi_{\rm B}' + \Phi_{\rm B}}{1+w} + \Phi_{\rm B}, \tag{36}$$

where $w \equiv p/\rho$ is the equation of state parameter, that is to say the energy density to pressure ratio of the dominant fluid. For instance, for the matter dominated era ($w = 0$), during which recombination takes place (at a redshift of $z_{\ell ss} \simeq 1100$), on large scales, one simply has $\zeta \simeq 5\Phi_{\rm B}/3$ since the Bardeen potential is constant. The importance of ζ lies in the fact that it is a conserved quantity on large scales [86,95]. Therefore, its spectrum, calculated at the end of inflation, can directly be propagated to the recombination time as it is not sensitive to the details of the cosmological evolution, in particular to those of the complicated reheating era [96–100]. The curvature perturbation can also be expressed in terms of the Mukhanov-Sasaki variable as

$$\zeta = \frac{1}{a\sqrt{2\epsilon_1}}\frac{v}{M_{\rm Pl}}. \tag{37}$$

Finally, in the framework of the theory of inflationary cosmological perturbations of quantum-mechanical origin, we have seen that v is in fact an operator. This implies that ζ and $\delta T/T$ are also quantum operators and, for this reason, from now on, we will denote them with a hat.

Since the operator $\widehat{\delta T}/T$ lives on the celestial sphere, it can be expanded over the spherical harmonic basis according to

$$\frac{\widehat{\delta T}}{T}(\boldsymbol{e}) = \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{m=\ell}\hat{a}_{\ell m}Y_{\ell m}(\theta,\phi), \tag{38}$$

where θ and ϕ are the angles defining the direction along which the vector $\boldsymbol{e}$ is pointing. Then, the angular two-point correlation function can be expressed in terms of the multipole moments C_ℓ as

$$\langle\Psi|\hat{a}_{\ell m}\hat{a}^*_{\ell' m'}|\Psi\rangle = C_\ell\delta_{\ell\ell'}\delta_{mm'}, \tag{39}$$

and, as a consequence, the two-point correlation function of the temperature fluctuations operator can be written as

$$\left\langle\Psi\left|\frac{\widehat{\delta T}}{T}(\boldsymbol{e}_1)\frac{\widehat{\delta T}}{T}(\boldsymbol{e}_2)\right|\Psi\right\rangle = \frac{1}{4\pi}\sum_{\ell=2}^{\infty}(2\ell+1)C_\ell P_\ell(\boldsymbol{e}_1\cdot\boldsymbol{e}_2), \tag{40}$$

the quantity P_ℓ denoting Legendre polynomials.

In order to pursue our demonstration that the CMB anisotropies are entirely determined by the quantum fluctuations, let us now express the multipole moments in terms of the cosmological perturbation power spectrum. Upon using Eqs. (35) and (38), one obtains

$$\hat{a}_{\ell m} = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}\Omega_{\boldsymbol{e}}\mathrm{d}\boldsymbol{k}\frac{\hat{\zeta}_{\boldsymbol{k}}(\eta_{\ell ss})}{5}\mathrm{e}^{-i\boldsymbol{k}\cdot[\boldsymbol{e}(\eta_{\ell ss}-\eta_0)-\boldsymbol{x}_0]}Y^*_{\ell m}(\boldsymbol{e}) \tag{41}$$

and, from this expression, it is easy to show that

$$C_\ell = \frac{1}{2a^2M_{\rm Pl}^2\epsilon_1}\frac{4\pi}{25}\int\frac{\mathrm{d}k}{k}j_\ell^2[k(\eta_0-\eta_{\ell ss})]\mathcal{P}_v(k), \tag{42}$$

where j_ℓ is a spherical Bessel function and where we used Eq. (34) to show that

$$\langle\Psi|\hat{\zeta}_{\boldsymbol{k}}\hat{\zeta}^*_{\boldsymbol{p}}|\Psi\rangle = \frac{1}{2a^2M_{\rm Pl}^2\epsilon_1}\frac{2\pi^2}{k^3}\mathcal{P}_v(k)\delta(\boldsymbol{k}-\boldsymbol{p}). \tag{43}$$

We see that C_ℓ is given by an integral over wave numbers of the Mukhanov-Sasaki power spectrum times a quantity that can be viewed as a "transfer matrix $j_{lk} \equiv j_\ell^2[k(\eta_0-\eta_{\ell ss})]$" which allows us to "translate" a three dimensional spatial frequency $\boldsymbol{k}$ into a two-dimensional spatial frequency ℓ on the celestial sphere. We emphasize again that the above result is valid on large scales only; otherwise the integral in Eq. (42) contains another transfer function $T_\zeta(k)$ which takes into account the subsequent evolution of the modes when they reenter the Hubble radius after inflation. Since ζ is a conserved quantity, we have $T_\zeta(k\to 0) = 1$.

Finally, let us also notice that Eq. (41) implies that $\langle\Psi|\hat{a}_{\ell m}|\Psi\rangle = 0$ since $\langle\Psi|\hat{\zeta}_{\boldsymbol{k}}|\Psi\rangle = 0$. Of course, this also means that $\langle\Psi|\widehat{\delta T}/T|\Psi\rangle = 0$.

E. Inflationary predictions

We have just seen that, in order to calculate the CMB multipole moments, we need to evaluate the curvature perturbation power spectrum. In this section, we calculate this quantity for power-law inflation.

The first step consists in solving the equation of motion (12). Upon using Eq. (11), one obtains the time dependence of the frequency of the parametric oscillator, which reads

$$\omega^2(\eta,\boldsymbol{k}) = k^2 - \frac{\beta(\beta+1)}{\eta^2}. \tag{44}$$

From this expression, one sees that there are two regimes depending on whether the first term is dominant or subdominant. The Hubble radius is given by $\ell_{\rm H} \equiv 1/H = a\eta/(1+\beta)$ and the Fourier mode wavelength can be expressed in terms of the comoving wave number as $\lambda = 2\pi a/k$. The first term dominates if $|k\eta| \gg 1$ or, equivalently, $\lambda \ll \ell_{\rm H}$. In this case $\omega \simeq k$ and we expect the mode function to oscillate as it would in Minkowski spacetime since, at those scales, spacetime curvature is negligible for the mode evolution. On the contrary, if $|k\eta| \ll 1$, or $\lambda \gg \ell_{\rm H}$, one has $\omega \sim 1/\eta$, so curvature dominates and one obtains one growing mode and one decaying mode. These arguments are confirmed when one studies the exact solution for the mode function $f_{\boldsymbol{k}}$. It can be expressed in terms of Bessel functions $J_\nu(z)$ as [101,102]

$$f_{\boldsymbol{k}} = (-k\eta)^{1/2}[C_{\boldsymbol{k}}J_{\beta+1/2}(-k\eta) + D_{\boldsymbol{k}}J_{-(\beta+1/2)}(-k\eta)], \tag{45}$$

where $C_{\boldsymbol{k}}$ and $D_{\boldsymbol{k}}$ are two integration constants. In order to match the initial vacuum behavior (24), one must choose

$$C_{\boldsymbol{k}} = -D_{\boldsymbol{k}} e^{i\pi(\beta+1/2)}, \qquad D_{\boldsymbol{k}} = \frac{i}{2}\sqrt{\frac{\pi}{k}}\frac{e^{-i\pi/4-i\pi(\beta+1/2)/2}}{\sin[\pi(\beta+1/2)]}. \tag{46}$$

In particular, one notices that both coefficients $C_{\boldsymbol{k}}$ and $D_{\boldsymbol{k}}$ scale as $1/\sqrt{k}$.

Since we want to evaluate the power spectrum on large scales, it is sufficient to take the limit $k\eta \to 0$ in Eq. (45). Then, one is led to

$$\begin{aligned}\mathcal{P}_{\zeta}|_{\rm stand} &= \frac{1}{2a^2 M_{\rm Pl}^2 \epsilon_1}\mathcal{P}_v(k)\\ &= \frac{1}{\pi\epsilon_1 m_{\rm Pl}^2 \ell_0^2} f(\beta) k^{2\beta+4} \equiv A_{\rm S} k^{n_{\rm S}-1},\end{aligned} \tag{47}$$

where $M_{\rm Pl} = m_{\rm Pl}/\sqrt{8\pi}$ and the function $f(\beta)$ is defined by [90]

$$f(\beta) \equiv \frac{1}{\pi}\left[\frac{\Gamma(-\beta-1/2)}{2^{1+\beta}}\right]^2, \tag{48}$$

where $\Gamma(z)$ is the Euler integral of the first kind [101,102]. This function is such that, for the de Sitter case $\beta = -2$, one has $f(\beta=-2) = 1$. The scalar spectral index $n_{\rm S} = 2\beta + 5$ and, for solutions close to the de Sitter solutions, one has $n_{\rm S} \simeq 1$, i.e., we have an almost scale invariant power spectrum. As discussed before, the deviations from scale invariance are related to the deviation from the de Sitter case $\beta = -2$. This conclusion is in fact valid for any slow-roll models. The amplitude $A_{\rm S}$ determines the level of the temperature fluctuations observed in the sky, namely $\delta T/T \sim 10^{-5}$.

Finally, let us evaluate the multipole moments explicitly. Upon using Eq. (42) and the expression of the power spectrum established above, one arrives at

$$C_\ell = \frac{\pi^{3/2}\Gamma[(3-n_{\rm S})/2]\Gamma[\ell+(n_{\rm S}-1)/2]}{\Gamma[(4-n_{\rm S})/2]\Gamma[\ell+2-(n_{\rm S}-1)/2]}(r_{\ell{\rm ss}})^{1-n_{\rm S}}\frac{A_{\rm S}}{25}, \tag{49}$$

where we have defined $r_{\ell{\rm ss}} \equiv \eta_0 - \eta_{\ell{\rm ss}}$. Since this equation has been derived for large scales, roughly speaking one can estimate it to be valid in the regime $\ell \ll 20$. For $n_{\rm S} \simeq 1$, the above expression implies that $C_\ell \propto 1/[\ell(\ell+1)]$.

Of course, in the real world, the argument goes the other way around. From measurements of the CMB anisotropies, we observe that, on large scales, $C_\ell \propto 1/[\ell(\ell+1)]$ and, therefore, we deduce that the corresponding power spectrum is close to scale invariance, i.e., $n_{\rm S} \simeq 1$. Obviously, this also means that a spectrum that is not very close to scale invariance is now ruled out (more precisely, the WMAP data indicate that $1 - n_{\rm S} = 0.018^{+0.019}_{-0.02}$ [20–22]). As already emphasized, the great success of inflation is that it precisely leads to such a power spectrum.

It should also be clear that the above discussion, although perfectly correct at the level of principles, is oversimplified at the technical level. The multipole moments are in fact computed at any scale (i.e., for any value of ℓ) by means of numerical calculations (since, in the most general case, they are solutions of more involved differential equations) [103]. Moreover, their shape is not only determined by the spectral index but is also affected by the other cosmological parameters. The constraints on the different inflationary models are then obtained by a Markov chain exploration of the parameter space [104]. But these technical considerations do not affect the considerations presented in this paper. Once again, as far as physical principles are concerned, the discussion presented in this section is accurate.

III. THE COSMOLOGICAL MEASUREMENT PROBLEM

A. Squeezed state

In this section, we study in more detail the properties of the quantum state in which the cosmological perturbations are placed [29,31,34,105]. As already mentioned around Eq. (19), it is described by the wave function

$$\Psi_{\boldsymbol{k}}(\eta, v_{\boldsymbol{k}}^{\rm R}, v_{\boldsymbol{k}}^{\rm I}) = \left(\frac{2\Re e\Omega_{\boldsymbol{k}}}{\pi}\right)^{1/2} e^{-\Omega_{\boldsymbol{k}}[(v_{\boldsymbol{k}}^{\rm R})^2+(v_{\boldsymbol{k}}^{\rm I})^2]} \tag{50}$$

$$= \left(\frac{2\Re e\Omega_{\boldsymbol{k}}}{\pi}\right)^{1/2} e^{-2\Omega_{\boldsymbol{k}}(\eta)v_{\boldsymbol{k}}v_{\boldsymbol{k}}^*}. \tag{51}$$

We see that this quantum state is completely known once the time dependence of $\Omega_{\boldsymbol{k}}(\eta)$ has been determined. The differential equation controlling the evolution of $\Omega_{\boldsymbol{k}}(\eta)$ is given by the second part of Eq. (20). This equation is a Ricatti equation (i.e., a first order, nonlinear, differential equation). As is well known, it can always be reduced to a second order but linear differential equation. As already mentioned, this is achieved through the change of variable $\Omega_{\boldsymbol{k}} = -if_{\boldsymbol{k}}'/(2f_{\boldsymbol{k}})$. The function $f_{\boldsymbol{k}}(\eta)$ obeys $f_{\boldsymbol{k}}'' + \omega^2 f_{\boldsymbol{k}} = 0$ and has been solved in Eq. (45). In the small-scale limit, one has $\Omega_{\boldsymbol{k}} \to k/2$ and the wave function (50) is the ground state of an harmonic oscillator. In the large-scale limit, a lengthy but straightforward calculation leads to

$$\begin{aligned}\frac{\Omega_{\boldsymbol{k}}(\eta)}{k} &= -\frac{i}{2k\eta}(1+\beta) - \frac{i}{4(\beta+3/2)}(-k\eta)\\ &\quad -\frac{i}{\pi}2^{2\beta}\sin(2\pi\beta)\Gamma^2\left(\beta+\frac{3}{2}\right)(-k\eta)^{-2\beta-2}\\ &\quad + \frac{\pi 2^{2\beta+1}}{\Gamma^2(-\beta-1/2)}(-k\eta)^{-2\beta-2} + \cdots.\end{aligned} \tag{52}$$

From this expression, one deduces that

$$\Re e\Omega_{\boldsymbol{k}}(\eta) = \frac{k\pi 2^{2\beta+1}}{\Gamma^2(-\beta-1/2)}(-k\eta)^{-2\beta-2} + \cdots \to 0, \tag{53}$$

and

$$\Im m\Omega_{\boldsymbol{k}}(\eta) = -\frac{1}{2\eta}(1+\beta) + \cdots = -\frac{a'}{2a} \to \infty, \tag{54}$$

where the limits are taken in the super-Hubble regime in which $k\eta \to 0$.

We have mentioned above that the Ricatti equation (20) can always be reduced to a linear second order differential equation. Of course, it can also be expressed as two linear, first order, differential equations. Therefore, one can introduce the functions $u_k(\eta)$ and $v_k(\eta)$ such that $f_k \equiv (u_k + v_k^*)/\sqrt{2k}$, the normalization $1/\sqrt{2k}$ being introduced for convenience. Then it is easy to show that these two functions obey

$$u_k' = iku_k + \frac{(a\sqrt{\epsilon_1})'}{a\sqrt{\epsilon_1}} v_k^*, \tag{55}$$

$$v_k' = ikv_k + \frac{(a\sqrt{\epsilon_1})'}{a\sqrt{\epsilon_1}} u_k^*. \tag{56}$$

The Wronskian $W = f_k' f_k^* - f_k'^* f_k$ can be straightforwardly evaluated as $W = i(|u_k|^2 - |v_k|^2)$. This means that, if we want to work with the choice $W = i$, one must have $|u_k|^2 - |v_k|^2 = 1$. This suggests to introduce the following parametrization:

$$u_k(\eta) = \mathrm{e}^{i\theta_k} \cosh r_k, \tag{57}$$

$$v_k(\eta) = \mathrm{e}^{-i\theta_k + 2i\phi_k} \sinh r_k. \tag{58}$$

The three functions $r_k(\eta)$, $\theta_k(\eta)$ and $\phi_k(\eta)$ are called the squeezing parameter, rotation angle and squeezing angle, respectively. It is clear that the knowledge of these three functions is equivalent to that of the function $\Omega_k(\eta)$ and, therefore, of the wave function. Upon using Eqs. (57) and (58), it is easy to show that

$$r_k' = \frac{(a\sqrt{\epsilon_1})'}{a\sqrt{\epsilon_1}} \cos(2\phi_k), \tag{59}$$

$$\phi_k' = k - \frac{(a\sqrt{\epsilon_1})'}{a\sqrt{\epsilon_1}} \coth(2r_k) \sin(2\phi_k), \tag{60}$$

$$\theta_k' = k - \frac{(a\sqrt{\epsilon_1})'}{a\sqrt{\epsilon_1}} \tanh r_k \sin(2\phi_k). \tag{61}$$

The explicit relation between Ω_k and the three squeezing parameters is given by

$$\Omega_k = \frac{k}{2} \frac{\cosh r_k - \mathrm{e}^{-2i\phi_k} \sinh r_k}{\cosh r_k + \mathrm{e}^{-2i\phi_k} \sinh r_k} - i\frac{a'}{2a}, \tag{62}$$

from which one deduces that

$$\Re\mathrm{e}\Omega_k = \frac{k}{2} \frac{1}{\cosh(2r_k) + \cos(2\phi_k)\sinh(2r_k)}, \tag{63}$$

$$\Im\mathrm{m}\Omega_k = \frac{k}{2} \frac{\sin(2\phi_k)\sinh(2r_k)}{\cosh(2r_k) + \cos(2\phi_k)\sinh(2r_k)} - \frac{a'}{2a}. \tag{64}$$

Equations (59)–(61), are highly nonlinear differential equations and cannot be solved in general. We notice that Eqs. (59) and (60) are in fact decoupled from Eq. (61). Therefore, they can be solved in a first step and then the solutions can be inserted in Eq. (61) to find the behavior of θ_k. In the case of power-law inflation, one can find explicit solutions for the de Sitter case, $\beta = -2$. Although this is not a solution for an arbitrary value of β, it is sufficient to understand the main features of the phenomenon of squeezing. One obtains

$$r_k(\eta) = -\mathrm{argsinh}\left(\frac{1}{2k\eta}\right), \tag{65}$$

$$\phi_k(\eta) = \frac{\pi}{4} + \frac{1}{2}\arctan\left(\frac{1}{2k\eta}\right). \tag{66}$$

Therefore, we see that, initially in the sub-Hubble limit, $r_k = 0$ (and $\phi_k = \pi/4$) while the super-Hubble limit corresponds to the limit of strong squeezing $r_k \to +\infty$ (and $\phi_k \to 0$).

Based on the previous considerations, it is clear that the super-Hubble limit is always associated with strong squeezing, even if we do not deal with the exact de Sitter solution. Indeed, now for an arbitrary β, Eq. (60) can be written as $\phi_k' \simeq -(\beta + 1)\sin(2\phi_k)/\eta$ which can be integrated and leads to $\phi_k \simeq \arctan[C|\eta|^{-2(\beta+1)}]$. For $\beta \lesssim -2$, this confirms the fact that $\phi_k \to 0$. In the same limit, one has $r_k' \simeq 1/\eta$ from which one obtains $r_k \propto (1+\beta)\ln a$. This confirms that the super-Hubble limit is the strong squeezing limit and, given the fact that modes of astrophysical interest today leave the Hubble scale 50–60 e-folds before the end of inflation, one can deduce that $r_k \simeq 120$ for those modes [29,30]. Compared to what can be achieved in the laboratory in quantum optics, this is a very large value [106].

In order to understand better the features of the quantum state (50), it is also interesting to calculate the mean values and dispersion of various quantities. First of all, it is clear that

$$\langle\Psi|\hat{v}_k^{\mathrm{R,I}}|\Psi\rangle = \langle\Psi|\hat{p}_k^{\mathrm{R,I}}|\Psi\rangle = 0. \tag{67}$$

Second, we also have

$$\langle\Psi|(\hat{v}_k^{\mathrm{R,I}})^2|\Psi\rangle = \frac{1}{4\Re\mathrm{e}\Omega_k}, \tag{68}$$

$$\langle\Psi|(\hat{p}_k^{\mathrm{R,I}})^2|\Psi\rangle = \Re\mathrm{e}\Omega_k + \frac{(\Im\mathrm{m}\Omega_k)^2}{\Re\mathrm{e}\Omega_k}. \tag{69}$$

Finally, the cross products can be expressed as

$$\langle\Psi|\hat{v}_k^{\mathrm{R}}\hat{p}_k^{\mathrm{R}}|\Psi\rangle = \frac{i\Omega_k}{2\Re\mathrm{e}\Omega_k}, \tag{70}$$

$$\langle\Psi|\hat{p}_k^{\mathrm{R}}\hat{v}_k^{\mathrm{R}}|\Psi\rangle = -i + \frac{i\Omega_k}{2\Re\mathrm{e}\Omega_k}, \tag{71}$$

and, of course, similar expressions for the operators $\hat{v}_k^{\mathrm{I}}$ and $\hat{p}_k^{\mathrm{I}}$. It is also interesting to notice that $\langle\Psi|\hat{v}_k^{\mathrm{R}}\hat{p}_k^{\mathrm{I}}|\Psi\rangle = \langle\Psi|\hat{v}_k^{\mathrm{I}}\hat{p}_k^{\mathrm{R}}|\Psi\rangle = 0$.

At this point, it is worth digressing about the definition of the conjugate momentum. The action (6) is of course defined up to a total derivative. In Ref. [15], it was shown that adding the term $\mathrm{d}[(a'/a)(v_{\boldsymbol{k}}^{\mathrm{R}})^2 + (a'/a)(v_{\boldsymbol{k}}^{\mathrm{I}})^2]/(2\mathrm{d}\eta)$ can also be viewed as a canonical transformation. This generates an additional term $(a'/a)(p_{\mathrm{k}}^{\mathrm{R,I}} v_{\mathrm{k}}^{\mathrm{R,I}*} + p_{\mathrm{k}}^{\mathrm{R,I}*} v_{\boldsymbol{k}}^{\mathrm{R,I}})$ in the Hamiltonian. A complete study was presented in Ref. [15] and, here, we only quote the main results. It was shown that, at the quantum level, this canonical transformation leaves the amplitude $\hat{v}_{\boldsymbol{k}}^{\mathrm{R,I}}$ invariant but induces the following transformations for the momentum: $\hat{p}_{\boldsymbol{k}}^{\mathrm{R,I}} \to \hat{\pi}_{\boldsymbol{k}}^{\mathrm{R,I}}$ with

$$\hat{\pi}_{\boldsymbol{k}}^{\mathrm{R,I}} = \hat{p}_{\boldsymbol{k}}^{\mathrm{R,I}} - \frac{a'}{a}\hat{v}_{\boldsymbol{k}}^{\mathrm{R,I}}. \tag{72}$$

On the other hand, the wave function is also modified, $\Psi_{\boldsymbol{k}} \to \bar{\Psi}_{\boldsymbol{k}}$, and the function $\Omega_{\boldsymbol{k}}$ changes according to $\Omega_{\boldsymbol{k}} \to \bar{\Omega}_{\boldsymbol{k}}$, where

$$\bar{\Omega}_{\boldsymbol{k}} = \Omega_{\boldsymbol{k}} + i\frac{a'}{2a}. \tag{73}$$

In particular, we see that the canonical transformation is such that the term $ia'/(2a)$ in the expression (62) of the function $\Omega_{\boldsymbol{k}}(\eta)$ is exactly canceled. The factor $N_{\boldsymbol{k}}$ of the wave function is not modified and is still given by the first of Eq. (21) (but of course should be used either with $\Omega_{\boldsymbol{k}}$ or $\bar{\Omega}_{\boldsymbol{k}}$ according to which set of variables is used). This also means that when the averages (67)–(71) are computed in the state $|\bar{\Psi}\rangle$, one obtains exactly the same expression, $\Omega_{\boldsymbol{k}}$ being just replaced with $\bar{\Omega}_{\boldsymbol{k}}$ (of course, $|\Psi\rangle$ and $|\bar{\Psi}\rangle$, being related by a canonical transformation, represent the same physical state).

