# Exact cosmological solutions of models with an interacting dark sector

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In this work we extend the first order formalism for cosmological models that present an interaction between a fermionic and a scalar field. Cosmological exact solutions describing universes filled with interacting dark energy and dark matter have been obtained. Viable cosmological solutions with an early period of decelerated expansion followed by late acceleration have been found, notably one which presents a dark matter component dominating in the past and a dark energy component dominating in the future. In another one, the dark energy alone is the responsible for both periods, similar to a Chaplygin gas case. Exclusively accelerating solutions have also been obtained.

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## I. INTRODUCTION

The recent improvements in the observational techniques available for the measurement of cosmological parameters show, with increasing accuracy, that the Universe is composed mainly of two mysterious entities, the so-called dark energy and dark matter. The first one is believed to be the cause of the observed accelerated expansion of the Universe [1–3] and corresponds to approximately 70% of its total energy density. The latter corresponds to almost 25% of the energy density of the Universe and plays an important role in large structure formation. The reader is referred to Refs. [4–15] for a review on the theoretical developments that followed these observations.

An interaction between these two dark sectors is rather plausible, and can be even considered to be a necessary feature in cosmological models based on quantum field theory. An extensive literature in the last years has shown both theoretical and phenomenological aspects of the coupling between dark matter and dark energy (see, e.g., Refs. [16–36]). The present work is an attempt to solve the equations of motion for a theoretical model based on an interaction Lagrangian, in which dark energy is represented by a canonical scalar field, and dark matter takes the form of a fermionic field.

Solving these equations, even for the case in which no interaction is taken into account, is very often an extremely arduous task, if not totally impossible, without the use of certain approximations. Many papers rely on intensive computer simulations to gather some insight about cosmological parameters, while others tackle the difficulties by analyzing the phase space dynamics represented by the generic equations of motion for the cosmological model in question.

In all cases, the viable cosmological solution including dark energy must obey that the energy density of scalar field must remain subdominant during radiation and matter dominant eras to allow for structure formation, becoming dominant in late times accounting for the current acceleration of the Universe. This can be obtained by *scaling solutions* where the energy density of dark energy mimics the background fluid (radiation or matter) [37]. The field must also exit this scaling solution in order to describe the late accelerated solution. Models with interaction between the dark sectors can account for the exit of scaling, by dynamically modifying the scalar field potential, and satisfy all the requirements to describe a viable dark energy scenario [38,39].

Another method used to obtain exact solutions in cosmology is the first order formalism (FOF), developed by Bazeia *et al.* [40], where cosmological solutions with dark energy modeled by a single scalar field were obtained exactly. The central point of that formalism is to assume the Hubble's factor to be a function of the scalar field $\phi$, thus reducing the second order equations to first order ones. In another work Bazeia *et al.* [41] generalized the first order formalism describing a universe filled with dust and dark energy. It is interesting to point that, in their work, a coupling between dark matter and dark energy arose naturally. The authors argued that this interaction is an effect of the applied first order formalism.

The aim of this work is to apply the FOF for a model where an interaction between the dark sectors is present. We consider a universe filled only with dark matter and dark energy, which means that our models only aim to model the late universe evolution. We have generalized Bazeias's cosmological models by explicitly adding a fermionic field $\psi$, which plays the role of dark matter, and an interaction term between the fermionic and the scalar parts.

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The imposition of such interaction renders a more involved Friedmann equation and equations of motion, increasing the number of variables of the problem and reducing the constraints of the system. Since we have this extra freedom in the model, we impose some restrictions to the scale factor, via an *ansatz* that relates it to the scalar field itself.

Even though these restrictions may reduce the possibilities of expansion of the Universe, we show in this paper that we can still construct a large class of exact solutions by applying the FOF for the interacting models with the restriction of the solution for the scalar field as a function of time $\phi(t)$ to be invertible. This aspect is actually essential for solving the equations exactly and is taken for granted in much of the literature on this subject. A few works involving dynamical analysis discard this requirement, but they are able to find exact solutions after making some different restrictions and approximations based on the structure of the phase space for the models [37].

The solutions obtained present only accelerated, and decelerated and accelerated periods. This is in accordance with the phase space analysis, where in most cases we obtain nonviable cosmological solutions, and the scaling solution, which describes the right cosmological evolution, is harder to find, obtaining only a few for each model. We also note that, together with the dark sectors interaction proposed as a hypothesis, the method introduces an interaction, as pointed out in Ref. [41]. This happens because of the choice that the Hubble parameter and scale factor depend only on the scalar field, which renders an effective interaction between dark energy and dark matter. We also account for this interaction and verify how this influences the solutions.

The plan of this paper is as follows: in Sec. II we introduce our model of interacting dark energy and dark matter introducing a fairly general interaction form. In Sec. III we present the first order formalism and present the *ansatz* necessary to reduce the order of the differential equations. In Sec. IV we show the exact solutions obtained for different parameters of the formalism. The last section shows our final remarks.

## II. INTERACTING DARK ENERGY AND DARK MATTER MODEL

First, let us show the model we are going to study. We use the Friedmann-Lemaître-Robertson-Walker metric with null curvature

$$ds^2 = dt^2 - a(t)^2(dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2), \quad (1)$$

where $a(t)$ is the scale factor. The action that describes this geometry and the material content is

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big(& -\frac{R}{4} + \ell\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \\ &+ \frac{i}{2}\Big[\bar{\psi}\gamma^\mu\nabla_\mu\psi - \bar{\psi}\overleftarrow{\nabla}_\mu\gamma^\mu\psi\Big] - M\bar{\psi}\psi \\ &+ \beta F(\phi)\bar{\psi}\psi\Big), \end{aligned} \quad (2)$$

where $\phi$ is the scalar field and $V(\phi)$ its potential, $\psi$ is the fermionic field, and $F(\phi)$ is an interaction. The constants $M$ and $\beta$ are the mass of the fermionic field and the coupling constant, respectively, and $\ell$ is a constant that can assume the values $\pm 1$, if one wants the action to also accommodate the possibility of a phantom field. The functions $F(\phi)$ and $V(\phi)$ are not fixed in the beginning.

We have to solve Friedmann's equations, the equation of motion for the scalar field, and the Dirac equation. We know that the homogeneous Dirac equation in a homogeneous and isotropic spacetime has a simple solution in terms of the scale factor, and we can see that $\bar{\psi}\psi = \alpha/a^3$, where $\alpha$ is a constant related to the energy density of the dark matter at the present phase of the history of the Universe [30]. Therefore, the system of equations that remains to be solved is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{V'}{\ell} = \frac{\beta F'\bar{\psi}\psi}{\ell}, \quad (3)$$

$$H^2 = \frac{8\pi}{3M_p^2}\Big\{\ell\frac{\dot{\phi}^2}{2} + V(\phi) + [M - \beta F(\phi)]\bar{\psi}\psi\Big\}, \quad (4)$$

$$\dot{H} = -\frac{4\pi}{M_p^2}\Big\{\ell\dot{\phi}^2 + [M - \beta F(\phi)]\bar{\psi}\psi\Big\}, \quad (5)$$

where $H = \frac{\dot{a}}{a}$ is Hubble's factor and $M_p$ the Planck's mass. A prime represents the derivative with respect to the scalar field $\phi$.

