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Relativistic virialization in the spherical collapse model for Einstein-de Sitter and ΛCDM cosmologies

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Spherical collapse has turned out to be a successful semianalytic model to study structure formation in different dark energy models and theories of gravity. Nevertheless, the process of virialization is commonly studied on the basis of the virial theorem of classical mechanics. In the present paper, a fully general-relativistic virial theorem based on the Tolman-Oppenheimer-Volkoff solution for homogeneous, perfect-fluid spheres is constructed for the Einstein–de Sitter and ΛCDM cosmologies. We investigate the accuracy of classical virialization studies on cosmological scales and consider virialization from a more fundamental point of view. Throughout, we remain within general relativity and the class of Friedmann-Lemaître-Robertson-Walker models. The virialization equation is set up and solved numerically for the virial radius, $y_{\rm vir}$, from which the virial overdensity Δ_V is directly obtained. Leading order corrections in the post-Newtonian framework are derived and quantified. In addition, problems in the application of this formalism to dynamical dark energy models are pointed out and discussed explicitly. We show that, in the weak field limit, the relative contribution of the leading-order terms of the post-Newtonian expansion are of the order of $10^{-3}\%$ and the solution of Wang and Steinhardt [Astrophys. J. **508**, 483 (1998)] is precisely reproduced. Apart from the small corrections, the method could provide insight into the process of virialization from a more fundamental point of view.

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I. INTRODUCTION

The question of how structures form in the Universe is a long-standing topic in theoretical cosmology and provides a lot of room for discussion. Since the fully nonlinear regime cannot be accessed analytically, huge N-body simulations have been set up to describe structure formation by gravitationally interacting particles in an expanding background. However, these attempts are computationally costly and, therefore, perturbative approaches have been developed in order to keep the continuous character of general relativity and the Friedmann-Lemaître-Robertson-Walker (FLRW) model and make use of methods from fluid mechanics. A very simple semianalytic model of this kind is the spherical collapse. A spherical, overdense patch evolves with the background expanding universe, slows down due to its self-gravity, turns around and collapses. The object is stabilized by virialization which prevents it from collapsing into a singularity. Despite its simplicity and idealizations, this model gives a first insight into the formation of spherical halos at all mass scales. The underlying formalism dates back to Gunn and Gott in 1972 [1] but has been rediscovered and continuously extended in recent years (see Refs. [2–15]).

In this work, we are going to use the results of Pace *et al.* [12] to investigate the process of virialization and try to find answers to some remaining questions in this context. The virial theorem provides a powerful tool to study systems in equilibrium, but in order to clarify its role in the framework of general relativity, a relativistic version is needed. After giving an overview of the classical concepts and the requirements of relativistic calculations in Secs. II and III, we will derive a relativistic version of the virial theorem based on the Tolman-Oppenheimer-Volkhoff (TOV) equation (see Refs. [16,17] for the original references) in an Einstein-de Sitter and ΛCDM universe (see Secs. IV and V). In the following, this will be applied to the virialization equation in the spherical collapse model and a post-Newtonian expansion will be performed (see Secs. VII and VIII). The relativistically corrected results for the virial radius and virial overdensity will be discussed and leading-order corrections are worked out in particular (Sec. X). We will also dedicate a section to the problems occurring when this formalism is applied to general dark energy (DE) models and point out possible ways to solve them (see Sec. IX). Throughout the paper, we will make use of natural units, i.e., $c = 1$.

II. VIRIALIZATION IN THE CLASSICAL SPHERICAL COLLAPSE

In the classical treatment of virialization, there are two major ingredients that have to be well understood. First of all, structure formation in the present universe is highly

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nonlinear on scales less than 10 Mpc, and an evolution equation for the spherical patch is needed that takes this nonlinearity into account. Secondly, the virial theorem has to be combined with energy conservation to a virialization condition that allows determining the time when collapse stops and the system reaches an equilibrium. The key quantities assigned to it are the virial radius normalized to the turn-around one, $y_{\rm vir} = R_{\rm vir}/R_{\rm ta}$, and the virial overdensity with respect to the background, $\Delta_V = \rho(R_{\rm vir})/\rho_{\rm b}(a_{\rm vir})$. These are general functions of redshift and provide a characterization of the equilibrium state of the halo.

The nonlinear evolution equation of a spherical overdensity of pressureless dark matter has already been treated in detail by many authors (see for example Refs. [2,12,18]). The resulting equation

$$\delta'' + \left(\frac{3}{a} + \frac{E'}{E}\right)\delta' - \frac{4}{3}\frac{\delta'^2}{1+\delta} - \frac{3}{2}\frac{\Omega_{\rm m,0}}{a^5E^2(a)}\delta(1+\delta) = 0, \tag{2.1}$$

describes the nonlinear evolution of a spherical, top-hat density contrast $\delta(a) = \frac{\rho-\rho_b}{\rho_b}$ with respect to the background dark matter density ρ_b. $E(a)$ contains all the dynamics of the background cosmological model and is related to the Hubble function via the expression $H(a) = H_0E(a)$. It will be important to express the virial overdensity Δ_V as a function of the turn-around one denoted by ζ:

$$\Delta_V = 1 + \delta(a_{\rm vir}) = \zeta\left(\frac{x_{\rm vir}}{y_{\rm vir}}\right)^3 \quad \text{with}$$
$$x_{\rm vir} = \frac{a_{\rm vir}}{a_{\rm ta}}, \qquad y_{\rm vir} = \frac{R_{\rm vir}}{R_{\rm ta}}. \tag{2.2}$$

The virial radius, $y_{\rm vir}$, is obtained from the virialization equation in which the classical virial theorem $\bar{T} = \overline{\frac{R}{2}\frac{\partial U}{\partial R}}$ is combined with the assumption of energy conservation during collapse.[1] It should be mentioned that energy conservation is a very common assumption in the literature and it is not proven whether it can actually be applied. Maor and Lahav [8,9], as well as Wang [14], pointed out that a homogeneous DE component with $w \neq -1, -1/3$ clearly violates energy conservation between turn-around and collapse:

$$\left[U + \frac{\partial U}{\partial R}\right]_{\rm vir} = U_{\rm ta}. \tag{2.3}$$

In case of an Einstein-de Sitter universe, one simply obtains $y_{\rm vir} = \frac{1}{2}$, whereas the corresponding virial overdensity (evaluated at collapse scale factor) is given by $\Delta_V = 18\pi^2 \sim 178$ (see Ref. [14]). In the case of ΛCDM and dynamical DE models, two major classical results have been proposed in the literature:

(i) Wang and Steinhardt (WS) (see Ref. [13])[2]:

$$y_{\rm vir} = \frac{1-\frac{\eta_v}{2}}{2+\eta_t-\frac{3}{2}\eta_v}. \tag{2.4}$$

(ii) Wang (PW) (see Ref. [14]):

$$y_{\rm vir} = \frac{1-(1+3w)\frac{\eta_t}{2}}{2-(1+3w)\frac{\eta_t}{2}} \underset{(w=-1)}{=} \frac{1+\eta_t}{2+\eta_t}, \tag{2.5}$$

with the WS parameters η_v and η_t and the equation-of-state-parameter w given by

$$\eta_t = 2\zeta^{-1}\frac{\Omega_\Lambda(a_{\rm ta})}{\Omega_{\rm m}(a_{\rm ta})}$$
$$\eta_v = 2\zeta^{-1}\left(\frac{a_{\rm ta}}{a_{\rm vir}}\right)^3\frac{\Omega_\Lambda(a_{\rm vir})}{\Omega_{\rm m}(a_{\rm vir})}$$
$$w = \frac{p_\Lambda}{\rho_\Lambda} = -1.$$

The corresponding virial overdensities become functions of the collapse (virial) scale factor a_c ($a_{\rm vir}$) and reach the Einstein-de Sitter (EdS) value asymptotically for small scale factors ($a_c < 10^{-1}$) corresponding to the matter-dominated era.[3]

III. REQUIREMENTS FOR RELATIVISTIC CALCULATIONS

Relativistic treatment of virialization in the same way as done in the classical case causes some trouble, because energy conservation is not global in general relativity. A second problem has been addressed by Komar (see Refs. [20,21]), stating that isolated bodies like a spherical halo can only be described exactly in asymptotically flat spacetimes which is generally not given in the case of FLRW models. A promising way out of these problems is assuming that the scale r of the halo is much smaller than the typical length scale of the background universe given by the Hubble radius R_H. If the Killing vector field of the FLRW spacetime is considered with respect to this assumption, its timelike component greatly exceeds its spatial components, allowing us to neglect the latter and, at good approximation, consider the Killing vector as timelike.

Figure 1 illustrates that neighboring Killing vectors are approximately parallel on the scale of the object, which means that the radial component of K is extremely small on these scales and thus negligible. A detailed quantitative analysis of this issue is given in Appendix A.

[1]In the following, we will drop the bars over the time-averaged quantities and implicitly assume time averaging.

[2]Horellou and Berge [19] have proposed a generalization due to dynamical DE models, but in the ΛCDM model both results agree.

[3]It has to be mentioned for completeness that this is only true for dynamical DE models that have negligible contribution in the matter- dominated era. Counterexamples are early DE models (see results in Ref. [12] and references therein).

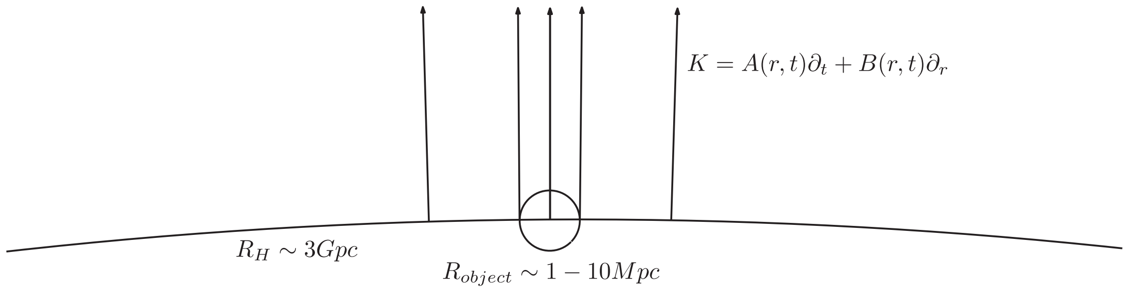

FIG. 1. Killing vector field on the spacelike hypersurface of the universe compared to an object of much smaller radius.

