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Discrete symmetries in the three-Higgs-doublet model

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N-Higgs-doublet models (NHDM) are among the most popular examples of electroweak symmetry breaking mechanisms beyond the Standard Model. Discrete symmetries imposed on the NHDM scalar potential play a pivotal role in shaping the phenomenology of the model, and various symmetry groups have been studied so far. However, in spite of all efforts, the classification of finite Higgs-family symmetry groups realizable in NHDM for any $N > 2$ is still missing. Here, we solve this problem for the three-Higgs-doublet model by making use of Burnside's theorem and other results from pure finite group theory which are rarely exploited in physics. Our method and results can also be used beyond high-energy physics, for example, in the study of possible symmetries in three-band superconductors.

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I. INTRODUCTION

The nature of electroweak symmetry breaking remains one of the hottest issues in high-energy physics. The experimental quest for the Higgs boson, which was suggested back in 1964 [1], has very recently passed the first checkpoint: the CMS and ATLAS collaborations at the LHC announced the discovery of the Higgs-like resonance at 126 GeV [2]. Already their first measurements indicate intriguing deviations from the Standard Model expectations. Whether these data signal that a nonminimal Higgs mechanism is indeed at work and, if so, what it is are among the hottest questions in particle physics these days.

Many different variants of a nonminimal Higgs mechanism have been proposed so far [3]. One conceptually simple and phenomenologically attractive class of models involves several Higgs doublets with identical quantum numbers. The scalar potential in these N-Higgs-doublet models (NHDM) is often assumed to be symmetric under a group of unitary (Higgs-family) or antiunitary (generalized CP) transformations acting in the space of doublets. These symmetries play a pivotal role in the phenomenology of the model, both in the scalar and in the fermionic sectors [4], and they often bear interesting astrophysical consequences. In fact, in many phenomenological models, one often starts by picking up a symmetry group and then deriving phenomenological consequences.

In this situation, it is often very desirable to know which symmetry groups can be incorporated in a given model, and how they affect the phenomenological consequences. Discrete symmetries are of special interest here for a number of reasons. First, unlike spontaneously broken continuous symmetries, they do not produce unwanted Goldstone bosons. Second, finite symmetry groups with multidimensional irreducible representations often lead to remarkable degeneracy patterns in the physical Higgs boson spectrum. The simplest example here is an S_3-symmetric three-Higgs-doublet model (3HDM) with the 2HDM-like Higgs spectrum. Third, finite symmetry groups can lead to so-called geometric CP violation [5–7], in which the calculable phases of vacuum expectation values are protected by the symmetry arguments. Finally, there is a quest for derivation of the patterns observed in the fermion mixing matrices from symmetry arguments, and finite groups are also at work here [4]. Although these groups are introduced in the fermionic sector of the model, they might be related to symmetry groups in the Higgs sector, and the search for a convenient realization of this link continues.

Given the high importance of symmetries for the NHDM phenomenology, it is natural to ask the question: which symmetry groups can be implemented in the scalar sector of NHDM for a given N?

In the two-Higgs-doublet model (2HDM), this question has been answered several years ago [8] (see also Ref. [9] for a review). Focusing on discrete symmetries, the only realizable group of unitary symmetries are $\mathbb{Z}_2$ and $(\mathbb{Z}_2)^2$. If antiunitary transformations are included, then $(\mathbb{Z}_2)^3$ is also realizable. For each group, the corresponding potential was written and phenomenological consequences were studied in detail (for example, an investigation of the $(\mathbb{Z}_2)^3$-symmetric 2HDM can be found in Ref. [10]).

With more than two doublets, the problem remains open. Variants of NHDM based on several finite groups have been studied [11], with an emphasis on A_4 [12] and $\Delta(27)$ or $\Delta(54)$ [5,6]. Also, several attempts have been made to classify at least some symmetries in NHDM [13]. In particular, a classification of all realizable Abelian symmetry groups in NHDM for any N was recently given in Ref. [14]. However, the full list of non-Abelian finite groups which can be symmetry groups in the NHDM scalar sector is not yet known. We stress that this task is different from just classifying all finite subgroups of $SU(3)$ [15], because invariance of the Higgs potential places strong and nontrivial restrictions on possible symmetry groups.