We also evaluate the energy-momentum tensor. We evaluate it separately for each component, the scalar field and the fermionic field, and for the interaction term [42]. We then obtain the energy density, pressure, and equation of state in the Friedmann-Lemaître-Robertson-Walker spacetime. As the interaction term, $\mathcal{L}_{\mathrm{int}} = \beta F(\phi)\bar{\psi}\psi$, depends on both the scalar field and the fermionic field, we choose not to include the interaction as a part of any of those fluids. Instead, we separate it as a different "component" so we can see how it evolves and how dominant this term is, since $F(\phi)$ is not known and the method can introduce new interactions.

The energy density $\rho$, the pressure $p$, and the equation of state $\omega$ for the scalar field are given by

$$\rho_{\mathrm{DE}} = \frac{\ell\dot{\phi}^2}{2} + V(\phi), \quad p_{\mathrm{DE}} = \frac{\ell\dot{\phi}^2}{2} - V(\phi), \quad w_{\mathrm{DE}} = \frac{p}{\rho}, \quad (6)$$

where for the standard field case $\ell = 1$ and for the phantom field case $\ell = -1$, leading to a change in the sign of the kinetic term in the energy density and pressure and resulting in an equation of state parameter such that $w \leq -1$. For dark matter, we have

$$\rho_{\mathrm{DM}} = M\frac{\alpha}{a^3}, \quad p_{\mathrm{DM}} = -\beta\phi\frac{\alpha}{a^3}, \quad \omega_{\mathrm{DM}} = \frac{-\beta\phi}{M}. \quad (7)$$

We can see that in the case with interaction, the pressure of dark matter does not vanish for all times, but it depends on both fields and the interaction constant. This happens because when we have interaction, the components do not conserve alone, but the sum of all components is what is conserved. However, we can stress that depending on the solutions, $p_{\rm DM}$ and $\omega_{\rm DM}$ can vanish for a certain time. It is important to notice that this choice of separation of the components breaks the standard distinction between the dark matter and dark energy, since now both components can evolve as matter, dark energy or both, depending on the solution obtained. This is fundamental to understand the mixed role played by each component in the solutions obtained in Sec. IV.

For the interaction component, the energy density, pressure, and equation of state are given by

$$\rho_{\rm int} = -\beta\phi\frac{\alpha}{a^3}, \quad p_{\rm int} = \beta\phi\frac{\alpha}{a^3}, \quad \omega_{\rm int} = -1. \tag{8}$$

The constant equation of state equal to $-1$ shows us that the interaction component also makes the role of accelerating the Universe, as the dark energy does. Therefore, if the interaction is too strong in comparison with the other components, this would act to accelerate the Universe.

We can also define, as usual, the density parameter $\Omega = \rho/\rho_{\rm crit}$, where $\rho_{\rm crit} = 3H_0^2\Gamma/2$ with $\Gamma = M_p^2/4\pi$.

The above system of Eqs. (3)–(5) describes the dynamics of the variables of our interacting system. We must solve this system in order to obtain the evolution of our components. We can see that the interaction term brings an explicit dependence on the scale factor in the equations adding a new dynamical variable in the differential equations. We can notice, for example, that the Friedmann Eq. (4) can no longer be used as a constraint equation, as it is in the case without interaction.

Because of these differences, the system is more complicated to be analyzed. If one wants to compute the solutions numerically, then one needs to perform a phase space analysis, which can only be done for a power law-like potential in the case of Yukawa couplings [44], requiring the imposition of assumptions in the variables to be studied for general potentials. Because of these difficulties, we explore a different route to study this model based on Refs. [40,41], where we rely on important simplifications to obtain exact solutions. We want to check if this method and these simplifications can be used and give good results in the context of interacting models.

## III. FIRST ORDER FORMALISM (FOF)

Now, we are going to introduce the formalism that we are going to use. Following the central idea of FOF, we make the simplifying assumption that

$$H(t) = W(\phi(t)) \Rightarrow \dot{H} = W_\phi\dot{\phi}, \tag{9}$$

where the dot stands for time derivative and $W_\phi \equiv \partial W/\partial\phi$. We require that $\phi$ must be invertible, so solutions will only be possible through this method if the scalar field and its time derivatives are smooth, monotonic functions [45]. Other approaches can present solutions that are not invertible (see, e.g., Refs. [46,47]), but they are usually valid only for very specific regions of the phase space of the cosmological models. This invertibility requirement restricts the solutions that can be obtained by the method.

With this in mind, we rewrite Eqs. (3)–(5) substituting $H$ by $W(\phi)$. Equation (5) becomes

$$-W_\phi\dot{\phi}\Gamma = \ell\dot{\phi}^2 + [M - \beta F(\phi)]\frac{\alpha}{a^3}, \tag{10}$$

with $\Gamma = \frac{Mp^2}{4\pi}$. By inspection of Eq. (9), we see that it is necessary to impose some restrictions on the form of the scale factor. In order to solve this equation in terms of $\dot{\phi}$, we make an extra assumption and choose to rewrite the scale factor using the *ansatz*

$$a(t)^{-3} = \sigma\dot{\phi}^n J(\phi), \tag{11}$$

where $\sigma$ is a real constant, $n$ is an integer, and $J(\phi)$ is an arbitrary function of the scalar field. This expression has a general form in a way that allows us to obtain a large class of exact solutions with interacting dark energy and dark matter, by choosing convenient integers $n$ and functions $J(\phi)$ that reduce the order of the equations of motion. A more direct approach, without assuming this *ansatz*, can also be done as shown in the Appendix.

Thus, substituting (11) in (10) we obtain

$$\dot{\phi}^{n-1} + \frac{[\ell\dot{\phi} + W_\phi\Gamma]}{[M - \beta F(\phi)]\alpha\sigma J(\phi)} = 0. \tag{12}$$

The potential $V_n(\phi)$ associated to the scalar field is calculated using (4), resulting in

$$V_n(\phi) = \frac{3\Gamma W^2}{2} - \ell\frac{\dot{\phi}^2}{2} - [M - \beta F(\phi)]\alpha\sigma\dot{\phi}^n J(\phi). \tag{13}$$

We can solve Eq. (12) as an algebraic equation for $\dot{\phi}$ for each value of $n$. Some of the roots of this equation can be imaginary, and will be discarded. We also restrict the values for $n$ in our calculation to $n = 0, 1, 2, \ldots$ Such a procedure reduces the order of the equations that need to be solved and transforms the equation of motion for the scalar field into a constraint equation. The functions $W(\phi)$, $J(\phi)$, and $F(\phi)$ need to satisfy the constraint equation:

$$3W\dot{\phi} + \frac{3\Gamma W W_\phi}{\ell} - \frac{[M - \beta F(\phi)]\alpha\sigma}{\ell}\frac{d}{d\phi}[\dot{\phi}^n J(\phi)] = 0, \tag{14}$$

where we have used Eqs. (12) and (13) and the relation $\ddot{\phi} = \dot{\phi}d\dot{\phi}/d\phi$ valid whenever the function $\phi(t)$ is invertible. This is the point where it is crucial for this function to be a one to one map, what is not always true.

As we can see, the power of this method is that you obtain two important quantities. The scale factor is exactly what it is not known and we would like to obtain (to verify, for example, that a scalar field with a potential yields acceleration) and not to choose (as it is done in the direct approach); and the form of the interaction, parametrized by $F(\phi)$ [48], the most important feature of the interacting models that is, in principle, unknown. You only impose the form of the Hubble parameter and that the scale factor depends on the scalar field given the form of this dependency. The method then provides you the form of the evolution of the system as well as the interaction between dark matter and dark energy.