We now come back to our calculation of the dispersion of amplitude operator and its conjugate momentum. Upon using Eqs. (68) and (63), one obtains

$$\langle\bar{\Psi}|(\hat{v}_{\boldsymbol{k}}^{\mathrm{R,I}})^2|\bar{\Psi}\rangle = \frac{1}{2k}[\cosh(2r_{\boldsymbol{k}}) + \cos(2\phi_{\boldsymbol{k}})\sinh(2r_{\boldsymbol{k}})]. \tag{74}$$

In the same manner, the dispersion of the operator $\hat{\pi}_{\boldsymbol{k}}^{\mathrm{R,I}}$ is given by

$$\langle\bar{\Psi}|(\hat{\pi}_{\boldsymbol{k}}^{\mathrm{R,I}})^2|\bar{\Psi}\rangle = \frac{k}{2}\frac{1+\sin^2(2\phi_{\boldsymbol{k}})\sinh^2(2r_{\boldsymbol{k}})}{\cosh(2r_{\boldsymbol{k}}) + \cos(2\phi_{\boldsymbol{k}})\sinh(2r_{\boldsymbol{k}})}. \tag{75}$$

Let us now consider two new operators $\hat{\mathcal{A}}_{\boldsymbol{k}}^{\mathrm{R,I}}$ and $\hat{\mathcal{B}}_{\boldsymbol{k}}^{\mathrm{R,I}}$, defined from $\hat{\pi}_{\boldsymbol{k}}^{\mathrm{R,I}}/\sqrt{k}$ and $\sqrt{k}\hat{v}_{\boldsymbol{k}}^{\mathrm{R,I}}$ through a rotation by the squeezing angle $\phi_{\boldsymbol{k}}$,

$$\hat{\mathcal{A}}_{\boldsymbol{k}}^{\mathrm{R,I}} = \frac{\hat{\pi}_{\boldsymbol{k}}^{\mathrm{R,I}}}{\sqrt{k}}\cos\phi_{\boldsymbol{k}} + \sqrt{k}\hat{v}_{\boldsymbol{k}}^{\mathrm{R,I}}\sin\phi_{\boldsymbol{k}}, \tag{76}$$

$$\hat{\mathcal{B}}_{\boldsymbol{k}}^{\mathrm{R,I}} = \frac{\hat{\pi}_{\boldsymbol{k}}^{\mathrm{R,I}}}{\sqrt{k}}\sin\phi_{\boldsymbol{k}} - \sqrt{k}\hat{v}_{\boldsymbol{k}}^{\mathrm{R,I}}\cos\phi_{\boldsymbol{k}}. \tag{77}$$

It is easy to check that $[\hat{\mathcal{A}}_{\boldsymbol{k}}, \hat{\mathcal{B}}_{\boldsymbol{k}}] = i$. Then, a lengthy but straightforward calculation leads to

$$\langle\bar{\Psi}|\hat{\mathcal{A}}_{\boldsymbol{k}}^{\mathrm{R,I}}|\bar{\Psi}\rangle = \frac{\mathrm{e}^{-2r_{\boldsymbol{k}}}}{2}, \tag{78}$$

$$\langle\bar{\Psi}|\hat{\mathcal{B}}_{\boldsymbol{k}}^{\mathrm{R,I}}|\bar{\Psi}\rangle = \frac{\mathrm{e}^{2r_{\boldsymbol{k}}}}{2}. \tag{79}$$

Therefore, we see that there exists a direction in the plane $(\pi_{\boldsymbol{k}}, v_{\boldsymbol{k}})$ where the dispersion is extremely small. This is why the corresponding state is called a squeezed state. In order to satisfy the Heisenberg inequality, the dispersion along the direction perpendicular to the previous one becomes very large. As already mentioned, the phenomenon of squeezing is widely studied in many different branches of physics, in particular in quantum optics. Squeezing occurs each time the quantization of a parametric oscillator is carried out. It is remarkable that the quantization of small fluctuations on top of an expanding universe also leads to that concept (squeezing here, i.e., $r_{\boldsymbol{k}} \neq 0$, does not require an accelerated expansion, only a dynamical background is necessary).

B. The classical limit

We have seen in the last section that the super-Hubble limit corresponds to a limit where the squeezing parameter $r_{\boldsymbol{k}}$ is large. In the literature, this regime is very often described as a regime where the cosmological perturbations have classicalized [31,32,39,39,107]. Since this concept is subtle in quantum mechanics (and particularly when quantum mechanics is applied to cosmology), we need to come back to this issue and to describe accurately what is meant by a "classical limit" in this context. In particular, it may seem strange at first sight that a quantum system placed in a strongly squeezed state can be described as a classical state since, in the context of, say, quantum optics, a similar situation would precisely be described as a nonclassical situation [108,109].

A convenient tool to study this question is the Wigner function, defined by

$$W(v_{\boldsymbol{k}}^{\mathrm{R}}, v_{\boldsymbol{k}}^{\mathrm{I}}, p_{\boldsymbol{k}}^{\mathrm{R}}, p_{\boldsymbol{k}}^{\mathrm{I}}) = \frac{1}{(2\pi)^2}\int \mathrm{d}x\mathrm{d}y\Psi^*\left(v_{\boldsymbol{k}}^{\mathrm{R}} - \frac{x}{2}, v_{\boldsymbol{k}}^{\mathrm{I}} - \frac{y}{2}\right) \times \mathrm{e}^{-ip_{\boldsymbol{k}}^{\mathrm{R}}x - ip_{\boldsymbol{k}}^{\mathrm{I}}y}\Psi\left(v_{\boldsymbol{k}}^{\mathrm{R}} + \frac{x}{2}, v_{\boldsymbol{k}}^{\mathrm{I}} + \frac{y}{2}\right). \tag{80}$$

Indeed, it is well known that the Wigner function can be understood as a classical probability distribution function whenever it is positive definite. Then, upon using the quantum state (50), the following explicit form is obtained

$$W(v_{\boldsymbol{k}}^{\mathrm{R}}, v_{\boldsymbol{k}}^{\mathrm{I}}, p_{\boldsymbol{k}}^{\mathrm{R}}, p_{\boldsymbol{k}}^{\mathrm{I}}) = \Psi\Psi^*\frac{1}{2\pi\Re\mathrm{e}\Omega_{\boldsymbol{k}}}\exp\left[-\frac{1}{2\Re\mathrm{e}\Omega_{\boldsymbol{k}}}(p_{\boldsymbol{k}}^{\mathrm{R}} + 2\Im\mathrm{m}\Omega_{\boldsymbol{k}}v_{\boldsymbol{k}}^{\mathrm{R}})^2\right] \times \exp\left[-\frac{1}{2\Re\mathrm{e}\Omega_{\boldsymbol{k}}}(p_{\boldsymbol{k}}^{\mathrm{I}} + 2\Im\mathrm{m}\Omega_{\boldsymbol{k}}v_{\boldsymbol{k}}^{\mathrm{I}})^2\right]. \tag{81}$$

The following remark is in order at this stage. One could have calculated the Wigner function with the state $\bar{\Psi}_{\boldsymbol{k}}$.

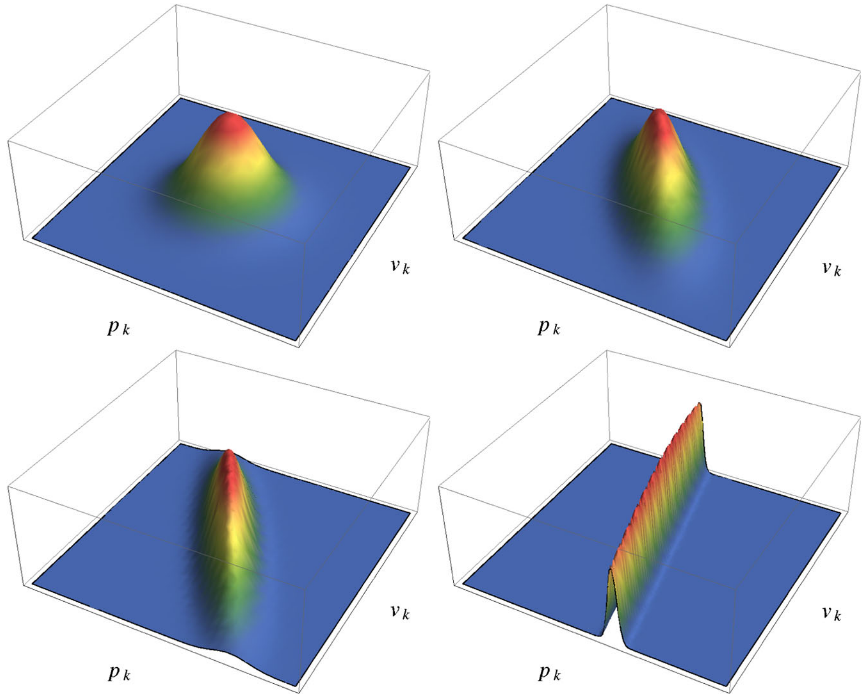

FIG. 1 (color online). Wigner function of a squeezed quantum state at different times during inflation. Only the two-dimensional function corresponding to the set of variables $(v_{\boldsymbol{k}}^{\mathrm{R}}, p_{\boldsymbol{k}}^{\mathrm{R}})$ has been represented, see Eq. (81). The time evolution of $\Re\mathrm{e}\Omega_{\boldsymbol{k}}$ and $\Im\mathrm{m}\Omega_{\boldsymbol{k}}$ has been expressed in terms of the two squeezing parameters $r_{\boldsymbol{k}}$ and $\phi_{\boldsymbol{k}}$. These ones are given by the solutions (65) and (66). The left upper panel corresponds to $r_{\mathrm{k}} = 0.0005$ and the corresponding state is almost a coherent one. The right upper panel corresponds to $r_{\boldsymbol{k}} = 0.48$, the left bottom one to $r_{\boldsymbol{k}} = 0.88$ and, finally, the right bottom one to $r_{\boldsymbol{k}} = 2.31$. The effect of the squeezing and the cigar shape of Eq. (82) are clearly visible.

Obviously, one would have obtained exactly the same expression except that all the $\Omega_{\boldsymbol{k}}$ terms would have been replaced with $\bar{\Omega}_{\boldsymbol{k}}$ and $p_{\boldsymbol{k}}^{\mathrm{R,I}}$ with $\pi_{\boldsymbol{k}}^{\mathrm{R,I}}$. In particular, this means that the term in parenthesis in the argument of the exponentials would have read $\pi_{\boldsymbol{k}}^{\mathrm{R,I}} + 2\Im\mathrm{m}\bar{\Omega}_{\boldsymbol{k}} v_{\boldsymbol{k}}^{\mathrm{R,I}}$. But, thanks to Eqs. (72) and (73), this is precisely $p_{\boldsymbol{k}}^{\mathrm{R,I}} + 2\Im\mathrm{m}\Omega_{\boldsymbol{k}} v_{\boldsymbol{k}}^{\mathrm{R,I}}$ since the two terms proportional to a'/a exactly cancel out. This is of course related to the fact that the Wigner function is invariant under a canonical transformation.

The Wigner function (81) is represented in Fig. 1 at different times or, equivalently, at different values of $r_{\boldsymbol{k}}$ ($r_{\boldsymbol{k}} = 0.0005$, 0.48, 0.88 and 2.31). The effect of the strong squeezing is clearly visible. Initially, in the sub-Hubble regime, $r_{\boldsymbol{k}}$ is small and the Wigner function is peaked over a small region in phase space. As inflation proceeds, the modes become super Hubble and $r_{\boldsymbol{k}}$ increases. As a consequence, the Wigner function spreads and acquires a cigar shape typical of squeezed states. In fact, in the strong squeezing limit, one has $\Re\mathrm{e}\bar{\Omega}_{\boldsymbol{k}} \to 0$ and $\Im\mathrm{m}\bar{\Omega}_{\boldsymbol{k}} \to k\sin\phi_{\boldsymbol{k}}/(2\cos\phi_{\boldsymbol{k}}) \to 0$, see Eqs. (63) and (64). Let us notice in passing that this last equation is consistent with Eq. (54). On the other hand, if one considers $\Im\mathrm{m}\bar{\Omega}_{\boldsymbol{k}}$, then the leading term $a'/(2a)$ is absent and one has to go to the next order in Eq. (64). This one is given by $k/[4(\beta+3/2)](-k\eta)$ and represents the leading term of $\Im\mathrm{m}\bar{\Omega}_{\boldsymbol{k}}$. It goes to zero in agreement with the fact that $\phi_{\boldsymbol{k}} \to 0$ in the strong squeezing limit. In this regime, the Wigner function can be written as

$$W(v_{\boldsymbol{k}}^{\mathrm{R}}, v_{\boldsymbol{k}}^{\mathrm{I}}, p_{\boldsymbol{k}}^{\mathrm{R}}, p_{\boldsymbol{k}}^{\mathrm{I}}) \to \Psi\Psi^* \delta\left(p_{\boldsymbol{k}}^{\mathrm{R}} + k\frac{\sin\phi_{\boldsymbol{k}}}{\cos\phi_{\boldsymbol{k}}} v_{\boldsymbol{k}}^{\mathrm{R}}\right) \times \delta\left(p_{\boldsymbol{k}}^{\mathrm{I}} + k\frac{\sin\phi_{\boldsymbol{k}}}{\cos\phi_{\boldsymbol{k}}} v_{\boldsymbol{k}}^{\mathrm{I}}\right). \tag{82}$$

This last equation represents the mathematical formulation of the cigar shape mentioned above.

It is important to notice that the behavior described above is very different from the behavior of the Wigner function of a coherent state. The coherent states are usually considered as the "most classical" states and their Wigner function is given by

$$W(v_{\boldsymbol{k}}^{\mathrm{R}}, p_{\boldsymbol{k}}^{\mathrm{R}}) = \frac{1}{\pi} \mathrm{e}^{-k[v_{\boldsymbol{k}}^{\mathrm{R}} - v_{\boldsymbol{k}}^{\mathrm{R,cl}}(\eta)]^2} \mathrm{e}^{-[p_{\boldsymbol{k}}^{\mathrm{R}} - p_{\boldsymbol{k}}^{\mathrm{R,cl}}(\eta)]^2/k}, \tag{83}$$

where $v_{\boldsymbol{k}}^{\mathrm{R,cl}}$ and $p_{\boldsymbol{k}}^{\mathrm{R,cl}}$ represent the classical solutions. The typical shape is plotted in Fig. 2. One sees that the Wigner functions remain peaked over a small region in phase space and that this packet follows the classical trajectory (an ellipse in this context). Comparing Figs. 1 and 2, we understand why a coherent state is usually considered as classical while a squeezed state is considered as highly nonclassical. In the case of the coherent state, if one is given, say, the value of $v_{\boldsymbol{k}}^{\mathrm{R}}$, then one obtains a value for the

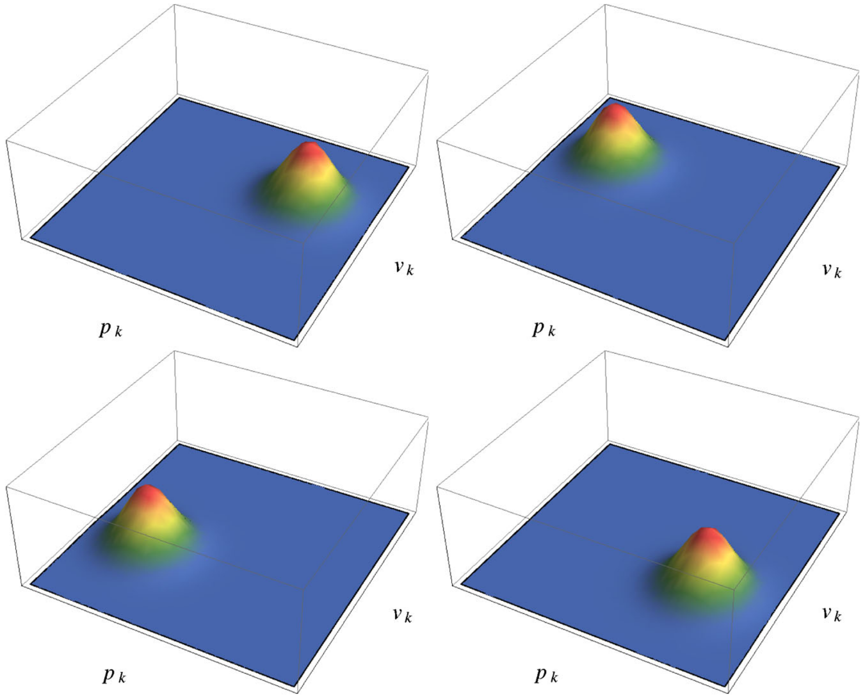

FIG. 2 (color online). Wigner function of a coherent state (83), represented at different times during inflation. Contrary to the Wigner function of a squeezed state of Fig. 1, the shape remains unchanged during the cosmological evolution. The Wigner function just follows the classical trajectory, an ellipse here since we deal with an harmonic oscillator. This justifies the fact that a coherent state can be viewed as the "most classical quantum state."

momentum, $p_{\boldsymbol{k}}^{\mathrm{R}}$, which is very close to the one we would have inferred in the classical case. This is of course due to the fact that the Wigner function follows the classical trajectory and has minimal spread around it in all phase space directions. On the contrary, in the case of the squeezed state, if one is given $p_{\boldsymbol{k}}^{\mathrm{R}}$ then the value of $v_{\boldsymbol{k}}^{\mathrm{R}}$ is very uncertain since the Wigner function is spread over a large region in phase space. Therefore, we conclude that the cosmological perturbations do not behave classically in the usual sense.

Given the previous discussion, it may seem relatively easy to observe genuine quantum effects in the CMB. Unfortunately this is not so, essentially because, in the strong squeezed limit, all quantum predictions can be in fact obtained from averages performed by mean of a classical stochastic process.

Let us first study how this question is usually treated. For this purpose, let us consider again the expectation of the operator $(\hat{\pi}_{\boldsymbol{k}}^{\mathrm{R}})^2$ [of course, one could also treat the case of $(\hat{\pi}_{\boldsymbol{k}}^{\mathrm{R}})^n$]. The quantum average is given by Eq. (69), namely

$$\langle\bar{\Psi}|(\hat{\pi}_{\boldsymbol{k}}^{\mathrm{R}})^2|\bar{\Psi}\rangle = \Re\mathrm{e}\bar{\Omega}_{\boldsymbol{k}} + \frac{(\Im\mathrm{m}\bar{\Omega}_{\boldsymbol{k}})^2}{\Re\mathrm{e}\bar{\Omega}_{\boldsymbol{k}}}. \tag{84}$$

On the other hand, if one computes the quantity

$$\int \mathrm{d}v_{\boldsymbol{k}}^{\mathrm{R}}\mathrm{d}\pi_{\boldsymbol{k}}^{\mathrm{R}} W_{r_k\to\infty}(v_{\boldsymbol{k}}^{\mathrm{R}}, \pi_{\boldsymbol{k}}^{\mathrm{R}})(\pi_{\boldsymbol{k}}^{\mathrm{R}})^2, \tag{85}$$

where $W_{r_k\to\infty}(v_{\boldsymbol{k}}^{\mathrm{R}}, \pi_{\boldsymbol{k}}^{\mathrm{R}})$ refers to the Wigner function in the strong squeezing limit (82), then one obtains $(\Im\mathrm{m}\bar{\Omega}_{\boldsymbol{k}})^2/\Re\mathrm{e}\bar{\Omega}_{\boldsymbol{k}}$, which coincides with Eq. (84) in the limit $r_{\boldsymbol{k}} \to \infty$. This result is often taken as a proof that a strongly squeezed state can be described as a classical stochastic process. However, this argument is not very convincing since it is a theorem [31] that the exact Wigner function [we stress again that, in Eq. (85), we have not used the general Wigner function but its limit when $r_{\boldsymbol{k}}$ is large] satisfies the following property:

$$\langle\hat{A}(\hat{v}_{\boldsymbol{k}}^{\mathrm{R}}, \hat{\pi}_{\boldsymbol{k}}^{\mathrm{R}})\rangle = \int \mathrm{d}v_{\boldsymbol{k}}^{\mathrm{R}}\mathrm{d}\pi_{\boldsymbol{k}}^{\mathrm{R}} W(v_{\boldsymbol{k}}^{\mathrm{R}}, \pi_{\boldsymbol{k}}^{\mathrm{R}})A(v_{\boldsymbol{k}}^{\mathrm{R}}, \pi_{\boldsymbol{k}}^{\mathrm{R}}), \tag{86}$$

where $\hat{A}$ is an arbitrary operator. Therefore, it does not come as a surprise that an expression like Eq. (85) reproduces the corresponding quantum average in the limit $r_{\boldsymbol{k}} \to \infty$.

In fact, as was discussed in Refs. [34,39,110], what makes the situation so peculiar is something different. The point is that, in the limit $r_{\boldsymbol{k}} \to \infty$, all the quantum predictions can be reproduced if one assumes that the system always followed classical laws but had random initial conditions with a given probably density function. This can be easily understood on the example of a free particle [34,39,110]. Let us assume that, initially (at $t = 0$), the probability to find the particle at x is given by

$$|\Psi(x, 0)|^2 = \sqrt{\frac{2}{\pi b^2}}\,\mathrm{e}^{-2x^2/b^2}, \tag{87}$$

where b is a parameter that characterizes the width of the distribution. At time t, this probability is given by

$$|\Psi(x,t)|^2=\sqrt{\frac{2}{\pi b^2}}\frac{1}{\sqrt{1+4t^2/(m^2b^4)}}\times\exp\left[-\frac{2b^2(x-k_0t/m)^2}{b^4+4t^2/m^2}\right], \tag{88}$$

where m is the mass of the particle and k_0 the center of the Gaussian wave packet in Fourier space.

Now let us consider a situation where we repeat many times an experiment consisting in sending a classical particle from the origin with a velocity v (equivalently, instead of repeating the experiments many times, one could also consider an ensemble of classical particles) and detecting it at a position $x \neq 0$. By definition, the particle follows the laws of classical physics which means that its motion can be described by the equation: $x = vt$ (they all start from $x = 0$ at $t = 0$). Then, let us assume that the velocities are classical random variables with a probability distribution function given by

$$P(v)=\frac{1}{\sqrt{\pi}\Delta v}\,\mathrm{e}^{-v^2/(\Delta v)^2}. \tag{89}$$

This means that according to the particle considered, the velocity is in fact not always the same. But because different particles have different velocities, they will not reach the position x at the same time. It is important to stress that, here, only the initial conditions are random and that the trajectory is purely classical. From the above distribution, we can easily infer that the probability of finding a particle at x, at time t, is

$$P(x,t)=\frac{1}{\sqrt{\pi}t\Delta v}\,\mathrm{e}^{-(x-vt)^2/(t\Delta v)^2}. \tag{90}$$

This distribution is in fact exactly $|\Psi(x,t)|^2$ in the limit $t\to\infty$ provided we identify $v=k_0/m$ and $\Delta v=\sqrt{2}/(mb)$. Let us notice that this last relation is exactly what is obtained at the quantum level since x and v are conjugate variables. As a matter of fact, Eqs. (87) and (89) are Fourier transforms of each other. We conclude that, provided we detect the particles far from the origin, the quantum predictions for the particles can be completely mimicked by means of a classical stochastic process.

As discussed in Ref. [110], the situation is exactly similar for the inflationary perturbations. The limit $r_{\boldsymbol{k}}\to\infty$ is in fact equivalent to the limit of large times in the example above. One can even calculate the Wigner function of the free particle described by the wave function (88) and show that it takes the same form as the one of Eq. (81). Therefore, the inflationary perturbations are said to be classical in the sense explained before: they can be described by a classical stochastic process. In practice, for instance, one can consider the $\hat{a}_{\ell m}$ in Eqs. (38) and (39) as classical random variables with probability density functions given by

$$P(a_{\ell 0}^{\mathrm{R}})=\frac{1}{\sqrt{2\pi}C_\ell}\,\mathrm{e}^{-(a_{\ell 0}^{\mathrm{R}})^2/(2C_\ell)}, \tag{91}$$

$$P(a_{\ell m}^{\mathrm{R}})=\frac{1}{\sqrt{\pi}C_\ell}\,\mathrm{e}^{-(a_{\ell m}^{\mathrm{R}})^2/C_\ell}, \qquad m\neq 0, \tag{92}$$

$$P(a_{\ell m}^{\mathrm{I}})=\frac{1}{\sqrt{\pi}C_\ell}\,\mathrm{e}^{-(a_{\ell m}^{\mathrm{I}})^2/C_\ell}, \qquad m\neq 0. \tag{93}$$

Of course one can check that $\langle a_{\ell m}^{\mathrm{R,I}}a_{\ell' m'}^{\mathrm{R,I}}\rangle = C_\ell\delta_{\ell\ell'}\delta_{mm'}$ where, now, the bracket means a classical average calculated by means of the above distributions.

Finally, we conclude this section by a few words on the density matrix $\hat{\rho}_{\boldsymbol{k}}^{\mathrm{R}}$. In fact, the density matrix is nothing but the Fourier transform of the Wigner function. Let us denote by $|v_{\boldsymbol{k}}^{\mathrm{R}}\rangle$ the eigenstates of the operator $\hat{v}_{\boldsymbol{k}}^{\mathrm{R}}$. Then, we have

$$\langle v_{\boldsymbol{k}}^{\mathrm{R}\prime}|\hat{\rho}_{\boldsymbol{k}}^{\mathrm{R}}|v_{\boldsymbol{k}}^{\mathrm{R}}\rangle=\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{e}^{iy(v_{\boldsymbol{k}}^{\mathrm{R}\prime}-v_{\boldsymbol{k}}^{\mathrm{R}})}W\left(\frac{v_{\boldsymbol{k}}^{\mathrm{R}\prime}+v_{\boldsymbol{k}}^{\mathrm{R}}}{2},y\right). \tag{94}$$

Upon using Eq. (81) in the above equation, one arrives at

$$\langle v_{\boldsymbol{k}}^{\mathrm{R}\prime}|\hat{\rho}_{\boldsymbol{k}}^{\mathrm{R}}|v_{\boldsymbol{k}}^{\mathrm{R}}\rangle=\left(\frac{2\Re\mathrm{e}\Omega_{\boldsymbol{k}}}{\pi}\right)^{1/2}\mathrm{e}^{-\Re\mathrm{e}\Omega_{\boldsymbol{k}}[(v_{\boldsymbol{k}}^{\mathrm{R}\prime})^2+(v_{\boldsymbol{k}}^{\mathrm{R}})^2]}\times\mathrm{e}^{-i\Im\mathrm{m}\Omega_{\boldsymbol{k}}[(v_{\boldsymbol{k}}^{\mathrm{R}\prime})^2-(v_{\boldsymbol{k}}^{\mathrm{R}})^2]}. \tag{95}$$

We notice that the off-diagonal terms, $v_{\boldsymbol{k}}^{\mathrm{R}\prime}\neq v_{\boldsymbol{k}}^{\mathrm{R}}$, oscillate very rapidly in the strong squeezing limit. This means that decoherence (defined as the disappearance of those off-diagonal terms) does not occur without taking into account an environment for the perturbations. Various discussions on what this environment may be can be found in Refs. [41–43].

