Generalized modified gravity with the second-order acceleration equation

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In the theories of generalized modified gravity, the acceleration equation is generally fourth order. So, it is hard to analyze the evolution of the Universe. In this paper, we present a class of generalized modified gravity theories which have the acceleration equation of a second-order derivative. Then both the cosmic evolution and the weak-field limit of the theories are easily investigated. We find that, not only the big bang singularity problem, but also that the current cosmic acceleration problem could be easily dealt with.

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I. MOTIVATION

In history, the motivation for modifying general relativity (GR) mainly comes from the fact that GR is not renormalizable. So, it cannot be conventionally quantized. In the first place, Utiyama and DeWitt showed that the renormalization at one loop requires the higher order curvature terms in the action of gravity theories [1]. Secondly, Stelle showed the corresponding gravity theories with these higher order terms are indeed renormalizable [2]. Finally, when quantum effects or string theory are taken into account, the effective low energy gravitational action also requires higher order curvature invariants [3–5]. So, it was generally believed that the modifications to GR would be important only at the scales of very close to the Planck length or Planck energy. Consequently, both the big bang singularity and black hole singularity are expected to be absent in the modified gravity theories [6–11]. This was the belief before 1998.

However, with the discovery of cosmic acceleration in 1998 [12,13], one realized that GR may also need to be modified on a very large scale or at very low energy (or very weak gravitational field). These constitute the infrared modifications to GR, for example, the Dvali-Gabadadze-Porrati (DGP) model [14], the $1/R$ modified gravity model [15], and so on. Here we shall not produce an exhaustive list of references on modified gravity, but we prefer the readers to see the review paper by Sotiriou and Faraoni [16] and the references therein. In general, the equations of motion for the generalized modified gravity are of fourth order and one can expect the particle content of the theory to have eight degrees of freedom: two for the massless graviton, one in a scalar excitation, and five in a ghostlike massive spin two field [17]. The presence of a ghost leads one to accept unphysical negative energy states in the theory and the property of unitarity is lost [18]. This ghost problem is closely related to the higher order property of the theories.

So the purpose of this paper is to seek the *second order* theories of gravity, at least in the background of a spatially flat Friedmann-Robertson-Walker (FRW) universe. Except for satisfying the requirement of second order, the theories also meet ghost-free conditions. Due to the property of the second order of the acceleration equation, the resulting Friedmann equation remains first order and the cosmic evolution of the universe is easily deal with.

The paper is organized as follows. In Sec. II, we briefly review the generalized modified gravity theories. The equations of motion are presented. In Sec. III, we propose the Lagrangian for the generalized modified gravity which are both ghost free and second order (in the background of a spatially flat FRW universe). In Sec. IV, we investigate the cosmic evolution of some specific models of a Lagrangian. In Sec. V, we investigate the weak-field limit of these models. Section IV gives the conclusion and discussion.

We use the system of units $G = c = \hbar = 1$ and the metric signature $(-, +, +, +)$ throughout the paper.

II. GENERALIZED MODIFIED GRAVITY

The generalized modified gravity theories have the action of the form [19]

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{16\pi}[R + f(R, P, Q)] + \mathcal{L}_m\right\}, \quad (1)$$

where f is a general function of the Ricci scalar R and two curvature invariants,

$$P \equiv R_{\mu\nu}R^{\mu\nu}, \qquad Q \equiv R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}, \quad (2)$$

which are of the lowest mass dimension and parity conserving. $R_{\mu\nu}$ and $R_{\mu\nu\sigma\lambda}$ are the Ricci tensor and the Riemann tensor, respectively. $\mathcal{L}_m$ is the Lagrangian density for matters.

If we define

$$f_R \equiv \frac{\partial f}{\partial R}, \qquad f_P \equiv \frac{\partial f}{\partial P}, \qquad f_Q \equiv \frac{\partial f}{\partial Q}, \quad (3)$$

then we obtain the generalized Einstein equations [19]

