Relativistic corrections to double charmonium production in high energy proton-proton interaction

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On the basis of perturbative QCD and the relativistic quark model, we calculate relativistic corrections to the process of pair J/ψ production in proton-proton collisions at LHC energy $\sqrt{S} = 7$ TeV. Relativistic terms in the production amplitude connected with the relative motion of heavy quarks and the transformation law of the bound state wave functions to the reference frame of moving J/ψ mesons are taken into account. For the gluon and quark propagators entering the amplitude, we use a truncated expansion in relative quark momenta up to the second order. Relativistic corrections to the quark bound state wave functions are considered by means of the Breit-like potential. It turns out that the examined effects decrease initial nonrelativistic cross section more than two times. The final result lies below the experimental value measured by LHCb.

I. INTRODUCTION

The production of heavy quarkonium states in different reactions is the subject of considerable interest during last years. The mechanism of heavy quarkonium production represents the long-standing problem of quantum chromodynamics [1–4]. Most current theoretical investigations are performed on the basis of nonrelativistic quantum chromodynamics (NRQCD) [5] and the quark models. According to these approaches, the production of heavy quarkonium is divided into two stages. On the first stage, one or several quark-antiquark pairs are produced at small distances of order $1/m_Q$. This short-range part in the production is associated with the basic interaction of free quarks and gluons and can be evaluated by perturbation theory. The subsequent nonperturbative transition from the intermediate states of quarks $QQ\ldots$ and antiquarks $\bar{Q}\bar{Q}\ldots$ to physical quarkonium states, on the second stage, involves long-distance scales of order of quarkonium size $1/(m_Q v)$. The formation of the quark bound states is parametrized by nonperturbative matrix elements in NRQCD and calculated by means of the bound state wave functions in the quark models. The finding of the correspondence between parameters of the quark models and NRQCD opens the way for better understanding of the quark-gluon dynamics. Both approaches complement each other and can reveal new aspects of the color dynamics of quarks and gluons.

One of the directions in this investigation is related with the pair production of double heavy quarkonium. The initial impulse to intensive investigations was given in

this field several years ago by the measurement of the pair charmonium production cross sections in electron-positron annihilation. The experimental data obtained at the Belle and *BABAR* detectors disagreed with the calculations based on NRQCD. The theoretical results were improved and adjusted in correspondence with the experiment after the account of one-loop perturbative corrections and relativistic corrections to the nonrelativistic cross section [6–8]. One of the learned lessons from this problem consists in the understanding that only sequential account of relativistic corrections in the heavy quarkonium production processes can lead to reliable theoretical results. It is necessary to point out that subsequent observation of numerous charmonium-like states by the Belle and *BABAR* collaborations with unclear nature demands further continuation of the investigations in this direction [9]. Recently, the first experimental result of the LHCb collaboration on the pair charmonium production in proton-proton interaction at a center-of-mass energy $\sqrt{S} = 7$ TeV was published [10]:

$$\sigma_{\text{LHCb}}^{\text{exp}} = 5.1 \pm 1.0 \pm 1.1 \text{ nb}, \tag{1}$$

where the first uncertainty is statistical and the second systematic. This production cross section for pairs with both J/ψ was obtained in the rapidity range $2 < y < 4.5$. There are several calculations of the pair charmonium production cross section in pp interaction in Refs. [11–15]. Theoretical estimations of the total cross section (integrated over all rapidities) are obtained in the leading order of QCD where the process of the gluon fusion is the dominant one: 18 nb [11], 22 nb [13]. In the framework of the k_t-factorization approach the prediction on the integral

production cross section range is 10–27 nb [14]. These calculations give the following total values of the cross section for the pair charmonium production in the kinematical region of the LHCb experiment (the region of rapidities $2 < y < 4.5$): 3.1 nb [11], 4.6 nb ($p_t > 3$ GeV) [15]. The theoretical uncertainty remains sufficiently large. To the present the calculations of the pair charmonium production in pp interaction were carried out in the leading order of QCD without inclusion of relativistic corrections. In addition to these permanent theoretical errors known from the experience of e^+e^- annihilation, we have in this task new specific uncertainty caused by parton distribution functions at small x values because the gluon contribution from the region of small x is dominant. There appears also another uncertainty related with the double parton interaction [16]. In this work we study one aspect of the improvement of the previous calculations connected with the account of relativistic corrections. Using the methods of the relativistic quark model [8,17–19], we perform new calculation of the cross section $\sigma(pp \to J/\psi J/\psi + X)$ accounting for relativistic corrections to the production amplitude and the bound state wave functions of heavy c quarks. So, the aim of this study consists in the relativistic description of the pair charmonium production at hadron colliders. It is important to note that the interest to the inclusive reactions $p + p \to 2J/\psi + X$, $pN \to 2J/\psi + X$ is not limited only by the investigation of the production mechanism. The study of the quarkonium production in the nuclear matter leads to new data about QCD at high density and temperature [20,21].

There exist different mechanisms for the pair charmonium production in pp collisions. At the collider energies, double quarkonium production occurs through the gluon-gluon interaction channel. In the color singlet model a pair of quark-antiquark ($c\bar{c}$) is created at short distances in color singlet state. Then it evolves nonperturbatively into an observed meson J/ψ. At small transverse momenta and small invariant masses of the J/ψ pair, the color singlet mechanism gives the main contribution to the cross section. Another possibility is to create a pair ($c\bar{c}$) with different color and angular momentum from that of the final meson. Then the color octet pair evolves to the color singlet charmonium emitting soft gluons. This color octet mechanism plays significant role in the region of high transverse momentum. Among the large number of Feynman diagrams describing the production of a pair of J/ψ, it has been separated into a class in which the J/ψ pair production is related with the double gluon fragmentation. In this study we analyze the total set of the production amplitudes in the color singlet model making primary emphasis upon relativistic effects.

II. GENERAL FORMALISM

The differential cross section $d\sigma$ for the inclusive double charmonium production in proton-proton

interaction can be presented in the form of the convolution of partonic cross section $d\sigma(gg \to J/\psi J/\psi)$ with the parton distribution functions in the initial protons [1,2]:

$$
\begin{aligned}
d\sigma[p &+ p \to J/\psi + J/\psi + X] \\
&= \int dx_1 dx_2 f_{g/p}(x_1, \mu) f_{g/p}(x_2, \mu) d\sigma[gg \to J/\psi J/\psi],
\end{aligned}
\tag{2}
$$

where $f_{g/p}(x, \mu)$ is the partonic distribution function for the gluon in the proton, $x_{1,2}$ are longitudinal momentum fractions of gluons, and μ is the factorization scale. Neglecting the proton mass and taking the c.m. reference frame of initial protons with the beam along the z axis, we can present the gluon on mass-shell momenta $k_{1,2} = x_{1,2} \frac{\sqrt{S}}{2}(1, 0, 0, \pm 1)$. $\sqrt{S}$ is the center-of-mass energy in proton-proton collision.