C. Ergodicity

Let us now discuss how, in practice, we can check the predictions of the theory previously reviewed. Initially, the system is placed in an eigenstate of the Hamiltonian (the vacuum state) of Eq. (19), which can also be expressed as a superposition in the basis of the states $|v_{\boldsymbol{k}}^{\mathrm{R}}\rangle$, namely

$$|\Psi\rangle=\int\mathrm{d}v_{\boldsymbol{k}}^{\mathrm{R}}N_{\boldsymbol{k}}(\eta)\,\mathrm{e}^{-\Omega_{\boldsymbol{k}}(\eta)(v_{\boldsymbol{k}}^{\mathrm{R}})^2}|v_{\boldsymbol{k}}^{\mathrm{R}}\rangle. \tag{96}$$

The corresponding mean value of the Hamiltonian operator can be expressed as

$$\langle\Psi|\mathcal{H}_{\boldsymbol{k}}^{\mathrm{R}}|\Psi\rangle=\frac{1}{2}\Re\mathrm{e}\Omega_{\boldsymbol{k}}+\frac{1}{2}\frac{(\Im\mathrm{m}\Omega_{\boldsymbol{k}})^2}{2\Re\mathrm{e}\Omega_{\boldsymbol{k}}}+\frac{\omega^2}{2}\frac{1}{4\Re\mathrm{e}\Omega_{\boldsymbol{k}}}. \tag{97}$$

Of course, initially $\Omega_{\boldsymbol{k}}=k/2$ and the energy is nothing but $\omega/2$ as expected for the vacuum state.

In the real world, we measure the temperature anisotropies. As we have seen (and as is appropriate for an observable in the quantum-mechanical framework), this quantity is represented by an operator. According to Eq. (38), measuring the temperature anisotropies is equivalent to measuring the observables $\hat{a}_{\ell m}$ which, in turn, according to

Eq. (41), is equivalent to measuring the observables $\hat{\zeta}_{\boldsymbol{k}}$ or $\hat{v}_{\boldsymbol{k}}$ (that is to say $\hat{v}_{\boldsymbol{k}}^{\mathrm{R}}$ and $\hat{v}_{\boldsymbol{k}}^{\mathrm{I}}$).

According to the postulates of quantum mechanics, measuring the observable $\hat{v}_{\boldsymbol{k}}^{\mathrm{R}}$ gives an eigenvalue $v_{\boldsymbol{k}}^{\mathrm{R}}$ (no hat, it is a number) with probability $|\langle v_{\boldsymbol{k}}^{\mathrm{R}}|\Psi\rangle|^2$ and, immediately after this measurement, the system is placed in the eigenstate $|v_{\boldsymbol{k}}^{\mathrm{R}}\rangle$. More concretely, after the measurement, we "see" a specific CMB map and we say that the measurement has produced a specific "realization." The result is given in terms of coefficients $a_{\ell m}$ (again, no hat) expressed in terms of the numbers $v_{\boldsymbol{k}}^{\mathrm{R}}$ through Eq. (41) (except, of course, that this equation should now be used with no hat on both sides). Equivalently, we see a specific temperature pattern $\delta T(\boldsymbol{e})/T$ (no hat) corresponding to the set of numbers $a_{\ell m}$, see also Eq. (38). In conclusion, the CMB map observed, say, by the WMAP satellite corresponds to one measurement (or one realization) of the operator $\widehat{\delta T}(\boldsymbol{e})/T$.

Then comes the question of how one can operationally verify these theoretical predictions. In quantum mechanics, in an ordinary laboratory situation, one would check that the theory is correct by repeating the experiment many times. In this way, one would generate many realizations of $\hat{v}_{\boldsymbol{k}}^{\mathrm{R}}$ (or, equivalently, of $\hat{a}_{\ell m}$ or $\widehat{\delta T/T}$) i.e., one would obtain N_{real} numbers $v_{\boldsymbol{k}}^{\mathrm{R}i}, i = 1, \cdots, N_{\mathrm{real}}$ [or $a_{\ell m}^i$ or $(\delta T/T)^i$] where N_{real} is the number of realizations (that is to say the number of times the experiments have been performed). With these N_{real} CMB maps, one could then check that the $v_{\boldsymbol{k}}^{\mathrm{R}i}$ are indeed distributed with a Gaussian probability density function in agreement with Eq. (50) or, with the N_{real} sets of numbers $a_{\ell m}^i$, one could infer whether they follow Eqs. (91)–(93), determine the corresponding variance and check that it is given by the C_ℓ predicted by the theory. Let us notice that the above discussion is independent from the fact that the perturbations can be described classically or not. If we are in the classical limit (in the restricted sense defined in the previous section), then we showed that measuring the observable $\hat{a}_{\ell m}$ can be viewed as measuring a classical system with random initial conditions but this does not change the fact that we need many realizations to check that the probability density function predicted by the theory is the correct one.

Clearly, in cosmology, the program described above cannot be carried out because one cannot repeat the experiment many times since we are given only one CMB map [33]. How, then, can we check the predictions of the theory of cosmological perturbations? To discuss this question, let us be more accurate about the operator $\widehat{\delta T}/T(\boldsymbol{e})$. In the large-squeezing limit, we have seen that it can be viewed as a classical stochastic process and, therefore, it is convenient to write it as

$$\frac{\delta T}{T}(\xi, \boldsymbol{e}), \tag{98}$$

where the symbol ξ labels the realizations. A given realization of a stochastic process is a function of $\boldsymbol{e}$. By contrast, a given realization of a random variable is not a function but a number. This is for instance the case of $a_{\ell m}(\xi)$. The idea is then to replace ensemble averages by spatial averages (i.e., averages over different directions $\boldsymbol{e}$) [33]. If the process is ergodic, these two types of averages are equal [33]. In that case, one can check the predictions of the theory even if one has only one realization at our disposal. Unfortunately, one can also show that a stochastic process living on a sphere (here, of course, the celestial sphere) cannot be ergodic [33]. Therefore, we are left with the task of constructing unbiased estimators with minimal variances. For instance, let us assume that we have calculated the number C_ℓ in some inflationary scenario and that we would like to compare its value to an actual measurement. How would we proceed? We would consider the random variable $\mathcal{C}_\ell(\xi)$ defined by the following expression [33]:

$$\mathcal{C}_\ell(\xi) = \frac{1}{4\pi}\int_{S^2} \mathrm{d}\Omega_1 \mathrm{d}\Omega_2 P_\ell(\cos\delta_{12})\frac{\delta T}{T}(\xi, \boldsymbol{e}_1)\frac{\delta T}{T}(\xi, \boldsymbol{e}_2), \tag{99}$$

where δ_{12} is the angle between the direction $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. As announced, the estimator $\mathcal{C}_\ell(\xi)$ is expressed as a spatial average of the stochastic process $\delta T/T$. It is easy to show that it is unbiased, $\langle\langle \mathcal{C}_\ell \rangle\rangle = C_\ell$ and has the minimum variance [33] (called the "cosmic variance") given by $\sqrt{2/(2\ell + 1)}C_\ell$. The double brackets $\langle\langle\rangle\rangle$ mean an ensemble average, which amounts to a quantum average in the high squeezing limit as mentioned before. One should be careful that this ensemble average has nothing to do with the one introduced below (denoted $\mathbb{E}$) for the CSL modifications of the Schrödinger equation, since these two stochasticities have completely different natures, the former being effective and the later intrinsic.

In practice, we would proceed as follows. From our CMB map $\delta T(\xi, \boldsymbol{e})/T$, we compute the integral in Eq. (99) and this gives a number representing one realization of the estimator $\mathcal{C}_\ell$, the only one we can have access to. It is unlikely that this number will be C_ℓ because it is unlikely that one realization of a random variable will be exactly equal to the mean value of that variable. However, if the variance is small (i.e., if the estimator is good), the corresponding probability density function will be sharply peaked around the mean value and any realization will therefore be close to the mean (and, in our case, it is not possible to decrease the value of the variance since we work with the best estimator). Therefore, we can study where the number we have obtained by following the above described procedure falls, compared to the interval $C_\ell \pm \sqrt{2/(2\ell + 1)}C_\ell$, where C_ℓ is the theoretically predicted multipole moment. Then, for instance, one can start a calculation of the χ^2 to assess to which confidence we have verified the theory. In fact, the cosmic variance can simply be seen as another source of error, besides those coming from the instruments.

Given the previous discussion, there is one issue that one can raise and which is the subject of the present paper. The question is how a specific outcome (a realization) is produced. Above, we have just assumed that this happens without discussing this point. According to the postulates of quantum mechanics in the Copenhagen interpretation, this macro-objectification takes place when a measurement is performed. Since the CMB anisotropies were produced at a redshift of $z_{\ell ss} \simeq 1100$, this means that it should have happened prior to that epoch (possibly during inflation itself). But, clearly, there was no observer at those early times. We face here the conventional measurement problem of quantum mechanics which is, in the context of cosmology, exacerbated.

IV. THE PEARLE-GHIRARDI-RIMINI-WEBER THEORY

A. A dynamical collapse model

Although one can manage to obtain, based on primordial vacuum quantum fluctuations, a set of correlation functions that are formally indistinguishable from a classical stochastic distribution, one still has to face the problem of reaching a specific realization before cosmological perturbations can start to grow in a classical way. This amounts to the question of the measurement problem in quantum mechanics, namely that there are two distinct evolution processes: the unitary and linear Schrödinger time evolution on the one hand, and the stochastic and nonlinear wave packet reduction on the other.

In what follows, we briefly present the collapse theories and explain how the Schrödinger equation can be modified in order to allow a dynamical description of the wave packet reduction. In fact, to be more precise, we shall restrict attention to the case of CSL [62,63,65,67].

The CSL model relies on the idea that an extra stochastic behavior should be added to the Schrödinger linear evolution, encoded through a Wiener process W_t, whose differential acts as a random square root of that of time, namely

$$\mathbb{E}(\mathrm{d}W_t) = 0, \quad \text{and} \quad \mathbb{E}(\mathrm{d}W_t \mathrm{d}W_{t'}) = \delta(t-t')\mathrm{d}t^2, \qquad (100)$$

where $\mathbb{E}$ stands for an ensemble average. One then expands the state vector variation $\mathrm{d}|\chi\rangle$ up to first order in time through

$$\mathrm{d}\,|\chi\rangle = (\hat{A}\mathrm{d}t + \hat{B}\mathrm{d}W_t)|\chi\rangle, \qquad (101)$$

where $\hat{A}$ and $\hat{B}$ are operators acting on the Hilbert space of available states. One then demands that, on average, the wave function will be normalized, i.e.,

$$\mathbb{E}\left(\langle\chi|\chi\rangle\right) = 1 \Rightarrow \mathbb{E}[\mathrm{d}(\langle\chi|\chi\rangle)] = 0, \qquad (102)$$

which, upon using Itô calculus[3] for the differentials and Eq. (100), yields

$$\hat{A}^\dagger + \hat{A} = -\hat{B}^\dagger\hat{B}, \qquad (103)$$

since the state $|\chi\rangle$ is arbitrary. The general solution of Eq. (103) is $\hat{A} = -i\hat{H} - \frac{1}{2}\hat{B}^\dagger\hat{B}$, where $\hat{H}$ is Hermitian and to be identified with the Hamiltonian leading to the usual Schrödinger dynamics.

In order to assign a probabilistic meaning to the norm of the wave function, it should be normalized. However, according to Eq. (101), although this is true on average, it varies stochastically according to

$$\mathrm{d}\|\chi\|^2 = \langle\chi|(\hat{B}+\hat{B}^\dagger)|\chi\rangle\mathrm{d}W_t = 2\langle\chi|\hat{B}|\chi\rangle\mathrm{d}W_t, \qquad (104)$$

where from now on we assume that $\hat{B}$ is Hermitian.

Equation (104) implies that the state $|\chi\rangle$ is not normalized, and one can define a normalized one whose probability distribution will thus be interpretable in terms of measurements. We then set

$$|\psi\rangle \equiv \frac{|\chi\rangle}{\|\chi\|}, \qquad (105)$$

whose dynamics can be computed using the previously derived rules. One finds

$$\mathrm{d}|\psi\rangle = \left\{\left[-i\hat{H} - \frac{1}{2}(\hat{B}-\langle\hat{B}\rangle)^2\right]\mathrm{d}t + (\hat{B}-\langle\hat{B}\rangle)\mathrm{d}W_t\right\}|\psi\rangle, \qquad (106)$$

where the quantum expectation value is taken on the normalized state vector and thus defined as

$$\langle\hat{B}\rangle \equiv \langle\psi|\hat{B}|\psi\rangle. \qquad (107)$$

The operator $\hat{B}$ can be decomposed as $\hat{B} = \sqrt{\gamma}\hat{Q}$. The coupling constant γ is the product of the localization rate with the width of the Gaussian wave function inducing the localizations [62], and sets the strength of the nonlinear effects and therefore the characteristic time scale over which these are measurable. The observable $\hat{Q}$, for instance the position operator, is the basis on which the states are to spontaneously collapse to (in the following, we also call the operator $\hat{Q}$, the "collapse operator").

As it turns out, and this is exemplified later in the case where the operator $\hat{Q}$ is identified with a cosmological perturbation Fourier mode (see Sec. VA), the natural evolution of Eq. (106) is to project an initial state $|\psi_0\rangle$ on an eigenstate $|\alpha\rangle$ of the operator $\hat{Q}$ setting

$$\hat{Q} = \sum_\alpha q_\alpha|\alpha\rangle\langle\alpha|, \qquad (108)$$

(the sum being replaced by an integral in the case of a continuous spectrum for $\hat{Q}$) such that $\hat{Q}|\alpha\rangle = q_\alpha|\alpha\rangle$, one finds that $\lim_{t\to\infty}|\Psi(t)\rangle = |\alpha\rangle$ for a given value of α, and this with a probability $P(\alpha) = |\langle\Psi|\alpha\rangle|^2$. In other words, the Born rule is naturally implemented as a dynamical consequence instead of being imposed as an extra hypothesis.

[3]This means that for two functions f and g of the stochastic variable W, one has $\mathrm{d}(fg) = f\mathrm{d}g + (\mathrm{d}f)g + \mathbb{E}[(\mathrm{d}f)(\mathrm{d}g)]$ and $\mathrm{d}f(W) = f'(W)\mathrm{d}W + \frac{1}{2}f''(W)\mathbb{E}[(\mathrm{d}W)^2]$, where a prime stands for ordinary derivative with respect to the argument W. It is necessary to expand up to second order in the noise because Eq. (100) means $\mathbb{E}(\mathrm{d}W_t^2) = \mathrm{d}t$.

Finally, defining the density operator as

$$\hat{\rho} \equiv \mathbb{E}(|\Psi\rangle\langle\Psi|), \tag{109}$$

one obtains, using Eq. (106) the so-called Lindblad equation, namely

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -i[\hat{H}, \hat{\rho}] - \frac{\gamma}{2}[\hat{Q}, [\hat{Q}, \hat{\rho}]], \tag{110}$$

providing its time development.

Let us now come to another very important aspect of the CSL theory and describe the so-called "amplification mechanism" which enables one to understand why the dynamics of microscopic systems is not much altered by the extra stochastic and nonlinear terms in Eq. (106). This is phenomenologically very important since this means that the laboratory experiments performed on "small" quantum systems are still accurately predicted by the standard Schrödinger equation while the macroscopic objects are quickly and efficiently localized. Let us consider an ensemble of N identical particles, assuming that, for each of them, the collapse operator is the physical position in space. Therefore, we can identify the operator and Wiener processes according to

$$\hat{B} \to \sqrt{\gamma}\sum_{i=1}^{N}\hat{x}_i \quad \text{and} \quad \mathrm{d}W_t \to \mathrm{d}W_t^{(i)} \tag{111}$$

in Eq. (106), with $\hat{x}_i$ the position operator for the ith particle. Note that in this case, one has as many independent Wiener processes as there are particles; they satisfy

$$\mathbb{E}[\mathrm{d}W_t^{(i)}\mathrm{d}W_{t'}^{(j)}] = \delta^{ij}\delta(t-t')\mathrm{d}t^2. \tag{112}$$

This naturally generalizes Eq. (106) to a set of operators and particles on which to project the relevant states.

We now assume that one can decompose the total wave vector $|\Psi\rangle$ in the form

$$|\Psi(\{x_i\})\rangle = |\Psi_{\mathrm{CM}}(R)\rangle \otimes |\Psi_{\mathrm{rel}}(\{r_i\})\rangle, \tag{113}$$

where the total wave function depends on the set of all of the position operators $\{x_i\}$, while the "macroscopic" part of it, $|\Psi_{\mathrm{CM}}\rangle$, depends only on the position $R \equiv N^{-1}\sum_i x_i$ of the center of mass, and the rest is a function only of the relative coordinates r_i defined through $x_i = R + r_i$.

Using Itô calculus to evaluate the differential of the tensor product in Eq. (113), it is easily checked that $|\Psi(\{x_i\})\rangle$ satisfies Eq. (106) with $\hat{B}$ and dW_t given by Eq. (111) if the components of the product respectively satisfy

$$\mathrm{d}|\Psi_{\mathrm{CM}}(R)\rangle = \left\{\left[-i\hat{H}_{\mathrm{CM}} - \frac{\gamma_{\mathrm{CM}}}{2}(\hat{R} - \langle\hat{R}\rangle)^2\right]\mathrm{d}t + \sqrt{\gamma_{\mathrm{CM}}}(\hat{R} - \langle\hat{R}\rangle)dW_t\right\}|\Psi_{\mathrm{CM}}(R)\rangle, \tag{114}$$

and

$$\mathrm{d}|\Psi_{\mathrm{rel}}(\{r_i\})\rangle = \left\{\left[-i\hat{H}_{\mathrm{rel}} - \frac{\gamma}{2}\sum_{i=1}^{N-1}(\hat{r}_i - \langle\hat{r}_i\rangle)^2\right]\mathrm{d}t + \sqrt{\gamma}\sum_{i=1}^{N-1}(\hat{r}_i - \langle\hat{r}_i\rangle)\mathrm{d}W_t^{(i)}\right\}|\Psi_{\mathrm{rel}}(\{r_i\})\rangle, \tag{115}$$

where we have assumed the total Hamiltonian could be split into $\hat{H} = \hat{H}_{\mathrm{CM}}(\hat{R}) + \hat{H}_{\mathrm{rel}}(\{\hat{r}_i\})$ and the new constant γ_{CM} appearing in Eq. (114) is given by $\gamma_{\mathrm{CM}} = N\gamma$. This illustrates the mechanism thanks to which localization is amplified for a macroscopic object containing a large number (in practice $N \sim 10^{23} \gg 1$ for usual classical systems) of particles, while the usual quantum spread is mostly conserved for the internal degrees of freedom. A recent inventory of all the constraints derived so far in various physical situations on the CSL parameter γ can be found in Ref. [111].

B. An illustrative example: The harmonic oscillator

In this section, we illustrate how the CSL theory works on the example of the harmonic oscillator, resetting the Planck constant $\hbar$ for easier comparison with previous works. This is an interesting case because it represents the prototypical example of a quantum system and, to our knowledge, this case has not been solved explicitly in the case of the CSL theory. Moreover, in cosmology, as explained before, we deal with a parametric oscillator, a case which shares some similarities with an harmonic oscillator, at least in some regimes. It is therefore important to understand first this simplest case in the CSL framework. In the following, we assume that the operator $\hat{B}$ introduced in the previous section is the position operator $\hat{x}$. As a consequence, the modified Schrödinger equation can be written as

$$\mathrm{d}\Psi = \left[-\frac{i}{\hbar}\hat{H}\mathrm{d}t + \sqrt{\gamma}(\hat{x} - \langle\hat{x}\rangle)\mathrm{d}W_t - \frac{\gamma}{2}(\hat{x} - \langle\hat{x}\rangle)^2\mathrm{d}t\right]\Psi, \tag{116}$$

where $\hat{H} = \hat{p}^2/(2m) + m\omega^2\hat{x}^2/2$ is the Hamiltonian. The parameter γ sets the strength of the collapse mechanism and, since we have chosen the position as the preferred basis, it has dimension $L^{-2} \times T^{-1}$. Following Ref. [73], the wave function can be taken as a Gaussian state and the most general form can be expressed as

$$\Psi(t, x) = |N(t)|\exp\{-\Re\mathrm{e}\Omega(t)[x - \bar{x}(t)]^2 + i\sigma(t) + i\chi(t)x - i\Im\mathrm{m}\Omega(t)x^2\}, \tag{117}$$

where, *a priori*, $|N|$, $\Re\mathrm{e}\Omega$, $\bar{x}$, σ, χ and $\Im\mathrm{m}\Omega$ are real stochastic variables. Introducing this wave function in Eq. (118), one obtains the following set of equations:

$$\frac{|N|'}{|N|} = \frac{1}{4}\frac{(\Re\mathrm{e}\Omega)'}{\Re\mathrm{e}\Omega} = \frac{\hbar}{m}\Im\mathrm{m}\Omega + \frac{\gamma}{4\Re\mathrm{e}\Omega}, \tag{118}$$

$$(\Re e\Omega)' = \gamma + 4\frac{\hbar}{m}(\Re e\Omega)(\Im m\Omega), \qquad (119)$$

$$(\Im m\Omega)' = -\frac{\hbar}{m}[2(\Re e\Omega)^2 - 2(\Im m\Omega)^2] + \frac{m}{\hbar}\frac{\omega^2}{2}, \qquad (120)$$

$$\bar{x}' = \frac{\hbar}{m}[\chi - 2(\Im m\Omega)\bar{x}] + \frac{\sqrt{\gamma}}{2\Re e\Omega}\frac{\mathrm{d}W_t}{\mathrm{d}t}, \qquad (121)$$

$$\sigma' = \frac{\hbar}{m}\left[-\Re e\Omega + 2(\Re e\Omega)^2\bar{x}^2 - \frac{1}{2}\chi^2\right], \qquad (122)$$

$$\chi' = -\frac{\hbar}{m}[4(\Re e\Omega)^2\bar{x} - 2\chi\Im m\Omega], \qquad (123)$$

where a prime means a derivative with respect to time. We see that the first equation can be integrated to give $|N| = (2\Re e\Omega/\pi)^{1/4}$, which ensures that the wave function is properly normalized. Then, the two following equations, Eqs. (119) and (120) "decouple" from the other equations and can be integrated separately. In particular, if we add them up, we arrive at

$$\Omega' = -2i\frac{\hbar}{m}\Omega^2 + \gamma + \frac{im}{2\hbar}\omega^2. \qquad (124)$$

This equation should be compared to Eq. (21). As expected, they are identical provided we take $\hbar = m = 1$ and $\gamma = 0$. Of course, in the present case, the frequency ω is constant since we deal with a harmonic oscillator rather than a parametric oscillator as is the case for cosmological perturbations. Equation (124) is a Ricatti equation and we have already seen that the appropriate change of variable to transform it into a linear second order differential equation is $\Omega = -imf'/(2\hbar f)$, where the function $f(t)$ obeys the equation

$$f'' + \left(\omega^2 - 2i\frac{\hbar}{m}\gamma\right)f = 0. \qquad (125)$$

This equation admits simple solutions that can be expressed in terms of exponentials, namely $f(t) \propto \exp(\pm\alpha t)$ where α is defined by

$$\alpha \equiv \sqrt{\frac{2i\gamma\hbar}{m} - \omega^2}. \qquad (126)$$

As a consequence, the solution for $\Omega(t)$ can be written as

$$\Omega(t) = -\frac{im}{2\hbar}\alpha\tanh(\alpha t + \phi), \qquad (127)$$

where ϕ is an integration constant that can be expressed in terms of the initial value of the function $\Omega(t)$

$$\phi = \mathrm{argtanh}\left[-\frac{2\hbar}{im}\frac{\Omega(t=0)}{\alpha}\right]. \qquad (128)$$

This solution resembles the formula obtained in the case of the free particle, see Ref. [73].

At this stage, we need to discuss the initial conditions. Our assumption is that, at $t = 0$, the quantum state is simply given by the ground state of the harmonic oscillator in conventional quantum mechanics. Technically, this means that we require the wave function to be

$$\Psi(t = 0, x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \mathrm{e}^{-m\omega x^2/(2\hbar)}, \qquad (129)$$

which implies that $\Re e\Omega = m\omega/(2\hbar)$ and $\Im m\Omega = 0$ or, equivalently, $\phi = \mathrm{argtanh}(i\omega/\alpha)$. Notice that this choice is fully compatible with the normalization established above, $|N| = (2\Re e\Omega/\pi)^{1/4}$. Of course, our choice also amounts to imposing $\bar{x}(t=0) = \sigma(t=0) = \chi(t=0) = 0$.