In this paper we solve this problem for the 3HDM. Starting from Abelian groups and applying several results from the finite group theory, we find the complete list of discrete symmetry groups of Higgs-family transformations realizable in 3HDM. Extension of our method to groups which include antiunitary transformations will be given elsewhere [16].

We draw the reader's attention to the somewhat nonstandard way we use group theory in our analysis. Usually, group-theoretical methods in physics are limited to representation theory. However, the formal group theory contains many powerful results beyond the representation theory which can be of great use for identifying symmetry properties of a model. These results can restrict possible symmetries of the model without the need to explicitly manipulate with degrees of freedom which transform under specific representations of these groups.

This paper can be viewed as a nontrivial example of how powerful the pure group theory can be in finding symmetries of a model. Specifically, we make use of Burnside's theorem and other results from finite group theory to solve the problem which seems to be out of reach for more traditional methods. Although we focus below on this specific application, we stress that the same method can also be used in various condensed matter problems which involve several interacting order parameters [17], in particular, in three-band superconductivity.

The structure of the paper is the following. In the next section we apply group-theoretic tools to find the general structure of the finite groups which can be realized as Higgs-family symmetry groups in 3HDM. This result allows us to restrict the search for possible symmetry groups to a very small set. Then, in Sec. III, we check all members of this set and see which groups can indeed be at work in 3HDM. We summarize our findings in Sec. IV.

II. STRUCTURE OF FINITE SYMMETRY GROUPS IN 3HDM

The most general renormalizable gauge-invariant scalar potential of 3HDM can be written as

$$V = Y_{ij}(\phi_i^\dagger \phi_j) + Z_{ijkl}(\phi_i^\dagger \phi_j)(\phi_k^\dagger \phi_l), \qquad (1)$$

where all indices run from 1 to 3. We are interested in unitary transformations mixing doublets ϕ_i that leave this potential invariant for some Y_{ij} and Z_{ijkl}. *A priori*, these transformations belong to the group $U(3)$. Multiplying the three doublets by a common phase factor, which trivially leaves the potential invariant, is already taken into account in the gauge group $U(1)_Y$. Therefore, we focus on additional transformations not reducible to overall phase rotations, which form the group $PSU(3) = SU(3)/\mathbb{Z}_3$, where $\mathbb{Z}_3$ is the center of $SU(3)$. Our task is therefore to find finite subgroups of $PSU(3)$ which can be the symmetry groups of the potential (1) for some choices of coefficients. We stress that we search for *realizable* symmetry groups, that is, for groups $G \subset PSU(3)$ such that there exists a G-symmetric potential which is not invariant under a larger symmetry group $G' \supset G$; see a fuller discussion in Ref. [14].

Abelian realizable symmetry groups for NHDM were characterized in Ref. [14]. For our task of classifying finite realizable symmetry groups in 3HDM, the following Abelian groups must be considered:

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2\times\mathbb{Z}_2, \quad \mathbb{Z}_3\times\mathbb{Z}_3. \qquad (2)$$

The first four are the only realizable finite subgroups of maximal tori in $PSU(3)$. The last group, $\mathbb{Z}_3\times\mathbb{Z}_3$, is on its own a maximal Abelian subgroup of $PSU(3)$, but it is not realizable because a $\mathbb{Z}_3\times\mathbb{Z}_3$-symmetric potential is automatically symmetric under $(\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_2$; see explicit expressions below. However, it still can appear as an Abelian subgroup of a finite non-Abelian realizable group; therefore, it must be taken into consideration. Trying to impose any other Abelian Higgs-family symmetry group on the 3HDM potential unavoidably makes it symmetric under a continuous group.

Let us denote by $G \subset PSU(3)$ a finite (non-Abelian) symmetry group in 3HDM. We shall now apply some results from the finite group theory to prove that G cannot be too large, and more specifically, we shall describe the generic structure of G.

All Abelian subgroups of G must be from the list (2). By Chauchy's theorem, if p is a prime divisor of the order of the group $|G|$, then G contains a subgroup $\mathbb{Z}_p$. Thus, the order of the group can have only two prime divisors: $|G| = 2^a 3^b$. Then, according to Burnside's $p^a q^b$ theorem, the group G is *solvable*. Solvability implies that G contains a normal Abelian subgroup, which belongs, of course, to the list (2). This is our first key group-theoretic step.