So, the setup to obtain the solution is the following. Given $W(\phi)$ and $J(\phi)$ and chosen a value for $n$, we write the potential (4) with the chosen form for these functions and solve the constraint Eq. (5) to obtain the form of the coupling $F(\phi)$. We plug this into (3) and solve this algebraic equation with the order given by the value chosen for $n$. With this, it is possible to obtain $\phi(t)$ and $a(t)$ and evaluate the energy density and pressure of both components.

As an example, we can show the form of the equations for $n = 0$. The first order equation will be given by

$$\dot{\phi} = -\frac{W_\phi \Gamma}{2\ell} \pm \frac{\sqrt{(W_\phi \Gamma)^2 - 4\ell[M - \beta F(\phi)]\alpha\sigma J(\phi)}}{2\ell} \tag{15}$$

with the potential $V_0$ being

$$V_0(\phi) = \frac{3\Gamma W^2}{2} - \ell\frac{\dot{\phi}^2}{2} - [M - \beta F(\phi)]\alpha\sigma J(\phi). \tag{16}$$

The constraint equation relating the functions $J$, $W$ and the interaction $F$ is given by

$$3W\dot{\phi} + \frac{3\Gamma W_\phi W}{\ell} - \frac{[M - \beta F(\phi)]\alpha\sigma}{\ell}\frac{d}{d\phi}(J(\phi)) = 0, \tag{17}$$

with the scale factor being given by $a(t)^{-3} = \sigma J(\phi)$.

As a check for this method, when we choose the function $J(\phi)$ to be constant (say, when $J(\phi) = 1$), we can see that these equations represent the special case of a Minkowski universe, known to be a fixed point in the phase space describing the dynamics of the system.

Another important check is to see if the case of a universe with an exponential expansion is recovered when we set $W(\phi) = \text{const}$, as it is expected since

$$H(\phi) = W(\phi) = H_0 \to \frac{\dot{a}}{a} = H_0 \to a(t) = a(0)e^{H_0 t}, \tag{18}$$

but without imposing a form for $a(t)$. We will try to find valid cosmological solutions using FOF for different values of $n$ and different forms of $W(\phi)$ and $J(\phi)$.

## IV. EXACT SOLUTIONS FOR THE INTERACTING DARK ENERGY AND DARK MATTER MODELS WITH FOF

In this section, we search for cosmologically viable solutions using the FOF. We show some examples of solutions giving different evolutions and couplings between the dark matter and dark energy. We intend to study how to obtain the cosmological solutions for different couplings and understand how well the method can describe the coupling scenario. As in the case of the phase space analysis, we expect to find at least one cosmologically viable solution presenting a period where the dark energy energy density becomes subdominant. In all cases $\ell = 1$.

After finding the solutions by the method, the free parameters of each of them must be constrained by the observational data in order to determine in which circumstances the solutions are viable. Some solutions required the introduction of parameters in order to have the right dimensions, that were also matched with the observations.

In the first two examples we describe an accelerated universe, given $W = \text{const}$, and see how the FOF for different $n$ and different $J(\phi)$ gives us different evolution and couplings between the dark energy and dark matter. This also serves as a test for the *ansatz* since we know that for that $W(\phi)$, the scale factor must evolve exponentially. We are interested in verifying if the solutions obtained are cosmologically viable by evaluating the cosmological parameters for each example.

### A. 1st example

Considering the case with $n = 0$, $\sigma = 1$ and $W(\phi) = H_0$, $J(\phi) = \frac{-\phi + M}{C}$, where $C$ is a constant with dimension of energy (MeV), Eqs. (11)–(14) result in

$$F(\phi) = \frac{M}{\beta} + \frac{9H_0^2 C(\phi - M)}{\alpha\beta}, \qquad V_0(\phi) = \frac{3}{2}\Gamma H_0^2 + \frac{9H_0^2}{2}(\phi - M)^2, \tag{19}$$

$$\phi(t) = M - De^{-3H_0 t}, \qquad a(t) = \left(\frac{C}{D}\right)^{1/3} e^{H_0 t}. \tag{20}$$

Such a universe develops a de Sitter expansion, therefore an accelerated expansion. We can also see this by calculating the acceleration parameter $q = \ddot{a}a/\dot{a}^2$. In this case $q$ is constant and equals to 1. The interaction, described by $F(\phi)$ and that also appears in $V(\phi)$, accounts for the interactions that were introduced between the dark sectors and the one introduced by the FOF, as pointed out in Ref. [41].

Here, the Hubble constant $H_0$ and the $\alpha$ parameter play the role of a coupling constant between dark matter and dark energy through the relation $\beta_F = -\frac{9H_0^2 C}{\alpha}$. These quantities also redefine the mass of the dark matter as $m_F = -\frac{9H_0^2 M}{\alpha}$, making it tachyonic.

In order to understand the role of such an extra interaction, the cosmological model obtained and to confirm the accelerated expansion seen in the acceleration parameter, we evaluate the energy density, the pressure, and the equation of state parameter for the scalar field,

$$\Omega_{\rm DE}(a) = 1 + \frac{6}{C^2\Gamma}\frac{1}{a^6}, \qquad p_{\rm DE}(a) = -\frac{3}{2}\Gamma H_0^2,$$
$$w_{\rm DE}(a) = \frac{-\frac{3}{2}\Gamma H_0^2}{\left[\frac{9H_0^2C^2}{a^6} + \frac{3}{2}\Gamma H_0^2\right]}. \tag{21}$$

We can also obtain these quantities for the dark matter,

$$\Omega_{\rm DM}(a) = \frac{2\alpha M}{3H_0^2\Gamma}\frac{1}{a^3}, \qquad p_{\rm DM}(a) = -\frac{9H_0^2}{C^2}\frac{1}{a^6} - \frac{\alpha M}{a^3},$$
$$\omega_{\rm DM}(a) = -1 - \frac{9CH_0^2}{\alpha M}\frac{1}{a^3}, \tag{22}$$

and for the interaction

$$\Omega_{\rm int}(a) = -\frac{6}{C^2\Gamma}\frac{1}{a^6} - \frac{2}{3}\frac{\alpha M}{H_0^2\Gamma}\frac{1}{a^3} = -\frac{p_{\rm int}(a)}{\rho_{\rm crit}},$$
$$\omega_{\rm int}(a) = -1. \tag{23}$$

We can readily see that $\Omega_{\rm tot} = \Omega_{\rm DE} + \Omega_{\rm DM} + \Omega_{\rm int} = 1$ as given by the Friedmann Eq. (4).

Given the form of the solution, we need to determine $\alpha M$ and $C$ by adjusting the solution to the observational data. We constrain some of the constants by fixing the density parameter of the dark matter today—$a_0 = 1$, the value of the scalar factor today—to be approximately 0.26, according to WMAP-7 data [49]. This fixes $\alpha M$.

We still need to constrain $C$. Since $C^2$ that appears in the energy density and in the equation of state cannot be negative so the scale factor is not imaginary, we can see from the expression for the density parameter of dark energy that this will always be greater than one. Adjusting it to be as close to one as possible, so we do not have an overdensity of dark energy, we were able to constrain $C$ to a value of the order of $10^{-8}$.

In the left panel of Fig. 1 we plot the density parameter of all the components. The interaction density is always negative and large. We can see that if we put this component together with any of the other components, how it is usual to do, this could influence a lot the final energy density of the components.