Since the evolution of the stochastic wave function is controlled by the function $\Omega(t)$, it is interesting to study how it evolves with time. Writing the number α as $\alpha \equiv \alpha^{\mathrm{R}} + i\alpha^{\mathrm{I}}$, where it is easy to show that

$$\alpha^{\mathrm{R}} = \frac{\omega}{\sqrt{2}}\left(\sqrt{1 + 4\frac{\hbar^2\gamma^2}{m^2\omega^4}} - 1\right)^{1/2}, \qquad (130)$$

$$\alpha^{\mathrm{I}} = \frac{\sqrt{2}}{\omega}\frac{\hbar\gamma}{m}\left(\sqrt{1 + 4\frac{\hbar^2\gamma^2}{m^2\omega^4}} - 1\right)^{-1/2}, \qquad (131)$$

and $\phi \equiv \phi^{\mathrm{R}} + i\phi^{\mathrm{I}}$, straightforward algebraic manipulations lead to the following expressions for $\Re e\Omega$ and $\Im m\Omega$:

$$\Re e\Omega(t) = \frac{m}{2\hbar}\frac{\alpha^{\mathrm{I}}\sinh[2(\alpha^{\mathrm{R}}t + \phi^{\mathrm{R}})] + \alpha^{\mathrm{R}}\sin[2(\alpha^{\mathrm{I}}t + \phi^{\mathrm{I}})]}{\cos[2(\alpha^{\mathrm{I}}t + \phi^{\mathrm{I}})] + \cosh[2(\alpha^{\mathrm{R}}t + \phi^{\mathrm{R}})]}, \qquad (132)$$

$$\Im m\Omega(t) = \frac{m}{2\hbar}\frac{\alpha^{\mathrm{I}}\sin[2(\alpha^{\mathrm{I}}t + \phi^{\mathrm{I}})] - \alpha^{\mathrm{R}}\sinh[2(\alpha^{\mathrm{R}}t + \phi^{\mathrm{R}})]}{\cos[2(\alpha^{\mathrm{I}}t + \phi^{\mathrm{I}})] + \cosh[2(\alpha^{\mathrm{R}}t + \phi^{\mathrm{R}})]}. \qquad (133)$$

In particular, the function $\Re e\Omega(t)$, with the initial condition specified above, is always positive. Notice also that there is a sign ambiguity in the definitions of the quantities α^{R} and α^{I} in Eqs. (130) and (131), but one can show that this does not affect the physical predictions of the model. It is also interesting to calculate the limit for large times of the two functions in Eqs. (132) and (133). One obtains

$$\lim_{t\to\infty}\Re e\Omega = \frac{m\alpha^{\mathrm{I}}}{2\hbar} \simeq \frac{m\omega}{2\hbar}\left(1 + \frac{1}{2}\frac{\hbar^2\gamma^2}{m^2\omega^4} + \cdots\right), \qquad (134)$$

$$\lim_{t\to+\infty}\Im m\Omega = -\frac{m\alpha^{\mathrm{R}}}{2\hbar} \simeq -\frac{\gamma}{2\omega}\left(1 - \frac{1}{2}\frac{\hbar^2\gamma^2}{m^2\omega^4} + \cdots\right), \qquad (135)$$

where the dots indicate an expansion in the small dimensionless parameter $\hbar\gamma/(m\omega^2)$. We see that, if $\gamma = 0$, we obtain the ground state given by Eq. (129). Deviations from that solution are controlled by the parameter $\hbar\gamma/(m\omega^2)$.

We are now in a position where one can investigate the physical properties of the quantum state (117). In particular, it is easy to show that $\langle\hat{x}\rangle = \bar{x}$ and $\langle\hat{p}\rangle = \chi - 2(\Im m\Omega)\bar{x}$. Initially, $\bar{x} = 0$ and the position operator has a

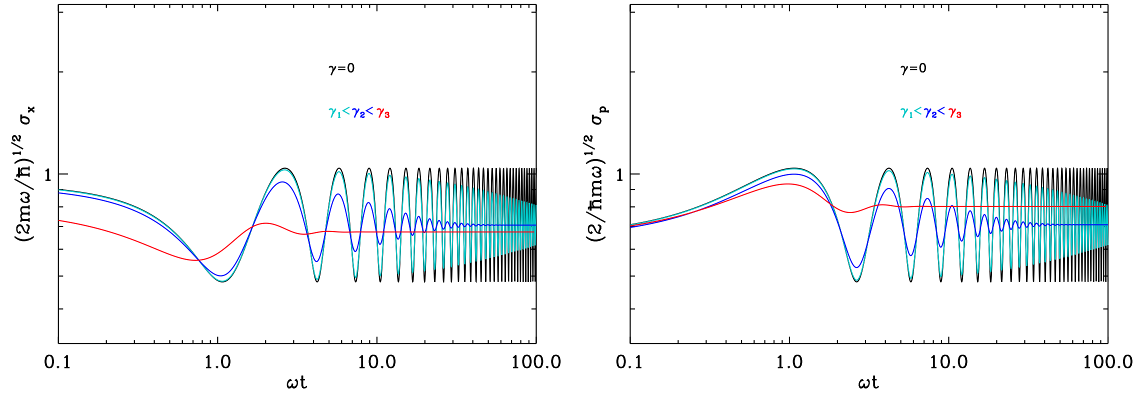

FIG. 3 (color online). Spread in position and momentum for different values of γ in the case of the harmonic oscillator; see Eq. (138). The conventional Schrödinger evolution corresponds to $\gamma = 0$ and is represented by the black curve which oscillates with constant amplitude. On the contrary, when the collapse mechanism is turned on, the oscillations are damped (blue and red curves), the spreads tend toward a constant value and localization occurs.

vanishing mean value as expected for the ground state of the harmonic oscillator but, at later times, due to the stochastic evolution of the wave function, it acquires a nonzero value. It is also possible to calculate the spread in position and momentum. One obtains

$$\sigma_x \equiv \sqrt{\langle \hat{x}^2\rangle - \langle \hat{x}\rangle^2} = \frac{1}{2}\frac{1}{\sqrt{\Re\mathrm{e}\Omega}}, \tag{136}$$

$$\sigma_p \equiv \sqrt{\langle \hat{p}^2\rangle - \langle \hat{p}\rangle^2} = \hbar\sqrt{\frac{(\Re\mathrm{e}\Omega)^2 + (\Im\mathrm{m}\Omega)^2}{\Re\mathrm{e}\Omega}}. \tag{137}$$

We see that these quantities only depend on $\Re\mathrm{e}\Omega$ and $\Im\mathrm{m}\Omega$. As a consequence, inserting Eqs. (132) and (133) in the above expressions of σ_x and σ_p, one arrives at

$$\sigma_x = \sqrt{\frac{\hbar}{2m}}\sqrt{\frac{\cos[2(\alpha^{\mathrm{I}}t + \phi^{\mathrm{I}})] + \cosh[2(\alpha^{\mathrm{R}}t + \phi^{\mathrm{R}})]}{\alpha^{\mathrm{I}}\sinh[2(\alpha^{\mathrm{R}}t + \phi^{\mathrm{R}})] + \alpha^{\mathrm{R}}\sin[2(\alpha^{\mathrm{I}}t + \phi^{\mathrm{I}})]}},$$
$$\sigma_p = \sqrt{\frac{m\hbar}{2}}\sqrt{(\alpha^{\mathrm{R}})^2 + (\alpha^{\mathrm{I}})^2} \times\sqrt{\frac{\cosh[2(\alpha^{\mathrm{R}}t + \phi^{\mathrm{R}})] - \cos[2(\alpha^{\mathrm{I}}t + \phi^{\mathrm{I}})]}{\alpha^{\mathrm{I}}\sinh[2(\alpha^{\mathrm{R}}t + \phi^{\mathrm{R}})] + \alpha^{\mathrm{R}}\sin[2(\alpha^{\mathrm{I}}t + \phi^{\mathrm{I}})]}}. \tag{138}$$

The time evolution of these quantities is displayed in Fig. 3. The black curves correspond to the conventional Schrödinger evolution, i.e., the case $\gamma = 0$. They show the usual oscillatory behavior. On the contrary, when $\gamma \neq 0$, we see that the oscillations are damped (see the smaller amplitude decaying red and blue curves). Then, the spreads converge towards a constant value, which only depends on γ, ω and m. This value is easy to evaluate and one finds

$$\lim_{t\to\infty}\sigma_x = \frac{1}{2^{3/4}}\sqrt{\frac{\omega}{\gamma}}\left(\sqrt{1 + 4\frac{\hbar^2\gamma^2}{m^2\omega^4}} - 1\right)^{1/4}, \tag{139}$$

$$\lim_{t\to\infty}\sigma_p = m\omega\left(1 + 4\frac{\hbar^2\gamma^2}{m^2\omega^4}\right)^{1/4} \times \lim_{t\to\infty}\sigma_x. \tag{140}$$

From these formulas one can see that the spread in position at infinity decreases with γ, from $\sqrt{\hbar/(2m\omega)}$ for $\gamma = 0$ to 0 for $\gamma \to \infty$. We see that the modified Schrödinger equation, as expected, implies a localization in position. We also notice that the microscopic behavior of the system is altered by the nonlinear and stochastic terms added to the theory. By contrast, in order to satisfy the Heisenberg uncertainty relation, the spread in momentum increases with γ, from $\sqrt{m\omega\hbar/2}$ for $\gamma = 0$ to infinity for $\gamma \to \infty$. For $\gamma = 0$ and at large times, one finds that the Heisenberg relation is saturated, $\sigma_x\sigma_p = \hbar/2$, as appropriate for a coherent state. In the limit $\gamma \to \infty$, one finds a larger value $\sigma_x\sigma_p = \hbar/\sqrt{2}$. Let us also remark that an exact eigenstate of the operator $\hat{x}$ is given by a Dirac function $\delta(x - \bar{x})$ centered at some value $\bar{x}$. On the other hand, we see that adding nonlinear and stochastic terms results in a spreading of the Dirac function into a Gaussian wave function with a finite width decreasing for increasing γ. Therefore, the modified Schrödinger equation does not exactly lead to an eigenstate of the position operator. In fact, the asymptotic value of σ_x obtained above defines the "precision" of the collapse and characterizes how close to an eigenstate of the collapse operator the final state is. In that sense, since σ_x decreases with γ, the bigger γ, the more "precise" the collapse.

To conclude this section, it is also interesting to calculate the time derivative of the quantum mean value of the Hamiltonian operator. One obtains

$$\frac{\mathrm{d}\langle\hat{H}\rangle}{\mathrm{d}t} = \frac{\hbar^2}{2m}\gamma - \frac{\hbar}{m}\sqrt{\gamma}\frac{\Im\mathrm{m}\Omega}{\Re\mathrm{e}\Omega}\langle\hat{p}\rangle\frac{\mathrm{d}W_t}{\mathrm{d}t} + \frac{1}{2}m\omega^2\langle\hat{x}\rangle\frac{\sqrt{\gamma}}{\Re\mathrm{e}\Omega}\frac{\mathrm{d}W_t}{\mathrm{d}t}. \tag{141}$$

This equation implies that

$$\frac{\mathrm{d}\mathbb{E}[\langle \hat{H} \rangle]}{\mathrm{d}t} = \frac{\hbar^2}{2m}\gamma. \tag{142}$$

As is well know, this formula expresses the nonconservation of energy in the CSL theory. From a phenomenological point of view, this increase of energy is usually so small (given the values of γ usually considered) that it cannot be detected. Put differently, the nonconservation of energy in the CSL theory cannot be used to rule out this theory [68].

V. THE INFLATIONARY CSL THEORY

The dynamical collapse model of the previous sections should apply to any quantum system, and hence in particular to cosmological perturbations as they arise from vacuum fluctuations. Spontaneously collapsing these happens to be a tremendously complicated task for many reasons discussed below, so in what follows, we suggest a much simplified modeling method which we then apply to the inflationary situation.

A. The modified Schrödinger equation for the Mukhanov-Sasaki variable

The first obvious problem one encounters when dealing with quantum cosmological perturbations is that the underlying theory ought to be relativistic. The straightforward relativistic generalization of the CSL model for quantum field theory, starting with the action (6) in the Tomonaga picture for instance, leads to unremovable divergences [65] (see however Ref. [69]), even more so when nonlinearities inherent to general relativity are taken into account.

The second option which happens to lead to a model in which calculations are actually possible consists in noting, as mentioned earlier in Sec. II D, that the spectrum of primordial perturbations depends on the wave number $\boldsymbol{k}$. In other words, once the Fourier spectrum is known, all of the observable quantities related with the CMB can be computed and compared with actual data. This means that mere knowledge of the modes $\hat{v}_{\boldsymbol{k}}$ ought to be enough in order for a complete description of the possible observations to be realized.

We shall therefore accordingly assume in what follows that the modified Schrödinger equation of motion for the wave function will be done at the level of the Fourier mode $\Psi_{\boldsymbol{k}}$, with spontaneous localization on the $\hat{v}_{\boldsymbol{k}}$ eigenmanifolds. This is consistent with previous approaches aimed at studying decoherence of cosmological perturbations where the pointer basis is often assumed to be precisely the Mukhanov-Sasaki operators, see Ref. [38]. Separating as before into real and imaginary parts, we shall thus assume the following basic equation:

$$\mathrm{d}\Psi_{\boldsymbol{k}}^{\mathrm{R}} = \Big[-i\hat{\mathcal{H}}_{\boldsymbol{k}}^{\mathrm{R}}\mathrm{d}\eta + \sqrt{\gamma}(\hat{v}_{\boldsymbol{k}}^{\mathrm{R}} - \langle \hat{v}_{\boldsymbol{k}}^{\mathrm{R}} \rangle)\mathrm{d}W_\eta - \frac{\gamma}{2}(\hat{v}_{\boldsymbol{k}}^{\mathrm{R}} - \langle \hat{v}_{\boldsymbol{k}}^{\mathrm{R}} \rangle)^2\mathrm{d}\eta\Big]\Psi_{\boldsymbol{k}}^{\mathrm{R}}, \tag{143}$$

and a similar equation for $\Psi_{\boldsymbol{k}}^{\mathrm{I}}$. Here, the quantity γ is a positive constant with mass dimension 2 if the scale factor is chosen to be dimensionless but is dimensionless if the scale factor is chosen to have mass dimension -1, which is the convention adopted here. As in Sec. IV, the parameter γ sets the strength of the collapse mechanism.

Let us now review all the limitations of postulating an ansatz equation such as Eq. (143). First, one should note that the constant γ in Eq. (143) cannot be the same as the one associated with the choice of the position operator as the collapse operator appearing in Eq. (116), despite our choice of the same notation. It is clear that each time one considers different collapse operators, this leads to different CSL parameters with different mass dimension. The same phenomenon is observed in Ref. [70] where the collapse operator is chosen to be a spin operator. In this case, it is clear that the corresponding CSL parameter cannot be the same as the one corresponding to the case where the collapse operator is the position (as it is for the case of the free particle [73]). This is unfortunate when it comes to a comparison of the constraints obtained in the laboratory with the constraints obtained in cosmology. In fact, what could be done is to consider the strict CSL theory where the collapse operator is usually taken to be the averaged density operator. In the language of cosmological perturbations, this amounts to assuming that there is spontaneous localization on the $\widehat{\delta\rho}(\eta, \boldsymbol{x})$ eigenmanifold, where $\delta\rho(\eta, \boldsymbol{x})$ is the perturbed energy density. This would have the advantage to introduce a universal γ with always the same dimension. Unfortunately, $\delta\rho(\eta, \boldsymbol{x})$ is a complicated functional of $v_{\boldsymbol{k}}$ and this would probably render the whole approach untractable. Let us also notice that γ could be taken as a function of the wave number $\boldsymbol{k}$, i.e., different CSL parameters for different modes. In this article, for simplicity, we do not follow this route.

Another issue is that we moved from real to reciprocal space while keeping the structure of the equation unchanged. In doing so, we also avoid from the outset any mode mixing that would be naturally arising from a real space modified Schrödinger equation: its stochastic version being nonlinear, one would expect a coupling of the Fourier modes, which is here automatically set to zero. Note this approximation is justified by data observations of the CMB.

Another important limitation of our treatment is the fact that the collapse concerns the modes independently. As a result, the amplification mechanism, so crucial to explain why the quantum behavior becomes increasingly less important for increasingly large systems (the effective collapse time being inversely proportional to the number of particles involved and, hence, to the size of the system), is simply not operating here. Therefore, even though one might consider cosmological size effects, the collapse will occur just as it would for an independent quantum particle. As we will see, that implies a severe constraint on the constant γ when comparison of the

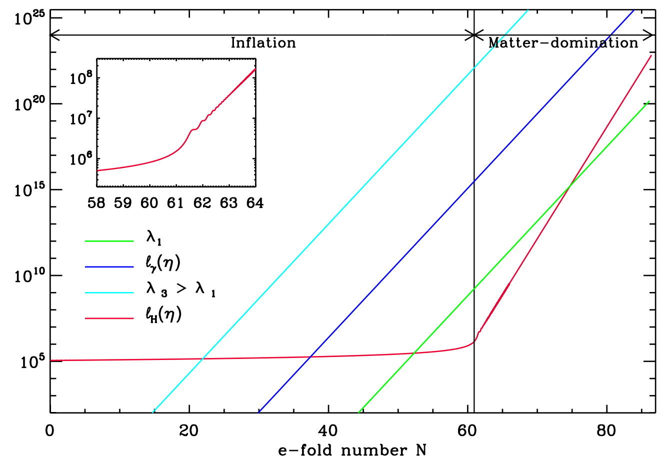

FIG. 4 (color online). Evolution of various physical length scales with time during the cosmic history in the CSL model described by Eq. (143) with a zoom on the transition from inflation to reheating inserted (see the concluding section). The solid line represents the Hubble radius ℓ_H and the dashed-dotted green and red lines, the physical wavelengths of two Fourier modes of cosmological relevance today. The solid blue line represents the built-in CSL scale ℓ_γ, see the discussion above Eq. (157). It is a preferred comoving scale and can also be viewed as a time-dependent preferred physical scale. Therefore, when a mode is below (above) ℓ_γ it remains so during the whole history of the Universe as is clear from the plot. This means that, contrary to the Hubble scale, there is no "ℓ_γ crossing" during the cosmic evolution. As a consequence, one expects the power spectrum to acquire a broken power-law shape, with two different branches, an expectation confirmed by the calculations in Sec. V D.

modified spectrum is made to actual observations on Hubble-size scales.

Finally, Eq. (143) is written in terms of the conformal Fourier mode of the original action. Because its normalization implies the equation be nonlinear, this means the constant γ can be translated, as we will show later, into a privileged *conformal* scale, and hence a time-dependent privileged length ℓ_γ, as shown in Fig. 4 and the discussion above Eq. (157). This is somehow similar to the fact, except at the perturbative and conformal levels, that considering nonflat spatial sections permits to define a curvature length and thus forbids to renormalize the scale factor arbitrarily. However, as shown in the Appendix, this last limitation does not affect the general conclusions drawn here.

B. Gaussian state

Our goal is now to solve Eq. (143). As was done for the standard case (50), one considers that the wave function assumes a Gaussian shape. Concretely, we take the most general form, namely

$$\Psi_{\boldsymbol{k}}^{\mathrm{R,I}}(\eta, v_{\boldsymbol{k}}^{\mathrm{R,I}}) = |N_{\boldsymbol{k}}(\eta)| \exp\{-\Re\mathrm{e}\Omega_{\boldsymbol{k}}(\eta)[v_{\boldsymbol{k}}^{\mathrm{R,I}} - \bar{v}_{\boldsymbol{k}}^{\mathrm{R,I}}(\eta)]^2 + i\sigma_{\boldsymbol{k}}^{\mathrm{R,I}}(\eta) + i\chi_{\boldsymbol{k}}^{\mathrm{R,I}}(\eta) v_{\boldsymbol{k}}^{\mathrm{R,I}} - i\Im\mathrm{m}\Omega_{\boldsymbol{k}}(\eta)(v_{\boldsymbol{k}}^{\mathrm{R,I}})^2\}, \tag{144}$$

where $\bar{v}_{\boldsymbol{k}}^{\mathrm{R,I}}$, $\sigma_{\boldsymbol{k}}^{\mathrm{R,I}}$ and $\chi_{\boldsymbol{k}}^{\mathrm{R,I}}$ are real numbers. The fact that one can assume $|N_{\boldsymbol{k}}|$ and $\Omega_{\boldsymbol{k}}$ to be independent of "R" or "I" will be justified in the following. Compared to Eq. (50), we see that Eq. (144) is more general and, therefore, contains more parameters. The case of Eq. (50) corresponds to $\bar{v}_{\boldsymbol{k}}^{\mathrm{R,I}} = 0$, $\chi_{\boldsymbol{k}}^{\mathrm{R,I}} = 0$ and $\arg N_{\boldsymbol{k}} = \sigma_{\boldsymbol{k}}$. Of course, the above Gaussian is similar to the wave function considered in the case of the harmonic oscillator of Eq. (117). The only difference is that the stochastic functions characterizing the wave function now depend on the wave number $\boldsymbol{k}$ and the role of the position is played by the Fourier amplitude of the Mukhanov-Sasaki variable.

The next step is to insert Eq. (144) into Eq. (143) in order to derive the differential equations obeyed by the functions parametrizing the Gaussian state. Straightforward manipulations making use of the Itô calculus lead to the following expressions:

$$\frac{|N_{\boldsymbol{k}}|'}{|N_{\boldsymbol{k}}|} = \frac{1}{4}\frac{(\Re\mathrm{e}\Omega_{\boldsymbol{k}})'}{\Re\mathrm{e}\Omega_{\boldsymbol{k}}} = \Im\mathrm{m}\Omega_{\boldsymbol{k}} + \frac{\gamma}{4\Re\mathrm{e}\Omega_{\boldsymbol{k}}}, \tag{145}$$

$$(\Re\mathrm{e}\Omega_{\boldsymbol{k}})' = \gamma + 4(\Re\mathrm{e}\Omega_{\boldsymbol{k}})(\Im\mathrm{m}\Omega_{\boldsymbol{k}}), \tag{146}$$

$$(\Im\mathrm{m}\Omega_{\boldsymbol{k}})' = -2(\Re\mathrm{e}\Omega_{\boldsymbol{k}})^2 + 2(\Im\mathrm{m}\Omega_{\boldsymbol{k}})^2 + \frac{1}{2}\omega^2(\eta, \boldsymbol{k}), \tag{147}$$

$$(\bar{v}_{\boldsymbol{k}}^{\mathrm{R,I}})' = \chi_{\boldsymbol{k}}^{\mathrm{R,I}} + \frac{\sqrt{\gamma}}{2\Re\mathrm{e}\Omega_{\boldsymbol{k}}}\frac{\mathrm{d}W_\eta}{\mathrm{d}\eta} - 2(\Im\mathrm{m}\Omega_{\boldsymbol{k}})\bar{v}_{\boldsymbol{k}}^{\mathrm{R,I}}, \tag{148}$$

$$(\sigma_{\boldsymbol{k}}^{\mathrm{R,I}})' = -\Re\mathrm{e}\Omega_{\boldsymbol{k}} + 2(\Re\mathrm{e}\Omega_{\boldsymbol{k}})^2(\bar{v}_{\boldsymbol{k}}^{\mathrm{R,I}})^2 - \frac{1}{2}(\chi_{\boldsymbol{k}}^{\mathrm{R,I}})^2, \tag{149}$$

$$(\chi_{\boldsymbol{k}}^{\mathrm{R,I}})' = -4(\Re\mathrm{e}\Omega_{\boldsymbol{k}})^2\bar{v}_{\boldsymbol{k}}^{\mathrm{R,I}} + 2\chi_{\boldsymbol{k}}^{\mathrm{R,I}}(\Im\mathrm{m}\Omega_{\boldsymbol{k}}). \tag{150}$$

Several remarks are in order at this point. First, we see that the evolution equations for $|N_{\boldsymbol{k}}|$, $\Re\mathrm{e}\Omega_{\boldsymbol{k}}$ and $\Im\mathrm{m}\Omega_{\boldsymbol{k}}$ are deterministic and independent of that of $\bar{v}_{\boldsymbol{k}}^{\mathrm{R,I}}$, $\sigma_{\boldsymbol{k}}^{\mathrm{R,I}}$ or $\chi_{\boldsymbol{k}}^{\mathrm{R,I}}$. This justifies the fact that one can assume these quantities to be independent on R, I provided similar initial conditions are chosen for R, I. This also means that these three quantities are not random (but their evolution is still explicitly modified by the stochastic dynamics when $\gamma \neq 0$). Second, Eq. (145) explicitly implies the conservation of the wave function norm: if one initially has a normalized state, i.e.,

$$|N_{\boldsymbol{k}}| = \left(\frac{2\Re\mathrm{e}\Omega_{\boldsymbol{k}}}{\pi}\right)^{1/4}, \tag{151}$$

it will remain so at any time. In fact, this equation is similar to Eq. (21) which is therefore not modified by the introduction of the nonlinear stochastic terms. Moreover, in the present case where the wave function is given by a single Gaussian, $\sigma_{\boldsymbol{k}}^{\mathrm{R,I}}$ is just an irrelevant global phase and can be ignored (this will no longer be the case when the quantum state is the sum of two Gaussians, see below). Third, it is easy to check that Eqs. (145)–(150), are the exact counter parts of Eqs. (118)–(123). The only difference is that ω is now a time-dependent quantity as expected since we deal with a parametric oscillator. We conclude that, instead of six coupled stochastic differential equations, we have in fact to solve two sets of two coupled differential equations, the first one being deterministic and the second one being stochastic. In particular Eqs. (146) and (147) can be combined and lead to the following Ricatti equation for the quantity $\Re\mathrm{e}\Omega_{\boldsymbol{k}} + i\Im\mathrm{m}\Omega_{\boldsymbol{k}} = \Omega_{\boldsymbol{k}}$:

$$\Omega_{\boldsymbol{k}}' = -2i\Omega_{\boldsymbol{k}}^2 + \gamma + \frac{i}{2}\omega^2(\eta, \boldsymbol{k}). \tag{152}$$

This equation is similar to Eq. (124) obtained for the harmonic oscillator. Of course, if $\gamma = 0$, then one exactly recovers the Ricatti equation (20). As discussed before, a Ricatti equation can always be reduced to a linear but second order differential equation: this is achieved through the transformation $\Omega_{\boldsymbol{k}} = -if_{\boldsymbol{k}}'/(2f_{\boldsymbol{k}})$, where $f_{\boldsymbol{k}}$ is a solution of the following linear differential equation:

$$f_{\boldsymbol{k}}'' + [\omega^2(\eta, \boldsymbol{k}) - 2i\gamma]f_{\boldsymbol{k}} = 0. \tag{153}$$

This equation is very similar to the equation for the mode function considered before. The only difference is the appearance of the term $-2i\gamma$ in the effective frequency. Obviously, if $\gamma = 0$, then one recovers the conventional case. Moreover, the fact that this is the counterpart of Eq. (125) is obvious.