Suppose A is the normal Abelian subgroup of G, $A \lhd G$. Obviously, $A \subseteq C_G(A)$, the centralizer of A in G (all elements $g \in G$ which commute with all $a \in A$). It turns out that this A can be chosen in such a way that it coincides with its own centralizer in G (that is, it is self-centralizing): $A = C_G(A)$ [16]. This means that elements $g \in G$, $g \notin A$, cannot commute with *all* elements of A. Therefore, they induce automorphisms (i.e., structure-preserving permutations) on A: $g^{-1}ag \in A$ for any $a \in A$, and these automorphisms are nontrivial. Even more, if g_1 and g_2 induce the same automorphism on A, $g_1^{-1}ag_1 = g_2^{-1}ag_2$ for all $a \in A$, then g_1 and g_2 belong to the same coset of A in G: $g_2 = g_1 a'$. Therefore, the homomorphism f: $G/A \to Aut(A)$, where $Aut(A)$ is the group of automorphisms on A, is *injective*. We conclude that

$$G/A = K, \qquad K \subseteq Aut(A). \qquad (3)$$

This is our second key group-theoretic step. It proves that G cannot be too large, and it also shows that G can be constructed as an extension of A by a subgroup of $Aut(A)$: $G = A.K$.

III. EXPLICIT CONSTRUCTION OF POSSIBLE SYMMETRY GROUPS

We now check all the candidates for A from the list (2) and see which extension can work in 3HDM. We use the explicit realization of each of the groups A [14] and search for additional transformations from $PSU(3)$ with the desired multiplication properties.

A. Extending $\mathbb{Z}_2$ and $\mathbb{Z}_3$

If $A = \mathbb{Z}_2$, then $Aut(\mathbb{Z}_2) = \{1\}$, so that $G = \mathbb{Z}_2$. This case was already considered in Ref. [14].

If $A = \mathbb{Z}_3$, then $Aut(\mathbb{Z}_3) = \mathbb{Z}_2$. The only nontrivial case to be considered is $G/A = \mathbb{Z}_2$, so that G is the dihedral group representing the symmetries of an equilateral triangle $G = D_6 = S_3$. If the $\mathbb{Z}_3$ group is generated by the phase rotations $a = \mathrm{diag}(\omega, \omega^2, 1)$ with $\omega = 2\pi/3$, then the transformation b generating $\mathbb{Z}_2$ and satisfying $b^{-1}ab = a^2$ must be of the form

$$b = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ e^{-i\delta} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad (4)$$

with arbitrary δ. The choice of the mixing pair of doublets (ϕ_1 and ϕ_2 in this case) is also arbitrary, so other b's with different pairs of mixing doublets are also allowed. The fact that b is not uniquely defined means that there is a whole family of D_6 groups parametrized by the value of δ even if we start with the fixed group $A = \mathbb{Z}_3$.

The generic $\mathbb{Z}_3$-symmetric potential contains the part invariant under any phase rotation

$$V_0 = -\sum_i m_i^2(\phi_i^\dagger \phi_i) + \sum_{i,j} \lambda_{ij}(\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j) + \sum_{i\neq j} \lambda'_{ij}(\phi_i^\dagger \phi_j)(\phi_j^\dagger \phi_i), \qquad (5)$$

and the following additional terms:

$$V_{\mathbb{Z}_3} = \lambda_1(\phi_2^\dagger \phi_1)(\phi_3^\dagger \phi_1) + \lambda_2(\phi_1^\dagger \phi_2)(\phi_3^\dagger \phi_2) + \lambda_3(\phi_1^\dagger \phi_3)(\phi_2^\dagger \phi_3) + \mathrm{H.c.}, \qquad (6)$$

with complex λ_1, λ_2, λ_3. If the parameters of V_0 satisfy

$$m_{11}^2 = m_{22}^2, \qquad \lambda_{11} = \lambda_{22}, \qquad \lambda_{13} = \lambda_{23}, \qquad \lambda'_{13} = \lambda'_{23}, \qquad (7)$$

and if, in addition, moduli of two among the three coefficients $\lambda_1, \lambda_2, \lambda_3$ coincide (for example, $|\lambda_1| = |\lambda_2|$), then the potential $V_0 + V_{\mathbb{Z}_3}$ becomes symmetric under one particular D_6 group constructed with b in (4) with the value of $\delta = (\arg\lambda_2 - \arg\lambda_1 + \pi)/3$.