From this argument, we can infer three major conclusions essential for the following considerations:

(i) Since $r \ll R_H$, it is possible to introduce a region that is huge compared to the scale of the object but still small with respect to the Hubble radius and scale of spatial curvature of the universe. Therefore, it can be claimed locally flat, and a coordinate frame can be found that covers this region such that energy-momentum conservation

$$\nabla_\mu T^{\mu\nu} = 0, \tag{3.1}$$

can be approximated by partial derivatives in a sufficiently large environment of the halo,

$$\partial_\mu T^{\mu\nu} = 0. \tag{3.2}$$

Since the virialization equation, $T_{\rm vir} + U_{\rm vir} = U_{\rm ta}$, represents energy conservation, this condition is essential. The last statement needs particular clarification. Starting from $\nabla_\nu T^{\mu\nu} = 0$, projection on the four-velocity u yields the continuity equation $u^\mu \nabla_\mu \rho + (\rho + p)\nabla \cdot u = 0$. Given Eq. (3.2) (which we require to hold within the locally flat region) and the local rest frame ($u^0 = 1$, $u^i = v^i$), this equation becomes $\dot\rho + (\rho + p)(\vec\nabla \cdot \vec v) = 0$. Integrating over a sufficiently large volume including the halo implies (such a volume can be found since the locally flat region is chosen bigger than r but still much smaller than R_H)

$$\int_V \dot\rho dV + \int_V (\rho + p)(\vec\nabla \cdot \vec v) dV = 0$$
$$\Rightarrow \frac{\partial}{\partial t}\int_V \rho dV + \int_V (\rho + p)(\vec\nabla \cdot \vec v) dV = 0. \tag{3.3}$$

Invoking Gauss's theorem and defining the total energy to be $E = \int_V \rho dV$, we have

$$\frac{\partial}{\partial t} E + \int_{\partial V} (\rho + p)\vec v \cdot d\vec A = 0. \tag{3.4}$$

Since $r \ll R_H$ and much smaller than the scale of spatial curvature, we can assume that the object can be approximately treated as perturbation in Minkowski spacetime such that for a volume V enclosing the total halo, the second integral can be claimed to vanish by virtue of this approximation. Thus, we can approximately infer $\dot E = 0$, supporting the statement above.

(ii) Since, in virtue of this approximation, a sufficiently large, local environment of the halo exists that can be considered flat, isolated objects can be defined in general relativity and the halo mass M is well defined in the sense of a Komar integral. For a detailed treatment of the spherical collapse in local coordinates in which perturbations due to the FLRW metric are damped with $(r/R_H)^2$, the reader is referred to Creminelli *et al.* [5] and Nicolis *et al.* [22].

(iii) On scales that are small compared to the Hubble radius, there exists an approximately timelike Killing vector field K of the FLRW metric that also fulfills the Frobenius condition[4] within an accuracy of at least 0.9%. Thus, we can insert and compare static solutions at turn-around and virial redshift in the virialization equation $T_{\rm vir} + U_{\rm vir} = U_{\rm ta}$. Since the virial theorem will be applied to the final state which is static by definition (within the timescales we consider), this approximation holds, in particular, for the turn-around being a critical point but not a static one in the exact treatment. For a detailed discussion of this issue, the reader is referred to Appendix A.

IV. TOV EQUATION FOR MATTER IN THE PRESENCE OF A COSMOLOGICAL CONSTANT

Relying on the assumptions of the previous section, we can set up a spherically symmetric, static spacetime with the metric:

$$ds^2 = -e^{2a(r)}dt^2 + e^{2b(r)}dr^2 + r^2 d\Omega^2. \tag{4.1}$$

In addition, we consider a system of two fluid components described by

$$T_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(\Lambda)}_{\mu\nu} = (\rho_T + p_T)u_\mu u_\nu + p_T g_{\mu\nu}, \tag{4.2}$$

in which

$$\rho_T = \rho_m + \rho_\Lambda, \qquad p_T = p_m + p_\Lambda.$$

[4]For ω being the corresponding dual vector to K, the Frobenius condition states that $\omega \wedge d\omega = 0$, which turns out to be equivalent to K being orthogonal to the spacelike sections spanned by suitably chosen spatial coordinates (see Ref. [23]).

The equation of state of the cosmological constant fluid corresponds to $w = \frac{p_\Lambda}{\rho_\Lambda} = -1$.

Energy and momentum are locally conserved for the total system as well as for each component separately, which means

$$\nabla_\nu T^{\mu\nu} = 0 \quad \text{and} \quad \nabla_\nu T^{\mu\nu(m)} = 0, \qquad \nabla_\nu T^{\mu\nu(\Lambda)} = 0.$$

Projecting the conservation equation of the (clustering) matter component onto the space perpendicular to the time direction leads to the relativistic Euler equation for the matter fluid

$$h_{\alpha\mu}\nabla_\nu T^{\mu\nu(m)} = 0 \quad \text{with} \quad h_{\alpha\mu} = g_{\alpha\mu} + u_\alpha u_\mu. \tag{4.3}$$

Working that out, one finds[5]

$$(\rho + p)\nabla_u u = -\nabla_\alpha p + u\nabla_u p. \tag{4.4}$$

In case of a static configuration ($\nabla_u p = 0$) and with the help of Eqs. (4.1) and (4.4) reduces to

$$a' = \frac{-p'}{\rho + p}. \tag{4.5}$$

The field equations for the given metric are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

$$\frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) = 8\pi G(\rho + \rho_\Lambda), \tag{4.6}$$

$$-\frac{1}{r^2} + e^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) = 8\pi G(p - \rho_\Lambda), \tag{4.7}$$

$$e^{-2b}\left(a'' - a'b' + a'^2 + \frac{a' - b'}{r}\right) = 8\pi G(p - \rho_\Lambda). \tag{4.8}$$

Combining Eqs. (4.6) and (4.7) with (4.5) leads to

$$e^{-b} = \frac{1}{\sqrt{1 - \frac{8\pi G}{3}(\rho + \rho_\Lambda)r^2}}, \tag{4.9}$$

$$-p' = \frac{4\pi G}{3} r \cdot \frac{(\rho + p)(\rho + 3p - 2\rho_\Lambda)}{1 - \frac{8\pi G}{3}(\rho + \rho_\Lambda)r^2}. \tag{4.10}$$

Equation (4.10) is the TOV equation for the ΛCDM model in the case of homogeneous densities. For solving it, we assume the matter pressure to vanish at the boundary $p(r = R) = 0$.[6]

This leads to

$$p(r) = \frac{\rho(f(\frac{1-Ar^2}{1-AR^2})^{1/2} - 1) + 2\rho_\Lambda}{3 - f(\frac{1-Ar^2}{1-AR^2})^{1/2}}, \tag{4.11}$$

where

$$A = \frac{8\pi G}{3}(\rho + \rho_\Lambda), \qquad f = 1 - \frac{2\rho_\Lambda}{\rho}, \qquad \rho_\Lambda = \frac{\Lambda}{8\pi G}. \tag{4.12}$$

Inserting Eqs. (4.10) and (4.11) into the hydrostatic equilibrium condition [Eq. (4.5)] and integrating with the boundary $a(R) = 1/2\ln(1 - AR^2)$ (Schwarzschild-de Sitter solution) gives

$$e^a = (1 - AR^2)^{1/2}\frac{3 - (\frac{1-Ar^2}{1-AR^2})^{1/2} f}{3 - f}, \tag{4.13}$$

and, thus, the full metric inside the sphere can be written as

$$ds^2 = -(1 - AR^2)\left(\frac{3 - (\frac{1-Ar^2}{1-AR^2})^{1/2} f}{3 - f}\right)^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - Ar^2} + r^2 d\Omega^2, \tag{4.14}$$

which represents the metric of the interior Schwarzschild-de Sitter spacetime.

The well-known exterior Schwarzschild-de Sitter solution

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}} + r^2 d\Omega^2, \tag{4.15}$$

matches continuously with Eq. (4.14) at $r = R$. In this particular case, it has to be mentioned that asymptotic flatness can only be reached approximately as discussed in Sec. III. Since the scale of the halo is much smaller than the Hubble radius ($r/R_H \sim 10^{-3}$), we can still assume the object to be nearly isolated. We decided to embed the sphere into the Schwarzschild-de Sitter spacetime instead of an FLRW spacetime, because spacetime around the object can be assumed to be approximately static as well (due to the approximated timelike Killing vector field on these scales). In the ordinary Tolman-Oppenheimer-Volkhoff solution (see Ref. [16]), the perfect fluid sphere is embedded into the vacuum described by the Schwarzschild solution. In order to be consistent with this approach, the generalization including a cosmological constant is embedded into the Schwarzschild-de Sitter spacetime. Nevertheless, it will turn out that the virial radius and overdensity can be predicted consistently with this approach, although a dark matter contribution outside the sphere is neglected (see Sec. X and the weak-field limits in Secs. VII and IX.

V. DERIVATION OF THE RELATIVISTIC VIRIAL THEOREM

The pressure profile in Eq. (4.11) contains the radial dependence of the pressure in a sphere consisting of a cosmological constant fluid and collapsed dark matter embedded into a background Schwarzschild-de Sitter spacetime. When virialization starts, the system can be approximately assumed to be in equilibrium, which means

[5]In the following sections, we define $\rho_m \equiv \rho$ and $p_m \equiv p$.

[6]In the derivation of Eq. (2.1), the assumption of pressureless dark matter is a crucial argument. Nevertheless, for consistency with the TOV equation, we have to allow a pressure profile for the interior of the sphere. This issue will be discussed below.

that it can really be described by Eq. (4.10).[7] In order to derive a virial theorem from that, one can take the first spatial moment which should usually lead to the virial theorem after time-averaging. This means that small fluctuations around the equilibrium state are averaged out over time such that only time-averaged quantities (energy expressions) are left in the virial theorem. Since the system is already in equilibrium and the TOV equation has no time-dependences, the time integral drops naturally and all quantities can be interpreted as time-averaged.

Eq. (4.10) is multiplied with r and integrated (averaged) over the spacetime volume element (hence, taking the spatial moment and time-averaging are performed in one step):

$$-\lim_{T\to\infty}\frac{1}{T}\int p' r\eta = \lim_{T\to\infty}\frac{1}{T}\int \eta\left[\frac{4\pi G}{3}r^2\frac{(\rho+p)(\rho+3p-2\rho_\Lambda)}{1-Ar^2}\right], \tag{5.1}$$

which becomes

$$-4\pi\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^R p' r^3 e^{a+b}\mathrm{d}t\mathrm{d}r = 4\pi\lim_{T\to\infty}\frac{1}{T}\times\int_0^T\int_0^R\left[\frac{4\pi G}{3}r^4\cdot\frac{(\rho+p)(\rho+3p-2\rho_\Lambda)}{1-Ar^2}e^{a+b}\right]\mathrm{d}t\mathrm{d}r. \tag{5.2}$$

Since all the quantities in the integral do not depend on time, the evaluation of the time integral cancels naturally and, while interpreting the given quantities as time-averaged, this becomes

$$-4\pi\int_0^R p' r^3 e^{a+b}\mathrm{d}r = 4\pi\int_0^R\left[\frac{4\pi G}{3}r^4\cdot\frac{(\rho+p)(\rho+3p-2\rho_\Lambda)}{1-Ar^2}e^{a+b}\right]\mathrm{d}r. \tag{5.3}$$

Looking at the LHS of this equation in Euclidean space and performing a partial integration, we see that

$$-4\pi\int_0^R p' r^3\mathrm{d}r = [-4\pi p r^3]_0^R + \int_0^R 12\pi r^2\mathrm{d}r = \int 3p\mathrm{d}V \equiv 2\overline{T}. \tag{5.4}$$

In consistency with the Euclidian case, we can propose that

$$2T \equiv 2\overline{T} = -4\pi\int_0^R p' r^3 e^{a+b}\mathrm{d}r. \tag{5.5}$$

Consequently, we define

$$2T = 4\pi\int_0^R\left[\frac{4\pi G}{3}r^4\cdot\frac{(\rho+p)(\rho+3p-2\rho_\Lambda)}{1-Ar^2}e^{a+b}\right]\mathrm{d}r. \tag{5.6}$$