In the quasipotential approach the invariant transition amplitude for the gluonic subprocess $g + g \to J/\psi + J/\psi$ can be presented as a convolution of a perturbative production amplitude of two c-quark and $\bar{c}$-antiquark pairs $\mathcal{T}(p_1, p_2; q_1, q_2)$ and the quasipotential wave functions of the final mesons $\Psi_{J/\psi}$ [8]:

$$
\begin{aligned}
\mathcal{M}[gg &\to J/\psi J/\psi](k_1, k_2, P, Q) \\
&= \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\mathbf{q}}{(2\pi)^3} \bar{\Psi}(p, P) \bar{\Psi}(q, Q) \otimes \mathcal{T}(p_1, p_2; q_1, q_2),
\end{aligned}
\tag{3}
$$

where p_1 and p_2 are four-momenta of c quark and $\bar{c}$ antiquark in the pair forming the first J/ψ particle, and q_2 and q_1 are appropriate momenta for quark and antiquark in the second meson J/ψ. They are defined in terms of total momenta $P(Q)$ and relative momenta $p(q)$ as follows:

$$
\begin{aligned}
p_{1,2} &= \frac{1}{2}P \pm p, & (pP) &= 0; \\
q_{1,2} &= \frac{1}{2}Q \pm q, & (qQ) &= 0.
\end{aligned}
\tag{4}
$$

In Eq. (3) we integrate over the relative three-momenta of quarks and antiquarks in the final state. The systematic account of all terms depending on the relative quark momenta p and q in (2) is important for increasing the accuracy of the calculation. $p = L_P(0, \mathbf{p})$ and $q = L_Q(0, \mathbf{q})$ are the relative four-momenta obtained by the Lorentz transformation of four-vectors $(0, \mathbf{p})$ and $(0, \mathbf{q})$ to the reference frames moving with the four-momenta P and Q. The parton-level differential cross section for $g + g \to J/\psi + J/\psi$ is expressed further through the Mandelstam variables s, t, and u:

$$s = (k_1 + k_2)^2 = (P + Q)^2 = x_1 x_2 S,$$

$$t = (P - k_1)^2 = (Q - k_2)^2$$

$$= M^2 - x_1 \sqrt{S}(P_0 - |\mathbf{P}| \cos\phi)$$

$$= M^2 - x_1 x_2 S + x_2 \sqrt{S}(P_0 + |\mathbf{P}| \cos\phi),$$

$$u = (P - k_2)^2 = (Q - k_1)^2$$

$$= M^2 - x_2 \sqrt{S}(P_0 + |\mathbf{P}| \cos\phi)$$

$$= M^2 - x_1 x_2 S + x_1 \sqrt{S}(P_0 - |\mathbf{P}| \cos\phi), \tag{5}$$

where M is the charmonium mass, and ϕ is the angle between $\mathbf{P}$ and the z axis. The transverse momentum P_T of J/ψ and its energy P_0 can be written as

$$P_T^2 = |\mathbf{P}|^2 \sin^2\phi = -t - \frac{(M^2 - t)^2}{x_1 x_2 S},$$

$$P_0 = \frac{x_1 x_2 \sqrt{S}}{x_1 + x_2} + \frac{x_1 - x_2}{x_1 + x_2}|\mathbf{P}| \cos\phi. \tag{6}$$

At leading order of perturbation theory in strong coupling constant α_s, there are 31 Feynman diagrams contributing to the amplitude of pair J/ψ production due to gluon fusion. The typical diagrams from this set are presented in Fig. 1. For the completeness, we show in Figs. 2 and 3 also the Feynman diagrams from two other subsets containing 5 and 8 Feynman diagrams which do not contribute to the production amplitude. Any Feynman amplitude shown in Fig. 2 has zero contribution because its color factor is equal to zero. The sum of four diagrams from the subset in Fig. 3 is equal zero with the account of relativistic corrections studied in this work. So, further we consider only the 31 Feynman amplitudes from Fig. 1.

Let us consider, for example, the transformation of the first amplitude in Fig. 1 which takes the form in the Feynman gauge:
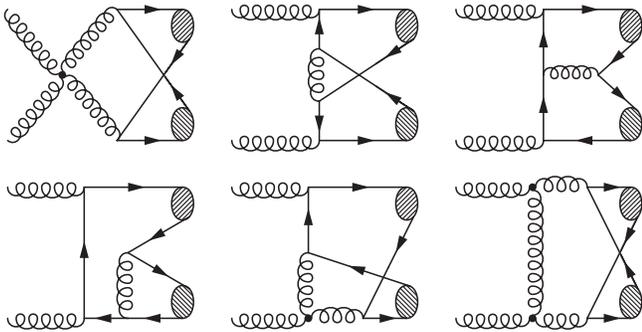

FIG. 1.   The typical LO diagrams contributing to the partonic process $g + g \rightarrow J/\psi + J/\psi$. The others can be obtained by reversing the quark lines or interchanging the initial gluons.
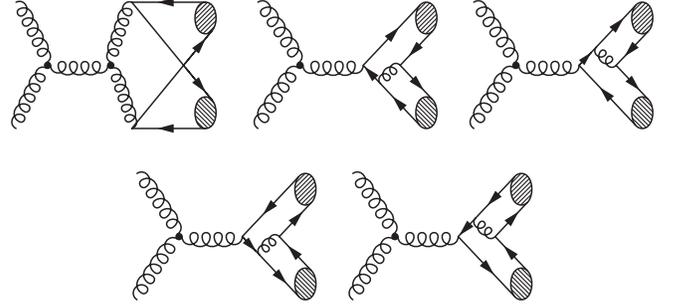

FIG. 2.   The subset of LO diagrams which give zero contribution to $g + g \rightarrow J/\psi + J/\psi$ because their color factor is equal zero.

$$\mathcal{T}_1^{ab}(p_1, p_2; q_1, q_2)$$

$$= 8\pi^2 \alpha_s^2 \delta^{ab} \varepsilon_1^\lambda(k_1) \varepsilon_2^\mu(k_2)$$

$$\times \frac{2g_{\lambda\mu}g_{\nu\sigma} - g_{\lambda\sigma}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\sigma}}{(\frac{P}{2} + \frac{Q}{2} + p + q)^2(\frac{P}{2} + \frac{Q}{2} - p - q)^2}$$

$$\times [\bar{u}(p_1)\gamma^\sigma v(q_1)][\bar{u}(q_2)\gamma^\nu v(p_2)], \tag{7}$$

where $\varepsilon_1(k_1)$ and $\varepsilon_2(k_2)$ are the polarization vectors of initial gluons. The amplitude (3) should be convoluted with two wave functions of J/ψ mesons taking in the reference frame moving with four momenta P and Q. The transformation law of the bound state wave function from the rest frame to the moving one with four momentum P was derived in the Bethe-Salpeter approach in Ref. [22] and in the quasipotential method in Ref. [23]. We use the last one and write the necessary transformation as follows:

$$\Psi_P^{\rho\omega}(\mathbf{p}) = D_1^{1/2,\rho\alpha}(R_{L_P}^W)D_2^{1/2,\omega\beta}(R_{L_P}^W)\Psi_0^{\alpha\beta}(\mathbf{p}),$$

$$\bar{\Psi}_P^{\lambda\sigma}(\mathbf{p}) = \bar{\Psi}_0^{\varepsilon\tau}(\mathbf{p})D_1^{+1/2,\varepsilon\lambda}(R_{L_P}^W)D_2^{+1/2,\tau\sigma}(R_{L_P}^W), \tag{8}$$

where R^W is the Wigner rotation, L_P is the Lorentz boost from the meson rest frame to a moving one, and the rotation matrix $D^{1/2}(R)$ is defined by
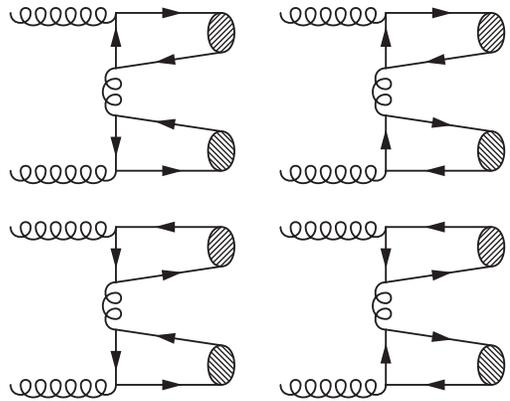