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$$\begin{aligned}\mathcal{G}_{\mu\nu} \equiv & G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f + f_R R_{\mu\nu} + 2 f_P R^\lambda{}_\mu R_{\lambda\nu} \\ & + 2 f_Q R_{\lambda\sigma\delta\mu} R^{\lambda\sigma\delta}{}_\nu + g_{\mu\nu} \nabla^2 f_R - \nabla_\mu \nabla_\nu f_R \\ & - 2 \nabla_\lambda \nabla_\sigma [f_P R^\lambda{}_{(\mu} \delta^\sigma{}_{\nu)}] + \nabla^2 (f_P R_{\mu\nu}) \\ & + g_{\mu\nu} \nabla_\lambda \nabla_\sigma (f_P R^{\lambda\sigma}) - 4 \nabla_\lambda \nabla_\sigma [f_Q R^\lambda{}_{(\mu\nu)}{}^\sigma] = 8\pi T_{\mu\nu}. \end{aligned} \quad (4)$$

$\mathcal{G}_{\mu\nu}$ is the generalized Einstein tensor, $G_{\mu\nu}$. The same as the Einstein tensor, it satisfies the Bianchi identity

$$\mathcal{G}_{\mu\nu}{}^{;\nu} = 0. \quad (5)$$

Different from the Einstein tensor, which is up to second-order derivative, $\mathcal{G}_{\mu\nu}$ is up to fourth order. So, Eq. (4) are usually a set of fourth-order differential equations except for the case of a cosmological constant $f = 2\Lambda$. The existence of higher derivatives in the equation of motion suggests that one would always find ghosts in a linearized analysis. Actually, it can be argued by considering the Cauchy problem, the gauge symmetries, and constraints on the theory (see for instance Ref. [17]), that the generalized modified gravity will contain at most eight degrees of freedom: two for the usual massless graviton, one for a scalar field, and five for ghostlike massive spin two excitation. The ghost problem leads one to accept negative energy states in the theory. So the property of unitarity is lost [18].

However, Comelli [20] and Navarro and Acoleyen [21] showed that with a suitable choice of parameters, the theory would be ghost free. Actually, they showed that the general Lagrangian of the form $\mathcal{L} = \mathcal{L}(R, P - 4Q)$ are ghost free. So in this case it is left with only an extra scalar degree of freedom to the gravitational sector. In the next section, we shall seek the Lagrangian, which gives the acceleration equation of the second-order derivative and the corresponding theories are ghost free.

III. THE LAGRANGIAN

The spatially flat FRW metric is given by

$$ds^2 = -dt^2 + a(t)^2 (dr^2 + r^2 d\Omega^2), \quad (6)$$

where $a(t)$ is the scale factor. Given the metric, we could calculate the Ricci scalar R and the curvature invariants, P, Q,

$$\begin{aligned} R &= 6(\dot{H} + 2H^2), \\ P &= 12[(\dot{H} + H^2)^2 + H^2(H^2 + \dot{H}) + H^4], \\ Q &= 12[(\dot{H} + H^2)^2 + H^4], \end{aligned} \quad (7)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and dot denotes the derivative with respect to the cosmic time t.

The second derivative of the scale factor, $\ddot{a}$ (in $\dot{H}$), is present in R, P, Q. If $f \propto R$, the corresponding equations of motion are the Einstein equations. They are second-order differential equations. In this scenario, $\dot{H}$ appears linearly in the Lagrangian. However, if $\dot{H}$ appears nonlinearly in the Lagrangian, just as contributed by R^2, P and Q, the corresponding equations of motion would be fourth-order differential equations.

It is not hard to conjecture that, if we are able to make $\dot{H}$ disappear in the Lagrangian f such that it is uniquely the function of Hubble parameter H, the resulting equation of motion must be of second-order differential equation. Then how do we make $\dot{H}$ disappear? We can examine the proper combination of R^2, P, Q.

To this end, let us calculate

$$\begin{aligned} I &\equiv \alpha R^2 + \beta P + \gamma Q \\ &= 12(3\alpha + \beta + \gamma)\dot{H}^2 + 12(12\alpha + 3\beta + 2\gamma)(H^2\dot{H} + H^4). \end{aligned} \quad (8)$$