We can see in the plot that the component that the "dark matter" dominates in the past, while dark energy dominates at late times. However, this does not mean that we observe deceleration in the past, as we already know it does not happen because of the acceleration parameter equal to 1. As we can see in the right panel of Fig. 1, the component we are calling dark matter actually accelerates the Universe since its equation of state asymptotes $-1$ towards the present, instead of being equal to zero as expected for dust. This happens for two reasons: first, because the mass term of dark matter is cancelled by the first factor in $F(\phi)$ in (21) and only the kinetic part plays a role; second, because of the *ansatz* chosen, the scalar field will play a role in all the terms. So, the evolution of the dark matter component contains the interaction and behaves differently than the usual dark matter.

For the dark energy component, we can see that the equation of state presents an initial period where this fluid would not accelerate the Universe, because it is larger than $-1/3$. So, this equation of state tends to a "matter" one in the past and then, around $a = 0.2$, it enters in the regime where it accelerates the Universe going to $-1$ in the present and behaving like dark energy. This feature of the equation of state can be found in the equation of state of a Chaplygin gas [50], where one component can mimic the behavior of matter and dark energy, and it is an alternative to the quintessence scenario of dark energy [51].

However, as the dark energy component is not dominant during the period where it does not accelerate the Universe, the period in which the dark matter component that accelerates is dominant; the effective behavior is of an accelerated universe during all times. This can be seen also to be the acceleration parameter, in the right panel of Fig. 1. It is important to notice that this solution describes a flat universe (the sum of the density parameter of the components is equal to one) that is accelerating.

Because this solution describes a universe that is only expanding in an accelerated way, this solution is not a good candidate for a viable cosmological solution for the late universe. The fact that we could only obtain an accelerated solution should not be used as a reason to discard it, since

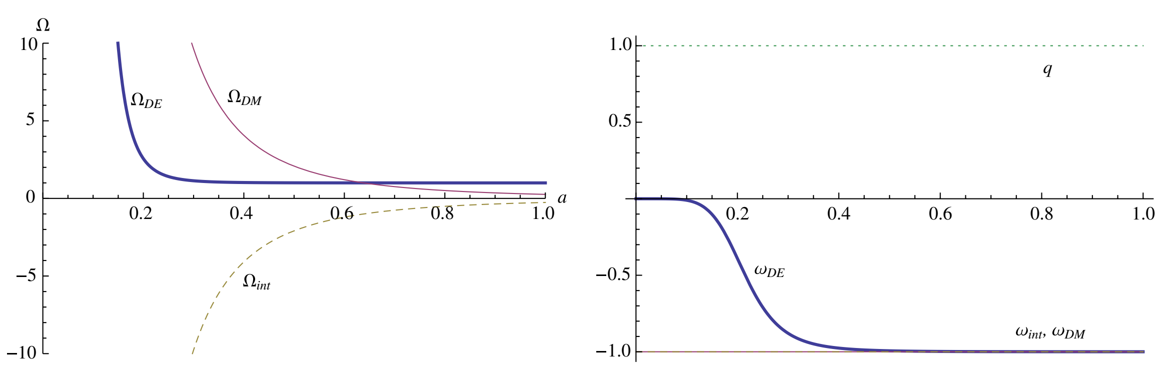


FIG. 1 (color online). Density parameter $\Omega$ (left panel) and equation of state parameter $w$ and acceleration parameter (right panel) for the model represented by Eqs. (21)–(23).

we imposed it by setting the Hubble parameter to be constant as an hypothesis. This solution could be a good one to describe a de Sitter inflationary period [52]with two different components responsible for the acceleration in different times of the evolution. This solution is an attractor solution in the phase space analysis of dark energy [37] and appears naturally in our formalism, as would have been expected.

We now show another solution, for a different $n$ but with the same constant Hubble parameter.

### B. 2nd example

Considering the case with $n = 1$, $\sigma = 1$ and functions $W(\phi) = H_0$ and $J(\phi) = -\frac{P}{\alpha\phi^2(M-\beta F(\phi))}$, where $P$ is a parameter with dimension 4 in energy ($\mathrm{MeV}^4$), Eqs. (11)–(14) result in

$$F(\phi)=\frac{M}{\beta}+\frac{C_1 e^{H_0\phi^3/P}}{\phi^4}, \quad V_1(\phi)=\frac{3}{2}\Gamma H_0^2+\frac{P^2}{2\phi^4}, \tag{24}$$

$$\phi(t) = (3Pt)^{1/3}, \qquad a(t) = \left(\frac{\beta\alpha C_1}{P^2}\right)^{1/3} e^{H_0 t}. \tag{25}$$

As in the previous example, this universe also develops a de Sitter accelerated expansion with acceleration parameter equal to 1. This can be shown by evaluating the acceleration parameter that is constant and equal to one, as can be seen in the right panel of Fig. 2, showing that the solution is always accelerated.

The new coupling constant is $\beta_F = C_1\beta$, and the mass of the dark matter is cancelled by the first term of the interaction $F(\phi)$. In this case dark matter is represented by massless fermions, i.e., they contribute only with a kinetic term.

Another solution for $\phi(t)$ is possible but this choice results in a contracting universe. As we are not interested in this type of solution (although we find it interesting that the formalism also presents us this sort of situation), we discard it. Here, the influence of the dark matter is explicitly present in the scale factor.

For this model the energy density, the pressure, and the equation of state parameter are given by

$$\begin{aligned}\Omega_{\mathrm{DE}}(a) &= 1 + \frac{2}{3\Gamma}\left(\frac{P}{3H_0}\right)^{\frac{2}{3}}\frac{1}{[\ln(\gamma a)]^{\frac{4}{3}}},\\ p_{\mathrm{DE}}(a) &= -\frac{3}{2}\Gamma H_0^2,\\ w_{\mathrm{DE}}(a) &= \frac{-1}{1+\frac{2}{3\Gamma}\left(\frac{P}{3H_0}\right)^{\frac{2}{3}}\frac{1}{[\ln(\gamma a)]^{\frac{4}{3}}}},\end{aligned} \tag{26}$$

where $\gamma^3 = P^2/C_1\alpha\beta$.

We can also obtain these quantities for the dark matter and the interaction,

$$\begin{aligned}\Omega_{\mathrm{DM}}(a) &= \frac{2M\alpha}{3H_0^2\Gamma}\frac{1}{a^3},\\ p_{\mathrm{DM}}(a) &= -\left(\frac{PH_0^2}{3}\right)^{\frac{2}{3}}\frac{1}{[\ln(\gamma a)]^{\frac{4}{3}}} - \frac{M\alpha}{a^3},\\ \omega_{\mathrm{DM}}(a) &= -1 - \left(\frac{P}{3H_0}\right)^{\frac{2}{3}}\frac{H_0^2}{M\alpha}\frac{a^3}{[\ln(\gamma a)]^{\frac{4}{3}}},\end{aligned} \tag{27}$$

and for the interaction

$$\begin{aligned}\Omega_{\mathrm{int}}(a) &= -\frac{2}{3\Gamma}\left(\frac{P}{3H_0}\right)^{\frac{2}{3}}\frac{1}{[\ln(\gamma a)]^{\frac{4}{3}}} - \frac{2M\alpha}{3H_0^2\Gamma}\frac{1}{a^3} = -\frac{p_{\mathrm{int}}(a)}{\rho_{\mathrm{crit}}},\\ \omega_{\mathrm{int}}(a) &= -1.\end{aligned} \tag{28}$$

We can constrain the constants in this example, $P$, $\gamma$, and $\alpha M$, like we did previously by fixing the density parameter of the dark matter today to be approximately 0.26, according to WMAP-7 data [49], to fix $\alpha M$. Setting $a = 1$ today, we can constrain the coefficient $\gamma$ to be of order of 10. As the density parameter of dark energy is always greater than one, the remaining constant, namely $P$, cannot be fixed using the density parameter of today, since $P^2$ cannot be negative so the scale factor is not negative. To constrain this constant, we use the equation of state of dark energy that must be negative for all times, asymptotically approaching $-1$, as shown in the right panel of Fig. 2. However, in order to $\omega_{\mathrm{DE}} \propto -1$ at $a_0 = 1$, $P^2$ must be small. Hence, in order to impose this condition we fine-tune $P$ to be of order of $10^{-7}$ to give us a realistic cosmological evolution.