Inserting Eqs. (4.11) and (4.1) into Eq. (5.6), we obtain

$$2T = \frac{16\pi^2 G}{3(3-f)}(1-AR^2)^{1/2}(2\rho+2\rho_\Lambda)^2 \int_0^R\frac{r^4}{(1-Ar^2)^{3/2}}\cdot\frac{(\frac{1-Ar^2}{1-AR^2})^{1/2}f}{3-(\frac{1-Ar^2}{1-AR^2})^{1/2}f}\mathrm{d}r. \tag{5.7}$$

This is one version of a fully relativistic virial theorem for clustering dark matter in a ΛCDM-background model. Of course, other attempts exist in the literature to derive a relativistic virial theorem for several purposes. Chandrasekhar [24] derived a post-Newtonian version of the tensor virial theorem by investigating the post-Newtonian hydrodynamic equations consistently with Einstein's field equations. Bonazzola [25] has proposed an integral identity consistent with general relativity in an asymptotically flat, stationary and axisymmetric spacetime. Vilain [26] considers a scalar generalization of the virial theorem to general relativity which is valid for spherically symmetric, asymptotically flat spacetimes and has been successfully applied to stability studies of perfect fluid spheres. In addition, Vilain's work allows us to interpret the result of Bonnazzola geometrically in the spherical case. Bonazzola and Gourgoulhon [27] extended the work of 1973 to any stationary, asymptotically flat spacetime in general. Straumann [23] proposes a virial expression in the case of a spherically symmetric, static spacetime based on the Komar integral and asymptotic flatness. Except Ref. [24], these remarkable results have in common that asymptotical flatness is a crucial assumption to the spacetime which is necessary in order to define isolated objects in the sense of a Komar integral (see Refs. [20,21]). We want to emphasize at this point that, strictly speaking, this condition has to be valid in our case as well. However, we make use of the fact that an isolated object can be approximately defined in the FLRW spacetime by assuming the scale of the halo to be much smaller than the corresponding Hubble radius.

VI. RELATIVISTIC GRAVITATIONAL POTENTIAL ENERGY

The modified TOV solution can also be applied to find a relativistic expression for the gravitational potential energy of a spherical body. The derivation is inspired by the considerations of Straumann (see Ref. [23]), but since it is quite technical, we refer to Appendix B and quote here only the final result:

[7] The TOV equation represents the equation of motion of the system in equilibrium.

$$T = \int_0^R 4\pi r^2 \epsilon \frac{1}{\sqrt{1 - Ar^2}} \mathrm{d}r, \tag{6.1}$$

$$U = \int_0^R 4\pi r^2 \rho \left(1 - \frac{1}{\sqrt{1 - Ar^2}}\right) \mathrm{d}r, \tag{6.2}$$

with ϵ denoting the relativistic kinetic energy density which is defined in Eq. (B10) (see Appendix B).

In case of small gravitational fields given for an object having a radius r, which is much larger than its Schwarzschild radius (this corresponds to $Ar^2 \ll 1$), we can expand Eqs. (6.1) and (6.2) to first order:

$$\begin{aligned} T &= \frac{4\pi R^3}{3}\epsilon + \frac{2\pi\epsilon A R^5}{5} + \mathcal{O}(A^2) \\ &= M - M_0 + \frac{2\pi\epsilon A R^5}{5} + \mathcal{O}(A^2) \\ U &= -\frac{2\pi\rho A}{5} R^5 + \mathcal{O}(A^2) \\ &= -\frac{3}{5}\frac{GM^2}{R} - \frac{4\pi G}{5} M \rho_\Lambda R^2 + \mathcal{O}(A^2). \end{aligned}$$

The kinetic energy will reduce to the special-relativistic result if gravitational effects are neglected to zeroth order. The potential energy contains the Newtonian self-energy of a homogeneous sphere as a leading-order term. Thus, classical limits can be reproduced showing that Eqs. (6.1) and (6.2) are consistently defined.

VII. VIRIALIZATION EQUATION

Assuming that energy conservation still holds during collapse, the virialization equation states

$$[T + U]_{\mathrm{vir}} = U_{\mathrm{ta}}. \tag{7.1}$$

Let us now insert all derived energy expressions and perform a change of variable $r \to y \equiv r/R_{\mathrm{ta}}$. After simplifying the result, we end up with

$$T_{\mathrm{vir}} = \frac{8\pi^2 G R_{\mathrm{ta}}^5}{3(3-f)} (1 - A_{\mathrm{vir}} y_{\mathrm{vir}}^2 R_{\mathrm{ta}}^2)^{1/2} (2\rho_{\mathrm{vir}} + 2\rho_\Lambda)^2 \times \int_0^{y_{\mathrm{vir}}} \frac{y^4}{(1 - A_{\mathrm{vir}} y^2 R_{\mathrm{ta}}^2)^{3/2}} \cdot \frac{\left(\frac{1 - A_{\mathrm{vir}} y^2 R_{\mathrm{ta}}^2}{1 - A_{\mathrm{vir}} y_{\mathrm{vir}}^2 R_{\mathrm{ta}}^2}\right)^{1/2} f}{3 - \left(\frac{1 - A_{\mathrm{vir}} y^2 R_{\mathrm{ta}}^2}{1 - A_{\mathrm{vir}} y_{\mathrm{vir}}^2 R_{\mathrm{ta}}^2}\right)^{1/2} f} dy, \tag{7.2}$$

$$U_{\mathrm{vir}} = 4\pi \rho_{\mathrm{vir}} R_{\mathrm{ta}}^3 \int_0^{y_{\mathrm{vir}}} y^2 \left(1 - \frac{1}{\sqrt{1 - A_{\mathrm{vir}} y^2 R_{\mathrm{ta}}^2}}\right) dy, \tag{7.3}$$

$$U_{\mathrm{ta}} = 4\pi \rho_{\mathrm{ta}} R_{\mathrm{ta}}^3 \int_0^1 y^2 \left(1 - \frac{1}{\sqrt{1 - A_{\mathrm{ta}} y^2 R_{\mathrm{ta}}^2}}\right) dy, \tag{7.4}$$

with the definitions

$$A_{\mathrm{vir}} = \frac{8\pi G}{3}(\rho_{\mathrm{vir}} + \rho_\Lambda), \qquad A_{\mathrm{ta}} = \frac{8\pi G}{3}(\rho_{\mathrm{ta}} + \rho_\Lambda),$$
$$f = 1 - \frac{2\rho_\Lambda}{\rho_{\mathrm{vir}}}, \qquad \rho_{\mathrm{vir}} = \frac{3M}{4\pi y_{\mathrm{vir}}^3 R_{\mathrm{ta}}^3}, \qquad \rho_{\mathrm{ta}} = \frac{3M}{4\pi R_{\mathrm{ta}}^3}.$$

Equation (7.1) has to be solved numerically for y_{vir} at different redshifts (see Fig. 2 for the results). The turn-around radius, R_{ta}, can be obtained by using the solution of Eqs. (2.1) and (2.2):

$$\begin{aligned} \zeta &= \frac{\rho(R_{\mathrm{ta}})}{\rho_m^b(a_{\mathrm{ta}})} = 1 + \delta(a_{\mathrm{ta}}), \quad \rho(R_{\mathrm{ta}}) = \frac{3M}{4\pi R_{\mathrm{ta}}^3}, \\ \rho_m^b(a_{\mathrm{ta}}) &= \frac{3H_0^2}{8\pi G}\Omega_m^{(0)} a_{\mathrm{ta}}^3 \\ \Rightarrow R_{\mathrm{ta}} &= a_{\mathrm{ta}} \cdot \left(\frac{2GM}{H_0^2 \Omega_m^{(0)}(1 + \delta(a_{\mathrm{ta}}))}\right)^{1/3}. \end{aligned} \tag{7.5}$$

Let us consider the classical limit with respect to two assumptions:

(i) The sphere's radius is much larger than its Schwarzschild radius $R_S = AR^3 \ll R$, i.e., $AR^2 \ll 1$.

(ii) The cosmological-constant density is much smaller than the dark matter density inside the sphere. Since ρ_Λ is of the order of the critical density and $\rho = \Delta_V \rho_{\mathrm{cr}}$ with $\Delta_V \sim 95$–180,[8] this can be assumed safely in our case.

Expanding Eq. (7.2) to first order in AR^2 and ρ_Λ/ρ, and simplifying it, we end up with

$$y_{\mathrm{vir}}^2 = \frac{\rho_{\mathrm{ta}} + \rho_\Lambda}{\frac{1}{2}\rho_{\mathrm{vir}} + 2\rho_\Lambda}. \tag{7.6}$$

Writing this in terms of the Wang-Steinhardt parameters η_t and η_v,[9] this becomes

$$-2\eta_v y_{\mathrm{vir}}^3 + (2 + \eta_t) y_{\mathrm{vir}} - 1 = 0. \tag{7.7}$$

Equation (7.7) can be solved approximately by

$$y_{\mathrm{vir}} = \frac{1 - \frac{\eta_v}{2}}{2 + \eta_t - \frac{3}{2}\eta_v}. \tag{7.8}$$

Thus, Eq. (7.2) reduces to the WS limit under the given assumptions.

VIII. POST-NEWTONIAN EXPANSION OF THE VIRIALIZATION EQUATION

The post-Newtonian expansion of the virialization equation can be done by simply performing a Taylor expansion of Eq. (7.2); however, we choose a more elegant way

[8] See Pace *et al.* [12] for their results in the ΛCDM case.
[9] See Wang and Steinhardt [13] and Sec. II