FIG. 3.   The sum of LO diagrams from this subset gives zero contribution to $g + g \rightarrow J/\psi + J/\psi$ with the account of relativistic corrections.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D_{1,2}^{1/2}(R_{L_P}^W) = S^{-1}(\mathbf{p}_{1,2})S(\mathbf{P})S(\mathbf{p}), \tag{9}$$

where the explicit form for the Lorentz transformation matrix of the four-spinor is

$$S(\mathbf{p}) = \sqrt{\frac{\epsilon(p) + m}{2m}}\left(1 + \frac{(\boldsymbol{\alpha}\mathbf{p})}{\epsilon(p) + m}\right). \tag{10}$$

Omitting a number of transformations which can be performed with (7) in (3) as in Refs. [8,18], we write the contribution to the production amplitude in the form:

$$\mathcal{M}_1^{ab}(k_1, k_2, P, Q) = 2M\delta^{ab}\pi^2\alpha_s^2 \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\bar{\Psi}_0^{J/\psi}(\mathbf{p})}{\left[\frac{\epsilon(p)}{m}\frac{\epsilon(p)+m}{2m}\right]} \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{\bar{\Psi}_0^{J/\psi}(\mathbf{q})}{\left[\frac{\epsilon(q)}{m}\frac{\epsilon(q)+m}{2m}\right]}$$

$$\times \mathrm{Tr}\Bigg\{\left[\frac{\hat{v}_1 - 1}{2} + \hat{v}_1\frac{\mathbf{p}^2}{2m(\epsilon(p)+m)} - \frac{\hat{p}}{2m}\right]\hat{\varepsilon}_P^*(\hat{v}_1 + 1)\left[\frac{\hat{v}_1 + 1}{2} + \hat{v}_1\frac{\mathbf{p}^2}{2m(\epsilon(p)+m)} + \frac{\hat{p}}{2m}\right]\gamma^\sigma$$

$$\times \left[\frac{\hat{v}_2 - 1}{2} + \hat{v}_2\frac{\mathbf{q}^2}{2m(\epsilon(q)+m)} + \frac{\hat{q}}{2m}\right]\hat{\varepsilon}_Q^*(\hat{v}_2 + 1)\left[\frac{\hat{v}_2 + 1}{2} + \hat{v}_2\frac{\mathbf{q}^2}{2m(\epsilon(q)+m)} - \frac{\hat{q}}{2m}\right]\gamma^\nu\Bigg\}$$

$$\times \varepsilon_1^\lambda(k_1)\varepsilon_2^\mu(k_2)\frac{2g_{\lambda\mu}g_{\nu\sigma} - g_{\lambda\sigma}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\sigma}}{(\frac{P}{2} + \frac{Q}{2} + p + q)^2(\frac{P}{2} + \frac{Q}{2} - p - q)^2}, \tag{11}$$

where $v_1 = P/M$, $v_2 = Q/M$; $\varepsilon_{P,Q}$ are polarization vectors of final J/ψ mesons with $\varepsilon_P \cdot P = 0$ and $\varepsilon_Q \cdot Q = 0$; $\epsilon(p) = \sqrt{m^2 + \mathbf{p}^2}$, $M = 2m$ is the mass of J/ψ meson. The hat symbol means contraction of the four-vector with the Dirac gamma matrices. The spin projectors $v(0)\bar{u}(0) = \hat{\varepsilon}^*(1 + \gamma_0)/(2\sqrt{2})$ corresponding to J/ψ mesons are introduced as well as projectors $\delta_{ij}/\sqrt{3}$ onto a color singlet states. We explicitly extracted in (11) the

normalization factors $\sqrt{2M}$ of quasipotential bound state wave functions.

The same transformations can be carried out with all 31 Feynman amplitudes in Fig. 1. In view of the large volume of calculations we have used the package FEYNARTS [24] for the system MATHEMATICA and FORM [25]. To make the entry of final amplitude more compact, we introduce a number of vertex functions Γ_i. Then we can present the total amplitude (3) in the form:

$$\mathcal{M}[gg \to J/\psi J/\psi](k_1, k_2, P, Q) = \frac{1}{9}M\pi^2\alpha_s^2 \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\mathbf{q}}{(2\pi)^3} \mathrm{Tr}\mathfrak{M},$$

$$\mathfrak{M} = \mathcal{D}_1\gamma_\beta\bar{\Psi}_{q,Q}\Gamma_1^\beta\bar{\Psi}_{p,P}\hat{\varepsilon}_2\frac{m - \hat{k}_2 + \hat{p}_1}{(k_2 - p_1)^2 - m^2} + \mathcal{D}_2\gamma_\beta\bar{\Psi}_{q,Q}\Gamma_2^\beta\bar{\Psi}_{p,P}\hat{\varepsilon}_1\frac{m - \hat{k}_1 + \hat{p}_1}{(k_1 - p_1)^2 - m^2} + \mathcal{D}_3\bar{\Psi}_{q,Q}\Gamma_3^\beta\bar{\Psi}_{p,P}\gamma_\beta$$

$$+ \mathcal{D}_4\bar{\Psi}_{p,P}\Gamma_4^\beta\bar{\Psi}_{q,Q}\gamma_\beta + \mathcal{D}_1\bar{\Psi}_{q,Q}\Gamma_5^\beta\bar{\Psi}_{p,P}\gamma_\beta\frac{m + \hat{k}_2 - \hat{q}_1}{(k_2 - q_1)^2 - m^2}\hat{\varepsilon}_2 + \mathcal{D}_2\bar{\Psi}_{q,Q}\Gamma_6^\beta\bar{\Psi}_{p,P}\gamma_\beta\frac{m + \hat{k}_1 - \hat{q}_1}{(k_1 - q_1)^2 - m^2}\hat{\varepsilon}_1, \tag{12}$$

where inverse denominators of gluon propagators are defined as $\mathcal{D}_{1,2}^{-1} = (k_2 - p_{1,2} - q_{1,2})^2$ and $\mathcal{D}_{3,4}^{-1} = (p_{1,2} + q_{1,2})^2$. Perturbative amplitude $\mathcal{T}(p_1, p_2; q_1, q_2)$ in Eq. (3) describes production of two c-quark and $\bar{c}$-antiquark pairs with the momenta $p_{1,2}$ and $q_{2,1}$, respectively. The formation of observable bound states from quark–antiquark pairs is determined in the quark model by the quasipotential wave functions $\Psi_{J/\psi}(p, P)$ and $\Psi_{J/\psi}(q, Q)$. These wave functions are calculated initially in the meson rest frame and then are transformed to the reference frames moving with the four-momenta P and Q. As a result we obtain in (12) the following expressions for relativistic bound state wave functions which determine the transition of heavy quarks to the bound state:

$$\bar{\Psi}_{p,P} = \frac{\bar{\Psi}_0^{J/\psi}(\mathbf{p})}{\left[\frac{\epsilon(p)}{m}\frac{\epsilon(p)+m}{2m}\right]}\left[\frac{\hat{v}_1 - 1}{2} + \hat{v}_1\frac{\mathbf{p}^2}{2m(\epsilon(p)+m)} - \frac{\hat{p}}{2m}\right]$$

$$\times \hat{\varepsilon}_P^*(P, S_z)(1 + \hat{v}_1)$$

$$\times \left[\frac{\hat{v}_1 + 1}{2} + \hat{v}_1\frac{\mathbf{p}^2}{2m(\epsilon(p)+m)} + \frac{\hat{p}}{2m}\right], \tag{13}$$

$$\bar{\Psi}_{q,Q} = \frac{\bar{\Psi}_0^{J/\psi}(\mathbf{q})}{\left[\frac{\epsilon(q)}{m}\frac{\epsilon(q)+m}{2m}\right]}\left[\frac{\hat{v}_2 - 1}{2} + \hat{v}_2\frac{\mathbf{q}^2}{2m(\epsilon(q)+m)} + \frac{\hat{q}}{2m}\right]$$

$$\times \hat{\varepsilon}_Q^*(Q, S_z)(1 + \hat{v}_2)$$

$$\times \left[\frac{\hat{v}_2 + 1}{2} + \hat{v}_2\frac{\mathbf{q}^2}{2m(\epsilon(q)+m)} - \frac{\hat{q}}{2m}\right]. \tag{14}$$