So, if we let

$$12\alpha + 3\beta + 2\gamma = 0, \quad (9)$$

namely,

$$\gamma = -6\alpha - \frac{3}{2}\beta, \quad (10)$$

we would obtain

$$I = -(36\alpha + 6\beta)\dot{H}^2. \quad (11)$$

Is $\dot{H}$ negative or positive? In order to answer this question, let us resort to the Einstein equation in FRW universe,

$$\dot{H} \propto -(\rho + p), \quad (12)$$

where ρ, p are the total cosmic energy density and pressure, respectively. It is apparent $\dot{H}$ is nonpositive in the history of the Universe which mainly covers three epochs dominated by radiation, matter, and cosmological constant, respectively. Therefore, we assume

$$\dot{H} \le 0, \quad (13)$$

in the following. Thus,

$$\sqrt{I} = \sqrt{-36\alpha - 6\beta}(-\dot{H}). \quad (14)$$

It is apparent that α and β should obey

$$-36\alpha - 6\beta \ge 0. \quad (15)$$

If we define J as follows,

$$J \equiv \frac{1}{\sqrt{12}} \sqrt{R + \frac{6}{\sqrt{-36\alpha - 6\beta}} \sqrt{\alpha R^2 + \beta P + \gamma Q}}, \quad (16)$$

then we have

$$J = H,$$

in the background of FRW universe.

Now we could conclude that, for the general function of $f(J)$, the Lagrangian density

$$\mathcal{L} = \frac{1}{16\pi}[R + f(J)] + \mathcal{L}_m, \tag{17}$$

always leads to the second-order equation of motion in the background of a spatially flat FRW universe. Of course, we have assumed the energy-momentum tensor contributed by the matters is up to second order.

In general, the above theories of gravity would contain the massive spin two ghost field in addition to the usual massless graviton and the massive scalar [16]. But the $f(R)$ theories of gravity are found to be ghost free. Reference [22] and Refs. [20,21] showed that the models given by

$$\mathcal{L} = \frac{1}{16\pi}[R + f(R, 4P - Q)], \tag{18}$$

are also ghost free. In view of this point, we should let

$$\frac{\beta}{\gamma} = -4. \tag{19}$$

Taking account of Eq. (10), we have

$$\beta = -\frac{24}{5}\alpha, \qquad \gamma = \frac{6}{5}\alpha. \tag{20}$$

So, J is found to be

$$J = \frac{1}{\sqrt{12}}\sqrt{R + \sqrt{6(4P - Q) - 5R^2}}. \tag{21}$$

Using the Gauss-Bonnet invariant

$$G = R^2 - 4P + Q, \tag{22}$$

we have

$$J = \frac{1}{\sqrt{12}}\sqrt{R + \sqrt{R^2 - 6G}}. \tag{23}$$

Then we recognize that the Lagrangian

$$\mathcal{L} = \frac{1}{16\pi}[R + f(J)], \tag{24}$$

is actually the modified Gauss-Bonnet gravity which has been studied in Ref. [23]. In general, the acceleration equation in the modified Gauss-Bonnet gravity is fourth order. However, for our specific case [Eq. (23)], we would obtain a second-order acceleration equation from the Lagrangian Eq. (24). We note that, in Eq. (24), J is the function of R, P, Q or R, G according to Eq. (23). In the background of four-dimensional FRW universe, we have $J = H$. But it is not the case for other spacetimes.

Here, we would like to point out that the above conclusion is valid only for the four-dimensional FRW universe. If the dimension of spacetime is greater than four, the variation of Eq. (24) would lead to a fourth-order acceleration equation. The reason could be understood as follows. In the Gauss-Bonnet gravity, if the dimension of spacetime is four, the Gauss-Bonnet term turns out to be a topological invariant and so it makes no contribution to the equation of motion. In higher dimensions, the Gauss-Bonnet term is not topologically invariant and so it would contribute to the equation of motion. For the modified Gauss-Bonnet gravity, the equation of motion is usually fourth order even in the four dimensions.

Actually, the Gauss-Bonnet term is generalized in the Lovelock gravity theory [24]. It is found that the equations of motion in Lovelock gravity are of second order in any spacetime with any dimensions, needless to say in the background of the four-dimensional FRW universe. However, there is a notable difference between the Lovelock gravity and the generalized modified gravity in Eq. (24). The Lovelock gravity is an ultraviolet modification to general relativity and is unable to achieve the infrared modification to gravity.