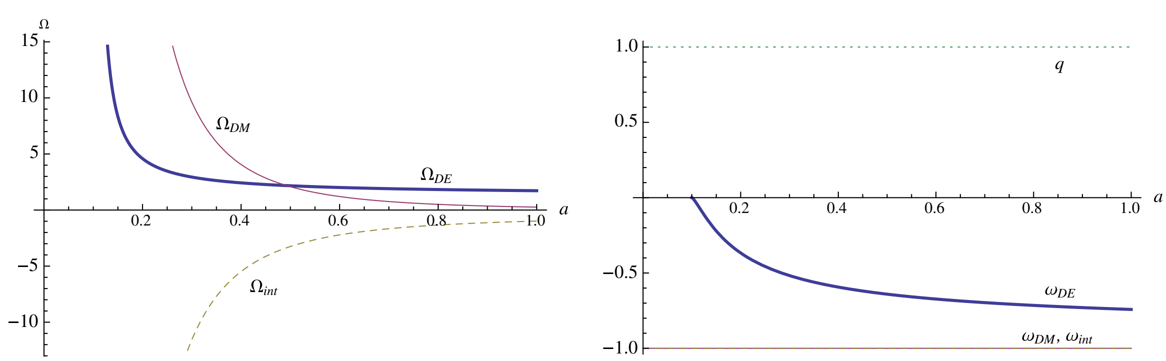


FIG. 2 (color online). Density parameter $\Omega$ (left panel) and equation of state parameter $w$ and acceleration parameter (right panel) for the model represented by Eqs. (26)–(28).

In the left panel of Fig. 2 we plot the density parameter of the component that we called dark matter, which dominates in the past, and of dark energy that dominates in the future. The density parameter of dark energy is always greater than one but it is compensated, as it is the dark matter one, by the interaction that it is always negative, giving a flat accelerated solution, with $\Omega_{\rm tot} = \Omega_{\rm DE} + \Omega_{\rm DM} + \Omega_{\rm int} = 1$. This interaction is strong and if included in any of the other components, it would greatly affect the evolution.

As in the first example, this does not mean that we have an early period of deceleration. As we can see in the right panel of Fig. 2, the equation of state of the dark matter is negative, tending to $-1$ through all the evolution. This component accelerates the Universe, different than what is expected from dust. Again, because in this example we have a cancellation of the dark matter mass and because the evolution is mixed with the scalar field one, the effective behavior is similar to the presence of a cosmological constant.

Also as the first example, the dark energy component decelerates in the past, starts to accelerate around $a = 0.15$ as $\omega$ reaches $-1/3$, and continues accelerating until the present with the equation of state tending to $-1$. This is also similar to the Chaplygin gas, since it evolves like dust and dark energy for different times.

The effective behavior is that we only observe acceleration, as seen in the acceleration parameter, since the dark matter component that accelerates the Universe dominates during the period where the dark energy decelerates. Then dark energy dominates and continues to accelerate the Universe.

This solution, again, is always accelerated, as expected since the Hubble parameter is constant. It could describe only the period of an inflationary universe with two accelerating components at different times, for example, and it is not a good description for the dark energy in the late universe.

We now present other models in the attempt to check realistic cosmological solutions for nonconstant Hubble parameters.

### C. 3rd example

Considering the case with $n = 1$, $\sigma = 1$ and functions $W(\phi) = \phi + H_0$ and $J(\phi) = -1/C^2$, where $C$ is a parameter with dimension of energy (MeV), Eqs. (11)–(14) result in

$$F(\phi) = \frac{3C^2\phi(\frac{\phi}{2} + H_0)}{\beta\alpha} + C_1, \tag{29}$$

$$\begin{aligned} V_1(\phi) = \frac{1}{8C^4}\{&4\alpha^2(-\beta C_1 + M)^2 + C^2[4C^2(3H_0^2\Gamma - \Gamma^2) \\ &+ 24H_0\phi(\alpha\beta C_1 + C^2\Gamma - \alpha M) \\ &+ 12\phi^2(C^2(3H_0^2 + \Gamma) + \alpha(\beta C_1 - M)) \\ &+ 36H_0C^2\phi^3 + 9C^2\phi^4]\}, \end{aligned} \tag{30}$$

$$\begin{aligned} \phi(t) &= -H_0 + \frac{\tilde{B}}{C}\tanh\Big(\frac{t\tilde{B}}{2C}\Big), \\ \tilde{B} &= \sqrt{9H_0^2C^2 + 6(-C_1\beta\alpha - C^2\Gamma + \alpha M)}, \end{aligned} \tag{31}$$

$$a(t) = \Big(\frac{-6C^4}{\tilde{B}}\cosh^2\Big(\frac{t\tilde{B}}{2C}\Big)\Big)^{1/3}. \tag{32}$$

In this case the mass of the fermions were not cancelled, and the interaction has two effective coupling constants. The acceleration parameter is given by

$$q(a) = -2 + \frac{3}{\delta a^3 - 1}, \tag{33}$$

where $\delta = -\tilde{B}/6C^4$. This acceleration parameter can present different behaviors depending on the value of $\delta$. This factor will be determined by the observational constraints on the parameters.

For this universe, the energy density for dark energy is

$$\begin{aligned} \Omega_{\rm DE}(a) = &\frac{2}{\Gamma}[C^2(2\Gamma - 3H_0^2) + 2\alpha(\beta C_1 - M)] \\ &- 24\frac{H_0\alpha\beta C_1}{\tilde{B}C}\sqrt{\frac{\delta a^3}{\delta a^3 - 1}} + \frac{\tilde{B}^2C^2}{3\Gamma}\Big[\frac{4}{\delta a^3 - 1} \\ &+ C^2\frac{\delta a^3 - 1}{\delta a^3}\Big] + \frac{1}{3\tilde{B}^2\Gamma}\{[9H_0^4 + 12(C^2 - 1)H_0^2\Gamma \\ &- 4\Gamma^2C^2] + 4\alpha^2(M - \beta C_1)^2 \\ &+ 12\alpha H_0^2(M - \beta C_1)\}\frac{\delta a^3}{\delta a^3 - 1}, \end{aligned} \tag{34}$$

where we do not show the other parameters because they are too lengthy.