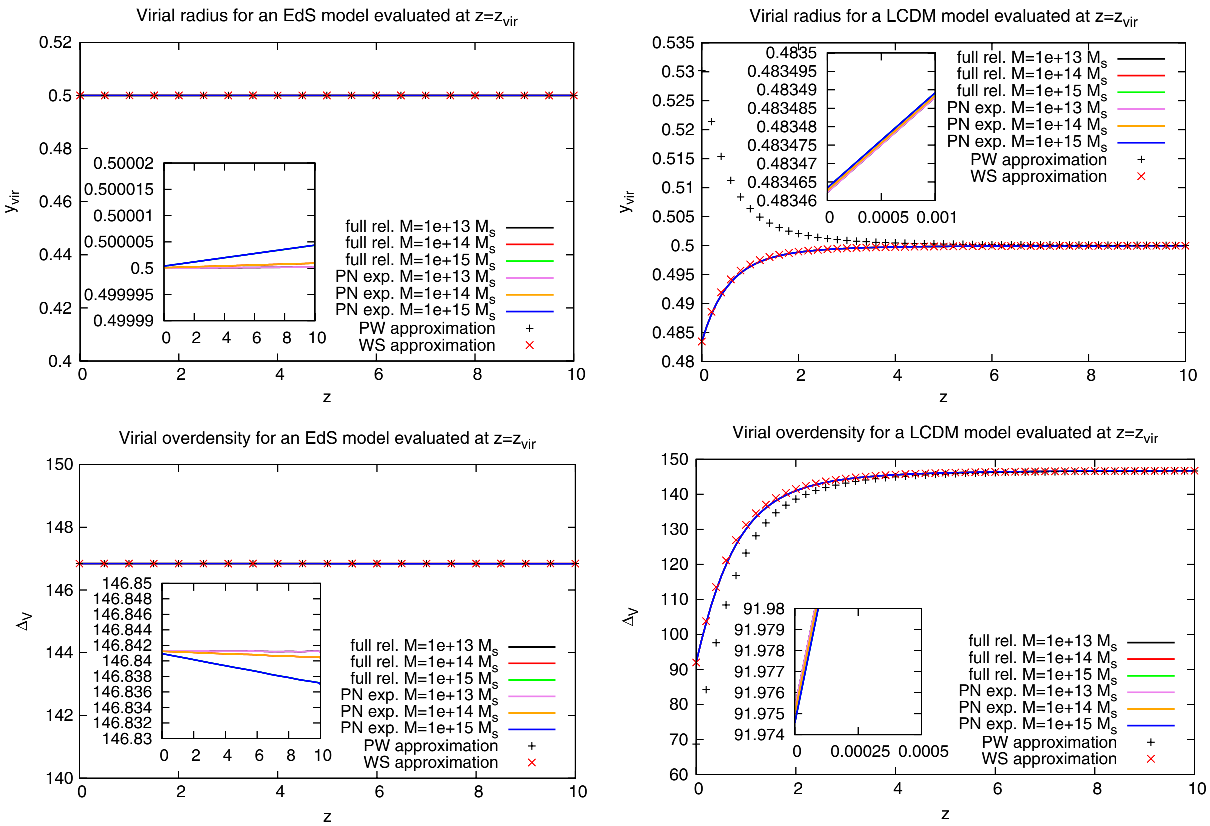

FIG. 2 (color online). Virial radius and virial overdensity obtained by the relativistic virialization equation and its post-Newtonian expansion as a function of collapse redshift for EdS and ΛCDM cosmologies. The upper panels show the virial radius for three different halo masses obtained by solving the full relativistic virialization equation [Eqs. (7.1), (7.2), (7.3), and (7.4)] and its post-Newtonian expansion [Eqs. (8.23), (8.24), (8.25), and (8.26)] using the two cosmological models. The lower panels show the corresponding virial overdensity obtained by Eq. (2.2). The classical solutions of Wang and Steinhardt (1998) and Wang (2006) are plotted for reference. The region close to $z_c = 0$ has been scaled up to sufficiently small redshifts in order to illustrate the dependence on the halo mass.

including the equation of motion of the collapsing sphere.[10]

We begin with a nonstatic, spherically symmetric spacetime described by

$$ds^2 = -e^{2a(r,t)}dt^2 + e^{2b(r,t)}dr^2 + r^2 d\Omega^2, \tag{8.1}$$

and use again the energy-momentum tensor of an ideal fluid with two components

$$T_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(\Lambda)}_{\mu\nu} = (\rho_m + \rho_\Lambda + p_m + p_\Lambda)u_\mu u_\nu + (p_m + p_\Lambda)g_{\mu\nu}. \tag{8.2}$$

The Λ component satisfies an equation of state given by

$$p_\Lambda = -\rho_\Lambda. \tag{8.3}$$

Consider a comoving reference frame in which the four-velocity u has the components

$$u^0 = e^{-a} \qquad u^i = 0 \quad \text{for } i = 1, 2, 3.$$

The relativistic Euler equation for the (clustering) matter component $h_{\alpha\mu}\nabla_\nu T^{\mu\nu,(m)} = 0$ states (in that frame)[11]:

$$a' = -\frac{p'}{\rho + p} \Rightarrow e^a = \frac{1}{\rho + p} \equiv \frac{1}{y}. \tag{8.4}$$

If we combine this relation with the field equations for the metric, we obtain the relativistic equation of motion for a spherically symmetric object (first derived by Misner and Sharp [28] in 1964 and applied by Collins [29] in 1978):

$$y\frac{d}{dt}\left(y\frac{dr}{dt}\right) = -\frac{1}{\rho + p}\frac{dp}{dr}\left(1 + y^2\dot{r}^2 - \frac{2GM(r)}{r}\right) - \frac{GM(r)}{r^2} - 4\pi G p r. \tag{8.5}$$

In the presence of a cosmological constant Λ with an equation of state like Eq. (8.3), this is slightly modified

[10]This equation was first derived by Misner and Sharp (see Ref. [28]).

[11]In the following, we define again $\rho_m \equiv \rho$, $p_m \equiv p$.

$$y\frac{d}{\mathrm{d}t}\left(y\frac{\mathrm{d}r}{\mathrm{d}t}\right) = -\frac{1}{y}\frac{dp}{\mathrm{d}r}\left(1 + y^2\dot{r}^2 - \frac{8\pi G}{3}(\rho + \rho_\Lambda)r^2\right) - \frac{4\pi G}{3}(\rho - 2\rho_\Lambda)r - 4\pi G p r. \tag{8.6}$$

In case of equilibrium, Eqs. (8.5) and (8.6) reduce to the TOV equations with or without a cosmological constant:

$$-p' = \frac{4\pi G}{3} r \frac{(\rho + p)(\rho + 3p)}{1 - \frac{8\pi G}{3}\rho r^2}, \tag{8.7}$$

$$-p' = \frac{4\pi G}{3} r \frac{(\rho + p)(\rho + 3p - 2\rho_\Lambda)}{1 - \frac{8\pi G}{3}(\rho + \rho_\Lambda)r^2}. \tag{8.8}$$

If only small oscillations of the system around its equilibrium are considered, we can assume that $\dot{r}^2/c^2 \ll 1$. Terms of this kind will be neglected in the following. After performing a Taylor-expansion up to the first post-Newtonian order $\mathcal{O}(\frac{1}{c^2})$ and inserting the zeroth-order expansion of the TOV equation [Eq. (8.8)],

$$-p' = \frac{4\pi G r}{3}\rho(\rho - 2\rho_\Lambda) + O\left(\frac{1}{c^2}\right), \tag{8.9}$$

we arrive at

$$\rho y^2 \ddot{r} = -p' - \frac{4\pi G r}{3}\rho(\rho - 2\rho_\Lambda) - \left(\frac{4\pi G r}{3}(\rho - 2\rho_\Lambda) + 4\pi G \rho r\right)p - \left(\frac{32\pi^2 G^2 r^3}{9}\rho(\rho^2 - \rho\rho_\Lambda - 2\rho_\Lambda^2)\right). \tag{8.10}$$

Since the derivation of the virial theorem requires an integration over the spacetime volume element, the metric has to be expanded as well. In our case, spacetime is described by the TOV metric given by

$$ds^2 = -(1 - AR^2)\left(\frac{3 - (\frac{1-Ar^2}{1-AR^2})^{1/2} f}{3 - f}\right)^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - Ar^2} + r^2 d\Omega^2, \tag{8.11}$$

where

$$A = \frac{8\pi G}{3}(\rho + \rho_\Lambda), \qquad f = 1 - \frac{2\rho_\Lambda}{\rho}.$$

An expansion up to $\mathcal{O}(1/c^2)$ leads to

$$ds^2 \approx -\left(1 - 2AR^2 - \frac{f}{(3 - f)}A(R^2 - r^2)\right)\mathrm{d}t^2 + (1 + AR^2)\mathrm{d}r^2 + r^2 d\Omega^2. \tag{8.12}$$

The canonical volume form becomes

$$\eta = \sqrt{-g}\mathrm{d}V_T = \sqrt{\left(\frac{1 - AR^2}{1 - Ar^2}\right)\left(\frac{3 - (\frac{1-Ar^2}{1-AR^2})^{1/2} f}{3 - f}\right)^2}\mathrm{d}V_T \approx \left(1 + \frac{A}{2}(r^2 - R^2) + \frac{f}{2(3 - f)}A(r^2 - R^2)\right)\mathrm{d}V_T, \tag{8.13}$$

with $\mathrm{d}V_T = \mathrm{d}t \wedge \mathrm{d}V$ being the total volume element for a flat spacetime in spherical polar coordinates

$$\mathrm{d}V_T = r^2 \sin\theta \cdot \mathrm{d}t \wedge \mathrm{d}r \wedge d\theta \wedge d\phi. \tag{8.14}$$

In the following, we will also apply the definition of the canonical volume form of the spacelike three-hypersurface Σ described by the spatial coordinates

$$\mathrm{d}V_\Sigma = \sqrt{-g|_\Sigma}\, r^2 \sin\theta \cdot \mathrm{d}r \wedge d\theta \wedge d\phi. \tag{8.15}$$

Taking the first spatial moment (multiplying with r and integrating over the spatial volume) leads to the post-Newtonian version of Lagrange's identity (see Ref. [29]):

$$\int \rho y^2 \ddot{r} r \mathrm{d}V_\Sigma = -\int r p' \mathrm{d}V_\Sigma - \int \frac{4\pi G r^2}{3}(\rho - 2\rho_\Lambda)\rho \mathrm{d}V_\Sigma - \int\left(\frac{4\pi G r^2}{3}(\rho - 2\rho_\Lambda) + 4\pi G \rho r^2\right)p\mathrm{d}V_\Sigma - \int \frac{32\pi^2 G^2 r^4}{9}(\rho^2 - \rho\rho_\Lambda - 2\rho_\Lambda^2)\rho \mathrm{d}V_\Sigma. \tag{8.16}$$

In analogy to the classical case, we interpret[12]

$$\frac{1}{2}\frac{d^2 I_r}{\mathrm{d}t^2} \equiv \int \rho y^2 \ddot{r} r \mathrm{d}V_\Sigma \qquad 2T \equiv -\int r p' \mathrm{d}V_\Sigma.$$

Using these definitions, Lagrange's identity becomes the familiar expression

$$\frac{1}{2}\frac{d^2 I_r}{\mathrm{d}t^2} = 2T - \int \frac{4\pi G r^2}{3}(\rho - 2\rho_\Lambda)\rho \mathrm{d}V_\Sigma - \int\left(\frac{4\pi G r^2}{3}(\rho - \rho_\Lambda) + 4\pi G \rho r^2\right)p\mathrm{d}V_\Sigma - \int \frac{32\pi^2 G^2 r^4}{9}(\rho^2 - \rho\rho_\Lambda - 2\rho_\Lambda^2)\rho \mathrm{d}V_\Sigma. \tag{8.17}$$

Dropping the corrections in $1/c^2$, the classical version of Lagrange's identity is

$$\frac{1}{2}\frac{d^2 I_r}{\mathrm{d}t^2} = 2T + U_m - 2U_\Lambda. \tag{8.18}$$

[12] I_r is defined to be the relativistic generalization of the classical moment of inertia. For a homogeneous sphere it is classically defined by $I = \frac{1}{2}\int_V \rho r^2 \mathrm{d}V$ (see Refs. [9,29]).

Performing the time average will lead to the post-Newtonian virial theorem, because motions like oscillations around the equilibrium configuration are averaged out. Since we have to apply

$$\lim_{T\to\infty}\int_0^T (\ldots)e^a \mathrm{d}t, \tag{8.19}$$

the volume element of the averaged form changes $\mathrm{d}V_\Sigma = e^b\mathrm{d}V \rightarrow e^{a+b}\mathrm{d}V = \sqrt{-g}\mathrm{d}V$ while the time integration is performed.