IV. SECOND-ORDER ACCELERATION EQUATION AND THE FRIEDMANN EQUATION

In this section, we shall derive the acceleration equation and the Friedmann equation from the Lagrangian density Eq. (17) or Eq. (24). In the background of the spatially flat FRW universe, the Lagrangian function is given by

$$L = a^3\left\{\frac{1}{16\pi}[R + f(J)] + \mathcal{L}_m\right\}. \tag{25}$$

Substituting the Lagrangian function into the Euler-Lagrange equation,

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{a}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right) + \frac{\partial L}{\partial a} = 0, \tag{26}$$

we obtain the acceleration equation

$$2\dot{H} + 3H^2 - \frac{1}{2}Hf' + \frac{1}{2}f - \frac{1}{6}\dot{H}f'' = -8\pi p, \tag{27}$$

where the prime denotes the derivative with respect to H. It is obvious that the acceleration equation belongs to the second-order differential equations.

Using the acceleration equation, we could obtain the Friedmann equation from the energy-conservation equation

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0 \tag{28}$$

as follows:

$$3H^2 + \frac{1}{2}f - \frac{1}{2}Hf' = 8\pi\rho. \tag{29}$$

The left-hand side of the Friedmann equation is only the function of Hubble parameter. It is remarkably simple in the investigation of the evolution of the Universe.

Equations (27)–(29) and the equation of state for matters,

$$p = p(\rho), \tag{30}$$

constitute the main equations which govern the evolution of the Universe. Among the four equations, only three of them are independent. For convenience, we always focus on Eqs. (28)–(30).

V. SOME EXAMPLES

In this section, we shall study some specific and interesting forms of $f(H)$.

A. ΛCDM Model

If $f = -16\pi\Lambda$ with Λ a constant, we obtain from Eq. (29),

$$3H^2 = 8\pi(\rho + \Lambda). \tag{31}$$

This is the Friedmann equation for a Λ cold dark matter (ΛCDM) universe. Although the ΛCDM model provides an excellent fit to the wealth of high-precision observational data, on the basis of a remarkably small number of cosmological parameters [25], it is plagued with the well-known cosmological constant problem and the cosmic coincidence problem which prompted cosmologists to look for other explanation for the observed accelerated expansion.

B. Power law for f

Assume the energy density contributed by f is the form of a power law ηH^n with η a positive constant and n an integer. Then from the Friedmann equation (29), we have

$$-\frac{1}{2}f + \frac{1}{2}Hf' = 8\pi\eta H^n. \tag{32}$$

Thus, f is derived as

$$f = 16\pi\eta\frac{H^n}{n-1}. \tag{33}$$

When $n = 1$, we have

$$f = 16\pi\eta H \ln H. \tag{34}$$

Now we have the conclusions as follows.

(1) When $f = 16\pi\eta J \ln J$ (for $n = 1$), we have the following Friedmann equation:

$$3H^2 = 8\pi(\rho + \eta H). \tag{35}$$

It is the same as the Friedmann equation given by the DGP modified gravity [26]. The equation can be rewritten as

$$3H^2 = 8\pi\rho + \frac{1}{6}\eta^2 + \frac{1}{6}\eta\sqrt{\eta^2 + 96\pi\rho}. \tag{36}$$

If we define

$$\eta \equiv \sqrt{6\rho_I/\pi}/4, \tag{37}$$

we have

$$3H^2 = 8\pi\left[\rho + \frac{\rho_I}{2} + \sqrt{\frac{\rho_I}{2}\left(2\rho + \frac{\rho_I}{2}\right)}\right]. \tag{38}$$

Here ρ_I is a constant energy density. The ρ_I terms are investigated as the candidate of dark energy in many literatures, for example, Ref. [27] and references therein. Different from the cosmological constant, this dark energy density increases with the increasing of background energy density ρ. So, the cosmic coincidence problem is greatly relaxed. But it is argued that the DGP model is disfavored by the history of cosmic structure formation [28] because of the fast increasing of dark energy density with redshifts.

(2) When $f = -\eta J^{-2}$ (for $n = -2$) with η a constant, we obtain from Eq. (29),

$$3H^2 - \frac{3}{2}\eta H^{-2} = 8\pi\rho. \tag{39}$$

The above equation can be rewritten as

$$3H^2 = 4\pi\rho + \sqrt{16\pi^2\rho^2 + \frac{9}{2}\eta}. \tag{40}$$

If we define

$$\eta \equiv \frac{128}{9}\pi^2\rho_I^2, \tag{41}$$

then Eq. (40) can be rewritten as

$$3H^2 = 4\pi(\rho + \sqrt{\rho^2 + 4\rho_I^2}). \tag{42}$$

It is apparent ρ_I plays the role of a constant energy density. When $\rho \gg \rho_I$, it restores to the standard Friedmann equation. When $\rho \ll \rho_I$, the Universe evolves into a de Sitter phase. We have shown that this model could interpret the current acceleration of the Universe [29]. Different from the DGP model, the dark energy density in this scenario decreases with the increasing of redshifts. So, the cosmic coincidence problem is also relaxed.