C. Evolution of the stochastic wave function during inflation

We now study the time evolution of the quantum state (144) in more detail. We start with the evolution of $\Re\mathrm{e}\Omega_{\boldsymbol{k}}$ and $\Im\mathrm{m}\Omega_{\boldsymbol{k}}$ since we have shown in the last section that it decouples from the other equations of motion. To derive the corresponding solutions, it is sufficient to solve Eq. (153). If the background is driven by a phase of power-law inflation, $\omega(\eta, \boldsymbol{k})$ is given by $\omega(\eta, \boldsymbol{k}) = k^2 - \beta(\beta+1)/\eta^2$ and the differential equation (153) reads

$$f_{\boldsymbol{k}}'' + \left[k^2 - \frac{\beta(\beta+1)}{\eta^2} - 2i\gamma\right]f_{\boldsymbol{k}} = 0. \tag{154}$$

We see that the only effect of the CSL term $-2i\gamma$ is to modify the comoving wave number $k^2 \to k^2 - 2i\gamma$. The solution of Eq. (154) can be written in terms of Bessel functions

$$f_{\boldsymbol{k}}(\eta) = (-z_{\boldsymbol{k}}k\eta)^{1/2}[C_{\boldsymbol{k}}J_{\beta+\frac{1}{2}}(-z_{\boldsymbol{k}}k\eta) + D_{\boldsymbol{k}}J_{-(\beta+1/2)}(-z_{\boldsymbol{k}}k\eta)], \tag{155}$$

where $C_{\boldsymbol{k}}$ and $D_{\boldsymbol{k}}$ are integration constants and where the complex number $z_{\boldsymbol{k}}$ is defined by

$$z_{\boldsymbol{k}} \equiv \sqrt{1 - i\frac{2\gamma}{k^2}} = \left(1 + 4\frac{\gamma^2}{k^4}\right)^{1/4} \mathrm{e}^{-\frac{i}{2}\arctan(2\gamma/k^2)}. \tag{156}$$

Equation (155) should be compared to its non-CSL counterpart, Eq. (45). The only difference is the appearance of the $z_{\boldsymbol{k}}$ factor. This is consistent with the remark made above since this factor always multiplies the expression $k\eta$ and can, therefore, be viewed as a "renormalization" of the wave number k. In the non-CSL case where $\gamma = 0$, one obviously has $z_{\boldsymbol{k}} = 1$ and Eq. (155) reduces to Eq. (45). It is interesting to remark that $z_{\boldsymbol{k}}$ for the parametric oscillator plays a role similar to that of α for the harmonic oscillator, see the definition (126). In fact, strictly following this last definition, one can introduce a mode-dependent $\alpha_{\boldsymbol{k}}$ parameter, namely $\alpha_{\boldsymbol{k}} \equiv \sqrt{2i\gamma - k^2}$ (using $\omega = k$ for massless perturbations) and, then, $z_{\boldsymbol{k}}$ appears to be just a rescaled $\alpha_{\boldsymbol{k}}$ parameter: $\alpha_{\boldsymbol{k}} = ikz_{\boldsymbol{k}}$. Finally, notice also that the sign ambiguity in the definition of $z_{\boldsymbol{k}}$ due to the presence of a square root has absolutely no impact on the results presented below.

Let us now discuss the solution $f_{\boldsymbol{k}}(\eta)$ and what this implies for the behavior of the wave function. In the presence of the CSL term, the problem is characterized by three scales: the wavelength of the Fourier mode given by $\lambda_{\boldsymbol{k}}(\eta) = a(\eta)/k$, the Hubble radius $\ell_{\mathrm{H}}(\eta) = a^2/a'$ and a new scale associated with the parameter γ defined by $\ell_\gamma \equiv a(\eta)/\sqrt{\gamma}$ or, in terms of mass scale, $M_\gamma \equiv \sqrt{\gamma}/a(\eta)$. Notice that ℓ_γ is a new, time-dependent, physical scale that is built in the inflationary CSL theory, see Fig. 4. In terms of these three physical scales, the quantity $z_{\boldsymbol{k}}k\eta$ which appears in Eq. (155) can be written as

$$z_{\boldsymbol{k}}k\eta = (1+\beta)\frac{\ell_{\mathrm{H}}}{\lambda_{\boldsymbol{k}}}\sqrt{1 - 2i\frac{M_\gamma^2}{k_{\mathrm{phys}}^2}}, \tag{157}$$

where $k_{\mathrm{phys}} = k/a$ is the physical wave number. At the beginning of inflation, the modes of cosmological interest today laid far inside the Hubble radius, which means $\lambda_{\boldsymbol{k}} \ll \ell_{\mathrm{H}}$, i.e., $k\eta \to -\infty$. Notice that these considerations are independent of the value of M_γ. Indeed, if $k_{\mathrm{phys}} \gg M_\gamma$, then $z_{\boldsymbol{k}} \simeq 1$ and the previous limit is not changed. On the contrary, if $k_{\mathrm{phys}} \ll M_\gamma$, then the condition $|z_{\boldsymbol{k}}| \gg 1$ is even better satisfied. It is also interesting to remark that, in this last case, $z_{\boldsymbol{k}}k\eta$ does not go to $-\infty$ along the real axis but along a direction that is inclined in the complex plane. However, this does not change the asymptotic behavior of the Bessel functions in this regime. Upon using Eq. (155), one obtains

$$\lim_{\lambda_{\boldsymbol{k}}/\ell_{\mathrm{H}}\to 0} f_{\boldsymbol{k}}(\eta) = \sqrt{\frac{2}{\pi}}\left[C_{\boldsymbol{k}}\sin\left(-z_{\boldsymbol{k}}k\eta - \frac{\pi}{2}\beta\right) + D_{\boldsymbol{k}}\cos\left(-z_{\boldsymbol{k}}k\eta + \frac{\pi}{2}\beta\right)\right]. \tag{158}$$

This expression can also be re-expressed in term of "plane wave" functions (writing $\alpha_{\boldsymbol{k}} \equiv \alpha_{\boldsymbol{k}}^{\mathrm{R}} + i\alpha_{\boldsymbol{k}}^{\mathrm{I}}$)

$$\lim_{\lambda_{\boldsymbol{k}}/\ell_{\mathrm{H}}\to 0} f_{\boldsymbol{k}}(\eta) = \frac{A_{\boldsymbol{k}}}{\sqrt{2\pi}}\mathrm{e}^{\alpha_{\boldsymbol{k}}^{\mathrm{R}}|\eta| - i\alpha_{\boldsymbol{k}}^{\mathrm{I}}\eta - i\pi/4} + \frac{B_{\boldsymbol{k}}}{\sqrt{2\pi}}\mathrm{e}^{-\alpha_{\boldsymbol{k}}^{\mathrm{R}}|\eta| + i\alpha_{\boldsymbol{k}}^{\mathrm{I}}\eta + i\pi/4}, \tag{159}$$

where the coefficients A_k and B_k can be expressed as linear combinations of C_k and D_k, namely

$$A_k = C_k\, e^{-i\pi(\beta+1/2)/2} + D_k\, e^{i\pi(\beta+1/2)/2} \tag{160}$$

$$B_k = C_k\, e^{i\pi(\beta+1/2)/2} + D_k\, e^{-i\pi(\beta+1/2)/2}. \tag{161}$$

The solution (159) is nothing but the Wentzel-Kramers-Brillouin (WKB) mode function $\exp(\pm i\int \omega \mathrm{d}\tau)/\sqrt{2\omega}$. The reason for this result is that, in the sub-Hubble regime, the WKB approximation is still valid even in presence of the CSL term. As is well known, this approximation is satisfied when the quantity $|Q/\omega^2| \ll 1$, where Q is given by

$$Q \equiv \frac{3}{4}\frac{1}{\omega^2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}\eta}\right)^2 - \frac{1}{2\omega}\frac{\mathrm{d}^2\omega}{\mathrm{d}\eta^2}. \tag{162}$$

Since, in the limit under consideration, ω^2 tends toward a constant, namely $\omega^2 = k^2 - 2i\gamma$, and since Q is given in terms of derivatives of ω, it is obvious that the criterion is satisfied. As already mentioned, the only effect of the CSL theory is to add the constant term $-2i\gamma$ to ω^2. Although this modifies the solution for the mode function, clearly, this cannot change the fact that WKB is valid at the beginning of inflation.

Let us now comment on Eq. (159). When $|\eta|$ goes to infinity, the second branch of the above solution is going to die away since $\alpha_k^{\mathrm{R}} > 0$. As a consequence, only the first branch remains and, since Ω_k is given in terms of a ratio, i.e., $-if_k'/(2f_k)$, the remaining constant A_k disappears from the final expression. Therefore, Ω_k becomes independent of the initial conditions and is given by $\Omega_k \simeq i\alpha_k/2$, which implies that $\Re\mathrm{e}\Omega_k \simeq -\alpha_k^{\mathrm{I}}/2 \simeq -k/2$. Returning to Eq. (144), this means that the wave function is not bounded at infinity and is not normalizable. The deep reason is that, in the CSL context, z_k (or α_k) is complex and this implies that the WKB solution acquires either a growing or a decaying exponential component which automatically kills one of the two branches. And, of course, z_k (or α_k) is complex because of the CSL term $-2i\gamma$.

Based on the previous discussion, it is clear that the only meaningful choice of initial conditions is to require that $A_k = 0$. From Eqs. (160) and (161), we see that this implies

$$C_k = -D_k\, e^{i\pi(\beta+1/2)}. \tag{163}$$

This choice exactly coincides with the Bunch-Davies initial conditions (46). From now on, we assume Eq. (163) but we will come back soon to this discussion. Then, one can rederive the behavior of Ω_k in the sub-Hubble regime. One obtains

$$\lim_{\lambda_k/\ell_{\mathrm{H}}\to 0} \Omega_k(\eta) = -\frac{i}{2}\alpha_k, \tag{164}$$

which is fully consistent with Eqs. (134) and (135). In particular, one can check that, now, $\Re\mathrm{e}\Omega_k \to k/2$ and the wave function becomes normalizable (of course, it tends to the ground state wave function). Therefore, we have proven that, as expected, the cosmological perturbations behave, in the sub-Hubble regime, exactly as the CSL harmonic oscillator.

Having studied the behavior of the stochastic wave function in the sub-Hubble regime, we now turn to the super-Hubble case. In the framework of CSL, and contrary to the sub-Hubble regime studied before, it is clear that this regime has no counter part in the case of the harmonic oscillator. It corresponds to the limit $\ell_{\mathrm{H}} \ll \lambda_k$ and, from Eq. (157), we see that this means $|z_k k\eta| \to 0$. Let us notice that one could also consider the case where $k_{\mathrm{phys}} \ll M_\gamma$ such that $M_\gamma/k_{\mathrm{phys}} \gg 1$ compensates the ratio $\ell_{\mathrm{H}}/\lambda_k$ in Eq. (157) resulting in a large $|z_k k\eta|$, even in the super-Hubble regime. Below, we briefly comment on this case. Here, we assume that M_γ is such that this does not happen. Then, upon using the asymptotic behavior of the Bessel functions for small values of their argument, one arrives at

$$\begin{aligned}\frac{\Omega_k}{k} =& -\frac{i(1+\beta)}{2k\eta} - \frac{i(-k\eta)}{4(\beta+3/2)} - \frac{(-k\eta)}{2(\beta+3/2)}\frac{\gamma}{k^2}\\ &+ i\frac{D_k}{C_k}\left(1-2i\frac{\gamma}{k^2}\right)^{-\beta-1/2} 2^{2\beta+1}\left(\beta+\frac{1}{2}\right)\\ &\times\frac{\Gamma(\beta+1/2)}{\Gamma(-\beta-1/2)}(-k\eta)^{-2\beta-2} + \cdots. \end{aligned}\tag{165}$$

This equation should be compared to the corresponding non-CSL formula (52). If $\gamma = 0$ and if one takes the Bunch-Davies initial conditions, $D_k = -C_k e^{-i\pi(\beta+1/2)}$, then the above equation exactly reduces to Eq. (52). Here, although we argued before that one should use the Bunch-Davies initial conditions (163), we temporarily keep the coefficients C_k and D_k arbitrary because, later on, we shall want to comment on their influence on the shape of the CSL power spectrum. Let us also notice that the last term of the above expression is in fact proportional to $z_k^{-(2\beta+1)}$. If we write z_k in polar form, $z_k \equiv |z_k| e^{i\theta_k}$ (of course, θ_k should not be confused with the squeezing angle) where the modulus and the phase can be read off directly from Eq. (156), and parametrize the initial conditions as $C_k = |C_k| e^{i\theta_c}$ and $D_k = |D_k| e^{i\theta_d - i\pi\beta + i\pi/2}$ (so that the Bunch-Davies limit is simply $\theta_d - \theta_c = 0$), then it is easy to determine the real and imaginary parts of the function Ω_k. One finds

$$\begin{aligned}\Re\mathrm{e}\Omega_k(\eta) =& -\frac{k}{2(\beta+3/2)}\frac{\gamma}{k^2}(-k\eta) + \frac{|D_k|}{|C_k|}|z_k|^{-(2\beta+1)}\\ &\times\cos[\pi\beta + (2\beta+1)\theta_k - \theta_d + \theta_c]\\ &\times\frac{k\pi 2^{2\beta+1}}{\Gamma^2(-\beta-1/2)\cos(\pi\beta)}\\ &\times(-k\eta)^{-2\beta-2} + \cdots, \end{aligned}\tag{166}$$

$$\Im\mathrm{m}\Omega_{\boldsymbol{k}}(\eta)=-\frac{k}{2k\eta}(1+\beta)-\frac{k}{4(\beta+3/2)}(-k\eta)-\frac{k}{\pi}\frac{|D_{\boldsymbol{k}}|}{|C_{\boldsymbol{k}}|}|z_{\boldsymbol{k}}|^{-(2\beta+1)}\times 2^{2\beta}\frac{1}{2}\sin[\pi\beta+(2\beta+1)\theta_{\boldsymbol{k}}-\theta_d+\theta_c]\times\cos(\pi\beta)\Gamma^2\left(\beta+\frac{3}{2}\right)(-k\eta)^{-2\beta-2}+\cdots. \tag{167}$$

These equations are the CSL counterparts of Eqs. (53) and (54). Of course, for $\gamma = 0$ and the Bunch-Davies initial conditions, they exactly reduce to those equations. We see that the main effect of the CSL theory is to strongly modify $\Re\mathrm{e}\Omega_{\boldsymbol{k}}$ since its leading term in the above expansion is a term which cancels if $\gamma = 0$. We also see that we still have $\Re\mathrm{e}\Omega_{\boldsymbol{k}} \to 0$ in the super-Hubble limit. In the absence of the CSL term, we would obtain the same limit but not with the same power. Compared to $\Re\mathrm{e}\Omega_{\boldsymbol{k}}$, $\Im\mathrm{m}\Omega_{\boldsymbol{k}}$ is much less modified since the first correction show up only in the third term of the expansion. As a consequence, we still have $\Im\mathrm{m}\Omega_{\boldsymbol{k}} \to \infty$ in the super-Hubble regime.

We now use the above results to discuss the collapse of the wave function in more detail. Since we have assumed in Eq. (143) that the "collapse operator" is $\hat{v}_{\boldsymbol{k}}$, we expect the nonlinear and stochastic terms in the modified Schrödinger equation to drive the initial Gaussian state to an eigenvector of $\hat{v}_{\boldsymbol{k}}$, that is to say to the Dirac function $\delta(v_{\boldsymbol{k}} - \bar{v}_{\boldsymbol{k}})$. However, in practice, as we learned from the harmonic oscillator example in Sec. IV B, this is not what happens. In practice, we find that the wave function tends towards a Gaussian state with a constant spread in position and that the larger the value of γ, the smaller the amplitude of this spread, i.e., $\lim_{t\to\infty}\sigma_x \to [\hbar/(4m\gamma)]^{1/4}$ when $\gamma \to \infty$. Therefore, strictly speaking, the exact localization is obtained only in the $\gamma \to \infty$ limit. Of course, if the spread is very small, then for all practical purposes, the collapse has been achieved. In fact, this is the essence of the amplification mechanism discussed in Sec. IV A. The effective value γ_{CM} of γ for a macroscopic object (or for its center of mass) is the fundamental γ times the number of particles in that object which results in a huge effective γ and, therefore, a very efficient localization. As a consequence, a collapse can occur for macroscopic objects while it does not happen for microscopic particles even if their behavior is slightly disturbed.

Let us now see how the previous discussion applies to inflation. The first difference is that the standard deviation, $1/(2\sqrt{\Re\mathrm{e}\Omega_{\boldsymbol{k}}})$, does not go to a constant as for the harmonic oscillator but to infinity since Eq. (166) implies that $\Omega_{\boldsymbol{k}} \propto \eta \to 0$. We remark that the divergence is less violent than when $\gamma \neq 0$ since, in that case, $\Omega_{\boldsymbol{k}} \propto \eta^2 \to 0$, according to Eq. (53). This is of course due to the influence of the nonlinear and stochastic terms. However, this influence is not sufficient to prevent the divergence of the variance and, therefore, to ensure an efficient localization. As a matter of fact, we see that, in the limit $\eta \to 0$, the main divergence in the Hamiltonian comes from the term $\propto \omega^2 v_{\boldsymbol{k}}^2$ while the CSL term goes like $\gamma v_{\boldsymbol{k}}^2$. Hence, it is because the term $\omega^2 \propto \eta^{-2}$ diverges at the end of inflation that the Hamiltonian strongly dominates the dynamics of the system, preventing the CSL terms $\propto \gamma v_{\boldsymbol{k}}^2$ to carry out its job and to localize $v_{\boldsymbol{k}}$ (however, see the Appendix). This is certainly a problem for the inflationary CSL theory. This issue can also be related to the fact that it is unclear how an amplification mechanism could be implemented in quantum field theory. As a consequence, the collapse mechanism is controlled by the parameter γ and no effective γ can be derived which would ensure a better localization.

Finally, let us mention that one could wonder whether the localization can be achieved during the radiation dominated era that takes place after inflation. In this case, the scale factor behaves as $a(\eta) \propto \eta$ and, therefore, $\epsilon_1 = 2$ and $(a\sqrt{\epsilon_1})''/(a\sqrt{\epsilon_1}) = 0$. As consequence, the mode equation for $f_{\boldsymbol{k}}$ is exactly that of a harmonic oscillator. This means that the variance now goes to a constant, see Sec. IV B, which seems to cure the problem discussed above. However, one can show that the corresponding value remains large for modes of astrophysical interest today. Therefore, this remains an unsatisfactory solution.

D. The CSL power spectrum

We now turn to one of the main goal of the present paper, namely the determination of the power spectrum predicted by the CSL theory. It was shown in Eqs. (29) and (37) that the power spectrum of the conserved quantity $\zeta_{\boldsymbol{k}}$ can be expressed as

$$\mathcal{P}_{\zeta}(k) = \frac{k^3}{16\pi^2 M_{\mathrm{Pl}}^2}\frac{1}{a^2\epsilon_1\Re\mathrm{e}\Omega_{\boldsymbol{k}}}. \tag{168}$$

Since we have determined the quantity $\Re\mathrm{e}\Omega_{\boldsymbol{k}}$ in Eq. (166), the calculation of $\mathcal{P}_{\zeta}$ becomes straightforward. One obtains

$$\mathcal{P}_{\zeta}(k) = g_\gamma(k,\beta)\left[1-\frac{\gamma}{k^2}g_\gamma(k,\beta)f(\beta)\times\frac{(-k\eta)^{2\beta+3}}{\beta+3/2}\right]^{-1}\mathcal{P}_{\zeta}(k)|_{\mathrm{stand}}, \tag{169}$$

where $\mathcal{P}_{\zeta}|_{\mathrm{stand}}$ is the standard power spectrum given by Eq. (47) and the function $f(\beta)$ has been defined in Eq. (48). The function $g_\gamma(k,\beta)$

$$g_\gamma(k,\beta) \equiv \frac{|C_{\boldsymbol{k}}|}{|D_{\boldsymbol{k}}|}|z_{\boldsymbol{k}}|^{2\beta+1}\frac{\cos(\pi\beta)}{\cos[\pi\beta+(2\beta+1)\theta_{\boldsymbol{k}}-\theta_d+\theta_c]} \tag{170}$$

is seen to depend on the choice of the initial conditions. It has the property that, for $\gamma = 0$ and the Bunch-Davies initial conditions, $g_{\gamma=0}(k,\beta) = 1$. In this case, and as expected, one can check that the modified power spectrum (169) reduces to the standard inflationary power spectrum. We also notice that the power spectrum (169) is still a time-dependent quantity, contrary to the conventional case where the time dependence cancels out. For this reason, it is convenient to evaluate it at the end of inflation. In that case, the quantity $-k\eta$ can be rewritten as

$$-k\eta = -\frac{k}{k_0}(1+\beta)\,e^{\Delta N_*/(1+\beta)}, \tag{171}$$

where k_0 is the comoving wave number of the Fourier mode, the wavelength of which equals the Hubble radius today, i.e., $k_0 = a_0 H_0$. The quantity ΔN_* denotes the number of e-folds spent by a mode of cosmological relevance today outside the Hubble radius during inflation; typically, one has $\Delta N_* \simeq$ 50–60. As a consequence, the power spectrum (169) can be reexpressed as

$$\mathcal{P}_\zeta(k) = g_\gamma(k,\beta)\left[1 - \frac{\gamma}{k_0^2} g_\gamma(k,\beta) f(\beta)\frac{|1+\beta|^{2\beta+3}}{(\beta+3/2)} \times e^{(2\beta+3)\Delta N_*/(1+\beta)}\left(\frac{k}{k_0}\right)^{2\beta+1}\right]^{-1} \mathcal{P}_\zeta(k)|_{\rm stand}. \tag{172}$$

Let us notice that, in Eq. (170), the quantities $|z_{\boldsymbol{k}}|$ of Eq. (156) and $\theta_{\boldsymbol{k}}$ must now be written as

$$|z_{\boldsymbol{k}}| = \left[1 + 4\frac{\gamma^2}{k_0^4}\left(\frac{k_0}{k}\right)^4\right]^{1/4}, \tag{173}$$

$$\theta_{\boldsymbol{k}} = -\frac{1}{2}\arctan\left[2\frac{\gamma}{k_0^2}\left(\frac{k_0}{k}\right)^2\right], \tag{174}$$

such that the amplitude of the CSL correction is controlled by the dimensionless ratio γ/k_0^2. The formula (172) is one of the main results of this article and the corresponding power spectra for different values of the ratio γ/k_0^2 are represented in Fig. 5.