Dropping all terms of $\mathcal{O}(1/c^4)$, the post-Newtonian virial theorem is

$$\begin{aligned}2T = &\int\frac{4\pi G r^2}{3}(\rho-2\rho_\Lambda)\rho\mathrm{d}V\\ &+\int\left(\frac{4\pi G r^2}{3}(\rho-2\rho_\Lambda)+4\pi G\rho r^2\right)p\mathrm{d}V\,(\mathrm{I})\\ &+\int\frac{32\pi^2G^2r^4}{9}(\rho^2-\rho\rho_\Lambda-2\rho_\Lambda^2)\rho\mathrm{d}V\,(\mathrm{II})\\ &+\int\frac{2\pi G r^2}{3}(\rho-2\rho_\Lambda)\left(1+\frac{f}{3-f}\right)A(r^2-R^2)\rho\mathrm{d}V\,(\mathrm{III}).\end{aligned} \tag{8.20}$$

It can be seen that the correction terms contain

(i) *pressure contributions* (I), since pressure acts as a source of gravity

(ii) *backreaction terms* (II) between the fluid components and the geometry of spacetime (due to the nonlinearity of general relativity)

(iii) *metric expansion terms* (III), since a nonvanishing energy-momentum tensor changes the metric (due to the field equations)

The potential energy given by Eq. (6.2) can be expanded in the same way:

$$U = -4\pi\rho\int_0^R\left(\frac{A}{2}r^4+\frac{3}{8}A^2r^6\right)\mathrm{d}r. \tag{8.21}$$

Performing the angular integration for the kinetic energy expression leads to

$$\begin{aligned}T = &\int_0^R\frac{8\pi^2Gr^4}{3}(\rho-2\rho_\Lambda)\rho\mathrm{d}r\\ &+\int_0^R\left(\frac{8\pi^2Gr^4}{3}(\rho-2\rho_\Lambda)+8\pi^2G\rho r^4\right)p\mathrm{d}r\\ &+\int_0^R\frac{64\pi^3G^2r^6}{9}(\rho^2-\rho\rho_\Lambda-2\rho_\Lambda^2)\rho\mathrm{d}r\\ &+\int\frac{32\pi^3G^2r^4}{9}(\rho-2\rho_\Lambda)(\rho+\rho_\Lambda)\\ &\times\rho\left(1+\frac{f}{3-f}\right)(r^2-R^2)\mathrm{d}r.\end{aligned} \tag{8.22}$$

Now, we rewrite some variables[13]

$$r = y\cdot R_{\mathrm{ta}},\qquad R_{\mathrm{vir}} = y_{\mathrm{vir}}\cdot R_{\mathrm{ta}},$$

and the virialization equation becomes

$$[T+U]_{\mathrm{vir}} = U_{\mathrm{ta}}, \tag{8.23}$$

with the terms

$$U_{\mathrm{vir}} = -4\pi\rho_{\mathrm{vir}}\int_0^{y_{\mathrm{vir}}}\left(\frac{A_{\mathrm{vir}}}{2}y^4+\frac{3}{8}A_{\mathrm{vir}}^2y^6R_{\mathrm{ta}}^2\right)R_{\mathrm{ta}}^5dy, \tag{8.24}$$

$$U_{\mathrm{ta}} = -4\pi\rho_{\mathrm{ta}}\int_0^1\left(\frac{A_{\mathrm{ta}}}{2}y^4+\frac{3}{8}A_{\mathrm{ta}}^2y^6R_{\mathrm{ta}}^2\right)R_{\mathrm{ta}}^5dy, \tag{8.25}$$

and

$$\begin{aligned}T_{\mathrm{vir}} = &\int_0^{y_{\mathrm{vir}}}\frac{8\pi^2Gy^4R_{\mathrm{ta}}^5}{3}(\rho_{\mathrm{vir}}-2\rho_\Lambda)\rho_{\mathrm{vir}}dy+\int_0^{y_{\mathrm{vir}}}\left(\frac{8\pi^2Gy^4R_{\mathrm{ta}}^5}{3}(\rho_{\mathrm{vir}}-2\rho_\Lambda)+8\pi^2G\rho_{\mathrm{vir}}y^4R_{\mathrm{ta}}^5\right)p_{\mathrm{vir}}dy\\ &+\int_0^{y_{\mathrm{vir}}}\frac{64\pi^3G^2y^6R_{\mathrm{ta}}^7}{9}(\rho_{\mathrm{vir}}^2-\rho_{\mathrm{vir}}\rho_\Lambda-2\rho_\Lambda^2)\rho_{\mathrm{vir}}dy\\ &+\int_0^{y_{\mathrm{vir}}}\frac{32\pi^3G^2y^4R_{\mathrm{ta}}^7}{9}(\rho_{\mathrm{vir}}^3-\rho_{\mathrm{vir}}^2\rho_\Lambda-2\rho_{\mathrm{vir}}\rho_\Lambda^2)\left(1+\frac{f}{3-f}\right)(y^2-y_{\mathrm{vir}}^2)dy.\end{aligned} \tag{8.26}$$

In addition, we have applied the following definitions:

$$\rho_{\mathrm{vir}} = \frac{3M}{4\pi y_{\mathrm{vir}}^3R_{\mathrm{ta}}^3},\qquad \rho_{\mathrm{ta}} = \frac{3M}{4\pi R_{\mathrm{ta}}^3},\qquad A_{\mathrm{vir}} = \frac{8\pi G}{3}(\rho_{\mathrm{vir}}+\rho_\Lambda),\qquad A_{\mathrm{ta}} = \frac{8\pi G}{3}(\rho_{\mathrm{ta}}+\rho_\Lambda).$$

As for the fully relativistic version, Eq. (8.23) has to be solved for y_{vir} numerically.

[13] R_{ta} is again calculated using Eq. (7.5).

IX. THE RELATIVISTIC FORMALISM AND DYNAMICAL DARK ENERGY

Even though we have spent some effort to generalize our method to dynamical DE models, certain problems occur which will be described in the following: Consider a two-component fluid described by

$$T^{(m)}_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{9.1}$$

$$T^{(Q)}_{\mu\nu} = (\rho_Q + p_Q)u_\mu u_\nu + p_Q g_{\mu\nu}, \tag{9.2}$$

$$T_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(Q)}_{\mu\nu}, \tag{9.3}$$

where the densities ρ and ρ_Q are assumed to be constant and the quintessence component has an equation of state $p_Q = w\rho_Q$ with constant w. Energy-momentum conservation is separately fulfilled for each fluid component

$$\nabla_\mu T^{\mu\nu,m} = 0, \tag{9.4}$$

$$\nabla_\mu T^{\mu\nu,Q} = 0. \tag{9.5}$$

The static, spherically symmetric field equations for this setup are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

$$\frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) = 8\pi G(\rho + \rho_Q), \tag{9.6}$$

$$-\frac{1}{r^2} + e^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) = 8\pi G(p + w\rho_Q), \tag{9.7}$$

$$e^{-2b}\left(a'' - a'b' + a'^2 + \frac{a' - b'}{r}\right) = 8\pi G(p + w\rho_Q). \tag{9.8}$$

It turns out that, in case of $w \neq -1$, Eqs. (9.6), (9.7), and (9.8) are no longer consistently solvable and lead to contradictory solutions (see Appendix C for a detailed proof). This means that a model based on

(i) *staticity*
(ii) *spherical symmetry*
(iii) *matter model* (two-component fluid consisting of clustering dark matter and homogeneous DE with equation of state $p_Q = w\rho_Q$)

does not lead to a consistent description. Static, spherically symmetric solutions with $w \neq -1$ can, therefore, only be given approximately. In order to achieve exact solutions in the general case, we need to drop at least one of the model assumptions. Since spherical symmetry is dictated by the spherical collapse model and we want to fix the matter model, we can only stick to time-dependent problems and drop staticity.[14]

A physical explanation for this constraint on w is local energy-momentum conservation. Since we assume that it has to be valid for each component separately and the field equations are constructed in a way that energy and momentum are locally conserved, a static quintessence component with $w \neq -1$ violates this requirement.

Consider a perfect fluid with $T_{\mu\nu} = (\rho_Q + p_Q)u_\mu u_\nu + p_Q g_{\mu\nu,}$ which has to obey local energy-momentum conservation

$$\nabla_\mu T^{\mu\nu,Q} = 0. \tag{9.9}$$

Projecting this onto the three-space perpendicular to the mean-fluid flow, we obtain the relativistic Euler equation

$$(g_{\alpha\nu} + u_\alpha u_\nu)\nabla_\mu T^{\mu\nu,Q} = 0, \tag{9.10}$$

which leads to

$$(\rho_Q + p_Q)\nabla_u u = -\mathrm{grad}\, p_Q - u\nabla_u p_Q. \tag{9.11}$$

In the case of a static configuration [$u = (1, 0, 0, 0)$, $p_Q(r, t) = p_Q(r)$] and the metric ansatz from Eq. (4.1), we are left with

$$-p'_Q = a'(\rho_Q + p_Q), \tag{9.12}$$

which becomes for the given equation of state $p_Q = w\rho_Q$:

$$0 = a'\rho_Q(1 + w). \tag{9.13}$$

Since a' is nonzero in general, we have to require $w = -1$ in order to satisfy local energy-momentum conservation. Thus, a quintessence fluid with constant density and $w \neq -1$ violates local energy-momentum conservation in case of a static configuration.

Energy and momentum are certainly conserved for a time-dependent DE density which scales like[15] $\rho_Q = \rho_Q^{(0)} a^{-3(1+w)}$. In this case, the only time-independent DE density is the cosmological constant representing the only possible static, quintessence fluid configuration with constant equation of state that satisfies local energy-momentum conservation. In our approach, we restricted ourselves to homogeneous DE that evolves independently from the matter component. In fact, it can be seen by our consideration that, on relevant halo scales, DE cannot be treated consistently as a homogeneous fluid. Therefore, as an alternative to a nonstatic solution, one might also

[14] It has to be mentioned for completeness that there exist static exterior solutions describing a Schwarzschild black hole surrounded by a quintessence fluid (see Refs. [30,31]). However, in these cases ρ_Q is constrained to be radial dependent: $\rho_Q(r) \sim r^{-3(1+w)}$.