(3) When $f = 16\pi\eta J^4/3$ (for $n = 4$), we have the Friedmann equation as follows:

$$3H^2 = 8\pi(\rho + \eta H^4). \tag{43}$$

It is a quadratic equation of H^2. Mathematically, we would have two roots for H^2. But physically, only one of which could reduce to the standard Friedmann equation in the limit of small ρ. So, the physical root takes the form of

$$3H^2 = \frac{3}{16\pi\eta}(3 - \sqrt{9 - 256\pi^2\rho\eta}). \tag{44}$$

Define

$$\eta \equiv \frac{9}{128\pi^2\rho_U}, \tag{45}$$

with ρ_U some positive constant. The Friedmann equation (44) turns out to be

$$3H^2 = 8\pi\rho_U\left(1 - \sqrt{1 - \frac{2\rho}{\rho_U}}\right). \tag{46}$$

Here ρ_U plays the role of a constant energy density. It is apparent ρ should obey $\rho \le \rho_U$. So, to the zeroth order of ρ/ρ_U, we obtain the standard Friedmann equation. To the first order of ρ/ρ_U, we obtain the Friedmann equation in the Randall-Sundrum brane world model [30]

$$3H^2 = 8\pi\left(\rho + \frac{\rho^2}{2\rho_U}\right). \tag{47}$$

Putting

$$\rho_U = 1, \qquad \rho = \frac{1}{a^4}, \tag{48}$$

and using Eq. (46), we plot the evolution of the scale factor a and the Hubble parameter H in Fig. 1, respectively. It shows that the Universe is created in finite time with finite a scale factor and a finite Hubble parameter. So, the big bang singularity is avoided.

It is apparent the energy density is also finite from Eq. (46). Then, how about the pressure and its higher derivatives? If they are irregular, some weak singularities would appear. We find that they are regular and that there are no weak singularities. The proof is as follows. Taking account of Eq. (48) and the energy conservation equation, we obtain the pressure $p = \rho/3$. Since ρ is regular, p is also regular. Then, with the help of energy conservation equation, we find the higher derivatives of pressure are regular.

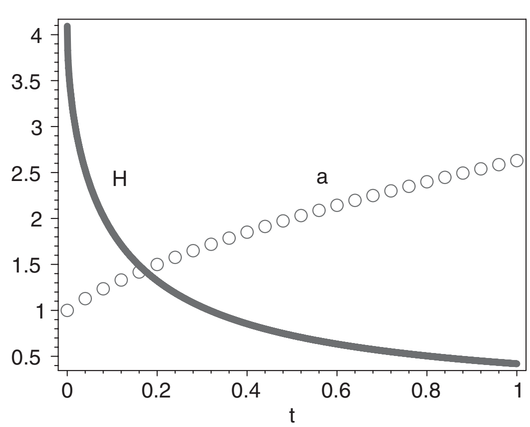

FIG. 1. The evolution of the scale factor a and the Hubble parameter H with respect to the cosmic time t. It shows that the Universe is created in finite time with a finite scale factor and a finite Hubble parameter. So the big bang singularity is avoided.

C. Past de Sitter universe

In the last part of Sec. V B, we find that the Universe could be created in finite time with a finite scale factor and a finite Hubble parameter. So, it seems there exists a starting point of time. Different from this case, in this section, we present a past de Sitter universe. In this scenario, the Universe starts from a de Sitter phase. So the history of the Universe is infinite. In other words, there does not exist a starting point of cosmic time.

In order to construct a past de Sitter universe, we could explore

$$f = 6J^2 - 4J\sqrt{6\pi\rho_U}\tanh^{-1}\left(\sqrt{\frac{3J^2}{8\pi\rho_U}}\right), \tag{49}$$

where ρ_U is a constant energy density. Substituting it into Eq. (29), we obtain

$$3H^2 = \left(1 - \frac{3H^2}{8\pi\rho_U}\right)8\pi\rho, \tag{50}$$

or

$$3H^2 = 8\pi\frac{\rho}{1 + \rho/\rho_U}. \tag{51}$$

To the zero order of $\rho/\rho_U \ll 1$, it restores to the Friedmann equation in general relativity. To the first order of ρ/ρ_U, we have

$$3H^2 = 8\pi\left(\rho - \frac{\rho^2}{\rho_U}\right). \tag{52}$$

It is the Friedmann equation in the theory of loop quantum gravity [31,32].