Leading order vertex functions in (12) are calculated in the Feynman gauge and can be presented as follows:

$$\Gamma_1^\beta = \hat{\varepsilon}_1 \frac{m - \hat{k}_1 + \hat{q}_2}{(k_1 - q_2)^2 - m^2} \gamma^\beta - 8\gamma^\beta \frac{m + \hat{k}_1 - \hat{p}_2}{(k_1 - p_2)^2 - m^2} \hat{\varepsilon}_1,$$

$$\Gamma_2^\beta = \hat{\varepsilon}_2 \frac{m - \hat{k}_2 + \hat{q}_2}{(k_2 - q_2)^2 - m^2} \gamma^\beta - 8\gamma^\beta \frac{m + \hat{k}_2 - \hat{p}_2}{(k_2 - p_2)^2 - m^2} \hat{\varepsilon}_2,$$

$$\Gamma_3^\beta = \hat{\varepsilon}_1 \frac{m - \hat{k}_1 + \hat{q}_2}{(k_1 - q_2)^2 - m^2} \left[\gamma^\beta \frac{m + \hat{k}_2 - \hat{p}_2}{(k_2 - p_2)^2 - m^2} \hat{\varepsilon}_2 - 8\hat{\varepsilon}_2 \frac{m - \hat{p}_1 - \hat{p}_2 - \hat{q}_1}{(p_1 + p_2 + q_1)^2 - m^2} \gamma^\beta \right]$$

$$+ \hat{\varepsilon}_2 \frac{m - \hat{k}_2 + \hat{q}_2}{(k_2 - q_2)^2 - m^2} \left[\gamma^\beta \frac{m + \hat{k}_1 - \hat{p}_2}{(k_1 - p_2)^2 - m^2} \hat{\varepsilon}_1 - 8\hat{\varepsilon}_1 \frac{m - \hat{p}_1 - \hat{p}_2 - \hat{q}_1}{(p_1 + p_2 + q_1)^2 - m^2} \gamma^\beta \right]$$

$$- 8\gamma^\beta \frac{m + \hat{p}_1 + \hat{q}_1 + \hat{q}_2}{(p_1 + q_1 + q_2)^2 - m^2} \left[\hat{\varepsilon}_2 \frac{m + \hat{k}_1 - \hat{p}_2}{(k_1 - p_2)^2 - m^2} \hat{\varepsilon}_1 + \hat{\varepsilon}_1 \frac{m + \hat{k}_2 - \hat{p}_2}{(k_2 - p_2)^2 - m^2} \hat{\varepsilon}_2 \right]$$

$$+ 18\gamma_\alpha \left[\mathcal{D}_1 \frac{m - \hat{k}_1 + \hat{q}_2}{(k_1 - q_2)^2 - m^2} \varepsilon_1^\alpha \gamma_\mu \mathfrak{C}_2^{\beta\mu}(p_1 + q_1) - \mathcal{D}_1 \frac{m + \hat{k}_1 - \hat{p}_2}{(k_1 - p_2)^2 - m^2} \hat{\varepsilon}_1 \mathfrak{C}_2^{\beta\alpha}(p_1 + q_1) \right.$$

$$\left. + \mathcal{D}_2 \frac{m - \hat{k}_2 + \hat{q}_2}{(k_2 - q_2)^2 - m^2} \varepsilon_2^\alpha \gamma_\mu \mathfrak{C}_1^{\beta\mu}(p_1 + q_1) - \mathcal{D}_2 \frac{m + \hat{k}_2 - \hat{p}_2}{(k_2 - p_2)^2 - m^2} \hat{\varepsilon}_2 \mathfrak{C}_1^{\beta\alpha}(p_1 + q_1) \right],$$

$$\Gamma_4^\beta = \hat{\varepsilon}_1 \frac{m - \hat{k}_1 + \hat{p}_1}{(k_1 - p_1)^2 - m^2} \left[\gamma^\beta \frac{m + \hat{k}_2 - \hat{q}_1}{(k_2 - q_1)^2 - m^2} \hat{\varepsilon}_2 - 8\hat{\varepsilon}_2 \frac{m - \hat{p}_2 - \hat{q}_1 - \hat{q}_2}{(p_2 + q_1 + q_2)^2 - m^2} \gamma^\beta \right]$$

$$+ \hat{\varepsilon}_2 \frac{m - \hat{k}_2 + \hat{p}_1}{(k_2 - p_1)^2 - m^2} \left[\gamma^\beta \frac{m + \hat{k}_1 - \hat{q}_1}{(k_1 - q_1)^2 - m^2} \hat{\varepsilon}_1 - 8\hat{\varepsilon}_1 \frac{m - \hat{p}_2 - \hat{q}_1 - \hat{q}_2}{(p_2 + q_1 + q_2)^2 - m^2} \gamma^\beta \right]$$

$$- 8\gamma^\beta \frac{m + \hat{p}_1 + \hat{p}_2 + \hat{q}_2}{(p_1 + p_2 + q_2)^2 - m^2} \left[\hat{\varepsilon}_2 \frac{m + \hat{k}_1 - \hat{q}_1}{(k_1 - q_1)^2 - m^2} \hat{\varepsilon}_1 + \hat{\varepsilon}_1 \frac{m + \hat{k}_2 - \hat{q}_1}{(k_2 - q_1)^2 - m^2} \hat{\varepsilon}_2 \right]$$

$$+ 18\gamma_\alpha \left[\mathcal{D}_2 \frac{m - \hat{k}_1 + \hat{p}_1}{(k_1 - p_1)^2 - m^2} \varepsilon_1^\alpha \gamma_\mu \mathfrak{C}_2^{\beta\mu}(p_2 + q_2) - \mathcal{D}_2 \frac{m + \hat{k}_1 - \hat{q}_1}{(k_1 - q_1)^2 - m^2} \hat{\varepsilon}_1 \mathfrak{C}_2^{\beta\alpha}(p_2 + q_2) \right.$$

$$\left. + \mathcal{D}_1 \frac{m - \hat{k}_2 + \hat{p}_1}{(k_2 - p_1)^2 - m^2} \varepsilon_2^\alpha \gamma_\mu \mathfrak{C}_1^{\beta\mu}(p_2 + q_2) - \mathcal{D}_1 \frac{m + \hat{k}_2 - \hat{q}_1}{(k_2 - q_1)^2 - m^2} \hat{\varepsilon}_2 \mathfrak{C}_1^{\beta\alpha}(p_2 + q_2) \right]$$

$$+ 18\mathcal{D}_3 \gamma_\nu [2\varepsilon_1 \varepsilon_2 g^{\nu\beta} - \varepsilon_1^\nu \varepsilon_2^\beta - \varepsilon_1^\beta \varepsilon_2^\nu + \mathcal{D}_2 \mathfrak{F}^{\nu\beta}(p_1 + q_1, p_2 + q_2) + \mathcal{D}_1 \mathfrak{F}^{\beta\nu}(p_2 + q_2, p_1 + q_1)],$$

$$\Gamma_5^\beta = \gamma^\beta \frac{m + \hat{k}_1 - \hat{p}_2}{(k_1 - p_2)^2 - m^2} \hat{\varepsilon}_1 - 8\hat{\varepsilon}_1 \frac{m - \hat{k}_1 + \hat{q}_2}{(k_1 - q_2)^2 - m^2} \gamma^\beta, \quad \Gamma_6^\beta = \gamma^\beta \frac{m + \hat{k}_2 - \hat{p}_2}{(k_2 - p_2)^2 - m^2} \hat{\varepsilon}_2 - 8\hat{\varepsilon}_2 \frac{m - \hat{k}_2 + \hat{q}_2}{(k_2 - q_2)^2 - m^2} \gamma^\beta, \quad (15)$$