[15] Of course, this is only true for constant equation-of-state parameter w. In the general case, ρ_Q evolves like $\rho_Q = \rho_Q^{(0)} g(a)$ with $g(a) = \exp(-3\int_1^a (1 + w(a'))d\ln a')$ (see, for example, Ref. [3]).

consider DE to cluster or interact with dark matter on these scales. In case of clustering DE (see Refs. [4,5,8,9,11]) or even models considering interactions between the two fluids (see Refs. [7,15]), our formalism will significantly change and might allow an application to these models. In contrast to the homogeneous case, clustering DE models require a coupled system of two evolution equations for the density contrasts δ_m and $\delta_{\rm DE}$ and allow virialization of the DE component as well. The TOV equation, therefore, has to be formulated in the total pressure $p = p_m + p_Q$ and solved with the boundary condition $p(r = R) = w\rho_Q$. The same would be required in models that include interaction between dark matter and DE, because energy-momentum conservation will no longer be fulfilled for each fluid component separately but only for the total fluid. In this case, we would have

$$\nabla_\mu T^{\mu\nu,m} = Q^\nu, \tag{9.14}$$

$$\nabla_\mu T^{\mu\nu,Q} = -Q^\nu, \tag{9.15}$$

with the interaction four-vector Q^ν but still

$$\nabla_\mu T^{\mu\nu} = 0 \quad \text{with} \quad T^{\mu\nu} = T^{\mu\nu,m} + T^{\mu\nu,Q}. \tag{9.16}$$

Nevertheless, while treating DE as homogeneous at small scales, it might be possible to quantify the error made by using this assumption in the spherical collapse approach. Even though a solution obtained from Eqs. (9.6) and (9.7) is inconsistent with Eq. (9.8), a TOV equation and a virial theorem can be derived for $w \neq -1$ in the exact same manner as for the cosmological constant case. Having generalized our formalism to make it applicable to the clustering case, comparison of the unphysical, homogeneous DE fluid with the physical, inhomogeneous one might be possible on relevant halo scales.

X. RESULTS

The relativistic virialization equation and its post-Newtonian expansion are solved for the virial radius. Figure 2 shows the virial radius and overdensity as a function of the collapse redshift z_c for three different halo masses in EdS and ΛCDM cosmologies. All quantities at virialization are evaluated at the exact virial redshift $z_{\rm vir}$. It is a common approximation in the spherical collapse model to insert the potential energy evaluated at the collapse redshift. As already investigated analytically by Lee and Ng [6], the result of the virial overdensity changes significantly[16] by inserting the exact virial redshift instead. We have developed and applied an iterative method to obtain $z_{\rm vir}$ numerically and the results of Lee and Ng are nicely reproduced. The derivation is postponed to Appendix D.

TABLE I. The three relative post-Newtonian contribution terms with respect to the classical Newtonian term are considered in EdS and ΛCDM cosmology for $z_c = 0$ and $M = 10^{15} M_\odot$.

Einstein-de Sitter	
Pressure term	$1.274132394 \times 10^{-5}$
Backreaction term	$3.18533352 \times 10^{-5}$
Metric expansion	9.5560225×10^{-6}
ΛCDM	
Pressure term	1.2893437×10^{-5}
Backreaction term	$2.7275899168 \times 10^{-5}$
Metric expansion	$7.073241606 \times 10^{-6}$

It can be seen clearly in both figures that the Wang-Steinhardt limit is precisely recovered. This is expected, because the expressions for the potential energy and the kinetic energy derived in Sec. V contain the WS solution as limit to zeroth order. Since the halo mass no longer cancels out naturally on both sides of the virialization equation, the spherical collapse becomes mass dependent. Therefore each result is plotted for three different masses, namely $10^{13} M_\odot$, $10^{14} M_\odot$ and $10^{15} M_\odot$. Nevertheless, it can be seen that the results for $M = 10^{13} M_\odot$ and $M = 10^{15} M_\odot$ differ from each other by only $10^{-3}\%$.

Since, averaged over sufficiently large timescales, the final state of a virialized cluster is static,[17] it can be considered as a homogeneous, static perfect-fluid sphere such that the Tolman-Oppenheimer-Volkoff solution can be applied. We have constructed the virialization equation based on the TOV solution instead of using the classical approach from Friedmann's equations. A few points have to be discussed concerning this method:

(i) As an important approximation, we have assumed that the Killing vector field K of the FLRW universe is timelike on halo scales. Since the final state is static anyway, this is most relevant for the turn-around, which is described by a static solution as well. As shown in Appendix A, the spacelike component of K is, by at least two orders of magnitude, smaller than the timelike one. Looking at the small corrections in first post-Newtonian order in Table I being five orders of magnitude smaller than the classical term, it remains an open question whether a time-dependent approach that does not obey that approximation would have a non-negligible effect on the results.

(ii) As can be seen in Table I, the normalized contributions to the first post-Newtonian order are of about $10^{-3}\%$, being almost independent of the type of contribution. This can be expected due to a simple

[16] In the EdS case $\Delta_V(z_c) \sim 178$ changes into $\Delta_V(z_{\rm vir}) \sim 146$.

[17] Oscillations around the virial radius can be expected to average out.

estimate. Let us assume a typical massive galaxy cluster with a mass of $10^{15}M_{\odot}$ and a virial radius of 1 Mpc. The gravitational potential ϕ/c^2, being the ratio of its Schwarzschild radius and virial radius, has the value

$$\frac{\phi}{c^2} = \frac{GM}{c^2 R_{\mathrm{vir}}} \sim 5 \cdot 10^{-5}. \tag{10.1}$$

Thus, the post-Newtonian terms which are of the order $(\phi/c^2)^2$ must have absolute values of about 10^{-10} which corresponds to a relative contribution of 10^{-5} ($10^{-3}\%$) with respect to the classical term.

The key question remains why the relativistic calculation reduces to the WS limit instead of the result of PW. The ansatz of a static, spherically symmetric metric reduces Einstein's field equations to a coupled system of three ordinary differential equations with respect to the radius r [see Eqs. (4.6), (4.7), and (4.8)]. Equation (4.6) is already decoupled from Eqs. (4.7) and (4.8) such that the rr component of the metric is constrained to be

$$g_{rr} = e^{2b} = \frac{1}{1 - Ar^2}, \qquad A = \frac{8\pi G}{3}(\rho + \rho_{\Lambda}). \tag{10.2}$$

It has to be mentioned that Eq. (10.2) does not contain any pressure term. Straumann's self energy expression derived in Appendix B is based on the number density $n(r)$ of dark matter particles that is integrated over the covariant volume element restricted to the hypersurface Σ spanned by the spatial coordinates:

$$\eta|_{\Sigma} = \sqrt{-g|_{\Sigma}} \mathrm{d}r d\theta d\phi = \frac{r^2 \sin\theta}{\sqrt{1 - Ar^2}} \mathrm{d}r d\theta d\phi. \tag{10.3}$$

Finally, the result becomes

$$\begin{aligned} U &= \int_0^R 4\pi r^2 \rho \left(1 - \frac{1}{\sqrt{1 - Ar^2}}\right) \mathrm{d}r \\ &\approx -\frac{3}{5}\frac{GM^2}{R} - \frac{4\pi G}{5} M \rho_{\Lambda} R^2 + \mathcal{O}(A^2), \end{aligned} \tag{10.4}$$

which perfectly reproduces the potential energy used by Wang and Steinhardt in their model.

The TOV equation for the pressure profile $p(r)$ is a generalization of the Newtonian hydrostatic equation. Pressure is contained as an additional source of gravity, but the post-Newtonian expansion of the resulting virial theorem in Eq. (8.20) shows clearly that the limit of WS is reproduced to zeroth order. Horellou and Berge have shown in 2005 (see Ref. [19]) that, from a classical point of view, the Wang-Steinhardt solution is exactly valid in the ΛCDM case. This means that for the classical self-energy expression of a matter sphere in a ΛCDM background inserted into the classical virialization equation (see Sec. II), the solution of Wang and Steinhardt is recovered.

To conclude this section, we can state the following: Based on the assumptions of a static, spherically symmetric spacetime and separately fulfilled energy-momentum conservation equations for matter and cosmological constant,[18] the field equations and the resulting TOV equation provide a relativistic virial theorem that contains the Wang-Steinhardt result as limit to zeroth order. It remains an open question whether a nonstatic approach to relativistic gravitational collapse that might also be adapted to dynamical DE models would still recover this result.

XI. REMARKS ON THE REQUIRED PRESSURE PROFILE

As already mentioned by Oppenheimer and Snyder in 1939 (see Ref. [32]), there does not exist any static, spherically symmetric solution of the field equations with vanishing pressure. If no positive pressure profile is present, nothing will prevent the spherical object from collapsing into a singularity. Therefore, in the classical spherical collapse, the virialization condition is introduced to define an equilibrium. In our case, gravity forces the system to build up a pressure profile to prevent the sphere from collapsing into a singularity. In fact, the process of virialization must convert ordered motion from the collapse into unordered motion and the kinetic energy associated with the unordered motion corresponds to pressure. Thus, an equilibrium is reached by a positive pressure profile that can be related directly to the mean kinetic energy of the system [see Eq. (5.5). This might be a first step toward a more fundamental theory of virialization avoiding the interpretation of an *enforced* equilibrium.

The nonlinear density evolution equation [see Eq. (2.1)] is still based on extended Newtonian theory.[19] In order to achieve full consistency, a relativistic evolution equation for either the density or the radius is recommended. Nevertheless, these equations will retain their full validity if dark matter is assumed to be pressureless during the evolution and the pressure profile is built up instantaneously at virialization.[20] The appearance of a pressure profile in this context is a direct consequence of the spherical collapse model, itself. Strictly speaking, virialization is a tool in the spherical collapse model in order to achieve an equilibrium state. The pressure is directly related to the mean kinetic energy of the system, which is again related to the potential energy by the virial theorem. Since the last

[18]The cosmological constant fluid trivially fulfills energy-momentum conservation, since the continuity equation reduces to $\dot{\rho}_{\Lambda} = 0$.

[19]Velocities and gravitational potentials are assumed to be small compared to c^2 and pressure is included as a source of gravity.

[20]Nevertheless, we have to admit that the assumption of an instantaneously appearing pressure profile at virial redshift is a highly idealized concept.

two quantities cannot vanish, in general, neither can the pressure. Thus, the virial theorem, which is combined with, but in no way related to, the nonlinear density evolution, requires a pressure profile, even if the latter is based on cold dark matter.

XII. CONCLUSION AND OUTLOOK

We propose a way to set up a fully relativistic method to obtain the virial radius and the virial overdensity for the EdS and ΛCDM cosmology. Within the assumption of an approximately timelike Killing vector field of the FLRW metric, static solutions for perfect fluid spheres in general relativity (namely the Tolman-Oppenheimer-Volkoff equation) have been successfully applied to extend the virial theorem in a consistent manner. The result has been inserted into the virialization equation which can be solved for the virial radius to find the corresponding virial overdensity. It turns out that the solution of Wang and Steinhardt [13] for the virial radius in a ΛCDM cosmology is almost perfectly reproduced by our formalism which can also be shown analytically by performing the weak-field limit. The first-order post-Newtonian expansion has been investigated, and the leading-order corrections have been worked out and calculated numerically. We found out that they have a relative contribution of $10^{-3}\%$ with respect to the classical term, which is of very small but expected size. Although these corrections are of limited astrophysical interest, the concept, itself, is a small step toward a more fundamental understanding of virialization of spherical halos in the presence of DE. In addition, an iterative method has been set up to calculate the exact virial redshift numerically. The results of Lee and Ng [6] are reproduced extremely well.

Naturally, galaxies and galaxy clusters are far more complex than homogeneous and isotropic spheres, but the spherical collapse model provides a very simple semi-analytic method that already suffices to estimate important parameters like the virial radius and virial overdensity. The process of virialization, itself, is an additional condition that has been introduced to prevent a spherical overdensity of pressureless dark matter from collapsing into a singularity. A pressure profile, which a relaxed continuous spherical object must obey due to general relativity, can provide new insight into the process of virialization, itself.