On the other hand, when $\rho/\rho_U \gg 1$, we obtain

$$3H^2 = 8\pi\rho_U. \tag{53}$$

This is for a de Sitter universe. Taking into account all matter sources which include relativistic matter (radiation), baryon matter, dark matter, and the cosmological constant, we can rewrite the Friedmann equation (51) as follows:

$$h^2 = \frac{\frac{\Omega_{r0}}{a^4} + \frac{\Omega_{m0}}{a^3} + \Omega_\lambda}{1 + \frac{\frac{\Omega_{r0}}{a^4} + \frac{\Omega_{m0}}{a^3} + \Omega_\lambda}{\Omega_U}}. \tag{54}$$

Observations show that $\Omega_{r0} = 8.1 \cdot 10^{-5}$, $\Omega_{m0} = 0.27$, $\Omega_\lambda = 0.73$, which are the ratios of energy density for radiation, matter (including baryon matter and dark matter), and cosmological constant in the present-day Universe. The dimensionless Hubble parameter h is defined by

$$h = \frac{H}{H_0}, \tag{55}$$

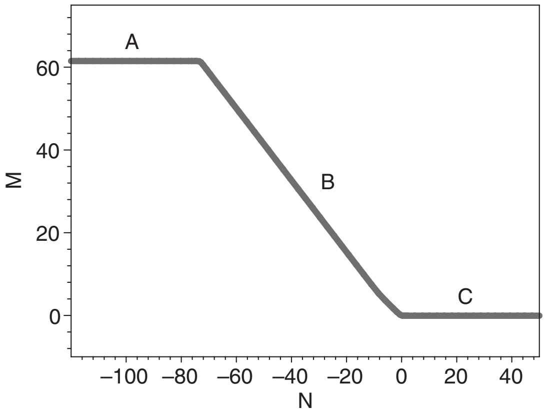

FIG. 2. There are three epochs A, B, C in the total life of the Universe. The epoch of A corresponds to the de Sitter phase. The Universe exponentially expands (inflating) in this period. Then the inflation stops around the redshift of $z \sim 10^{30}$. The epoch of B is dominated firstly by the radiation and then by the matter. It stops around the redshift of $z \sim 0$. The epoch of C is dominated by a small cosmological constant. C is the future de Sitter phase.

with H_0 the present-day Hubble parameter. Ω_U is the dimensionless energy density contributed by ρ_U.

Taking $\Omega_U = 10^{123}$ (which represents ρ_U is Planck energy density), we plot the evolution of the rescaled dimensionless Hubble parameter $M \equiv \log_{10} h$ with respect to the rescaled scale factor $N \equiv \ln a$ in Fig. 2. There are three epochs A, B, C in the total life of the Universe. The epoch of A corresponds to the de Sitter phase. The Universe exponentially expands (inflating) in this period. Then the inflation stops around the redshift of $z \sim 10^{30}$. The epoch of B is dominated firstly by the radiation and then by the matter. It stops around the redshift of $z \sim 0$. The epoch of C is dominated by a small cosmological constant. It is the future de Sitter phase.

VI. WEAK-FIELD LIMIT

In this section, we shall study the weak-field limit of the above modified gravity theories. Following Refs. [20,21], we expand the action in powers of the curvature perturbations. Then it can be shown that at the bilinear level, the linearization of the theory over a maximally symmetric spacetime, will be the same as the theory obtained in Refs. [17,33],

$$S = \int d^4x\sqrt{-g}\frac{1}{16\pi}\left(R - \Lambda + \delta R + \frac{1}{6m_0^2}R^2 - \frac{1}{2m_2^2}C_{\mu\nu\sigma\lambda}C^{\mu\nu\sigma\lambda}\right), \tag{56}$$

where $C_{\mu\nu\sigma\lambda}$ is the Weyl tensor and we have defined

$$\Lambda \equiv \langle f - Rf_R + R^2(f_{RR}/2 - f_P/4 - f_Q/6) + R^3(f_{RP}/2 + f_{RQ}/3) + R^4(f_{PP}/8 + f_{QQ}/18 + f_{PQ}/6)\rangle_0, \tag{57}$$