where we introduce the following tensors:

$$\mathfrak{C}_{1,2}^{\alpha\beta}(x) = \frac{1}{2}(2x\varepsilon_{1,2}g^{\alpha\beta} - (k_{1,2}^\beta + x^\beta)\varepsilon_{1,2}^\alpha + (2k_{1,2}^\alpha - x^\alpha)\varepsilon_{1,2}^\beta),$$

$$\mathfrak{F}^{\alpha\beta}(x, y) = 4(x\varepsilon_1)(y\varepsilon_2)g^{\alpha\beta} + (k_1 + x)(k_2 + y)\varepsilon_1^\alpha \varepsilon_2^\beta + \varepsilon_1 \varepsilon_2 (2k_1^\alpha - x^\alpha)(2k_2^\beta - y^\beta) + 2x(\varepsilon_1 \varepsilon_2^\alpha - \varepsilon_2 \varepsilon_1^\alpha)(2k_2^\beta - y^\beta)$$

$$- 2y(\varepsilon_1 \varepsilon_2^\beta - \varepsilon_2 \varepsilon_1^\beta)(2k_1^\alpha - x^\alpha) - x\varepsilon_1(x^\alpha + 4y^\alpha)\varepsilon_2^\beta - y\varepsilon_2(4x^\beta + y^\beta)\varepsilon_1^\alpha. \quad (16)$$

Our expressions for the amplitude (12) and vertex functions (15) contain relative momenta p and q in exact form. In order to take into account relativistic corrections of the second order in p and q we expand all inverse denominators of the quark and gluon propagators. Such expansions look as follows:

$$\frac{1}{(p_1 + q_1)^2} = \frac{4}{s} - \frac{16}{s^2}[(p + q)^2 + pQ + qP] + \cdots, \qquad \frac{1}{(k_2 - q_2)^2 - m^2} = \frac{2}{t - M^2} - \frac{4}{(t - M^2)^2}[q^2 + 2qk_2] + \cdots, \quad (17)$$

where the Mandelstam variables for the gluonic subprocess s and t are defined in (5). There are 16 quark and gluon propagators in the amplitude (12) which have to be expanded in the same way as in (17). All denominators of these propagators in the nonrelativistic limit take one of the following forms: $(t - M^2)/2$, $(M^2 - s - t)/2$, $\pm s/4$ or $s/2$. Then, the inequalities

$$4M^2 \le s, \qquad \left| t + \frac{s}{2} - M^2 \right| \le \frac{s}{2}\sqrt{1 - \frac{4M^2}{s}} \tag{18}$$

mean that in the case of the most unfavorable values of the variables $x_{1,2}$ and t, we can roughly estimate expansion parameters in (17) as $2p^2/M^2$ and $2q^2/M^2$. Preserving in the expanded amplitude terms up to the second order in the relative momenta p and q, we can perform angular integration using the following relations for $\mathcal{S}$-wave charmonium:

$$\int \frac{\Psi_0^{\mathcal{S}}(\mathbf{p})}{\left[\frac{\epsilon(p)}{m}\frac{\epsilon(p)+m}{2m}\right]}\frac{d\mathbf{p}}{(2\pi)^3} = \frac{1}{\sqrt{2}\pi}\int_0^\infty \frac{p^2 R_{\mathcal{S}}(p)}{\left[\frac{\epsilon(p)}{m}\frac{\epsilon(p)+m}{2m}\right]}dp,$$

$$\int p_\mu p_\nu \frac{\Psi_0^{\mathcal{S}}(\mathbf{p})}{\left[\frac{\epsilon(p)}{m}\frac{\epsilon(p)+m}{2m}\right]}\frac{d\mathbf{p}}{(2\pi)^3} = -\frac{1}{3\sqrt{2}\pi}(g_{\mu\nu} - v_{1\mu}v_{1\nu})\int_0^\infty \frac{p^4 R_{\mathcal{S}}(p)}{\left[\frac{\epsilon(p)}{m}\frac{\epsilon(p)+m}{2m}\right]}dp, \tag{19}$$

where $R_{\mathcal{S}}(p)$ is the radial charmonium wave function.

To illustrate the described transformations, we present here the result of the calculation of the first amplitude in Fig. 1:

$$\mathcal{M}_1^{ab} = \frac{32\alpha_s^2 \delta^{ab}}{9ms^4}\int \frac{m + \epsilon(p)}{2\epsilon(p)}R(p)p^2 \int \frac{m + \epsilon(q)}{2\epsilon(q)}R(q)q^2\{3s^2[\varepsilon_1 \cdot \varepsilon_2(s\varepsilon_P^* \cdot \varepsilon_Q^* - 2\varepsilon_P^* \cdot Q\varepsilon_Q^* \cdot P)$$
$$- 2\varepsilon_P^* \cdot \varepsilon_Q^*(\varepsilon_1 \cdot P\varepsilon_2 \cdot Q + \varepsilon_1 \cdot Q\varepsilon_2 \cdot P) + 2\varepsilon_P^* \cdot Q(\varepsilon_1 \cdot P\varepsilon_2 \cdot \varepsilon_Q^* + \varepsilon_1 \cdot \varepsilon_Q^*\varepsilon_2 \cdot P) - \varepsilon_1 \cdot \varepsilon_P^* \times (s\varepsilon_2 \cdot \varepsilon_Q^* - 2\varepsilon_2 \cdot Q\varepsilon_Q^* \cdot P)$$
$$- \varepsilon_2 \cdot \varepsilon_P^*(s\varepsilon_1 \cdot \varepsilon_Q^* - 2\varepsilon_1 \cdot Q\varepsilon_Q^* \cdot P)](3(1 - c_p - c_q - c_p^2 - c_q^2) + c_p c_q \times (67 + 3c_p + 3c_q) + 3c_p^2 c_q^2)$$
$$- 64m^2 s[\varepsilon_P^* \cdot Q(\varepsilon_1 \cdot P\varepsilon_2 \cdot \varepsilon_Q^* + \varepsilon_1 \cdot \varepsilon_Q^*\varepsilon_2 \cdot P) + \varepsilon_Q^* \cdot P(\varepsilon_1 \cdot Q\varepsilon_2 \cdot \varepsilon_P^* + \varepsilon_1 \cdot \varepsilon_P^*\varepsilon_2 \cdot Q)](3(c_p + c_q)$$
$$+ c_p c_q(194 - 3c_p - 3c_q)) + 16m^2 s^2[\varepsilon_1 \cdot \varepsilon_P^*\varepsilon_2 \cdot \varepsilon_Q^* + \varepsilon_1 \cdot \varepsilon_Q^*\varepsilon_2 \cdot \varepsilon_P^*] \times (9(c_p + c_q) + c_p c_q(380 - 9c_p - 9c_q))$$
$$+ 192m^2 s\varepsilon_P^* \cdot \varepsilon_Q^*[\varepsilon_1 \cdot P\varepsilon_2 \cdot Q + \varepsilon_1 \cdot Q\varepsilon_2 \cdot P](c_p + c_q + c_p c_q(62 - c_p - c_q))$$
$$+ 16m^2 s\varepsilon_1 \cdot \varepsilon_2(\varepsilon_P^* \cdot \varepsilon_Q^*[32m^2(3(c_p + c_q) + c_p c_q(329 - 3c_p - 3c_q)) - 3s \times (3 - 2c_p - 2c_q - 3c_p^2 - 3c_q^2)$$
$$+ sc_p c_q(613 + 6c_p + 6c_q) + 9sc_p^2 c_q^2] + 4\varepsilon_P^* \cdot Q\varepsilon_Q^* \cdot P[3(c_p + c_q) + c_p c_q(202 - 3c_p - 3c_q)])$$
$$+ 512m^2 c_p c_q(2\varepsilon_P^* \cdot Q\varepsilon_Q^* \cdot P(\varepsilon_1 \cdot Q\varepsilon_2 \cdot P + \varepsilon_1 \cdot P\varepsilon_2 \cdot Q) - 2m^2 \varepsilon_P^* \cdot \varepsilon_Q^* \times [1064m^2 \varepsilon_1 \cdot \varepsilon_2 + \varepsilon_1 \cdot P(125\varepsilon_2 \cdot Q - 8\varepsilon_2 \cdot P)$$
$$+ \varepsilon_1 \cdot Q(125\varepsilon_2 \cdot P - 8\varepsilon_2 \cdot Q)] - m^2[266\varepsilon_1 \cdot \varepsilon_2\varepsilon_P^* \cdot Q\varepsilon_Q^* \cdot P + \varepsilon_1 \cdot \varepsilon_P^*(131s\varepsilon_2 \cdot \varepsilon_Q^* + 2\varepsilon_Q^* \cdot P(4\varepsilon_2 \cdot P - 129\varepsilon_2 \cdot Q))$$
$$+ \varepsilon_1 \cdot \varepsilon_Q^* \times (131s\varepsilon_2 \cdot \varepsilon_P^* + 2\varepsilon_P^* \cdot Q(4\varepsilon_2 \cdot Q - 129\varepsilon_2 \cdot P)) + 2\varepsilon_1 \cdot P(4\varepsilon_2 \cdot \varepsilon_P^*\varepsilon_Q^* \cdot P - 129\varepsilon_2 \cdot \varepsilon_Q^*\varepsilon_P^* \cdot Q)$$
$$+ 2\varepsilon_1 \cdot Q(4\varepsilon_2 \cdot \varepsilon_Q^*\varepsilon_P^* \cdot Q - 129\varepsilon_2 \cdot \varepsilon_P^*\varepsilon_Q^* \cdot P)])\}dpdq, \tag{20}$$