This construction allows us to write down an example of the D_6 potential. In order to prove that D_6 is indeed a realizable group, we need to show that the resulting potential is not symmetric under any other Higgs-family transformation. This is proved by the mere observation that all other possible groups to be discussed below which could contain D_6 lead to *stronger* restrictions on the potential than (7) and $|\lambda_1| = |\lambda_2|$. Therefore, not satisfying those stronger restrictions will yield a potential symmetric only under D_6. Finally, one can also show that the potential we obtained does not have any continuous symmetry. The same logic applies to other realizable groups below.

B. Extending $\mathbb{Z}_4$

If $A = \mathbb{Z}_4$ (generated by a), then $Aut(\mathbb{Z}_4) = \mathbb{Z}_2$, so that $G = \mathbb{Z}_4.\mathbb{Z}_2$. The two non-Abelian possibilities for G are the dihedral group D_8, representing symmetries of the square, and the quaternion group Q_8. In both cases, $b^{-1}ab = a^3$, with the only difference that $b^2 = 1$ for D_8 while $b^2 = a^2$ for Q_8. Representing a by $\mathrm{diag}(i, -i, 1)$, we find

$$b(D_8) = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ e^{-i\delta} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
$$b(Q_8) = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ -e^{-i\delta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Again, in each case we obtain a family of b's parametrized by phase δ. The $\mathbb{Z}_4$-symmetric potential is $V_0 + V_{\mathbb{Z}_4}$, where

$$V_{\mathbb{Z}_4} = \lambda_1(\phi_3^\dagger \phi_1)(\phi_3^\dagger \phi_2) + \lambda_2(\phi_1^\dagger \phi_2)^2 + \mathrm{H.c.} \qquad (8)$$

An explicit analysis shows that to make it D_8 invariant, we only need to satisfy conditions (7). Then, the potential is symmetric under $b(D_8)$ with the phase $\delta = \arg\lambda_2/2$. Since any larger group that could possibly contain D_8 leads to stronger restrictions on the potential, we conclude that D_8 is realizable in 3HDM.

Now, if instead of D_8 we try to make the potential symmetric under Q_8, we unavoidably need to set $\lambda_1 = 0$. Removing one term from (8) immediately makes it symmetric under a *continuous* group of phase rotations [14]. Therefore, Q_8 is not realizable in 3HDM.

C. Extending $\mathbb{Z}_2 \times \mathbb{Z}_2$

If $A = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $Aut(\mathbb{Z}_2 \times \mathbb{Z}_2) = GL_2(2) = S_3$. $\mathbb{Z}_2 \times \mathbb{Z}_2$ can be realized as the group of independent sign flips of the three doublets with generators $a_1 = \mathrm{diag}(1, -1, -1)$ and $a_2 = \mathrm{diag}(-1, 1, -1)$. The potential symmetric under this group contains V_0 and additional terms

$$V_{\mathbb{Z}_2\times\mathbb{Z}_2} = \tilde\lambda_{12}(\phi_1^\dagger \phi_2)^2 + \tilde\lambda_{23}(\phi_2^\dagger \phi_3)^2 + \tilde\lambda_{31}(\phi_3^\dagger \phi_1)^2 + \mathrm{H.c.} \qquad (9)$$

The coefficients $\tilde\lambda_{ij}$ can be complex; we denote their phases as ψ_{ij}.

There are three possibilities to extend A: by $\mathbb{Z}_2$, by $\mathbb{Z}_3$, and by S_3. The first extension, $(\mathbb{Z}_2 \times \mathbb{Z}_2).\mathbb{Z}_2$, leads to D_8, which was already constructed above.