Let us now discuss in more detail the CSL power spectrum (172). First, we notice that, in the short-wavelength regime $k/k_0 \to \infty$, the power spectrum reduces to $\mathcal{P}_\zeta(k) \simeq g_\gamma(k,\beta)\mathcal{P}_\zeta|_{\rm stand}$. Moreover, in this limit, we see that $|z_{\boldsymbol{k}}| \to 1$ and $\theta_{\boldsymbol{k}} \to 0$. As a consequence, an almost scale invariant (namely, $n_{\rm S} = 2\beta + 4$ with $\beta \lesssim -2$) power spectrum is recovered if one assumes the Bunch-Davies initial conditions, $|C_{\boldsymbol{k}}| = |D_{\boldsymbol{k}}|$ and $\theta_d - \theta_c = 0$ since, in that case, $g_\gamma(k,\beta) = 1$. This almost scale invariant branch of the power spectrum is clearly seen in Fig. 5. Second, there is clearly another regime which corresponds to the case where the second term in the square brackets in Eq. (172) starts playing a role. If we neglect factors of order one, this happens at $k = k_\gamma$, where k_γ solves

$$\frac{\gamma}{k_0^2} g_\gamma(k_\gamma,\beta)\,e^{(2\beta+3)\Delta N_*/(1+\beta)}\left(\frac{k_\gamma}{k_0}\right)^{2\beta+1} \simeq 1. \tag{175}$$

The value of g_γ is mainly controlled by the value of $|z_{\boldsymbol{k}}|$ which is always close to unity provided that $k \ll k_z$ with

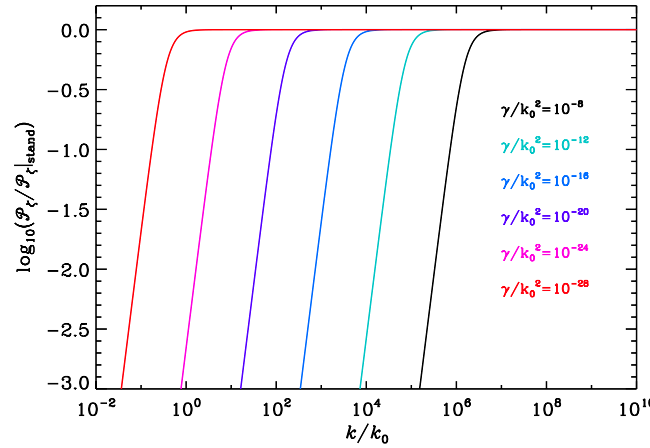

FIG. 5 (color online). Ratio of the power spectrum given by Eq. (172) to the standard power spectrum given by Eq. (47) for different values of the parameter γ/k_0^2 (and for $\beta = -2.01$, a value leading to a standard power spectrum close to scale invariance). The number of e-folds between Hubble radius crossing and the end of inflation (for the modes of cosmological interest today) has been taken to $\Delta N_* = 60$ and the initial conditions have been chosen to be the adiabatic vacuum.

$$\frac{k_z}{k_0} \equiv \sqrt{2}\left(\frac{\gamma}{k_0^2}\right)^{1/2}. \tag{176}$$

Then, let us assume that $g_\gamma \simeq 1$ when the condition (175) is met. In this case, the scale k_γ can be expressed as

$$\frac{k_\gamma}{k_0} \sim \left(\frac{\gamma}{k_0^2}\right)^{-1/(1+2\beta)} \exp\left[-\frac{2\beta+3}{(\beta+1)(2\beta+1)}\Delta N_*\right]. \tag{177}$$

Choosing the fiducial value $\beta \simeq -2$ leads to $k_\gamma/k_0 \sim (\gamma/k_0^2)^{1/3}\exp(\Delta N_*/3)$. One can check that, indeed, $k_\gamma \gg k_z$ and, therefore, assuming $g_\gamma \simeq 1$ was, in retrospect, valid. As a consequence, in the range $k \ll k_\gamma$, the spectrum approximately behaves as $\propto k^{2\beta+4}/k^{2\beta+1} = k^3$, that is to say with a spectral index of $n_{\rm S} \simeq 4$. This second branch is also clearly visible in Fig. 5. In addition, the dependence in g_γ is canceled out which means that this prediction is actually independent of the choice of the initial conditions, a remarkable property indeed (this also means that, even if $k \ll k_z$, the spectral index remains the same). Moreover, we see that this spectral index is also independent of β which is also remarkable. In this sense, the CSL branch of the power spectrum can be said to be "universal" (unfortunately not scale invariant).

We are now in a position where we can discuss the cosmological constraints on the parameter γ. From the high accuracy measurements of the CMB anisotropies [20–22], we know that the power spectrum is almost scale invariant, $n_{\rm S} \simeq 1$, and that a spectral index $n_{\rm S} = 4$ is completely excluded. This means that the CSL branch must correspond to scales much larger than the present Hubble radius, in other words $k_\gamma/k_0 \ll 1$. This condition means that, for $\beta \simeq -2$, one has

$$\frac{\gamma}{k_0^2} \ll \mathrm{e}^{-\Delta N_*} \simeq 10^{-28}. \tag{178}$$

To our knowledge, this is the first time that a constraint on the parameter γ is obtained from cosmological considerations (see, however, Ref. [76]). We see that the constraint is expressed as a limit not on γ itself but on the combination γ/k_0^2 where we remind the reader that k_0 is the comoving wave number of the Hubble radius today. Looking at Eqs. (143) and (153), this was expected since the CSL modification amounts to a redefinition of the comoving wave number $k^2 \to k^2 - 2i\gamma$. This means that, in order to characterize the amplitude of the modification, one has to compare the comoving wave number to γ, hence the ratio γ/k_0^2. The appearance of the comoving wave number in the observational constraint reflects the fact that the theory contains a built-in "time-dependent physical preferred scale" $\ell_\gamma(\eta)$. In terms of physical scales, the constraint (178) can be rewritten as

$$\frac{\ell_{\mathrm{H}}}{\ell_\gamma}\bigg|_{\mathrm{today}} \ll 10^{-13}. \tag{179}$$

Clearly, the constraint is very strong and means that the scale ℓ_γ is very large in comparison to the Hubble radius today. This is another illustration of the fact that squeezed states are fragile and easily perturbed. For the CSL theory itself, this probably means that, in order to be compatible with cosmological inflation, an important fine-tuning is required. Of course, this conclusion should be toned down given the uncertainty that exists on a CSL formulation of quantum field theory as discussed in Sec. V A. One might argue for instance that the above result could be due to the fact that our modified Schrödinger equation is not necessarily the appropriate one in the context of quantum field theory. It would also be interesting to compare the cosmological constraint with the other constraints on γ derived in the literature. But, as explained before, because we assumed $\hat{v}_{\boldsymbol{k}}$ to be the preferred basis for the collapse, our parameter γ is actually different from the parameter γ considered elsewhere, in particular it has a different dimension. This complicates tremendously any comparison with other systems.

Finally, before closing this section, let us discuss the following question. In this article, we have defined the power spectrum in the CSL theory by means of the formula $\mathbb{E}(\langle\hat{v}_{\boldsymbol{k}}^2\rangle) - \mathbb{E}(\langle\hat{v}_{\boldsymbol{k}}\rangle^2)$. However, there is an issue regarding this definition. Indeed, it is clear that it does not go to zero when the parameter γ vanishes. Actually, it tends towards the standard result when the Schrödinger equation is recovered. However, it was argued in Ref. [112] that the power spectrum should go to zero in the limit where $\gamma \to 0$ and, therefore, cannot be given by the definition used above. The reason advocated by Ref. [112] is that, without a collapse, the theory remains homogeneous and isotropic and, as a consequence, there is simply no perturbations at all. This has led Ref. [112] to define the CSL power spectrum by $\mathbb{E}(\langle\hat{v}_{\boldsymbol{k}}\rangle^2) - \mathbb{E}^2(\langle\hat{v}_{\boldsymbol{k}}\rangle)$, a quantity which indeed vanishes when $\gamma \to 0$ and differs from the previous one. In this last paragraph, we explore the difference between these two alternative definitions. At any time, the wave function can always be expanded as

$$\Psi(\eta, v_{\boldsymbol{k}}) = \int \Psi(\eta, \bar{v}_{\boldsymbol{k}})\delta(v_{\boldsymbol{k}} - \bar{v}_{\boldsymbol{k}})\mathrm{d}\bar{v}_{\boldsymbol{k}}, \tag{180}$$

where the superscripts "R, I" have been ignored for convenience. If a dynamical collapse of the wave function takes place then Ψ is projected (collapsed) on an eigenstate of the operator $\hat{v}_{\boldsymbol{k}}$, namely

$$\Psi \to \Psi_{\mathrm{col}} \equiv \delta(v_{\boldsymbol{k}} - \bar{v}_{\boldsymbol{k}}), \tag{181}$$

where $\bar{v}_{\boldsymbol{k}}$ depends on the specific realization under consideration, then one obviously has

$$\begin{aligned}\langle\Psi_{\mathrm{col}}|\hat{v}_{\boldsymbol{k}}^2|\Psi_{\mathrm{col}}\rangle &= \langle\Psi_{\mathrm{col}}|\hat{v}_{\boldsymbol{k}}|\psi_{\mathrm{col}}\rangle^2,\\ &= \bar{v}_{\boldsymbol{k}}^2.\end{aligned} \tag{182}$$

Therefore, for each realization, one has $\langle\hat{v}_{\boldsymbol{k}}^2\rangle = \langle\hat{v}_{\boldsymbol{k}}\rangle^2$, once the wave function has collapsed. Since this is true for all realizations, it remains the case after taking the stochastic average. Therefore, after the collapse, one can write

$$\mathbb{E}(\langle\hat{v}_{\boldsymbol{k}}^2\rangle) = \mathbb{E}(\langle\hat{v}_{\boldsymbol{k}}\rangle^2), \tag{183}$$

and this remains true for any Hermitian operator. Note that this argument strongly depends on the fact that the wave function has actually collapsed to an eigenstate of the operator $\hat{v}_{\boldsymbol{k}}$. For instance, in the case of a harmonic oscillator studied in Sec. IV B, it was shown that the asymptotic state is not exactly a Dirac wave function, but a Gaussian state the spread of which does not vanish for finite values of γ. In that situation, the two above expressions are not identical.

On the other hand, the second terms in both definitions of the power spectrum differ

$$\mathbb{E}(\langle\hat{v}_{\boldsymbol{k}}\rangle^2) \neq \mathbb{E}^2(\langle\hat{v}_{\boldsymbol{k}}\rangle), \tag{184}$$

so the two spectra do not coincide even after the collapse. The difference ultimately boils down to the fact that it is built out of a standard deviation which is not a Hermitian operator. This is a generic question for the predictions of any theory mixing different kinds of averages (in the case at hand, quantum and stochastic) whenever nonlinear combinations of Hermitian operators are involved.

VI. THE COLLAPSE OF COSMOLOGICAL PERTURBATIONS

In this section, we investigate the collapse mechanism and its dynamics in more detail. In particular, we calculate the collapse time and compare it with the cosmological characteristic times. For this purpose, we now consider the following double Gaussian quantum state [73]:

$$\begin{aligned}\Psi_k(\eta, v_k) = &|N_k^{(1)}(\eta)|\exp\{-\Re e\Omega_k^{(1)}(\eta)[v_k - \bar{v}_k^{(1)}(\eta)]^2 \\ &+ i\sigma_k^{(1)}(\eta) + i\chi_k^{(1)}(\eta)v_k - i\Im m\Omega_k^{(1)}(\eta)(v_k)^2\} \\ &+ |N_k^{(2)}(\eta)|\exp\{-\Re e\Omega_k^{(2)}(\eta)[v_k - \bar{v}_k^{(2)}(\eta)]^2 \\ &+ i\sigma_k^{(2)}(\eta) + i\chi_k^{(2)}(\eta)v_k - i\Im m\Omega_k^{(2)}(\eta)(v_k)^2\},\end{aligned} \tag{185}$$

where, as before, $|N_k^{(1,2)}|$, $\Re e\Omega_k^{(1,2)}$, $\bar{v}_k^{(1,2)}$, $\sigma_k^{(1,2)}$, $\chi_k^{(1,2)}$ and $\Im m\Omega_k^{(1,2)}$ are real, possibly stochastic, numbers. The superscripts "R, I" have not been written for convenience but it should be remembered that they are of course present. Inserting the above state into the modified Schrödinger equation leads to the following set of formulas:

$$\begin{aligned}\frac{|N_k^{(1,2)}|'}{|N_k^{(1,2)}|} = &\Im m\Omega_k^{(1,2)} + \frac{\gamma}{4\Re e\Omega_k^{(1,2)}} - \sqrt{\gamma}[\langle\hat{v}_k\rangle - \bar{v}_k^{(1,2)}] \\ &\times\frac{dW_\eta}{d\eta} - \frac{\gamma}{2}[\langle\hat{v}_k\rangle - \bar{v}_k^{(1,2)}]^2,\end{aligned} \tag{186}$$

$$[\Re e\Omega_k^{(1,2)}]' = \gamma + 4[\Re e\Omega_k^{(1,2)}][\Im m\Omega_k^{(1,2)}], \tag{187}$$

$$[\Im m\Omega_k^{(1,2)}]' = -2[\Re e\Omega_k^{(1,2)}]^2 + 2[\Im m\Omega_k^{(1,2)}]^2 + \frac{1}{2}\omega^2(\eta, \boldsymbol{k}), \tag{188}$$

$$\begin{aligned}[\bar{v}_k^{(1,2)}]' = &\chi_k^{(1,2)} + \frac{\sqrt{\gamma}}{2\Re e\Omega_k^{(1,2)}}\frac{dW_\eta}{d\eta} - 2[\Im m\Omega_k^{(1,2)}]\bar{v}_k^{(1,2)} \\ &+ \frac{\gamma}{\Re e\Omega_k^{(1,2)}}[\langle\hat{v}_k\rangle - \bar{v}_k^{(1,2)}],\end{aligned} \tag{189}$$

$$\begin{aligned}[\sigma_k^{(1,2)}]' = &-\Re e\Omega_k^{(1,2)} + 2[\Re e\Omega_k^{(1,2)}]^2[\bar{v}_k^{(1,2)}]^2 \\ &- \frac{1}{2}[\chi_k^{(1,2)}]^2,\end{aligned} \tag{190}$$

$$[\chi_k^{(1,2)}]' = -4[\Re e\Omega_k^{(1,2)}]^2\bar{v}_k^{(1,2)} + 2\chi_k^{(1,2)}[\Im m\Omega_k^{(1,2)}]. \tag{191}$$

These equations should be compared to Eqs. (145)–(150). They are obviously very similar except the two last terms of Eq. (186) and the last term of Eq. (189) which are new. In the case of a single Gaussian, one has $\langle\hat{v}_k\rangle = \bar{v}_k$ and these terms disappear. In the present case, the expression of $\langle\hat{v}_k\rangle$ is a very complicated function of all the parameters describing the wave function. Let us also notice that, since the evolution of $\sigma_k^{(1,2)}$ and $\chi_k^{(1,2)}$ depends on $\bar{v}_k^{(1,2)}$, these quantities also feel the coupling between the two Gaussian components. However, one can see that the equations of motion for $\Re e\Omega_k^{(1,2)}$ and $\Im m\Omega_k^{(1,2)}$ decouple from the other equations of motion and form an independent and closed subsystem. This means that the evolution of these two functions is identical to that of their counterpart in the simple Gaussian case and, moreover, that, if the initial conditions are chosen to be the same, $\Omega_k^{(1)} = \Omega_k^{(2)}$ at any subsequent time. From now on, for this reason, the superscripts "(1)" and/or "(2)" on these quantities will be dropped.

It should be clear that the above system of differential equations is rather complicated to study. However, as we shall see, the most relevant properties of the evolution of the double Gaussian quantum state can be analyzed in a rigorous way. In particular, it is interesting to introduce the function $\Gamma_k(\eta) \equiv \ln[|N_k^{(2)}|/|N_k^{(1)}|]$, see Ref. [73]. This quantity characterizes the relative importance of one Gaussian component to the other and, therefore, provides a criterion to decide whether the collapse has taken place. The superposition of the two Gaussian quantum states reduces to one of them when $|\Gamma_k|$ goes to infinity. In practice, the collapse will be said to have occurred when $|\Gamma_k| > b$ with, say, $b \sim 10$ [73]. Then, by subtracting the two equations (186), one arrives at the following evolution equation for Γ_k

$$\begin{aligned}\frac{d\Gamma_k}{d\eta} = &\sqrt{\gamma}[\bar{v}_k^{(2)} - \bar{v}_k^{(1)}]\frac{dW}{d\eta} \\ &- \gamma[\bar{v}_k^{(2)} - \bar{v}_k^{(1)}][\bar{v}_k^{(1)} + \bar{v}_k^{(2)} - 2\langle\hat{v}_k\rangle].\end{aligned} \tag{192}$$

This equation remains complicated because of the presence of the term $\langle\hat{v}_k\rangle$. However, the calculation can be simplified if one assumes that the two Gaussian components of the wave function do not overlap, i.e., have separate supports. Technically, this means that $\Re e\Omega_k[\bar{v}_k^{(2)} - \bar{v}_k^{(1)}]^2 \gg 1$, leading to the following simple formula:

$$\langle\hat{v}_k\rangle \simeq \frac{|N_k^{(1)}|^2\bar{v}_k^{(1)} + |N_k^{(2)}|^2\bar{v}_k^{(2)}}{|N_k^{(1)}|^2 + |N_k^{(2)}|^2}. \tag{193}$$

Inserting this formula into Eq. (192) and defining X_k by $X_k \equiv \bar{v}_k^{(2)} - \bar{v}_k^{(1)}$, one obtains the following expression:

$$\frac{d\Gamma_k}{d\eta} = \sqrt{\gamma}X_k\frac{dW_\eta}{d\eta} + \gamma X_k^2\tanh(\Gamma_k). \tag{194}$$

This stochastic differential equation can be further simplified. Indeed, using the new timelike variable [73]

$$s_k \equiv \gamma\int_{\eta_{\rm ini}}^{\eta} X_k^2(u)du, \tag{195}$$

Eq. (194) can be rewritten as

$$\frac{d\Gamma_k}{ds_k} = \frac{dW_s}{ds_k} + \tanh(\Gamma_k), \tag{196}$$

where

$$W_s = \sqrt{\gamma}\int_0^{s_k} X_k dW_\eta \tag{197}$$

is another Wiener process with respect to the time variable s_k.

A. Collapse time: Definition

Let us now study the stochastic differential equation driving the evolution of Γ_k in more detail. In particular, we would like to know how much time it takes for the wave function to collapse or, in technical terms, we would like to determine the value of s_k such that $|\Gamma_k| > b$. The quantity Γ_k being stochastic, two complications arise. First, once it has reached a value larger than b, there is no guarantee that it will stay in this region. The random behavior of Γ_k could temporally bring it back to the region $|\Gamma_k| \leq b$. However, since the average trend is clearly to have a collapse, this would happen for a limited amount of time only before Γ_k returns in the regime where $|\Gamma_k| \geq b$. For this reason, we will consider that the wave function has collapsed when Γ_k has crossed the value $\pm b$ for the first time. Technically, this means that we are led to define the "collapse time," S_k, as $S_k \equiv \inf(s_k)$ such that $|\Gamma_k(s_k)| > b$, see also Ref. [73]. A second issue is that, clearly, the value of S_k will differ from one realization to the other or, in other words, that S_k is still a random variable. Therefore, we will rather define the collapse time as the ensemble average value of S_k but we will also be interested in calculating its higher order momenta.

We now seek an explicit expression for the quantity S_k. It can be obtained in the following manner. Let us consider a function $c(\Gamma_k)$ that we do not characterize in more detail for the moment (but see below). It can always be Taylor expanded in $\mathrm{d}\Gamma_k$. At second order, the result reads

$$c(\Gamma_k + \mathrm{d}\Gamma_k) = c(\Gamma_k) + c'(\Gamma_k)\mathrm{d}\Gamma_k + \frac{1}{2}c''(\Gamma_k)\mathrm{d}\Gamma_k^2 + \mathcal{O}(\mathrm{d}\Gamma_k^3), \tag{198}$$

where $\mathrm{d}\Gamma_k$ is given by Eq. (196). At first order in $\mathrm{d}s_k$, this leads to

$$\mathrm{d}c[\Gamma_k(s_k)] = c'[\Gamma_k(s_k)]\mathrm{d}W_s + c'[\Gamma_k(s_k)]\tanh[\Gamma_k(s_k)]\mathrm{d}s_k + \frac{1}{2}c''[\Gamma_k(s_k)]\mathrm{d}s_k. \tag{199}$$

Then, integrating the above expression between $s_k = 0$ where $\Gamma_k(s_k = 0) = b_0$ and $s_k = S_k$ where $\Gamma_k(s_k = S_k) = \pm b$, one gets the following (Itô) formula:

$$c(\pm b) - c(b_0) = \int_0^{S_k} c'[\Gamma_k(s_k)]\mathrm{d}W_s + \int_0^{S_k}\Big\{c'[\Gamma_k(s_k)] \times \tanh[\Gamma_k(s_k)] + \frac{1}{2}c''[\Gamma_k(s_k)]\Big\}\mathrm{d}s_k. \tag{200}$$

At this stage, we now specify the function c. We require it to be the solution of the differential ordinary equation

$$\frac{1}{2}c''(x) + \tanh(x)c'(x) = -1, \tag{201}$$

with boundary conditions $c(-b) = c(+b) = 0$. It is easy to show that $c(x) = b\tanh(b) - x\tanh(x)$. This means that the first term on the left-hand side of Eq. (200) vanishes and that the integrand of the second term on the right-hand side is just -1. Therefore, Eq. (200) can be rewritten as

$$S_k = c(b_0) + \int_0^{S_k} c'[\Gamma_k(s_k)]\mathrm{d}W_s, \tag{202}$$

and this gives an (implicit) expression for the quantity S_k. Finally, by averaging over all realizations, one obtains [73]

$$\mathbb{E}(S_k) = c(b_0) = b\tanh(b) - b_0\tanh(b_0). \tag{203}$$

The fact that the stochastic average of the integral in Eq. (202) vanishes comes from the fact that $c'[\Gamma_k(s_k)]$ depends only on stochastic events occurring at $s_k' < s_k$. As a consequence, it can be expressed as an integration over $\mathrm{d}s_k'$ and $\mathrm{d}W_{s'}$ where $s_k' < s_k$. Since $\mathbb{E}(\mathrm{d}W_{s'}\mathrm{d}W_s) = \delta(s_k' - s_k)\mathrm{d}s_k^2$, at first order in $\mathrm{d}s_k$, the stochastic average of the integral term in Eq. (202) vanishes. Actually, things are slightly more complicated since the upper bound of this integral, S_k, is a stochastic quantity itself. Therefore, the averaging process should also be carried out on this upper bound, and a generalized demonstration which includes this case can be found in Ref. [113] (theorem 1 on p. 28).

In order to characterize better the properties of this collapse time, it is also important to determine its variance. Interestingly enough, the same technique described above can be used in order to calculate iteratively higher orders of S_k. Upon using Eq. (202) one has

$$\mathbb{E}(S_k^2) = c^2(b_0) + \int_0^{S_k} c'^2[\Gamma_k(s_k)]\mathrm{d}s_k. \tag{204}$$

We see that we now need to evaluate the integral in the above expression. For this purpose, we consider a new function $e(\Gamma_k)$. As was done before, it can be Taylor expanded and this leads exactly to Eq. (200) (with, of course, c replaced by e). Compared with the proof that allowed us to obtain $\mathbb{E}(S_k)$, at this point, the strategy changes. We now require the function $e(x)$ to be the solution of the following ordinary differential equation [compare with Eq. (201)]:

$$\frac{1}{2}e''(x) + \tanh(x)e'(x) = -e'^2(x), \tag{205}$$

with boundary conditions $e(-b) = e(b) = 0$. As before, one can use this differential equation into the Itô formula to simplify the second integral in Eq. (204) [more precisely, the integrand is replaced by $-e'^2(x)$]. Taking the stochastic average of the resulting equation, one gets

$$e(b_0) = \int_0^{S_k} e'^2[\Gamma_k(s_k)]\mathrm{d}s_k. \tag{206}$$

As a consequence, we deduce that

$$\mathbb{E}(S_k^2) = c^2(b_0) + e(b_0). \tag{207}$$

The only thing which remains to be done is to solve Eq. (205). In fact, it turns out to be more convenient to solve the slightly simpler differential equation satisfied by $e_1(x) \equiv c^2(x) + e(x)$, namely $e_1''(x)/2 + \tanh(x)e_1'(x) = -2c(x)$, with boundary conditions $e_1(-b) = e_1(b) = 0$. It is straightforward to show that $e_1(x) = x^2 - b^2 + [1 + 2b\tanh(b)][b\tanh(b) - x\tanh(x)]$. Then, the second moment of S can be simply expressed as $\mathbb{E}(S_{\boldsymbol{k}}^2) = e_1(b_0)$ which, therefore, gives an explicit expression for the variance of the collapse time. Since b is supposed to be a large number $b \gg 1$ and if we assume that the two Gaussians have comparable initial weights which implies that $b_0 \sim 0$, then one obtains, at leading order in b,

$$\mathbb{E}(S_{\boldsymbol{k}}) \simeq b, \tag{208}$$

$$\sqrt{\mathbb{E}(S_{\boldsymbol{k}}^2) - \mathbb{E}^2(S_{\boldsymbol{k}})} \simeq \sqrt{b}. \tag{209}$$

These two equations tell us that the relative standard deviation scales as $1/\sqrt{b}$ and, therefore, that the distribution of $S_{\boldsymbol{k}}$ becomes more peaked as b increases. For this reason, in the following, we will simply estimate the collapse time by means of the sloppy requirement that $s_{\boldsymbol{k}} = b$. Finally, let us mention that one could also apply the technique used in this section in order to determine the higher order correlation functions of the process $S_{\boldsymbol{k}}$.