where we introduce the relativistic parameter $c_p = \frac{m - \epsilon(p)}{m + \epsilon(p)}$. Extracting relativistic factors $(\epsilon + m)/2\epsilon$ in the integrals over both relative momenta $\mathbf{p}$ and $\mathbf{q}$, we observe that the amplitude $\mathcal{M}_1^{ab}$ is a power-like expansion in relativistic parameters c_p and c_q. Due to the presence of four different polarization vectors, which correspond to incoming gluons and outcoming J/ψ particles, the result (20) appears to be sufficiently lengthy. We have also obtained analogous expressions for remained 30 diagrams, but due to the bulkiness of the total amplitude they are not presented here.

To calculate the cross section, we have to sum the squared modulus of the amplitude upon all polarizations using the following relations for final J/ψ mesons and initial gluons correspondingly:

$$\sum_\lambda \varepsilon_P^\mu \varepsilon_P^{*\nu} = v_1^\mu v_1^\nu - g^{\mu\nu},$$

$$\sum_\lambda \varepsilon_Q^\mu \varepsilon_Q^{*\nu} = v_2^\mu v_2^\nu - g^{\mu\nu}, \tag{21}$$

$$\sum_\lambda \varepsilon_{1,2}^\mu \varepsilon_{1,2}^{*\nu} = \frac{k_1^\mu k_2^\nu + k_1^\nu k_2^\mu}{k_1 \cdot k_2} - g^{\mu\nu}.$$

We find it useful to present the differential cross section for double charmonium production in the proton-proton interaction in the following form:

$$\frac{d\sigma}{dt}[gg \to J/\psi J/\psi](t, s)$$
$$= \frac{\pi m^2 \alpha_s^4}{2304 s^2}|\tilde{R}(0)|^4 \sum_{i=0}^3 \omega_i F^{(i)}(t, s), \tag{22}$$

where the function $F^{(0)}$ describes the leading-order (LO) contribution. It coincides with the nonrelativistic analytical expression for the cross section obtained for the studied process in Refs. [11,12,15,26]. The functions $F^{(i)}$ ($i = 1, 2, 3$) describe relativistic corrections. Explicit expressions for all functions $F^{(i)}$ entering the cross section (22) are written in the Appendix. A number of specific parameters ω_i appeared in (22) are defined as

$$\omega_0 = 1, \qquad \omega_1 = \frac{I_1}{I_0}, \qquad \omega_2 = \frac{I_2}{I_0}, \qquad \omega_3 = \omega_1^2. \tag{23}$$

They comprise the nonperturbative parameters in the relativistic quark model which determine the transition of quarks and antiquarks into the bound states. The parameter $\tilde{R}(0)$, which represents the relativistic generalization of radial wave function at the origin, is defined by the formula:

$$\tilde{R}(0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{m + \epsilon(p)}{2\epsilon(p)} R(p) p^2 dp. \tag{24}$$

The parameters ω_i are determined by integrals containing the bound state wave function in the following form:

$$I_0 = \int_0^\infty \frac{m + \epsilon(p)}{2\epsilon(p)} R(p) p^2 dp,$$
$$I_{1,2} = \int_0^m \frac{m + \epsilon(p)}{2\epsilon(p)} \left(\frac{m - \epsilon(p)}{m + \epsilon(p)}\right)^{1,2} R(p) p^2 dp. \tag{25}$$

Our basic relations for the cross section (22) evidently show that there exists another source of relativistic corrections connected with the charmonium wave functions. For their calculation with the desired accuracy, we suppose that the dynamics of a $c\bar{c}$ pair is determined by the QCD generalization of the standard Breit Hamiltonian [27], which in the c.m. reference frame can be written as

$$H = H_0 + \Delta U_1 + \Delta U_2 + \Delta U_3,$$
$$H_0 = 2\sqrt{\mathbf{p}^2 + m^2} - 2m - \frac{C_F \tilde{\alpha}_s}{r} + Ar + B, \tag{26}$$

$$\Delta U_1(r) = -\frac{C_F \alpha_s^2}{4\pi r}[2\beta_0 \ln(\mu r) + a_1 + 2\gamma_E \beta_0], \qquad a_1 = \frac{31}{3} - \frac{10}{9}n_f, \qquad \beta_0 = 11 - \frac{2}{3}n_f,$$

$$\Delta U_2(r) = -\frac{C_F \alpha_s}{2m^2 r}\left[\mathbf{p}^2 + \frac{\mathbf{r(rp)p}}{r^2}\right] + \frac{\pi C_F \alpha_s}{m^2}\delta(\mathbf{r}) + \frac{3C_F \alpha_s}{2m^2 r^3}(\mathbf{SL}) - \frac{C_F \alpha_s}{2m^2}\left[\frac{\mathbf{S}^2}{r^3} - 3\frac{(\mathbf{Sr})^2}{r^5} - \frac{4\pi}{3}(2\mathbf{S}^2 - 3)\delta(\mathbf{r})\right] - \frac{C_A C_F \alpha_s^2}{2m r^2},$$

$$\Delta U_3(r) = f_V\left[\frac{A}{2m^2 r}\left(1 + \frac{8}{3}\mathbf{S}_1\mathbf{S}_2\right) + \frac{3A}{2m^2 r}\mathbf{LS} + \frac{A}{3m^2 r}\left(\frac{3}{r^2}(\mathbf{S}_1\mathbf{r})(\mathbf{S}_2\mathbf{r}) - \mathbf{S}_1\mathbf{S}_2\right)\right] - (1 - f_V)\frac{A}{2m^2 r}\mathbf{LS}, \tag{27}$$