We can also obtain these quantities for the dark matter,

$$\begin{aligned} \Omega_{\rm DM}(a) &= -\frac{2\alpha M}{C^2\Gamma}\frac{1}{\delta a^3 - 1}, \\ p_{\rm DM}(a) &= \frac{\tilde{B}^2}{36C^4}\frac{1}{\delta a^3}\Big[6\alpha\beta C_1 - 9c^2H_0^2 + \tilde{B}^2C^4\frac{(\delta a^3 - 1)}{\delta a^3}\Big], \\ \omega_{\rm DM}(a) &= -\frac{1}{6\alpha M}\Big[6\alpha\beta C_1 - 9C^2H_0^2 + \tilde{B}^2C^4\frac{(\delta a^3 - 1)}{\delta a^3}\Big], \end{aligned} \tag{35}$$

and for the interaction,

$$\begin{aligned} \Omega_{\rm int}(a) = &-\frac{2}{3C^2\Gamma}\frac{(\delta a^3)^{\frac{1}{2}}}{\delta a^3 - 1} \\ &\times\Big[6\alpha\beta C_1 - 9C^2H_0^2 + B^2C^4\frac{(\delta a^3 - 1)}{\delta a^3}\Big] \\ = &-\frac{p_{\rm int}(a)}{\rho_{\rm crit}}, \quad \omega_{\rm int}(a) = -1. \end{aligned} \tag{36}$$

We have to fix the parameters $\beta C_1$, $\alpha M$, and $C$ using the observational data. From (33), depending on the value of $\delta$ we can have periods of only acceleration and only

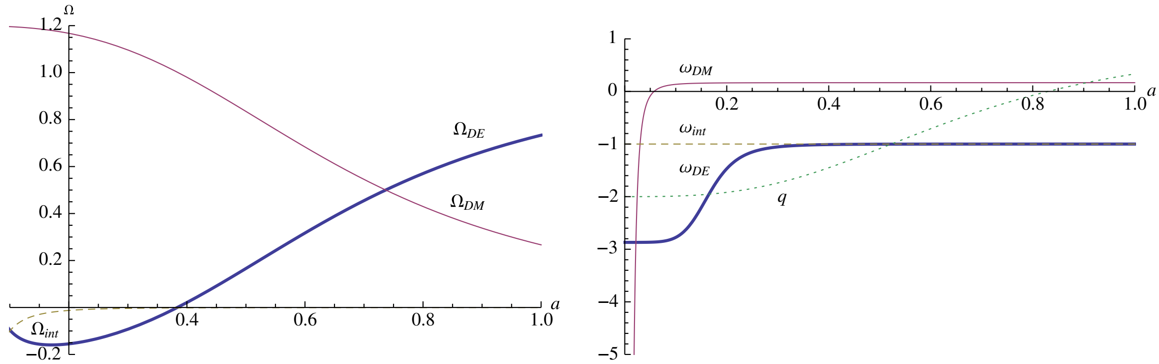


FIG. 3 (color online). Density parameter $\Omega$ (left panel) and equation of state parameter $\omega$ and acceleration parameter (right panel) for the model represented by Eqs. (29)–(36). The equation of state of dark matter is normalized in order to make the graph easier to see in a way still holds its main features.

deceleration or periods that combine both. We choose a $\delta$ parameter that gives us an early period of deceleration followed by a period of acceleration, as it can be seen in the right panel of Fig. 3. This is chosen because this is the expected behavior for the late evolution of our Universe (with the restriction that here it is only composed only by dark energy and dark matter).

To help fixing the remaining parameters, we have adjusted the density parameter of the dark matter today to be approximately 0.26 and the density parameter of dark energy to be approximately 0.74, according to WMAP-7 data [49] adjusted for only two fluids. This way we were able to fine-tune the combinations of the constants $\beta C_1$, $\alpha M$ and $C$ in (34) and (35) so the cosmological parameters agree with the observations. We plotted the density parameter and equation of state.

In the left panel of Fig. 3 we can see that dark matter dominates in the past, reaching its today value as $a = 1$, followed by a late period of dark energy domination, that also has the right density value today. As seen in the acceleration parameter in the right panel of Fig. 3, where we have deceleration followed by acceleration, we can see from the equation of state that the dark matter component decelerates the Universe during all periods, while the dark energy one accelerates the Universe during all times. The only exception is during the very early universe where the equation of state of dark matter, that is plotted normalized for comparison reasons, is negative. However, this phase is very short and our simplified model with only two components is not a good approximation for this early period. The interaction density is again always negative. However, different from the other examples, it is small compared to the others. In this case the interaction would not spoil the behavior of other components.

So, this model is a viable cosmological solution, since we have an early period of dark matter domination followed by a late period of dark energy acceleration. It describes a flat universe since $\Omega_{\rm tot} = \Omega_{\rm DE} + \Omega_{\rm DM} + \Omega_{\rm int} = 1$, similar to the standard cold dark matter plus a quintessence field scenario. This was obtained by fine-tuning the free parameters of the solution. However, the early evolution is different from this standard scenario which may indicate that with this choice of parameters the interaction is important only in the early universe. This shows the power of the method to provide good candidate models to our Universe.

As we can see in this example, the role of the parameters is decisive in the behavior of the solution. The fact that the interaction gives large contributions in the first and second examples could be what causes the solution to not exhibit any other behavior than acceleration, since the formalism gives a higher weight in the scalar field component, by making all the assumptions on $H$ and $a(t)$ as depending on $\phi$. This contribution is imposed by the FOF, as pointed in Ref. [41], making the interaction large when adjusted to the observational data and spoiling the cosmological solution. In this third example, we obtained a good cosmological solution and the contribution from the interaction was small.

### D. 4th example

We will further investigate good cosmological solutions given by the method. We present now an example that exhibits a viable cosmological evolution but with different features than the previous one.

Considering the case with $n = 3$, $\sigma = 1$ and functions $W(\phi) = \frac{P}{\phi\Gamma}$ and $J(\phi) = -\frac{\phi^2}{4\alpha P(M - \beta F(\phi))}$, where $P$ is a parameter with dimension four in energy (MeV$^4$), Eqs. (11)–(14) result in

$$F(\phi) = \frac{M}{\beta} + C_1 \frac{e^{\frac{3\phi^2}{4\Gamma}}}{\phi^4}, \quad V_3(\phi) = \frac{3P^2}{2\Gamma\phi^2}, \tag{37}$$

$$\phi(t) = (6Pt)^{1/3}, \qquad a(t) = \left(\frac{\alpha\beta C_1}{2P^2}\right)^{1/3} e^{\frac{(6Pt)^{2/3}}{4\Gamma}}. \tag{38}$$

This is also a case presenting massless fermionic dark matter, and an interaction displaying a product of exponential and inverse power-law functions, with coupling constant $\beta_F = C_1\beta$. We can obtain from this solution the acceleration parameter

$$q(a) = 1 - \frac{\Gamma}{3}\left(\frac{6}{(Pt)^2}\right)^{1/3} \tag{39}$$

$$= 1 - \frac{1}{2\ln(\xi a)}, \tag{40}$$

where $\xi = (2P^2/\alpha\beta C_1)^{1/3}$. We can see from this expression that for

$$t > \frac{(2\Gamma)^{3/2}}{6P}, \tag{41}$$

the expansion is accelerated, given that $q > 0$. So, this solution presents a transition between decelerated and accelerated expansion as we can see in Fig. 4, which is expected by the observations.