B. Collapse time in the sub-Hubble regime

In the last section, we have explained how to determine the collapse time in terms of the variable $s_{\boldsymbol{k}}$. In order to translate this result in terms of a more physical time (conformal time or, better, number of e-folds), we need to use Eq. (195) which, in turn, requires the knowledge of the function $X_{\boldsymbol{k}}$. This one cannot be determined in full generality but it is easy to characterize it in the sub- and super-Hubble regimes. In this section, we investigate the sub-Hubble regime.

Let us define $K_{\boldsymbol{k}} \equiv \chi_{\boldsymbol{k}}^{(2)} - \chi_{\boldsymbol{k}}^{(1)}$. This quantity measures the shift in momentum between the two Gaussian components of the wave function (185) (we recall that $X_{\boldsymbol{k}}$ measures the shift in position). Then, taking the difference between the versions "(1)" and "(2)" of Eq. (189) on the one hand, and versions "(1)" and "(2)" of Eq. (191) on the other hand, we arrive at a closed system which can be written in a matrix form, namely

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\begin{pmatrix} X_{\boldsymbol{k}} \\ K_{\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} -2\Im\mathrm{m}\Omega_{\boldsymbol{k}} - \frac{\gamma}{\Re\mathrm{e}\Omega_{\boldsymbol{k}}} & 1 \\ -4(\Re\mathrm{e}\Omega_{\boldsymbol{k}})^2 & 2\Im\mathrm{m}\Omega_{\boldsymbol{k}} \end{pmatrix}\begin{pmatrix} X_{\boldsymbol{k}} \\ K_{\boldsymbol{k}} \end{pmatrix}. \tag{210}$$

At this stage, there is no approximation and the above equation is general. In the sub-Hubble regime, one can use Eq. (164) to simplify the expressions of $\Re\mathrm{e}\Omega_{\boldsymbol{k}}$ and $\Im\mathrm{m}\Omega_{\boldsymbol{k}}$. Moreover, we are mainly interested in computing the collapse time for the modes that correspond to the (almost) scale invariant part of the power spectrum since it is clearly less interesting to compute this quantity in a regime that is already excluded by the data. As was discussed before, this amounts to considering that $\gamma/k^2 \ll 1$. Under those conditions, one has $\Re\mathrm{e}\Omega_{\boldsymbol{k}} \to k/2$ and $\Im\mathrm{m}\Omega_{\boldsymbol{k}} \to -\gamma/(2k)$ and Eq. (210) can be reexpressed as

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\begin{pmatrix} X_{\boldsymbol{k}} \\ K_{\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} -\gamma/k & 1 \\ -k^2 & -\gamma/k \end{pmatrix}\begin{pmatrix} X_{\boldsymbol{k}} \\ K_{\boldsymbol{k}} \end{pmatrix}. \tag{211}$$

This system of differential equations can be integrated and the solution reads

$$K_{\boldsymbol{k}}(\eta) = \mathrm{e}^{-\gamma(\eta-\eta_{\mathrm{ini}})/k}\{K_{\boldsymbol{k},\mathrm{ini}}\cos[k(\eta-\eta_{\mathrm{ini}})] - kX_{\boldsymbol{k},\mathrm{ini}}\sin[k(\eta-\eta_{\mathrm{ini}})]\}, \tag{212}$$

$$X_{\boldsymbol{k}}(\eta) = \mathrm{e}^{-\gamma(\eta-\eta_{\mathrm{ini}})/k}\left\{X_{\boldsymbol{k},\mathrm{ini}}\cos[k(\eta-\eta_{\mathrm{ini}})] + \frac{K_{\boldsymbol{k},\mathrm{ini}}}{k}\sin[k(\eta-\eta_{\mathrm{ini}})]\right\}, \tag{213}$$

where $K_{\boldsymbol{k},\mathrm{ini}}$ and $X_{\boldsymbol{k},\mathrm{ini}}$ are two integration constants conveniently chosen to be the values of $K_{\boldsymbol{k}}$ and $X_{\boldsymbol{k}}$ at initial time $\eta = \eta_{\mathrm{ini}}$. For simplicity, we now consider a situation such that $K_{\boldsymbol{k},\mathrm{ini}} = 0$. Upon using Eq. (195), one finds that

$$s_{\boldsymbol{k}} = -\frac{k}{4}X_{\boldsymbol{k},\mathrm{ini}}^2[\mathrm{e}^{-2\gamma(\eta-\eta_{\mathrm{ini}})/k} - 1] - \frac{\gamma^2}{k^3}X_{\boldsymbol{k},\mathrm{ini}}^2\frac{1}{1+4\gamma^2/k^4}\mathrm{e}^{-2\gamma(\eta-\eta_{\mathrm{ini}})/k}\{\cos[2k(\eta-\eta_{\mathrm{ini}})] - \sin[2k(\eta-\eta_{\mathrm{ini}})] - 1\}. \tag{214}$$

If we expand the above result in γ/k^2 for the reason discussed before then, at leading order, one obtains an approximated expression for the mapping between the variables η and $s_{\boldsymbol{k}}$

$$s_{\boldsymbol{k}} \simeq \frac{kX_{\boldsymbol{k},\mathrm{ini}}^2}{4}[1 - \mathrm{e}^{-2\gamma(\eta-\eta_{\mathrm{ini}})/k}]. \tag{215}$$

This expression means that $s_{\boldsymbol{k}}$ runs from 0 to $kX_{\boldsymbol{k},\mathrm{ini}}^2/4$ when η runs from η_{ini} to infinity. Therefore, the time $s_{\boldsymbol{k}}$ evolves in a finite range. However, in order to be consistent, one must have $\eta < \eta_* = -1/k$ since the equations that have been used in order to derive $s_{\boldsymbol{k}}$ are valid only in the sub-Hubble regime. As a consequence, we have in fact $s_{\boldsymbol{k}} \in [0, s_*]$ where $s_* \equiv kX_{\boldsymbol{k},\mathrm{ini}}^2/4\{1 - \exp[(2\gamma/k^2)(1 + k\eta_{\mathrm{ini}})]\}$. Since we have $|k\eta_{\mathrm{ini}}| \gg 1$, one can thus write $s_* \simeq kX_{\boldsymbol{k},\mathrm{ini}}^2/4[1 - \exp(2\gamma\eta_{\mathrm{ini}}/k)]$. If $s < s_*$, then Eq. (215) can be inverted in order to evaluate the (total) number of e-folds in terms of the time variable $s_{\boldsymbol{k}}$. One finds

$$N_{\boldsymbol{k}} = (1+\beta)\ln\left[1 - \frac{k^2}{2\gamma}\frac{1}{k\eta_{\mathrm{ini}}}\ln\left(1 - \frac{4s_{\boldsymbol{k}}}{kX_{\boldsymbol{k},\mathrm{ini}}^2}\right)\right], \tag{216}$$

and one checks that if $s_{\boldsymbol{k}} = 0$ then $N_{\boldsymbol{k}} = 0$, if $s_{\boldsymbol{k}} = s_*$ then $N \to \infty$, and that the condition $s < s_*$ is sufficient to guarantee that the above expression is well defined.

Let us now discuss the above results in more detail. First, we notice in Eqs. (212) and (213) that the functions $K_{\boldsymbol{k}}(\eta)$

and $X_k(\eta)$ tend to zero when $\eta - \eta_{\rm ini} \gg 1$. When this happens, the two Gaussians have the same mean in position and momentum; in other words the two Gaussians have merged. This "merging phenomenon" seems to be a generic feature and can also be observed for the free particle [73] and/or the harmonic oscillator in Minkowski spacetime. Therefore, it does not come as a surprise that it also shows up in the sub-Hubble regime where the Fourier mode under consideration does not feel spacetime curvature. This also means that it is not a peculiar property of inflation.

The free particle situation can be studied [73] by returning to Eqs. (118)–(123). It is sufficient to consider that $\omega = 0$ in those equations to obtain this case. This means that the mode equation (125) now reads $f_k'' - \alpha^2 f_k = 0$, where the quantity α, defined in Eq. (126) for the harmonic oscillator, now reads $\alpha = \sqrt{2i\gamma\hbar/m} = \sqrt{\gamma\hbar/m}(1+i)$ and is obtained from Eq. (126) by taking $\omega = 0$. As a consequence, the solution for $\Omega(t)$ has exactly the same form as in Eq. (127) but now with the new α given above. This implies that $\Re e\Omega \to \sqrt{\gamma m/\hbar}/2$ and $\Im m\Omega \to \sqrt{\gamma m/\hbar}/2$ when $t \to \infty$. These formulas should be compared to Eqs. (134) and (135). Then, considering the equations of motion for a double Gaussian state, and defining $X \equiv \bar{x}_2 - \bar{x}_1$ and $K \equiv \chi_2 - \chi_1$, upon using Eq. (210), one obtains the following set of equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X \\ K \end{pmatrix} = \begin{pmatrix} -\sqrt{\gamma\hbar/m} & \hbar/m \\ -\gamma & -\sqrt{\gamma\hbar/m} \end{pmatrix}\begin{pmatrix} X \\ K \end{pmatrix}. \tag{217}$$

This equation should be compared to Eq. (211). In particular, one notices that, here, the free particle case is not simply obtained from this equation by considering $k = \omega = 0$. If we assume that $K(0) = 0$, then the solution for $X(t)$ is given by $X(t) = X(0)\exp(-t\sqrt{\hbar\gamma/m})\cos(t\sqrt{\hbar\gamma/m})$. We see that this solution resembles solutions (213) and (212) obtained before. Therefore, the merging is indeed already present for a free particle in flat spacetime and is not a specific feature of inflation. The exponential factor is mainly responsible for the merging and this means that the "merging time" of the free particle is given by

$$T_{\rm merge}^{\rm fp} = \sqrt{\frac{m}{\hbar\gamma}}. \tag{218}$$

This expression is consistent with the merging time derived in Ref. [73].

In order to discuss our inflationary result, one should consider the merging time of the harmonic oscillator instead of that of the free particle since this is the appropriate limit in the sub-Hubble regime. Following the same logic as before, it is easy to show that, for the harmonic oscillator, Eq. (217) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X \\ K \end{pmatrix} = \begin{pmatrix} -\gamma\hbar/(m\omega) & \hbar/m \\ -m\omega^2/\hbar & -\gamma\hbar/(m\omega) \end{pmatrix}\begin{pmatrix} X \\ K \end{pmatrix}. \tag{219}$$

We see that it is indeed similar to Eq. (211) if we take $\omega = k$ (and $m = \hbar = 1$). The solution for $X(t)$ can be expressed as $X(t) = X(0)\exp[-\hbar\gamma/(m\omega)t]\cos(\omega t)$, assuming as before $K(0) = 0$. This solution is perfectly consistent with (212) and (213). Compared to the free particle case, one notices that the coefficient in the exponential is now different from the frequency of the trigonometric function. But the most important result that one can deduce from the above considerations is that the merging phenomenon is also present for the harmonic oscillator and that the corresponding merging time is given by

$$T_{\rm merge}^{\rm ho} = \frac{m\omega}{\hbar\gamma} = \omega(T_{\rm merge}^{\rm fp})^2. \tag{220}$$

Let us remark that the last expression could have been guessed on dimensional grounds.

In the case of inflation, the conformal merging time is given by [see Eqs. (212) and (213)]

$$k(\eta_{\rm merge} - \eta_{\rm ini}) = \frac{k^2}{\gamma}. \tag{221}$$

However, there is a new twist in the discussion. It is not obvious that the above equation admits a solution because, in some sense, we have a limited amount of time from $\eta_{\rm ini}$ to η_*, the time of Hubble horizon crossing (defined by $|k\eta_*| = 1$). For times such that $|k\eta| < 1$, we are no longer in the sub-Hubble regime and the above equation can no longer be used. But, given a value of k^2/γ, and an initial time $\eta_{\rm ini}$, it is not obvious that there exists a time $\eta_{\rm merge}$ such that Eq. (221) is satisfied. In fact, there exists a solution only if $|k\eta_{\rm ini}| > 1 + k^2/\gamma$. This condition means that, for a given k^2/γ, one can always give more time to the system to satisfy Eq. (221) by starting its evolution earlier (which is equivalent to increasing $|\eta_{\rm ini}|$). It is easy to show that the previous inequality is in fact a condition on the total number of e-folds during inflation ($\beta \lesssim -2$), namely

$$N_{\rm T} \gtrsim \Delta N_* + \ln\left(1 + \frac{k^2}{\gamma}\right), \tag{222}$$

where $\Delta N_* \simeq 50$ for the modes of cosmological interest today. If this condition is met, then the merging occurs after $N_k^{\rm merge}$ with

$$N_k^{\rm merge} = -\ln\left(1 + \frac{k^2}{\gamma k\eta_{\rm ini}}\right). \tag{223}$$

Moreover, the term $k^2/(\gamma k\eta_{\rm ini})$ is of the order $\sim k^2 e^{-N_{\rm T}+50}/\gamma$ and it seems reasonable to assume that it is small. Indeed, typically, the total number of e-folds during inflation is very large and, even if $k^2/\gamma \gg 1$, the factor $e^{-N_{\rm T}}$ will entirely compensate its influence (to be more concrete, we know that $k^2/\gamma \gtrsim 10^{28}$ but $N_{\rm T}$ can easily be larger than, say, 1000 and can even be as large as 10^8). Then, the merging time during inflation can be approximated by

$$N_{\boldsymbol{k}}^{\rm merge} \simeq -\frac{k^2}{\gamma k \eta_{\rm ini}} \ll 1. \tag{224}$$

We see that this expression scales as $\propto k/\gamma$, in full agreement with the previous considerations on the harmonic oscillator, see Eq. (220).

Let us now study the collapse time. First of all, the collapse can occur in the sub-Hubble regime only if $b < s_*$. If we use the expression of s_* and assume, as before, that $k^2/(\gamma k \eta_{\rm ini}) \ll 1$, then $s_* \simeq k X_{\boldsymbol{k},\rm ini}^2/4$ and the condition for having the collapse in the sub-Hubble regime can be simply rewritten as

$$b \ll \frac{k X_{\boldsymbol{k},\rm ini}^2}{4}. \tag{225}$$

If this condition is satisfied, then the "e-fold collapse number" of the mode under consideration is obtained by putting $s_{\boldsymbol{k}} = b$ in the above expression (216). Upon using the same assumptions as before, we obtain that

$$N_{\boldsymbol{k}}^{\rm col} \simeq -\frac{2b}{\gamma X_{\boldsymbol{k},\rm ini}^2 \eta_{\rm ini}} \ll 1. \tag{226}$$

At this point, several remarks are in order. First, we notice that $N_{\boldsymbol{k}}^{\rm col}/N_{\boldsymbol{k}}^{\rm merge} = 4b/(k X_{\boldsymbol{k},\rm ini}^2) \ll 1$. This means that the collapse occurs on a much smaller time scale than the merging. This property was also noticed in the case of a free particle in Ref. [73]. This means that the merging cannot be viewed as a substitute for the collapse. Second, we notice that $N_{\boldsymbol{k}}^{\rm col}$ is actually independent of k. We interpret this fact as meaning that, on sub-Hubble scales, the mode under consideration must behave as in flat spacetime. Indeed, for a free particle or the harmonic oscillator in Minkowski spacetime, the condition for the collapse to occur can be written as $s = \gamma \int X^2(\tau) d\tau \simeq \gamma X(0)^2 T_{\rm col}^{\rm fp,ho} = b$, where we have used $X(t) \simeq X(0)$ since we have shown that the merging takes place on a much longer time scale. This implies that

$$T_{\rm col}^{\rm fp,ho} \simeq \frac{b}{\gamma X(0)^2}, \tag{227}$$

and one verifies that it is similar to Eq. (226). Therefore, if the collapse occurs on sub-Hubble scales, its properties are, as expected, similar to what happens in flat spacetime. Finally, if one starts from an initial state made of several well-separated Gaussian wave functions, the previous calculation suggests that it will almost instantaneously turn into a single Gaussian state. As a matter of fact, it is a general property [64,114] of the CSL dynamics that it asymptotically leads to Gaussian states. *A posteriori*, this remark reinforces the assumption of using a Gaussian state for the calculation of the spectrum in Sec. V D.

When condition (225) is not satisfied, there will be no collapse on sub-Hubble scales. However, we can still hope it will happen on super-Hubble scales. In fact, the claim that the collapse has occurred depends on the value chosen for b. Before, we used $b \simeq 10$ and for this value, given that our working assumption is $k X_{\boldsymbol{k},\rm ini}^2 \gg 1$, condition (225) is probably always satisfied. Therefore, it is only if we are more demanding about the criterion that defines the collapse that this condition can be violated. It is clear that a more stringent criterion takes more time to be satisfied and, in this case, the time "at our disposal" in the sub-Hubble regime may not be sufficient. In this situation, we have to consider the super-Hubble regime. In the next section, we turn to this case and show that the collapse is less efficient on large scales than it is on small scales.

C. Collapse time in the super-Hubble regime

In this section, we repeat the previous discussion but now in the super-Hubble regime. Therefore, we restart from equations (210) but now use the super-Hubble limit (166) and (167) for $\Re\mathrm{e}\Omega_{\boldsymbol{k}}$ and $\Im\mathrm{m}\Omega_{\boldsymbol{k}}$. For the modes of cosmological interest today in the (almost) scale invariant branch of the CSL power spectrum, one has $\gamma/k^2 \lll 1$ and the solution for $X_{\boldsymbol{k}}(\eta)$ can be simply written as

$$X_{\boldsymbol{k}}(\eta) \simeq X_{\boldsymbol{k}*}(-k\eta)^{\beta+1}, \tag{228}$$

where $X_{\boldsymbol{k}*}$ is the value of $X_{\boldsymbol{k}}(\eta)$ when the mode under consideration $\boldsymbol{k}$ crosses the Hubble radius. One can see that $X_{\boldsymbol{k}}(\eta)$ increases with time contrary to what happens in the sub-Hubble regime. From this expression, it is easy to derive the relation between $s_{\boldsymbol{k}}$ and the conformal time. One obtains

$$s_{\boldsymbol{k}} = -\frac{\gamma}{k^2}\frac{k X_{\boldsymbol{k}*}^2}{2\beta+3}[(-k\eta)^{2\beta+3} - 1]. \tag{229}$$

The last formula is valid only on super-Hubble time, that is to say for $\eta > \eta_* = -1/k$. At $\eta = \eta_*$, $s_{\boldsymbol{k}} = 0$ and then $s_{\boldsymbol{k}} \to \infty$ as $\eta \to 0$. From this expression, it is also possible to relate the time variable $s_{\boldsymbol{k}}$ and the number of e-folds. One arrives at

$$N_{\boldsymbol{k}} = N_* + \frac{1+\beta}{2\beta+3}\ln\left(1 - \frac{k^2}{\gamma}\frac{2\beta+3}{k X_{\boldsymbol{k}*}^2}s_{\boldsymbol{k}}\right). \tag{230}$$

This expression is always well defined because $2\beta + 3 < 0$. One verifies that $s_{\boldsymbol{k}} = 0$ corresponds to $N_{\boldsymbol{k}} = N_*$.

Let us now derive the time of collapse. As usual, it is obtained by $s_{\boldsymbol{k}} = b$. As a consequence, it is simply given by

$$N_{\boldsymbol{k}}^{\rm col} = N_* + \frac{1+\beta}{2\beta+3}\ln\left(1 - \frac{k^2}{\gamma}\frac{2\beta+3}{k X_{\boldsymbol{k}*}^2}b\right). \tag{231}$$

As a first check of this equation, we notice that, when $\gamma \to \infty$, $N_{\boldsymbol{k}}^{\rm col} \simeq N_*$. Of course, this result is expected since a large value of γ means that the collapse mechanism is very efficient and, therefore, that the wave function almost instantaneously collapses. On the other hand, formula (231) can be further simplified. Indeed, if the collapse has not taken place on sub-Hubble scales, it is also the case for the merging since $N_{\boldsymbol{k}}^{\rm col}/N_{\boldsymbol{k}}^{\rm merge} \ll 1$. As a consequence, $X_{\boldsymbol{k}}(\eta)$ has not evolved much and one can replace $X_{\boldsymbol{k}*}$ by $X_{\boldsymbol{k},\rm ini}$. Moreover, for the same reason, one must have $b \gtrsim k X_{\boldsymbol{k},\rm ini}^2/4$, see also Eq. (225). In addition, we know that

$k^2/\gamma \gg 1$. Therefore, the first term in the argument of the logarithm in Eq. (231) can be neglected. For $\beta \simeq -2$, this equation can be rewritten as

$$N_{\boldsymbol{k}}^{\mathrm{col}} - N_* \simeq \ln\left(\frac{k^2}{\gamma}\right) + \ln\left(\frac{b}{kX_{\boldsymbol{k},\mathrm{ini}}^2}\right). \tag{232}$$

Of course the result will depend on what we require for b and what we assume for $X_{\boldsymbol{k},\mathrm{ini}}$. However, it seems reasonable to assume that the second logarithm will not lead to a dominant contribution. If this is the case, then our result simply says that the wave function collapses just $\ln(k^2/\gamma)$ e-folds after the Hubble radius crossing. Given the constraint obtained from the measurement of the power spectrum in Eq. (178), one already knows that $N_{\boldsymbol{k}}^{\mathrm{col}} - N_* \gtrsim 28$. Smaller values of γ/k^2 would of course lead to a larger number of e-folds. We conclude this section by noticing that the constraint (178) is compatible with a collapse occurring during inflation. Only for values of γ such that $\gamma/k^2 \ll 10^{-50}$ (and $b \gtrsim kX_{\boldsymbol{k},\mathrm{ini}}^2/4$) would the collapse happen after inflation.

D. The Born rule derived

Finally, we conclude with a section where we calculate the probabilities of collapsing to each of the two branches of the wave function. We show that these probabilities are given by the Born rule, which is of course expected since the CSL theory is precisely designed to reproduce this result, as already discussed in Sec. IV (see also Ref. [73]).

Let us denote by p_1 the probability that the system collapses on the first Gaussian branch of the wave function. This is also the probability that, from given initial conditions, the stochastic quantity $\Gamma_{\boldsymbol{k}}$ reaches first the region $\Gamma_{\boldsymbol{k}} < -b$ (i.e., before the region $\Gamma_{\boldsymbol{k}} > b$) and that, therefore, one has $\Gamma_{\boldsymbol{k}}(S_{\boldsymbol{k}}) = -b$. Clearly, the probability p_2 that the wave function collapses on the second branch is the probability that $\Gamma_{\boldsymbol{k}}(S_{\boldsymbol{k}}) = b$. Now, let us introduce a function $\psi(x)$ which is defined by

$$\psi(x) \equiv \frac{g(x) - g(b)}{g(-b) - g(b)}, \tag{233}$$

where $g(x)$ will be specified soon. By construction, one has $\psi(-b) = 1$ and $\psi(b) = 0$. Since, by definition, $\Gamma_{\boldsymbol{k}}(S_{\boldsymbol{k}})$ can only take two values (namely $\pm b$), one has

$$\mathbb{E}\{\psi[\Gamma_{\boldsymbol{k}}(S_{\boldsymbol{k}})]\} = p_1\psi(-b) + p_2\psi(b) = p_1, \tag{234}$$

and this gives us a method to calculate p_1. To do so, we follow what was explained in Sec. VI A, see in particular Eq. (200), and we write the corresponding Itô formula

$$\psi[\Gamma_{\boldsymbol{k}}(S_{\boldsymbol{k}})] - \psi(b_0) = \int_0^{S_k} \psi'[\Gamma_{\boldsymbol{k}}(s_{\boldsymbol{k}})]\mathrm{d}W_s + \int_0^{S_k}\left\{\psi'[\Gamma_{\boldsymbol{k}}(s_{\boldsymbol{k}})]\tanh[\Gamma_{\boldsymbol{k}}(s_{\boldsymbol{k}})] + \frac{1}{2}\psi''[\Gamma_{\boldsymbol{k}}(s_{\boldsymbol{k}})]\right\}\mathrm{d}s_{\boldsymbol{k}}. \tag{235}$$

Then, let us choose the function $g(x)$ such that it obeys the equation

$$\frac{1}{2}g''(x) + \tanh(x)g'(x) = 0, \tag{236}$$

or, equivalently, $g(x) = \tanh(x)$. Since Eq. (233) implies that $\psi(x)$ and $g(x)$ are linearly related, $\psi(x)$ also obeys the above differential equation. As a consequence, the second integral in Eq. (235) vanishes. Taking the stochastic average, one obtains

$$\mathbb{E}\{\psi[\Gamma_{\boldsymbol{k}}(S_{\boldsymbol{k}})]\} = p_1 = \psi(b_0), \tag{237}$$

which is explicitly known since $g(x)$ has been determined.