There are certainly some topics this paper cannot address, because they are far beyond its scope. Although the spherical collapse models are powerful tools to obtain estimates of the evolution of structures in the universe, they are limited by their simplicity. In particular, the fact that the spherical overdensity is in no way embedded continuously into the background Friedmann universe is still very idealized and dissatisfying. Secondly, DE cannot be described yet in a self-consistent way with general relativity, since local energy-momentum conservation, which is required by the theory, is not fulfilled and a coordinate representation of the two fluids is missing. Approaches based on the Lemaître-Tolman-Bondi models (see Refs. [33–38]) as well as the presented work of Misner and Sharp (see Ref. [28]) are promising candidates for following investigations.

APPENDIX A: APPROXIMATELY TIMELIKE KILLING VECTOR FIELD OF FLRW SPACETIME ON HALO SCALES

Let us start with the FLRW spacetime given by the metric

$$ds^2 = -\mathrm{d}t^2 + \frac{a^2(t)}{(1+\frac{kr^2}{4})^2}(\mathrm{d}r^2 + r^2 d\Omega^2). \tag{A1}$$

This model is based on isotropy and homogeneity of the three-dimensional spacelike hypersurfaces describing a space of constant curvature k.

Isotropy requires the existence of a coordinate frame in which spatial rotations are isometries. Given that frame, the most general ansatz for a Killing vector field of Eq. (A1) is:

$$K = A(r,t)\partial_t + B(r,t)\partial_r, \tag{A2}$$

with A and B being arbitrary scalar functions of radius and time.

Equation (A2) has to be inserted into the Killing equation to obtain any relation between A and B.

$$(\mathcal{L}_K g)_{\mu\nu} = 0. \tag{A3}$$

The Lie derivative of a rank (0, 2) tensor can be written as

$$K^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu K^\lambda + g_{\mu\lambda}\partial_\nu K^\lambda = 0. \tag{A4}$$

Inserting Eq. (A2) into Eq. (A4), the following three constraints can be obtained:

$$(\mathcal{L}_K g)_{00}\colon\ 2\dot{A} = 0, \tag{A5}$$

$$(\mathcal{L}_K g)_{11}\colon\ (2\dot{a}A + 2aB')\left(1 + \frac{k}{4}r^2\right) - kraB = 0, \tag{A6}$$

$$(\mathcal{L}_K g)_{22}\colon (2\dot{a}Ar^2 + 2arB)\left(1 + \frac{k}{4}r^2\right), - kr^3aB = 0, \tag{A7}$$

$$(\mathcal{L}_K g)_{01}\colon\ A' = \frac{a^2}{(1+\frac{kr^2}{4})^2}\dot{B}, \tag{A8}$$

$$(\mathcal{L}_K g)_{33} = \sin^2\theta(\mathcal{L}_K g)_{22} = 0. \tag{A9}$$

If Eqs. (A6) and (A7) are combined, we can obtain a differential equation in B

$$B' = \frac{B}{r}. \tag{A10}$$

Let us now apply the following approximation: At scale factors we consider

$$a_c, a_{\rm ta} \sim \mathcal{O}(1) - \mathcal{O}(10^{-1}), \tag{A11}$$

the Hubble radius $R_H = 1/H$ is much bigger than the scale r of the halo. From Friedmann's first equation we can infer

$$\Omega_k(a) = -\frac{k}{H^2(a)} = 1 - \Omega_m(a) - \Omega_\Lambda(a). \tag{A12}$$

Therefore,

$$\begin{aligned} kr^2 &= -H^2(a)r^2(1 - \Omega_m(a) - \Omega_\Lambda(a)) \\ &= -\left(\frac{r}{R_H(a)}\right)^2(1 - \Omega_m(a) - \Omega_\Lambda(a)). \end{aligned} \tag{A13}$$

Since $r/R_H \ll 1$ and cosmological parameter measurements from WMAP or SDSS[21] indicate $1 - \Omega_m(a) - \Omega_\Lambda(a) \lesssim 10^{-2}$, we can safely assume

$$r^2 \ll \left|\frac{1}{k}\right|. \tag{A14}$$

Given Eq. (A14), the constraints obtained from the Killing equation reduce to

$$\dot{A} = 0, \tag{A15}$$

$$B' = \frac{B}{r}, \tag{A16}$$

$$A' = a^2\dot{B}. \tag{A17}$$

Since we want to consider K on scales much smaller than R_H, we expand $A(r, t)$ and $B(r, t)$ to first order in rH:

$$A(r, t) = A_0 + A_1 rH + \mathcal{O}((rH)^2), \tag{A18}$$

$$B(r, t) = B_0 + B_1 rH + \mathcal{O}((rH)^2), \tag{A19}$$

with constant expansion coefficients A_0, A_1, B_0 and B_1. Inserting Eqs. (A18) and (A19) into Eqs. (A15)–(A17), the following constraints can be obtained:

$$A_1\dot{H}r = 0 \Rightarrow A_1 = 0, \tag{A20}$$

$$B_1 H = \frac{B_0}{r} + B_1 H \Rightarrow B_0 = 0, \tag{A21}$$

$$B_1 r\dot{H} = 0 \Rightarrow B_1 = 0. \tag{A22}$$

Thus, we can conclude that

$$A(r, t) = A_0 + \mathcal{O}((rH)^2), \tag{A23}$$

$$B(r, t) = \mathcal{O}((rH)^2). \tag{A24}$$

[21]See Refs. [39,40] for reference.

In the spherical collapse model, we consider parameters given within the following range:

$$r \sim 1 - 10\,\text{Mpc} \qquad R_H(a) \sim 3 \cdot 10^2\text{–}3 \cdot 10^3\ \text{Mpc}.$$

Thus, Eqs. (A23) and (A24) imply

$$\frac{|B|}{|A|} \sim 0.09\%\text{–}0.9\%, \tag{A25}$$

which means that the Killing vector field is timelike and approximately fulfills the Frobenius condition on relevant halo scales being small compared to the Hubble radius:

$$K \approx A_0\partial_t. \tag{A26}$$

At this point, we want to emphasize that we do not intend to give an exact timelike solution to the Killing equation. It is well known that, apart from a static or a de Sitter solution, there does not exist a timelike Killing vector field for a general FLRW metric. Consequently, Eq. (A2), even though motivated by spacetime symmetries, does not generally solve the Killing equation. Instead, we expand the time- and spacelike coefficients in r/R_H, derive an approximate, first-order solution and obtain the spacelike component to be smaller than the timelike one by $\mathcal{O}((r/R_H)^2)$.

To conclude this section, the main results will be summarized again:

(i) On scales being much smaller than the Hubble radius of the cosmological background model, we have negligible spatial curvature, and an approximate solution K of the Killing equation can be obtained to first order in r/R_H.

(ii) As a major property of this solution on small scales, the spatial component of K is suppressed by $(r/R_H)^2$. Thus, K is timelike

$$\langle K, K\rangle = g(K, K) = -A^2 + a^2B^2 < 0, \tag{A27}$$

and even points, approximately, into the direction of the cosmic time (with an accuracy of at least 0.9%). Thus, on halo scales, K is approximately orthogonal onto the underlying hypersurfaces spanned by the three spatial coordinates. This allows an approximate description of the halo by static solutions of Einstein's field equations.

APPENDIX B: DERIVATION OF THE RELATIVISTIC POTENTIAL ENERGY

The following derivation is essentially taken from Straumann [23] and will be only slightly modified for our purposes: Let us consider a particle representation of dark matter and compare the total mass M of the spherical halo with the rest mass of the gravitationally interacting dark matter particles. The total rest mass of all dark matter (DM) particles is given by

$$M_0 = Nm_N, \tag{B1}$$

with N being the total number of particles and m_N the rest mass of a single particle. Let us define a current density J as a one-form such that we can express the number of particles via a surface integral:

$$N = \int_{t=\mathrm{const}} *J. \tag{B2}$$

J can be expressed as

$$J = J_\mu \theta^\mu \Rightarrow *J = J_\mu * \theta^\mu = J_\mu \eta^\mu \quad \text{with} \quad \eta^\mu = *\theta^\mu, \tag{B3}$$

with respect to an arbitrary dual basis $\{\theta^\mu\}$.

We transform the integral by evaluating the Hodge dual explicitly:[22]

$$\begin{aligned}\int_{t=\mathrm{const}} *J &= \int_{t=\mathrm{const}} J_\mu \eta^\mu = \int_{t=\mathrm{const}} J_0 \eta^0 \\ &= \int_{t=\mathrm{const}} J_0 * \theta^0.\end{aligned} \tag{B4}$$

Assuming the metric ansatz given in Eq. (4.1) and defining the dual basis like

$$\begin{aligned}\theta^0 &= e^a \mathrm{d}t, \qquad \theta^1 = e^b \mathrm{d}r, \\ \theta^2 &= r^2 d\theta, \qquad \theta^3 = r^2 \sin^2\theta d\phi,\end{aligned} \tag{B5}$$

$*\theta^0$ can be evaluated:

$$*\theta^0 = e^a * \mathrm{d}t = e^b r^2 \sin\theta \mathrm{d}r \wedge d\theta \wedge d\phi = \theta^1 \wedge \theta^2 \wedge \theta^3. \tag{B6}$$

Since, in a static configuration, the θ^μ are an orthonormal system, the above result can be expected. Thus, we are left with

$$\begin{aligned}N &= \int_{t=\mathrm{const}} J_0 \theta^1 \wedge \theta^2 \wedge \theta^3 \\ &= \int_{t=\mathrm{const}} J_0 e^b r^2 \sin\theta \mathrm{d}r \wedge d\theta \wedge d\phi \\ &= \int_0^R 4\pi r^2 J_0 e^b \mathrm{d}r.\end{aligned} \tag{B7}$$

The number density $n(r)$ can be obtained by projection of J onto the four-velocity u^μ being $(1, 0, 0, 0)^T$ in the chosen coordinate frame:

$$n(r) = -u^\mu J_\mu = J_0, \tag{B8}$$

such that we get

$$N = \int_0^R 4\pi r^2 n(r) e^{b(r)} \mathrm{d}r. \tag{B9}$$

We can define the proper DM energy density (total energy density with subtracted particle rest energy density)

$$\epsilon(r) = \rho(r) - m_N n(r), \tag{B10}$$

which corresponds to an intuitive proper internal energy of

$$E = M - M_0 = M - N m_N. \tag{B11}$$

The proper internal energy can be decomposed into a total kinetic and a total potential energy of the system such that $T + V = M - M_0$. Let us insert the integral for the particle number given by Eq. (B7)

$$\begin{aligned}m_N N &= \int_0^R 4\pi r^2 e^b m_N n(r) \mathrm{d}r \\ &= \int_0^R (\rho(r) - \epsilon(r)) 4\pi r^2 e^b \mathrm{d}r = M - T - V \\ &= \int_0^R 4\pi r^2 \rho(r) \mathrm{d}r - T - V,\end{aligned} \tag{B12}$$

where we have assumed that the total mass is simply the volume integral of the density profile

$$M = \int_0^R 4\pi r^2 \rho(r) \mathrm{d}r. \tag{B13}$$

Solving Eq. (B12) for $T + V$, we obtain

$$T + V = \int_0^R 4\pi r^2 e^b \epsilon(r) \mathrm{d}r + \int_0^R 4\pi r^2 \rho(r)(1 - e^b) \mathrm{d}r. \tag{B14}$$