The extension by $\mathbb{Z}_3$ is necessarily split, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$, leading to the group $T \simeq A_4$, the symmetry group of a tetrahedron. To construct it, we need to find b acting on $\{a_1, a_2, a_1 a_2\}$ by cyclic permutations. Fixing the direction of permutations by $b^{-1} a_1 b = a_2$, we find that b must be of the form

$$b = \begin{pmatrix} 0 & e^{i\delta_1} & 0 \\ 0 & 0 & e^{i\delta_2} \\ e^{-i(\delta_1+\delta_2)} & 0 & 0 \end{pmatrix}, \quad (10)$$

with arbitrary δ_1, δ_2. It then follows that if coefficients in (9) satisfy

$$|\tilde{\lambda}_{12}| = |\tilde{\lambda}_{23}| = |\tilde{\lambda}_{31}|, \quad (11)$$

then $V_{\mathbb{Z}_2\times\mathbb{Z}_2}$ is symmetric under a particular b with

$$\delta_1 = \frac{2\psi_{12} - \psi_{31} - \psi_{23}}{6}, \qquad \delta_2 = \frac{2\psi_{23} - \psi_{31} - \psi_{12}}{6}.$$

By rephasing, one can bring (9) to the following form:

$$V_T = \tilde{\lambda}[(\phi_1^\dagger \phi_2)^2 + (\phi_2^\dagger \phi_3)^2 + (\phi_3^\dagger \phi_1)^2] + \text{H.c.}, \quad (12)$$

with complex $\tilde{\lambda}$. In addition, the symmetry under b places stronger conditions on the parameters of V_0, and the most general V_0 satisfying them is now

$$V_0 = -m^2 \sum_i (\phi_i^\dagger \phi_i) + \lambda\Big[\sum_i (\phi_i^\dagger \phi_i)\Big]^2 + \sum_{i\neq j}[\lambda'(\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j) + \lambda''|\phi_i^\dagger \phi_j|^2]. \quad (13)$$

The last extension, $(\mathbb{Z}_2 \times \mathbb{Z}_2).S_3$, leads to the group $O = S_4$, the symmetry group of an octahedron and a cube. It includes T as a subgroup; therefore, the most general O-symmetric potential is V_0 from (13) plus V_T from (12) with the additional condition that $\tilde{\lambda}$ is real.

D. Extending $\mathbb{Z}_3 \times \mathbb{Z}_3$

Finally, if $A = \mathbb{Z}_3 \times \mathbb{Z}_3$, then $K \subset Aut(\mathbb{Z}_3 \times \mathbb{Z}_3) = GL_2(3)$, the general linear group of transformations of two-dimensional vector space over the finite field $\mathbb{F}_3$, whose role is played by A. One can define an antisymmetric scalar product in this space and prove that K must include only transformations from $GL_2(3)$ that preserve this scalar product: $K \subseteq Sp_2(3) = SL_2(3)$.

The group $SL_2(3)$ has order 24 and contains elements of order 2, 3, 4, and 6. Elements of order 6 cannot be used for extension because they would generate the Abelian subgroup $\mathbb{Z}_6$, which is absent in (2). Besides, we will show below that K must always contain the subgroup $\mathbb{Z}_2$. There are three kinds of subgroups of $K \subset SL_2(3)$ containing $\mathbb{Z}_2$ but not containing $\mathbb{Z}_6$: $\mathbb{Z}_2$, $\mathbb{Z}_4$, and Q_8. Since, as we argued, the quaternion group Q_8 is not realizable in 3HDM, K can only be $\mathbb{Z}_2$ or $\mathbb{Z}_4$.

To show that $K \supseteq \mathbb{Z}_2$, consider first the subgroup of $SU(3)$ generated by

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \omega = \exp\Big(\frac{2\pi i}{3}\Big).$$

This subgroup is usually denoted as $\Delta(27)$ [18], and it is well known in model building with three Higgs doublets [5]. Since $[a, b] = aba^{-1}b^{-1} \in Z(SU(3))$, the image of $\Delta(27)$ under the canonical homomorphism $SU(3) \to PSU(3)$ becomes the desired Abelian group $\mathbb{Z}_3 \times \mathbb{Z}_3$. The true generators of $\mathbb{Z}_3 \times \mathbb{Z}_3$ are cosets $\bar{a} = aZ(SU(3))$ and $\bar{b} = bZ(SU(3))$ from $PSU(3)$. The $\mathbb{Z}_3 \times \mathbb{Z}_3$-invariant potential is