$$\delta \equiv \langle f_R - Rf_{RR} - R^2(f_{RP} + 2f_{RQ}/3) - R^3(f_{PP}/4 + f_{QQ}/9 + f_{PQ}/3)\rangle_0, \tag{58}$$

$$m_0^{-2} \equiv \langle 3f_{RR} + 2f_P + 2f_Q + R(3f_{RP} + 2f_{RQ}) + R^2(3f_{PP}/4 + f_{QQ}/3 + f_{PQ})\rangle_0, \tag{59}$$

$$m_2^{-2} \equiv -\langle f_P + 4f_Q\rangle_0. \tag{60}$$

Here $\langle\cdots\rangle_0$ represent the values of the corresponding quantities in the background of some spacetime and $F_{RR} \equiv \frac{\partial^2 f}{\partial R^2}$, etc. It is apparent for the action, Eq. (24), that the inverse of mass squared of the ghost is $m_2^{-2} = 0$. Thus, there is no ghost in the spectrum. But there is still an extra scalar with the mass m_0. The Λ term behaves as the vacuum energy and the δ term contributes the variation of the gravitational constant,

$$\frac{\delta G}{G} = \frac{-\delta}{1+\delta} \simeq -\delta. \tag{61}$$

In Tables I and II, we calculate the parameters for the five models:

(a) $f = -16\pi\Lambda$;
(b) $f = 16\pi\eta J\ln J$;
(c) $f = -\eta J^{-2}$;
(d) $f = 16\pi\eta J^4/3$;
(e) $f = 6J^2 - 4J\sqrt{6\pi\rho_U}\tanh^{-1}(\sqrt{\frac{3J^2}{8\pi\rho_U}})$,

in the background of de Sitter spacetime and Minkowski spacetime (by taking the limit of $\Lambda \to 0$). From Table I, we see that the models a, b, c contribute the nonvanishing cosmological constant terms. For models d and e, the Λ

TABLE I. The parameters of five models in the background of de Sitter spacetime.

Models	Λ	δ	m_0^{-2}	m_2^{-2}
(a)	Λ	0	0	0
(b)	$-\sqrt{\Lambda\rho_I}(9\ln 2 + 3\ln\pi + 3\ln\Lambda - 8 - 3\ln 3)/16$	$-\sqrt{\rho_I/\Lambda}(-3\ln 3 + 4 + 9\ln 2 + 3\ln\pi + 3\ln\Lambda)/32$	$m_0 = 0$	0
(c)	ρ_I^2/Λ	$-\rho_I^2/(4\Lambda^2)$	$m_0 = 0$	0
(d)	0	0	$m_0 = 0$	0
(e)	0	0	$m_0 = 0$	0

TABLE II. The parameters of five models in the background of Minkowski spacetime.

Models	Λ	δ	m_0^{-2}	m_2^{-2}
(a)	0	0	0	0
(b)	0	0	$m_0 = 0$	0
(c)	$+\infty$	$-\infty$	$m_0 = 0$	0
(d)	0	0	$m_0 = 0$	0
(e)	0	0	$m_0 = 0$	0

terms are zero because they are the ultraviolet modification to GR. In the column of δ, we see the models b, c contribute the nonvanishing gravitational constant terms. Except for the model of a, there exists a scalar degree of freedom with vanishing mass in b, c, d, e.

In the background of Minkowski spacetime, we see from the Table II that, except for the model of c, the other models make no contribution to vacuum energy and gravitational constant. The reason for this is that the model c is essentially a modification with inverse curvature invariants. So, the Minkowski spacetime does not solve the corresponding equations of motion.

VII. CONCLUSION AND DISCUSSION

In the theories of generalized modified gravity, the acceleration equation is generally fourth order. So, it is hard to analyze the evolution of the Universe [15]. On the other hand, these theories are also plagued with the ghost problem. So, the property of unitary of the theory is lost [18]. In view of this point, we present a class of generalized modified gravity theories which have the acceleration equation of a second-order derivative and are ghost free. Then we explore some specific examples for the Lagrangian function. We find that both the cosmic evolution and the weak-field limit of the theories are easily investigated. Furthermore, not only the big bang singularity problem, but also the current cosmic acceleration problem, could be easily dealt with.

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