This leads to the definition

$$T = \int_0^R 4\pi r^2 \epsilon(r) e^b \mathrm{d}r, \tag{B15}$$

$$U = \int_0^R 4\pi r^2 \rho(r)(1 - e^b) \mathrm{d}r. \tag{B16}$$

Consider a top-hat density profile and a two-component fluid consisting of DM and a cosmological constant such that

$$e^{2b} = \frac{1}{1 - Ar^2} \quad \text{with} \quad A = \frac{8\pi G}{3}(\rho + \rho_\Lambda). \tag{B17}$$

Thus, we finally end up with

$$T = \int_0^R 4\pi r^2 \epsilon \frac{1}{\sqrt{1 - Ar^2}} \mathrm{d}r, \tag{B18}$$

$$U = \int_0^R 4\pi r^2 \rho \left(1 - \frac{1}{\sqrt{1 - Ar^2}}\right) \mathrm{d}r. \tag{B19}$$

APPENDIX C: STATIC, SPHERICALLY SYMMETRIC FIELD EQUATIONS WITH HOMOGENEOUS DE

Consider a two-component fluid described by

$$T^{(m)}_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \tag{C1}$$

[22] Since we consider a static configuration, we can find a coordinate frame in which the J_i components vanish. We will assume that in the following without loss of generality.

$$T^{(Q)}_{\mu\nu} = (\rho_Q + p_Q)u_\mu u_\nu + p_Q g_{\mu\nu}, \tag{C2}$$

$$T_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(Q)}_{\mu\nu}, \tag{C3}$$

where the densities ρ and ρ_Q are assumed to be constant and the quintessence component has an equation of state $p_Q = w\rho_Q$ with constant w. Energy-momentum conservation is fulfilled separately for each fluid component:

$$\nabla_\mu T^{\mu\nu,m} = 0, \tag{C4}$$

$$\nabla_\mu T^{\mu\nu,Q} = 0. \tag{C5}$$

The static, spherically symmetric field equations of this setup are

$$\begin{aligned} G_{\mu\nu} &= 8\pi G T_{\mu\nu} \\ \frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) &= 8\pi G(\rho + \rho_Q), \end{aligned} \tag{C6}$$

$$-\frac{1}{r^2} + e^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) = 8\pi G(p + w\rho_Q), \tag{C7}$$

$$e^{-2b}\left(a'' - a'b' + a'^2 + \frac{a' - b'}{r}\right) = 8\pi G(p + w\rho_Q). \tag{C8}$$

We have to find out whether there are conditions for the solvability of the field equations without using any concrete solution for a and b such that effects resulting from boundary conditions are excluded. With the help of Eq. (C6), we can express b':

$$b' = \frac{1}{2r}(1 - (1 - 8\pi G(\rho + \rho_Q)r^2)e^{2b}). \tag{C9}$$

In the same way, Eq. (C7) can be solved for a':

$$a' = -\frac{1}{2r}(1 - (1 + 8\pi G(p + w\rho_Q)r)e^{2b}). \tag{C10}$$

If we add Eqs. (C6) and (C7), we will get

$$a' = -b' + 4\pi G(\rho + p + \rho_Q(1 + w))re^{2b}. \tag{C11}$$

If Eq. (C9) is inserted into that expression, we will obtain Eq. (C10), so the first two field equations are consistent. Energy-momentum conservation for the matter component of the fluid means

$$\nabla_\mu T^{\mu\nu,m} = 0. \tag{C12}$$

Projecting this onto the space perpendicular to the velocity flow, we obtain the relativistic Euler equation

$$(g_{\alpha\nu} + u_\alpha u_\nu)\nabla_\mu T^{\mu\nu} = 0, \tag{C13}$$

which becomes

$$(\rho + p)\nabla_u u = -\mathrm{grad}p - u\nabla_u p. \tag{C14}$$

In case of a static configuration, we are left with

$$-p' = a'(\rho + p), \tag{C15}$$

which is basically the hydrostatic equilibrium condition for the matter configuration of our system. Consider the derivative of Eq. (C10)

$$\begin{aligned} a'' = \frac{1}{2r^2}(1 + e^{2b}\{&-1 + 8\pi G(p + w\rho_Q)r^2 + 8\pi G p' r^3 \\ &+ 2b'r(1 + 8\pi G(p + w\rho_Q))r^2\}). \end{aligned} \tag{C16}$$

Inserting Eqs. (C9), (C10), and (C15) into Eq. (C16) and simplifying leads to

$$\begin{aligned} a'' = \frac{1}{2r^2} &+ \frac{1}{2r^2}e^{2b}4\pi G(5p + 4w\rho_Q + \rho)r^2 \\ &- \frac{1}{2r^2}e^{4b}(1 + 8\pi G(p + w\rho_Q)r^2) \\ &\times (1 - 4\pi G(\rho - p + 2\rho_Q)r^2). \end{aligned} \tag{C17}$$

On the other hand, a'' can be expressed with the help of Eq. (C8)

$$a'' = a'b' - a'^2 - \frac{a' - b'}{r} + 8\pi G(p + w\rho_Q)e^{2b}. \tag{C18}$$

If we plug in Eqs. (C9) and (C10), we can obtain (after some algebra)

$$\begin{aligned} a'' = \frac{1}{2r^2} &+ \frac{1}{2r^2}e^{2b}[4\pi G(\rho + \rho_Q)r^2 + 20\pi G(p + w\rho_Q)r^2] \\ &- \frac{1}{2r^2}e^{4b}(1 + 8\pi G(p + w\rho_Q)r^2) \\ &\times (1 - 4\pi G(\rho + \rho_Q - p - w\rho_Q)). \end{aligned} \tag{C19}$$

For Eqs. (C6)–(C8) being consistent, Eqs. (C17) and (C19) have to be identical which means that each coefficient belonging to the same order of e^{2b} has to be equal for each radius r.

(i) 0th order:

$$\frac{1}{2r^2} = \frac{1}{2r^2} \quad \text{trivially fulfilled.}$$

(ii) 1st order:

$$\begin{aligned} &\frac{1}{2r^2}4\pi G(5p + 4w\rho_Q + \rho)r^2 \\ &\quad = \frac{1}{2r^2}[4\pi G(\rho + \rho_Q)r^2 + 20\pi G(p + w\rho_Q)r^2] \\ &\quad \Rightarrow 5p + \rho + 4w\rho_Q = \rho + \rho_Q + 5p + 5w\rho_Q \\ &\quad \Rightarrow 4w = (1 + 5w) \Rightarrow w = -1. \end{aligned}$$

(iii) 2nd order:

$$\begin{aligned}&\frac{1}{2r^2}(1+8\pi G(p+w\rho_Q)r^2)\\&\quad\times(1-4\pi G(\rho-p+2\rho_Q)r^2)\\&=\frac{1}{2r^2}(1+8\pi G(p+w\rho_Q)r^2)\\&\quad\times(1-4\pi G(\rho-p+\rho_Q-w\rho_Q)r^2).\end{aligned}$$

If we require that $p \neq -w\rho_Q - 1/(8\pi G r^2)$, which we have to assume in the general case, we can say

$$\begin{aligned}4\pi G(\rho-p+2\rho_Q)&=4\pi G(\rho-p+\rho_Q-w\rho_Q)\\\Rightarrow 2\rho_Q&=\rho_Q(1-w)\\\Rightarrow w&=-1.\end{aligned}$$

Thus, Eqs. (C6)–(C8) necessarily require $w = -1$ in order to be consistently solvable.

APPENDIX D: ITERATIVE METHOD TO FIND THE VIRIAL REDSHIFT

Consider Eq. (2.1) and solve this for the nonlinear density contrast $\delta(a)$. Once this is done, Eq. (2.2) can be used

$$\Delta(r,a)\equiv\frac{\rho(r)}{\rho_b(a)}=\zeta\left(\frac{x}{y}\right)^3=1+\delta(a). \tag{D1}$$

This allows us to express $y(a)$ with the help of δ

$$y(a)=\frac{a}{a_{ta}}\cdot\left(\frac{\zeta}{1+\delta(a)}\right)^{1/3}. \tag{D2}$$

Using Eq. (D2) and the virialization equation [Eqs. (2.3), (7.1), and (8.23)], an iterative method can be constructed to find the virial scale factor a_{vir}.

Starting from $a^{(0)} = a_c$, we will proceed in the following way

$$\begin{aligned}&a^{(0)}\overset{(2.3)}{\longrightarrow}y_{vir}^{(0)}\overset{(D2)}{\longrightarrow}a(y_{vir}^{(0)})=a^{(1)}\\&a^{(1)}\overset{(2.3)}{\longrightarrow}y_{vir}^{(1)}\overset{(D2)}{\longrightarrow}a(y_{vir}^{(1)})=a^{(2)}\\&a^{(2)}\overset{(2.3)}{\longrightarrow}y_{vir}^{(2)}\overset{(D2)}{\longrightarrow}a(y_{vir}^{(2)})=a^{(3)}\\&\quad\vdots\qquad\quad\vdots\qquad\quad\vdots\\&a^{(n)}\overset{(2.3)}{\longrightarrow}y_{vir}^{(n)}\overset{(D2)}{\longrightarrow}a(y_{vir}^{(n)})=a^{(n+1)}\end{aligned} \tag{D3}$$

In each step, the quantity $a(y^{(i)})$ is needed which is given by the root of the equation

$$y(a)-y_{vir}^{(i)}=0\longrightarrow a(y_{vir}^{(i)}), \tag{D4}$$

and which can be found numerically in each step i.

It turns out that this method converges, so that the following condition can be applied to stop the iteration:

$$\left|\frac{a^{(n+1)}-a^{(n)}}{a^{(n)}}\right|\leq \text{tol}, \tag{D5}$$

with an appropriate tolerance *tol*. Given a tolerance of tol $\sim 10^{-7}$, the iteration can be stopped within three to four steps. Thus, the final result can be defined as

$$a^{(n+1)}\equiv a_{vir}\Rightarrow z_{vir}=\frac{1}{a_{vir}}-1. \tag{D6}$$

It will turn out that the virial overdensity changes significantly if the exact quantities are evaluated at $z = z_{vir}$: Let us calculate Δ_V at redshift $z_c = 0$ in the EdS model, which corresponds to a virial redshift of $z_{vir} = 0.065$ and a turn-around redshift of $z_{ta} = 0.587$.

(i) Case 1 ($z = z_c$):

$$\Delta_V=\left(\frac{3\pi}{4}\right)^2(1+z_{ta})^3\left(\frac{1}{y_{vir}}\right)^3=177.518. \tag{D7}$$

(ii) Case 2 ($z = z_{vir}$):

$$\Delta_V=\left(\frac{3\pi}{4}\right)^2\left(\frac{1+z_{ta}}{1+z_{vir}}\right)^3\left(\frac{1}{y_{vir}}\right)^3=146.958. \tag{D8}$$

This has also been predicted analytically by Lee and Ng (see Ref. [6]).

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