$$V = -m^2[(\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2) + (\phi_3^\dagger \phi_3)] + \lambda_0[(\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2) + (\phi_3^\dagger \phi_3)]^2 + \frac{\lambda_1}{\sqrt{3}}[(\phi_1^\dagger \phi_1)^2 + (\phi_2^\dagger \phi_2)^2 + (\phi_3^\dagger \phi_3)^2 - (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) - (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) - (\phi_3^\dagger \phi_3)(\phi_1^\dagger \phi_1)] + \lambda_2[|\phi_1^\dagger \phi_2|^2 + |\phi_2^\dagger \phi_3|^2 + |\phi_3^\dagger \phi_1|^2] + \lambda_3[(\phi_1^\dagger \phi_2)(\phi_1^\dagger \phi_3) + (\phi_2^\dagger \phi_3)(\phi_2^\dagger \phi_1) + (\phi_3^\dagger \phi_1)(\phi_3^\dagger \phi_2)] + \text{H.c.}, \quad (14)$$

with real m^2, λ_0, λ_1, λ_2 and complex λ_3, all values being generic. This potential is, however, symmetric under a larger group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \simeq \Delta(54)/\mathbb{Z}_3$, which is generated by $\bar{a}$, $\bar{b}$, $\bar{c}$ with the following relations:

$$\bar{a}^3 = \bar{b}^3 = 1, \qquad \bar{c}^2 = 1, \qquad [\bar{a}, \bar{b}] = 1, \qquad \bar{c}\,\bar{a}\,\bar{c}^{-1} = \bar{a}^2, \qquad \bar{c}\,\bar{b}\,\bar{c}^{-1} = \bar{b}^2.$$

In terms of the explicit transformation laws, $\bar{c}$ is the coset $cZ(SU(3))$, with c being the exchange of any two doublets, so that $\langle \bar{a}, \bar{c} \rangle = S_3$ is the group of arbitrary permutations of the three doublets. Thus, if $G = (\mathbb{Z}_3 \times \mathbb{Z}_3).K$, then a G-symmetric potential must be a restriction of (14), so that $K \supseteq \mathbb{Z}_2$.

Turning now to the extension $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ (which is also known as $\Sigma(36)$ [18]), we note that $SL_2(3)$ contains three distinct $\mathbb{Z}_4$ subgroups, which, however, intersect at the center of $SL_2(3)$. All three are conjugate inside $SL_2(3)$ and lead, up to isomorphism, to the same symmetry group. To give an example of a potential symmetric under $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, we choose an element $d \in SL_2(3)$ of order 4 that generates the cyclic permutation of generators $\bar{a}$, $\bar{b}$, $\bar{a}^2$, $\bar{b}^2$. It can be represented by the following $SU(3)$ transformation:

$$d = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad d^{-1} = d^*, \quad d^4 = 1. \qquad (15)$$

Then by analyzing how the potential changes under d, we obtain the following criterion: (14) becomes symmetric under $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, if λ_3 is real and is equal to $(\sqrt{3}\lambda_1 - \lambda_2)/2$. One can also show that the resulting potential is not invariant under any continuous symmetry group.

IV. SUMMARY

Because of the important phenomenological role the symmetries play in multi-Higgs-doublet models, the task of classifying all symmetries in NHDM is of much interest. Here we solved this problem for 3HDM. Focusing on groups of unitary transformations and including the finite Abelian groups found in Ref. [14], we obtain the following list of finite groups realizable as Higgs-family symmetry groups of the 3HDM scalar sector:

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad D_6, \quad D_8,$$
$$T \simeq A_4, \quad O \simeq S_4, \quad (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \simeq \Delta(54)/\mathbb{Z}_3,$$
$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4 \simeq \Sigma(36). \qquad (16)$$

This list is complete: trying to impose any other finite Higgs-family symmetry group on the 3HDM potential will unavoidably lead to a potential symmetric under a continuous group. Applying methods described in Ref. [14], one can also obtain the list of realizable groups in 3HDM which include antiunitary transformations. These results, as well as a study of symmetry breaking patterns for each of these groups, will be presented elsewhere [16].

We stress that we solved the classification problem in a rather nonstandard way, by applying results and tools from formal finite group theory and without using representation theory. We view this as an example of how powerful pure group theory can be in identifying symmetries of a model. We also stress that the same method can be applied to problems beyond particle physics—for example, for understanding symmetries of the order parameters in three-band superconductors.

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