For this model the energy density, the pressure, and the equation of state parameter for dark energy are given by

$$\begin{aligned}\Omega_{\rm DE}(a) &= 1 + \frac{1}{3\ln(\xi a)}\\ p_{\rm DE}(a) &= \frac{P^2}{8\Gamma^2}\frac{1}{\ln(\xi a)}(1 - 3\ln(\xi a)),\\ \omega_{\rm DE}(a) &= \frac{(1 - 3\ln(\xi a))}{(1 + 3\ln(\xi a))}.\end{aligned} \tag{42}$$

For the dark matter and the interaction,

$$\begin{aligned}\Omega_{\rm DM}(a) &= \frac{8\Gamma^2 M\alpha}{3P^2}\frac{\ln(\xi a)}{a^3},\\ p_{\rm DM}(a) &= -\frac{P^2}{8\Gamma^2}\frac{1}{(\ln(\xi a))^2} - \frac{\alpha M}{\xi a^3},\\ \omega_{\rm DM}(a) &= -1 - \frac{P^2}{8\Gamma^2\alpha M}\frac{a^3}{\ln(\xi a)},\end{aligned} \tag{43}$$

and for the interaction

$$\begin{aligned}\Omega_{\rm int}(a) &= -\frac{1}{3\ln(\xi a)} - \frac{8\Gamma^2 M\alpha}{3P^2}\frac{\ln(\xi a)}{a^3} = -\frac{p_{\rm int}(a)}{\rho_{\rm crit}},\\ \omega_{\rm int}(a) &= -1.\end{aligned} \tag{44}$$

The parameters we have to fix in this example are $\xi$, $P$, and $\alpha M$. First, we can fix the parameters by imposing that the transition from deceleration and acceleration happens in the near past, as expected for a good cosmological model, choosing $\xi \sim 2, 02$. We also adjusted the density parameter of the dark matter today to be approximately 0.26, fine-tuning the ratio $\alpha M/P$.

With that we can evaluate the density parameters, plotted in the left panel of Fig. 4. The density parameter of dark energy is always greater than one and dominates through all evolution. The dark matter parameter is also plotted and is subdominant, and the interaction is always negative and is small compared to the other components, except for very early times.

Although dark energy dominates during all times, as it can be seen by the equation of state of dark energy in the right panel of Fig. 4, the dark energy decelerates the expansion in early times. It changes from deceleration for acceleration when reaches $\omega = -1/3$, in the same time as the acceleration parameter changes sign. This shows that the density and equation of state evolutions are coherent with the evolution shown in the acceleration parameter.

As already pointed out, this dark energy component that behaves like matter and dark energy is analogous to the Chaplygin gas, and in this example it is responsible for the entire evolution of the Universe mimicking its components.

The so-called dark matter component has a large negative value. This happens because of the interaction and because in this example, as in the first and in the second, the dark matter mass is cancelled.

This solution represents a flat universe, as $\Omega_{\rm DE} + \Omega_{\rm DM} + \Omega_{\rm int} = 1$, presenting all the necessary requirements for a viable cosmological model and describes a universe where the same scalar field is responsible for the early inflationary acceleration and the late one.

We notice that in this model the interaction given by the FOF is small, after adjusting the parameter to fit the observational data, and it represents a viable cosmological solution. A possible reason for this is that the form one chooses for $W(\phi)$ and $J(\phi)$ leads to a coupling that can potentially spoil the cosmological solution, since only small couplings

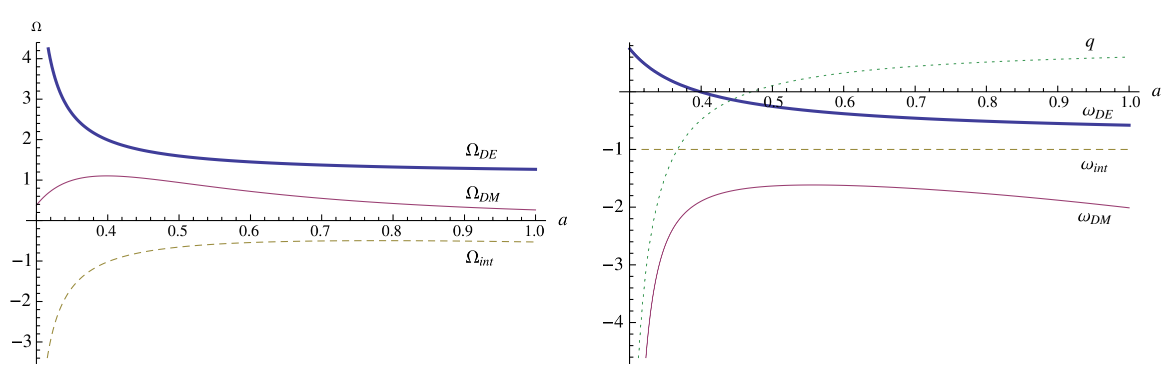


FIG. 4 (color online). Density parameter $\Omega$ (left panel) and equation of state parameter $w$ and acceleration parameter (right panel) for the model represented by Eqs. (37)–(44).

are cosmologically acceptable in models of interaction between dark matter and dark energy and since the FOF favors the energy density of the scalar field and large parameters enhance this component.

## V. CONCLUSIONS

Interacting cosmological models have attracted much attention in the last few years, both as a theoretical laboratory as well as a describing phenomenology. However, due to the enormous difficulties in obtaining solutions for the Friedmann equations in such an approach, the theoretical effort has not gone too far. The best description of the possible solutions to these problems comes from the dynamical systems analysis, which gives an overview of all the families of possible solutions. One could ask for specific exact solutions for a particular interaction, which is known to be very difficult, if not completely impossible, as is usual in the framework of general relativity.

The method described here has the purpose of simplifying the search for exact cosmological solutions for an interacting dark sector. Such a simplification takes place after a series of assumptions that, however, do not weaken the value of the solutions since they can be suitable for some specific situations.

We have obtained a series of solutions and have discussed their properties. As in the phase space analysis method, we have obtained solutions where the dark energy was the only dominant component and two solutions where we describe both decelerated and accelerated periods. They are viable cosmological solutions since the deceleration period will allow for structure formation, and the accelerated one will account for the observed late accelerated expansion of the Universe.

However, although they present the same behavior, these solutions accomplish that in a different way. The third solution has a massive dark matter component with an equation of state that decelerates the Universe, dominating in the past, and a dark energy component that accelerates in the late epoch. The fourth example has a nonmassive dark matter component always subdominant and a dark energy component that plays the hole of a matter-like component, decelerating the Universe in the past, and of dark energy, accelerating it in late times. This behavior is similar to the Chaplygin gas, although it does not present a phantom epoch.

The first and second solutions present an accelerating-only expansion. As a constant Hubble parameter has been imposed, they show that the method is robust. The interaction given by the method plus the imposed one are large in this example. In the third and fourth examples, the ones that give viable cosmological solutions, the interaction is small, at least in most of the evolution.

Hence, we have shown that the FOF is capable of giving viable cosmological solutions in the same way as the phase space analysis. It also gives the expected attractor solutions where only expansion is observed.

The analysis of specific models, with well-motivated interaction functions and coupling constants, and the comparison with observational data will be the subject of future work.

## ACKNOWLEDGMENTS

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## APPENDIX: DIRECT APPROACH FOR SOLVING THE SYSTEM OF EQUATIONS

It is worthy to stress the difference between the FOF and the more direct approach to the solution of the cosmological equations. We shall do this by presenting the following example.

We take $F(\phi) = \phi$ and choose the de Sitter solution ($\dot{a}/a = H_0$). In this case the equation of motion for the scalar field can be written as

$$\ell(\ddot{\phi} + 3H_0\dot{\phi}) + V' = \frac{\beta\alpha}{a^3}. \tag{A1}$$

Using Friedmann's equations, we can eliminate the potential $V$ and solve for the scalar field $\phi$, obtaining

$$\phi(t) = K_1 + K_2 e^{-3H_0 t} + K_3 e^{-\frac{3}{2}H_0 t}, \tag{A2}$$

where $K_1$, $K_2$, and $K_3$ are constants. Note that, since there are no restrictions on the sign of these constants, the requirement of invertibility for $\phi(t)$ is not fulfilled for all the solutions.