The probability p_2 can be deduced along the same lines, by introducing a new function ψ such that, this time, $\psi(-b) = 0$ and $\psi(b) = 1$. Another method, much simpler, is just to use the condition $p_1 + p_2 = 1$. The final result reads

$$p_1 = \frac{\tanh(b_0) - \tanh(b)}{\tanh(-b) - \tanh(b)}, \tag{238}$$

$$p_2 = \frac{\tanh(b_0) - \tanh(-b)}{\tanh(b) - \tanh(-b)}. \tag{239}$$

From the definition of $\Gamma_{\boldsymbol{k}}$, these two formula can be rewritten as [73]

$$p_1 = \frac{|N_1(\eta_{\mathrm{ini}})|^2}{|N_1(\eta_{\mathrm{ini}})|^2 + |N_2(\eta_{\mathrm{ini}})|^2}, \tag{240}$$

$$p_2 = \frac{|N_2(\eta_{\mathrm{ini}})|^2}{|N_1(\eta_{\mathrm{ini}})|^2 + |N_2(\eta_{\mathrm{ini}})|^2}, \tag{241}$$

which are exactly the Born rules of conventional quantum mechanics.

VII. CONCLUSION

Let us now summarize our main findings. In this paper, we have applied the CSL theory to inflation. Since the CSL scenario addresses the measurement problem in quantum mechanics, it is *a priori* relevant to explain how the wave packet reduction took place in the early Universe, in the absence of any observer. Assuming that the wave function has to collapse on an eigenstate of the Mukhanov-Sasaki operator, we have computed the scalar power spectrum of cosmological perturbations and studied the dynamics of the wave function collapse. We have found that, in order to preserve the scale invariance of the power spectrum, it is necessary to fine-tune the parameter γ which controls the amplitude of the CSL corrections. Typically, depending on which temporal gauge is chosen (see the Appendix), we have found that the dimensionless parameter that can be constructed out of γ must be smaller than $\exp(-\text{a few}\Delta N_*)$, where $\Delta N_* \simeq 50$–60 is the number of e-folds spent by the relevant modes outside the Hubble radius during inflation. We have also found that the time available during the inflationary phase is sufficient in order for the perturbations' wave function to collapse. However, due to the smallness of γ, the spread of the final wave function is too important, rendering the collapse process not sufficiently efficient. Therefore, under the assumptions made in this paper, it seems fair to claim that the collapse theories cannot solve the inflationary macro-objectification question.

The conclusions drawn above may not be as drastic as they appear at first sight, because they are subject to some assumptions, and in particular the choice of the collapse operator as the Fourier space Mukhanov-Sasaki $v_{\boldsymbol{k}}$ variable: all cosmological predictions made to date are based on this variable, rendering this choice very sensible, but it is by no means unique (see, e.g., the discussion in Sec. VA). Moreover, $v_{\boldsymbol{k}}$ can be understood as a quantum field living in a curved spacetime, so it should be treated by a quantum field theory version of the CSL mechanism. The present state of the art of this subject technically forbids such a direct treatment, hence our simplifying hypothesis. Could it be that a full relativistic version of CSL, reproducing the many successes of quantum field theory and of the ensuing particle physics, is needed before we can even embark in examining cosmological perturbations? We doubt so, because cosmology, contrary to ordinary quantum field theory, is endowed with a preferred timelike direction that renders the "time-dependent Minkowski approximation" accurate enough for all practical purposes. It is left for future investigations to verify that the potential problems raised and stringent constraints obtained in this work could be naturally solved in a more general, yet unknown, framework.

There are other questions that could be the subject of further works. In particular, there is the issue that energy is not conserved in the CSL theory. In the case of the harmonic oscillator, this is expressed through Eq. (142). In the case of cosmological perturbations, it is easy to show that this leads to

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\langle\hat{\mathcal{H}}_{\boldsymbol{k}}\rangle = \frac{\gamma}{2} + \omega\omega'\langle v_{\boldsymbol{k}}^2\rangle. \qquad (242)$$

The CSL contribution can easily be integrated and gives $\langle\hat{\mathcal{H}}_{\boldsymbol{k}}\rangle|_{\mathrm{CSL}} \simeq \gamma\eta_{\mathrm{ini}}/2$ at the end of inflation. Expressed in terms of the Hamiltonian rather than the Hamiltonian density, one arrives at

$$\langle\hat{H}\rangle|_{\mathrm{CSL}} \simeq -4\pi^2\frac{\gamma}{2}\eta_{\mathrm{ini}}\int k^2\mathrm{d}k, \qquad (243)$$

which is infinite. It does not come as a surprise as it is known that the CSL Tomanaga-Schwinger equation precisely leads to this type of divergence [67,68]. It could be regularized by introducing an ultraviolet cutoff although we notice on the above equation the weird property that the infinite integral is over comoving wave numbers rather than over physical ones. This energy nonconservation should cause a continuous increase of energy density during inflation. It is interesting to notice that it cannot occur at first order in the perturbations since $\mathbb{E}(\langle\widehat{\delta\rho}_{\boldsymbol{k}}\rangle) = 0$. This means that it will be important at second order only. Then, it would be important to quantify this effect and, in particular, to compare it to the background energy density $\simeq H^2M_{\mathrm{Pl}}^2$ in order to check whether this leads to a back-reaction problem.

Another point is that we have shown that the power spectrum, contrary to what happens in the standard case, remains a time-dependent quantity, i.e., still evolves with time on large scales during inflation. It is therefore not obvious that $\mathcal{P}_\zeta$ evaluated at the end of inflation is exactly the power spectrum that should be used at recombination. In fact, what happens just after the end of inflation is of great interest for the cosmological consequences of CSL. Indeed, just after inflation, the stages of preheating and reheating begin [96–98]; this is also shown in Fig. 4. During this phase of evolution, the inflaton field oscillates at the bottom of its potential, $\varphi(t) \propto \sin(mt+\Delta)/(mt)$ where Δ is a phase and m the mass of the inflaton (in the case of power-law inflation, the potential has no minimum and, therefore, can only be used to describe the slow-roll regime; here, we assume that the potential can be approximated by $m^2\varphi^2$ in the vicinity of the minimum). In this case, the equation of motion (12) for the Mukhanov-Sasaki variable takes the form of a Mathieu equation [99]. As is well known, this equation possesses unstable solutions when the parameters falls in the resonant bands. In the case of inflation, one can show that the large-scale perturbations are in the first instability band which makes $v_{\boldsymbol{k}}$ growing and $\zeta_{\boldsymbol{k}}$ staying constant [99,100]. In the CSL case, the corresponding Mathieu equation would read

$$\frac{\mathrm{d}^2v_{\boldsymbol{k}}}{\mathrm{d}z^2} + [\mathcal{A}_{\boldsymbol{k}} - 2q\cos(2z+2\Delta)]v_{\boldsymbol{k}} = 0, \qquad (244)$$

where $z \equiv mt + \pi/4$, a_{e}, t_{e} denoting the scale factor and the cosmic time at the end of inflation and with

$$\mathcal{A}_{\boldsymbol{k}} = 1 + \frac{k^2 - 2i\gamma}{m^2a^2}, \qquad (245)$$

$$q = \frac{2}{mt_{\mathrm{e}}}\left(\frac{a_{\mathrm{e}}}{a}\right)^{3/2}. \qquad (246)$$

Since $q \ll 1$, in the regular case when $\gamma = 0$, the condition to be in the first resonant bands, $1 - q < \mathcal{A}_{\boldsymbol{k}} < 1 + q$, is equivalent to $0 < k/a < \sqrt{3Hm}$. In the CSL case, the coefficient $\mathcal{A}_{\boldsymbol{k}}$ becomes complex. Therefore, in order to determine the corresponding Floquet index, it now becomes necessary to study the instability chart of the Mathieu equation in the complex domain. Although this is beyond the scope of this paper, this is certainly a subject worth investigating. In particular, it would be interesting to see whether the instability is enhanced in this case as one can, maybe naively, suspect. If so, maybe the preheating stage can put even more stringent constraints on the parameter γ.

We have seen that the study of the CSL cosmological perturbations is in fact equivalent to the study of the CSL parametric oscillator (i.e., a harmonic oscillator with a time-dependent frequency). The previous discussion suggests that it would be interesting to investigate the case of a parametric oscillator in the presence of a resonance in the CSL framework. In quantum field theory, this is a common situation and typical examples are the dynamical Schwinger effect [115] (the analogy between cosmological perturbations and the Schwinger effect was discussed in Ref. [15]) or the dynamical Casimir effect [116] which was recently observed for the first time [117] in the laboratory. In fact, if we want to avoid the objection that the quantum field CSL theory is not yet ready, it would be even more interesting to find a nonrelativistic system governed by a Mathieu equation and to investigate its behavior within the CSL theory. We believe that all of the equations presented in the present article can be straightforwardly applied to this case. Here, we suggest that a Paul trap [118] could be such an example. As for the inflationary preheating, we expect the coefficients of the Mathieu equation to become complex because of the $-2i\gamma$ term. This will probably make the system extremely unstable and, as a consequence, it will probably be possible to put very tight constraints on the value of γ. We hope that this case will be treated in details soon.

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APPENDIX: "GAUGE INVARIANCE" OF THE CSL POWER SPECTRUM

In section VA, we discussed the choice of the collapse operator, i.e., the operator that appears in the nonlinear and stochastic part of the modified Schrödinger equation. In principle, this operator should be determined by a more fundamental theory. However, the CSL model is just a phenomenological approach and the collapse operator is just put by hand in order to match what we observe when an experiment or an observation is performed (the position of a spot in a detector, the energy density of a field, etc.). In the case of the cosmological primordial perturbations, we have argued that the Mukhanov-Sasaki variable $\hat{v}_{\boldsymbol{k}}$ is the most sensible choice. But this variable often appears factorized by a background quantity, typically a power of the scale factor $a(\eta)$. Therefore, instead of $\hat{v}_{\boldsymbol{k}}$, one could very well choose the collapse operator to be $h(a)\hat{v}_{\boldsymbol{k}}$, where h is *a priori* an arbitrary function of the scale factor a. After all, $\hat{v}_{\boldsymbol{k}}$ and $h(a)\hat{v}_{\boldsymbol{k}}$ share the same eigenspectrum and drive the system towards the same target states with the same probabilities. But the point is that, *a priori* and as is discussed in detail below, this does not lead to the same solution for the mode function $f_{\boldsymbol{k}}(\eta)$ and, therefore, *a priori*, for the power spectrum.

In fact, this issue is related to an even more fundamental problem. Indeed, one could claim that the conformal time η used in this paper to write the modified Schrödinger equation is not the physical one and that one should use instead, say, the cosmic time t (of course, the discussion also applies to any other time variables related to η through a transformation that depends only on the background). In fact, a choice of time is equivalent to a choice of h since it has the same effect on the modified Schrödinger equation. And, of course, as already mentioned, one could worry that different choices lead to different predictions. Therefore, the phenomenological approach used in this article suffers from what can be called a temporal gauge problem. This problem probably originates from the fact that the CSL equation is not covariant under diffeomorphisms (contrary to the standard theory of cosmological perturbations).

In this appendix, we investigate this question, showing the remarkable property that the conclusions obtained in this paper for $h(a) = 1$ are in fact valid for any other functions h. It is true that the detailed shape of the power spectrum depends on the gauge but its global properties are independent of the choice of h. This means that, *a priori* for any h allowing meaningful initial conditions, the power spectrum of cosmological perturbations has a broken power-law shape, with $n_{\mathrm{S}} = 1$ at small wavelengths and $n_{\mathrm{S}} = 4$ at large wavelengths. As a consequence, the requirement of moving the non-scale-invariant part of the spectrum beyond the Hubble radius today always leads to extreme constraints on the parameter γ.

Let us now consider the modified Schrödinger equation of motion for $\Psi_{\boldsymbol{k}}$ in the CSL picture, with spontaneous localization on the $h(a)\hat{v}_{\boldsymbol{k}}$ eigenmanifolds. It reads

$$\mathrm{d}\,\Psi_{\boldsymbol{k}}^{\mathrm{R}} = \Big[-i\hat{\mathcal{H}}_{\boldsymbol{k}}^{\mathrm{R}}\mathrm{d}\eta + \sqrt{\gamma}h(a)(\hat{v}_{\boldsymbol{k}}^{\mathrm{R}} - \langle\hat{v}_{\boldsymbol{k}}^{\mathrm{R}}\rangle)\mathrm{d}W_\eta - \frac{\gamma}{2}h^2(a)(\hat{v}_{\boldsymbol{k}}^{\mathrm{R}} - \langle\hat{v}_{\boldsymbol{k}}^{\mathrm{R}}\rangle)^2\mathrm{d}\eta\Big]\Psi_{\boldsymbol{k}}^{\mathrm{R}}, \tag{A1}$$

and a similar equation for $\Psi_{\boldsymbol{k}}^{\mathrm{I}}$. This equation should be compared with Eq. (143), the only difference being that the operator $\hat{v}_{\boldsymbol{k}}$ is now multiplied by $h(a)$. Parametrizing $\Psi_{\boldsymbol{k}}$ as in Eq. (144) using again $\Omega_{\boldsymbol{k}} = -if_{\boldsymbol{k}}'/(2f_{\boldsymbol{k}})$, one is led to the following equation for the mode function

$$f_{\boldsymbol{k}}'' + [\omega^2(\eta, \boldsymbol{k}) - 2i\gamma h^2(a)]f_{\boldsymbol{k}} = 0. \tag{A2}$$

This expression should be compared with Eq. (153): as expected, the only difference is that an extra $h^2(a)$ appears in front of the γ term. For simplicity, let us choose h to be a simple power law and let us assume the inflationary dynamics to be close to a de Sitter universe $a(\eta) \simeq -\ell_0/\eta$. Then, the mode function can be reexpressed as

$$f_{\boldsymbol{k}}'' + \left(k^2 - \frac{2}{\eta^2} - 2i\gamma a^p\right)f_{\boldsymbol{k}} = 0. \tag{A3}$$

If $p < 0$, the Bunch-Davies vacuum state cannot be chosen at the onset of inflation since the k^2 term does not dominate in the parenthesis. This means that one must work with $p \geq 0$. In this paper the case $p = 0$ [i.e., $h(a) = 1$] has been studied, hence one only needs to study the $p > 0$ cases. It is interesting first to notice that the cases $p > 0$ provide a natural amplification phenomenon depending on the physical length of the mode since the amplitude of the term proportional to γ now increases as the mode is stretched by the growth of the scale factor. This is consistent with the physical intuition which tells us that the collapse should occur for macro extended objects only. If $p > 2$, the term proportional to γ dominates the dynamics at the end of inflation, when $k\eta$ goes to 0, and one can expect the power spectrum scale invariance to be destroyed. Therefore, if p is an integer, we are left with the cases $p = 1$ and $p = 2$ that we now study.

If $p = 1$, the general solutions of Eq. (A3) can be expressed in terms of Whittaker functions $W_{\mu,\kappa}(z)$ [101,102] as

$$f_{\boldsymbol{k}}(\eta) = C_{\boldsymbol{k}} W_{\gamma\ell_0/k,3/2}(2ik\eta) + D_{\boldsymbol{k}} W_{-\gamma\ell_0/k,3/2}(-2ik\eta), \tag{A4}$$

where $C_{\boldsymbol{k}}$ and $D_{\boldsymbol{k}}$ are integration constants that can be determined by choosing the Bunch-Davies vacuum state for the initial conditions. This leads to $C_{\boldsymbol{k}} = 0$. Then, in the limit where $k\eta$ goes to 0, $\Re\mathrm{e}\Omega_{\boldsymbol{k}}(\eta)$ can be Taylor expanded, and this provides a simple expression for this quantity. In particular, we find that $\Re\mathrm{e}\Omega_{\boldsymbol{k}}/k = \gamma\ell_0/(2k) + \mathcal{O}(k\eta)$, showing that, in this case, the spread does not diverge in the large-scale limit and that, as a consequence, the localization of the wave function becomes much more accurate. Moreover, since the inverse of $\Re\mathrm{e}\Omega_{\boldsymbol{k}}$ is basically $\mathcal{P}_\zeta$, this allows us to calculate the power spectrum, provided we push the expansion to higher orders. One obtains

$$\mathcal{P}_\zeta(k) = g\left(\frac{\ell_0\gamma}{k}\right)\left[1 + \frac{\ell_0\gamma}{k_0} g\left(\frac{\ell_0\gamma}{k}\right) \mathrm{e}^{2\Delta N_*}\left(\frac{k_0}{k}\right)^3 - 2\frac{\ell_0\gamma}{k} g\left(\frac{\ell_0\gamma}{k}\right)\left(1 - \frac{\ell_0^2\gamma^2}{k^2}\right)\ln\left(2\frac{k}{k_0}\,\mathrm{e}^{-\Delta N_*}\right)\right]^{-1} \times \mathcal{P}_\zeta(k)|_{\mathrm{stand}}, \tag{A5}$$

where $\mathcal{P}_\zeta|_{\mathrm{stand}}$ is the standard power spectrum (47), and where $g(x)$ is defined by

$$\frac{1}{g(x)} \equiv 1 + 3x - 3x^2 - x^3 - 2x(1 - x^2)[\psi(2 + x) - 2\psi(1)], \tag{A6}$$

$\psi(x)$ being the digamma Euler function [101,102]. Let us notice that, in Eq. (A5), we have sometimes introduced the quantity $\ell_0\gamma/k$. Of course, the most convenient way of dealing with this quantity is to express it as $(\ell_0\gamma/k_0)k_0/k$ such that the dimensionless small parameter $\ell_0\gamma/k_0$ explicitly appears. The spectrum given by Eq. (A5) should be compared with the one obtained in Eq. (172) with the choice

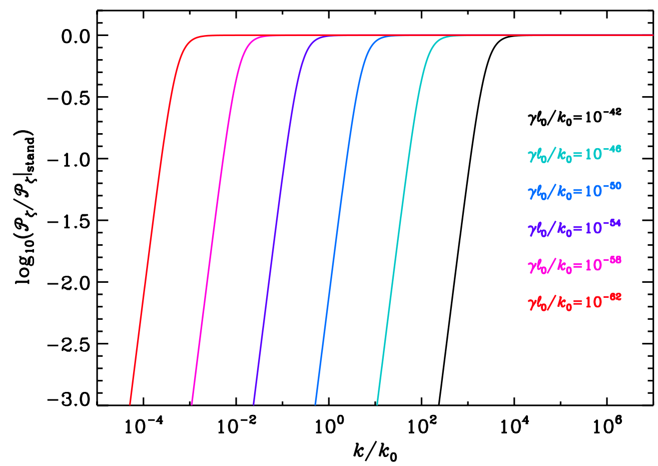

FIG. 6 (color online). Ratio of the power spectrum given by Eq. (A5) ($p = 1$) to the standard power spectrum given by Eq. (47) for different values of the parameter $\gamma\ell_0/k_0$.

$h = 1$. They share the same broken power-law structure, with a scale-invariant part $n_{\mathrm{S}} \simeq 1$ at small scales and a branch with $n_{\mathrm{S}} = 4$ on large scales. This spectrum is displayed in Fig. 6 for different values of the parameter $\ell_0\gamma/k_0$.

The break in the power spectrum appears at $k^3/k_0^3 \simeq \ell_0\gamma/k_0\mathrm{e}^{2\Delta N_*}$. Therefore, in order for the non-scale-invariant part of the power spectrum to be outside the Hubble radius, one must have

$$\frac{\gamma\ell_0}{k_0} \ll \mathrm{e}^{-2\Delta N_*} \simeq 10^{-53}. \tag{A7}$$

This equation should be compared to Eq. (178). We see that, in the present case, we also obtain a constraint that can be considered as "extreme". In other words, it seems that a very important fine-tuning is necessary to maintain the consistency of the CSL predictions with the CMB observations. We also notice that, instead of γ/k_0^2, it is now the combination $\gamma\ell_0/k_0$ that is constrained. Of course, this is just the consequence of the fact that, as already discussed, changing the collapse operator can change the dimension of the parameter γ. In some sense, we face again the discussion of the temporal gauge issue.

Let us now turn to the case $p = 2$ in Eq. (A3). The general solutions of this equation can be expressed in terms of Bessel functions with a complex order [101,102], namely

$$f_{\boldsymbol{k}}(\eta) = C_{\boldsymbol{k}}\sqrt{-k\eta}\,J_{\frac{3}{2}\sqrt{1+\frac{8}{9}i\gamma\ell_0^2}}(-k\eta) + D_{\boldsymbol{k}}\sqrt{-k\eta}\,J_{-\frac{3}{2}\sqrt{1+\frac{8}{9}i\gamma\ell_0^2}}(-k\eta), \tag{A8}$$

where $C_{\boldsymbol{k}}$ and $D_{\boldsymbol{k}}$ are integration constants that can be determined by requiring, as usual, the initial state to be the Bunch-Davies vacuum. This leads to $C_{\boldsymbol{k}} = -D_{\boldsymbol{k}}\mathrm{e}^{3i\pi/2\sqrt{1+8/9i\gamma\ell_0^2}}$. In the limit where $k\eta$ goes to 0,

$\Re\mathrm{e}\Omega_{\boldsymbol{k}}$ can be Taylor expanded and, at first order in the parameter $\gamma\ell_0^2$, the power spectrum reads

$$\mathcal{P}_\zeta(k) \simeq \left(1+\frac{2\pi}{3}\gamma\ell_0^2\right)\left[1+\frac{2\gamma\ell_0^2}{3}\,\mathrm{e}^{3\Delta N_*}\left(\frac{k_0}{k}\right)^3 + \frac{4}{3}\gamma\ell_0^2\frac{k_0}{k}\,\mathrm{e}^{\Delta N_*}\right]^{-1}\mathcal{P}_\zeta(k)|_{\mathrm{stand}}. \quad \text{(A9)}$$

The formula (A9) should be compared with Eqs. (172) and (A5). Again, the power spectrum has the same shape, with a scale invariant part on small scales and a noninvariant branch with $n_\mathrm{S} = 4$ on large scales. This is clearly seen in Fig. 7, where the spectrum (A9) is represented for different values of the parameter $\gamma\ell_0^2$. The break in the power spectrum appears at $k^3/k_0^3 \simeq \gamma\ell_0^2/3\mathrm{e}^{3\Delta N_*}$. Therefore, in order for the non-scale-invariant part of the power spectrum to be outside the observational window, one must require that

$$\gamma\ell_0^2 \ll \mathrm{e}^{-3\Delta N_*} \simeq 10^{-79}. \quad \text{(A10)}$$

Again, we can consider the above constraint as a fine-tuning. It is also interesting to notice that, contrary to Eqs. (178) or (A7) and (A10) involves physical quantities only. This is because, when $p = 2$, the CSL correction that should be compared to the comoving wave number squared is $\propto \gamma a^2$, see Eq. (A3). In other words, γ should now be compared to the physical wave number. If we take $\ell_0 \simeq 10^5\ell_\mathrm{Pl}$, which comes from the CMB normalization, then one arrives at $\gamma \ll 10^{-89}$.

Let us conclude this appendix by noticing that the above results are in fact generic and do not depend on the value of p. Technically, the power spectrum is obtained by taking the super-Hubble limit of the mode function $f_{\boldsymbol{k}}(\eta)$, by inserting it in the expression of $\Re\mathrm{e}\Omega_{\boldsymbol{k}} = \Re\mathrm{e}[-if_{\boldsymbol{k}}'/(2f_{\boldsymbol{k}})]$ and by retaining only the leading order in $k\eta$. In the standard case, the leading terms of the mode function expansion turn out to cancel out in $\Re\mathrm{e}\Omega_{\boldsymbol{k}}$, leaving an expression which precisely gives a scale-invariant power spectrum. This cancellation originates from the fact that the Wronskian is conserved. In the CSL case, the fact that $\gamma \neq 0$ implies that this symmetry no longer exists, and, as a consequence, the nice cancellations mentioned above no longer show up and scale invariance is immediately broken. In some sense, the fact that the γ term destroys the scale invariance of the power spectrum does not come from the fact that its presence modifies the time dependence of the effective frequency (the value of p or the choice of h), but is rather due to the fact that it makes the effective frequency a complex quantity. We conclude that modifying the definition of the collapse operator by multiplying it with a background function, despite changing the dimension of γ, always constrains this parameter to be extremely fine-tuned.

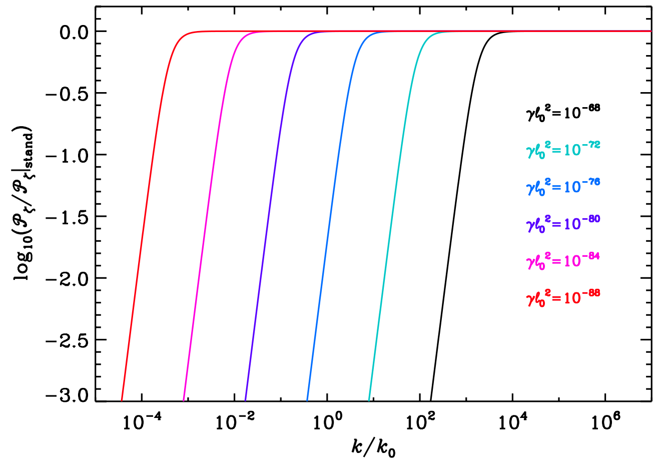

FIG. 7 (color online). Ratio of the power spectrum given by Eq. (A9) ($p = 2$) to the standard power spectrum given by Eq. (47) for different values of the parameter $\gamma\ell_0^2$.

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