where $\mathbf{L} = [\mathbf{r} \times \mathbf{p}]$, $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$, n_f is a number of flavors, $C_A = 3$ and $C_F = 4/3$ are the color factors of the SU(3) color group, and γ_E is the Euler constant. The parameter f_V of vector-exchange confining potential was set to be $f_V = 0.7$. The mass of heavy c quark in our model is equal to $m = 1.55$ GeV. For the dependence of the QCD coupling constant $\tilde{\alpha}_s(\mu)$ on the renormalization point μ in the pure Coulomb term in (26), we use the three-loop result [28]

$$\tilde{\alpha}_s(\mu) = \frac{4\pi}{\beta_0 \mathcal{L}} - \frac{4\pi b_1 \mathcal{L}}{(\beta_0 \mathcal{L})^2} + \frac{4\pi}{(\beta_0 \mathcal{L})^3}$$
$$\times [b_1^2(\ln^2 \mathcal{L} - \ln \mathcal{L} - 1) + b_2],$$
$$\mathcal{L} = \ln(\mu^2/\Lambda^2), \qquad b_1 = \frac{64}{9}, \qquad b_2 = \frac{3863}{54}; \tag{28}$$

whereas, in all other terms of the Hamiltonian (27) we use the one-loop approximation for the coupling constant α_s. The typical momentum transfer scale in a quarkonium is of order of the quark mass, so we

choose the renormalization scale $\mu = m = 1.55$ GeV and $\Lambda = 0.168$ GeV, which gives $\alpha_s = 0.314$ for the charmonium states. The parameters of the linear potential $A = 0.18$ GeV^2 and $B = -0.16$ GeV have the usual values of quark models. Starting with the Hamiltonian (26), we construct the effective potential model based on the Schrödinger equation and find its numerical solutions for J/ψ meson. Additional details of this model are contained in Appendix C of Ref. [18]. Note that the obtained charmonium wave function is strongly decreasing in the region of relativistic momenta $p \gtrsim m$. Our numerical evaluation gives the following values of S-wave charmonium masses: $M_{J/\psi}^{\text{th}} = 3.072$ GeV and $M_{\eta_c}^{\text{th}} = 2.988$ GeV, which lie within the reasonable accuracy in the comparison with their experimentally measured results [29] $M_{J/\psi}^{\text{exp}} = 3.097$ GeV and $M_{\eta_c}^{\text{exp}} = 2.980$ GeV.

III. NUMERICAL RESULTS AND DISCUSSION

In this work we investigate the role of relativistic effects in the production of a pair of charmonium mesons

in proton-proton interaction in the relativistic quark model. We have studied only the order α_s^4 parton process of gluon-gluon fusion in the color singlet model. At the calculation of the production amplitude (12), we keep relativistic corrections of two types. The first type is determined by several functions depending on the relative quark momenta **p** and **q** arising from the gluon propagators, the quark propagators, and relativistic meson wave functions. The second type of corrections originates from the perturbative and nonperturbative treatment of the quark-antiquark interaction operator (26) which leads to the essential modification of non-relativistic wave functions.

For the calculation of relativistic corrections in the bound state wave functions $\Psi_0(\mathbf{p})$, we take the Breit potential (27) and construct the effective potential model as in Refs. [18,30] by means of the rationalization of the kinetic energy operator. Using the program of numerical solution of the Schrödinger equation [31], we obtain the following values of all relativistic parameters entering the cross section (22): $\tilde{R}(0) = 0.57$ GeV$^{\frac{3}{2}}$, $\omega_1 = -0.051$, and $\omega_2 = 0.0047$.

As it is evident from Eq. (25), our definition of integral parameters $I_{1,2}$ describing relativistic contributions from the production amplitude contains the cutoff at relativistic momentum of order m. In spite of the convergence of integrals $I_{1,2}$, our relativistic model can not provide a reliable calculation of the wave functions in the region of relativistic momenta $p \gtrsim m$. So, we introduce a cutoff in (25) in order to avoid possible errors caused by the mentioned uncertainty. It is obviously that in the quark model we can calculate a number of nonperturbative parameters (23) only with certain accuracy. The way of further improvements in the calculation is related in the first place with more accurate construction of the bound state wave function at relativistic momenta. In the approach of NRQCD we encounter analogous difficulties connected with the determination of numerous nonperturbative matrix elements [6].

Let us note also that the cross section (22) contains the fourth power of the modified wave function at the origin $\tilde{R}(0)$ and the strong coupling constant α_s. Thus, small changes of the bound state wave function can lead to substantial changes in final results for the cross section. The value $\tilde{R}(0)$ is calculated with sufficiently high accuracy with the parameters and potential (26) of the relativistic quark model. The parameter $|\tilde{R}(0)|^4$ undergoes essential decrease in comparison with nonrelativistic value. But other relativistic corrections connected with the functions $F^{(i)}$ ($i = 1, 2, 3$) in (22) have the opposite effect on the cross section value (22). Analytical expression of nonrelativistic contribution to the cross section which is determined by $F^{(0)}$ coincides with previous calculations in Refs. [11–13,15,26]. In the evaluation of α_s we set the renormalization scale to be the transverse

mass $\mu = m_T = \sqrt{4m^2 + P_T^2}$, which is the generally accepted choice. For the running coupling constant $\alpha_s(\mu)$, we use the LO result with the initial value $\alpha_s(\mu = M_z) = 0.118$.

The basic expression (2) for the calculation of the differential cross section contains the gluon distribution functions in the proton because the leading contribution comes from a gluon fusion process. When the energy of colliding beams increases, the initial parton momentum fraction x_i needed to produce heavy quarkonium decreases. It reaches the region in x where the number of the gluons becomes much larger than the number of quarks. The gluon distribution function determines the probability to find a gluon in the proton with some momentum fraction. There exists a number of the parametrizations for partonic distribution functions [32]. We use the gluon parton distribution functions from the set CTEQ5L as in Ref. [11]. The total numerical value of the cross section obtained from (2) at the center-of-mass energy $\sqrt{S} = 7$ TeV is equal to

$$\sigma_{\text{total}} = 9.6 \text{ nb.} \tag{29}$$

Due to law-x behavior of gluon distribution functions, the main contribution to the integral cross section (29) results from the region $x_{1,2} \sim 10^{-3}$. To be more precise, this region is determined by the condition: $7.8 \times 10^{-7} < x_1 x_2 < 7.8 \times 10^{-6}$. In the frequently used nonrelativistic limit when $\psi(0) = 0.21$ GeV$^{\frac{3}{2}}$ and $R(0) = 0.74$ GeV$^{\frac{3}{2}}$ the total value of the cross section amounts $\sigma_{\text{nonrel}} = 18.3$ nb and agrees with the calculation in Ref. [11]. In the non-relativistic limit of our quark model based on Eqs. (26) and (27) we obtain slightly greater values $R(0) = 0.79$ GeV$^{\frac{3}{2}}$ and $\sigma_{\text{nonrel}} = 23.1$ nb. To obtain (29) the factorization scale in the parton distribution functions, $f_{g/p}(x, \mu)$ is taken equal to the transverse mass, too: $\mu = m_T$.