For a power-law scale factor $a = Kt^p$, with $K$ and $p$ positive constants, we have for $\phi(t)$

$$\phi(t) = Y_1 + Y_2\left[\frac{(\ln t)^2}{2} + Y_3 \ln t\right], \tag{A3}$$

where $Y_1$, $Y_2$, and $Y_3$ are constants. This solution is clearly noninvertible. Thus, the FOF and the direct method can be understood as complementary formalisms.

[1] A. Riess *et al.*, Astron. J., **116**, 1009 (1998).
[2] S. J. Perlmutter *et al.*, Astron. J. **517**, 565 (1999).
[3] N. A. Bahcall, J. P. Ostriker, S. J. Perlmutter, and P. J. Steinhardt, Science **284**, 1481 (1999).
[4] T. Padmanabhan, Phys. Rep. **380**, 235 (2003).
[5] M. Trodden and S. M. Carroll, TASI lectures, contributions to the TASI-02 and TASI-03 summer schools, arXiv: astro-ph/0401547.
[6] A. Linde, Prog. Theor. Phys. Suppl. **163**, 295 (2006).
[7] J. Frieman, C. Hill, A. Stebbins, and I. Waga, Phys. Rev. Lett. **75**, 2077 (1995).
[8] R. R. Caldwell, R. Dave, and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998).
[9] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003).
[10] S. M. Carroll, Phys. Rev. Lett. **81**, 3067 (1998).
[11] I. Zlatev, L. Wang, and P. J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999).
[12] K. Skenderis and P. K. Townsend, Phys. Lett. B **468**, 46 (1999).
[13] O. DeWolfe, D. Z. Freedman, S. S. Gubser, and A. Karch, Phys. Rev. D **62**, 046008 (2000).
[14] K. Skenderis and P. K. Townsend, Phys. Rev. Lett. **96**, 191301 (2006).
[15] R. Rosenfeld and J. A. Frieman, J. Cosmol. Astropart. Phys. 09 (2005) 003.
[16] C. Wetterich, Astron. Astrophys. **301**, 321 (1995).
[17] A. V. Macciò, C. Quercellini, R. Mainini, L. Amendola, and S. A. Bonometto, Phys. Rev. D **69**, 123516 (2004); L. Amendola, M. Baldi, and C. Wetterich, Phys. Rev. D **78**, 023015 (2008).
[18] M. Baldi, Mon. Not. R. Astron. Soc. **411**, 1077 (2011); **420**, 430 (2012).
[19] K. Koyama, R. Maartens, and Y.-S. Song, J. Cosmol. Astropart. Phys. 10 (2009) 017.
[20] L. P. Chimento, A. S. Jakubi, D. Pavon, and W. Zimdahl, Phys. Rev. D **67**, 083513 (2003); W. Zimdahl and D. Pavon, Phys. Lett. B **521**, 133 (2001); W. Zimdahl, Int. J. Mod. Phys. D **14**, 2319 (2005).
[21] B. Wang, Y. G. Gong, and E. Abdalla, Phys. Lett. B **624**, 141 (2005); B. Wang, C. Y. Lin, and E. Abdalla, Phys. Lett. B **637**, 357 (2006).
[22] B. Wang, J. Zang, C. Y. Lin, E. Abdalla, and S. Micheletti, Nucl. Phys. **B778**, 69 (2007).
[23] S. Das, P. S. Corasaniti, and J. Khoury, Phys. Rev. D **73**, 083509 (2006).
[24] G. Huey and B. Wandelt, Phys. Rev. D **74**, 023519 (2006).
[25] E. Abdalla and B. Wang, Phys. Lett. B **651**, 89 (2007).
[26] C. Feng, B. Wang, Y. Gong, and R.-K. Su, J. Cosmol. Astropart. Phys. 09 (2007) 005.
[27] Z. K. Guo, N. Ohta, and S. Tsujikawa, Phys. Rev. D **76**, 023508 (2007).
[28] E. Abdalla, R. Abramo, L. Sodre, and B. Wang, Phys. Lett. B **673**, 107 (2009).
[29] Z. Huang, B. Wang, E. Abdalla, and R. Su, J. Cosmol. Astropart. Phys. 05 (2006) 013; J. Shen, B. Wang, E. Abdalla, and R. Su, Phys. Lett. B **609**, 200 (2005).
[30] S. Micheletti, E. Abdalla, and B. Wang, Phys. Rev. D **79**, 123506 (2009).
[31] S. Micheletti, J. Cosmol. Astropart. Phys. 05 (2010) 009.
[32] S. Micheletti, Phys. Rev. D **85**, 123536 (2012).
[33] C. Feng, B. Wang, E. Abdalla, and R. Su, Phys. Lett. B **665**, 111 (2008).
[34] B. Wang, C. Lin, D. Pavon, and E. Abdalla, Phys. Lett. B **662**, 1 (2008).
[35] D. Pavon and B. Wang, Gen. Relativ. Gravit. **41**, 1 (2009).
[36] E. Abdalla, L. R. Abramo, and J. C. C. de Souza, Phys. Rev. D **82**, 023508 (2010).
[37] E. Copeland, M. Sami, and S. Tsujikawa, Int. J. Mod. Phys. D **15**, 1753 (2006).
[38] B. Gumjudpai, T. Naskar, M. Sami, and S. Tsujikawa, J. Cosmol. Astropart. Phys. 06 (2005) 007.
[39] L. Amendola, M. Quartin, S. Tsujikawa, and I. Waga, Phys. Rev. D **74**, 023525 (2006).
[40] D. Bazeia, C. B. Gomes, L. Losano, and R. Menezes, Phys. Lett. B **633**, 415 (2006).
[41] D. Bazeia, L. Losano, J. J. Rodrigues, and R. Rosenfeld, Eur. Phys. J. C **55**, 113 (2008).
[42] Similar separation can be seen in Ref. [43].
[43] S. Lepe, J. Lorca, F. Pena, and Y. Vasquez, Int. J. Mod. Phys. D **20**, 2543 (2011).
[44] E. G. M. Ferreira, and E. Abdalla (unpublished).
[45] R. Courant, *Differential and Integral Calculus* (John Wiley & Sons, New York, 1988), Vol. 1.
[46] A. Saa, E. Gunzig, L. Brenig, V. Faraoni, T. M. Rocha Filho, and A. Figueiredo, Int. J. Theor. Phys. **40**, 2295 (2001).
[47] E. Elizalde, S. Nojiri, S. D. Odintsov, D. Saez-Gomez, and V. Faraoni, Phys. Rev. D **77**, 106005 (2008).
[48] Although we put a general function in the interacting term that will cover a great range of possible interactions, we are already restricting that the interaction in the fermionic sector depends on $\bar{\psi}\psi$. If $F(\phi) = \phi$, we recover the Yukawa coupling.
[49] E. Komatsu *et al.*, Astrophys. J. Suppl. Ser. **192**, 18 (2011).
[50] N. Bilic, G. B. Tupper, and R. D. Viollier, Phys. Lett. B **535**, 17 (2002); A. Y. Kamenshchik, U. Moschella, and V. Pasquier, Phys. Lett. B **511**, 265 (2001); Phys. Rev. D **66**, 043507 (2002).
[51] Chaplygin gas can also develop a period of phantom evolution as seen in Ref. [50] that we do not observe here.
[52] A. Guth, Phys. Rev. D **23**, 347 (1981).