To compare the results of our calculation with the measured value of the cross section in Ref. [10], it is necessary to write the differential cross section in terms of the rapidity $y_P = \frac{1}{2} \ln \frac{P_0 + P_\parallel}{P_0 - P_\parallel}$. The rapidities of outcoming charmonia with momenta P and Q can be obtained in the form

$$y_{P,Q} = \frac{1}{2} \ln \frac{x_1}{x_2} \pm \frac{1}{2} \ln \left[\frac{s}{M^2 - t} - 1 \right]. \tag{30}$$

The differential cross section $d\sigma/dy_P$ for the reaction $pp \to 2J/\psi + X$ is shown in Fig. 4. It is clear from this plot that relativistic effects strongly influence on the rapidity distribution of the final charmonium. In the LHCb experiment [10] the rapidity lies in the range $2 < y_{P,Q} < 4.5$, so we should integrate the differential cross section (2) over rapidities from such interval in order to obtain the value corresponding to the experiment
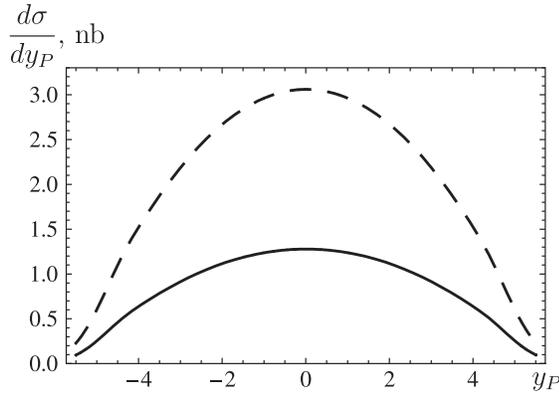
FIG. 4. The differential cross sections for $pp \to 2J/\psi + X$ at $\sqrt{S} = 7$ TeV as functions of rapidity y_P. Solid and dashed curves represent total and nonrelativistic results, respectively.

at the LHCb detector. Then we obtain at the center-of-mass energy $\sqrt{S} = 7$ TeV

$$\sigma^{\text{th}}(2 < y_{P,Q} < 4.5) = 1.6 \text{ nb}. \tag{31}$$

The result (31) is significantly smaller than the experimental value of the cross section (1). Different sources of relativistic corrections in (2) are differently directed. But we observe that combined action of all relativistic effects leads to essential decreasing of the production cross section. In this work we carry out the investigation only of one important source of corrections to the nonrelativistic cross section. Decreasing behavior of the cross section $\sigma(pp \to 2J/\psi + X)$ due to the account of relativistic contributions is noticeable clearly in spite of existing theoretical errors occurred in our calculation. In our analysis of the production amplitudes we correctly take into account relativistic contributions of order $O(v^2)$. Therefore, the first basic theoretical uncertainty

of our calculation is related with the omitted terms of order $O(v^4)$. Since the calculation of the charmonium mass is sufficiently accurate in our model (the error is less then 1%), we suppose that the uncertainty in the cross section calculation due to omitted relativistic corrections of order $O(v^4)$ in the quark interaction operator (the Breit Hamiltonian) is also very small. Taking into account that the average value of the heavy quark velocity squared in the charmonium is $\langle v^2 \rangle = 0.3$, we expect that relativistic corrections of order $O(v^4)$ to the cross section (31) coming from the production amplitude should not exceed 30% of the obtained relativistic result. As we mentioned above in the quasipotential approach, we cannot find precisely the bound state wave functions in the region of relativistic momenta $p \geq m$. Using indirect arguments related with the mass spectrum calculation, we estimate in 10% the uncertainty in the wave function determination. Larger value of the error will lead to the essential discrepancy between the experiment and theory in the calculation of the charmonium mass spectrum. Then the corresponding error in the cross section (31) is not exceeding 20%. We do not consider a part of theoretical error related with radiative corrections of order α_s because these corrections are omitted in our analysis. So, our total theoretical error is not exceeding 36%. To obtain this estimate, we add the above mentioned uncertainties in quadrature.

We show in Fig. 5 the distribution over transverse momentum of the J/ψ mesons integrated over all rapidities at $\sqrt{S} = 7$ TeV (left) and $\sqrt{S} = 14$ TeV (right). Previous investigations [11,13,15] of the pair charmonium production in pp interaction showed that the color singlet channel prevails in the differential cross section $d\sigma/dP_T(pp \to 2J/\psi + X)$ at small P_T, but the color octet channel dominates at large P_T. It can be seen in Fig. 5 at $\sqrt{S} = 7$ TeV that the account of relativistic corrections leads to the ratio of relativistic and
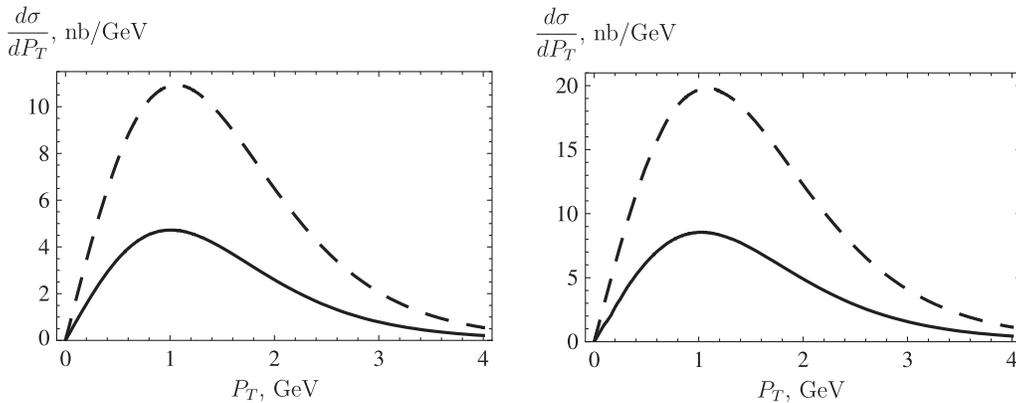

FIG. 5. The differential cross sections for $pp \to 2J/\psi + X$ at $\sqrt{S} = 7$ TeV (left) and $\sqrt{S} = 14$ TeV (right) as functions of transverse momentum P_T of the J/ψ pair integrated over the rapidity. Solid and dashed curves represent total and nonrelativistic results, respectively.

nonrelativistic cross sections $\sigma_{\rm rel}/\sigma_{\rm nr} \approx 0.4$ near the peak. This trend remains unchanged in the region of high transverse momenta. So, the color-octet contribution retains the dominance at large P_T. We investigate also the relative value of relativistic corrections in the production rate with the growth of the energy $\sqrt{S}$. Our calculation show that at $\sqrt{S} = 14$ TeV (see the right plot in Fig. 5) the ratio of relativistic and nonrelativistic cross sections is retained without essential modifications. The cross section increases with the growth of the energy and reaches the value $\sigma^{\rm th}(2 < y_{P,Q} < 4.5) = 2.98$ nb at $\sqrt{S} = 14$ TeV. It is appropriate to mention here one result regarding to the study of relativistic effects in single J/ψ production at hadron collisions [33]. It was shown in that paper that relativistic corrections to the color-singlet J/ψ hadroproduction of order $O(v^2)$ are at a level of about 1% for sufficiently large P_T: $5 \le P_T \le 50$ GeV. Our calculation demonstrates that in the region of transverse momenta $5 \le P_T \le 50$ GeV the value of relativistic corrections in the cross section of the pair charmonium production reaches 60%. Relativistic corrections which we study in this work include not only the terms of order $O(v^2)$ in the production amplitude but also the same order effects in the long-distance matrix elements. In spite of the difference between (1) and (31), we consider that at present it is difficult to state that there is the discrepancy between the theory and experiment in double charmonium production. Indeed, it is known that next-to-leading order (NLO) in α_s contributions have large value in inclusive single-quarkonium production at hadron colliders [4,34,35]. The example is found in the inclusive J/ψ production where the NLO corrections to the color-singlet contribution increase the total cross section by a factor of about 2 and the production rate of J/ψ is much increased for larger transverse momentum P_T. Therefore, one can expect that the NLO corrections to the double charmonium production in proton-proton interaction can smooth the appeared difference between (1) and (31). Moreover, as we mentioned above there exists a new mechanism through the double parton scattering which gives the contribution comparable with the standard nonrelativistic result: $\sigma_{\rm DPS}(pp \to 2J/\psi + X) = 2$ nb [16]. Accounting for this result and our value of the cross section (31), we obtain the summary value $\sigma(pp \to 2J/\psi + X) = 3.6$ nb. Then, taking into account the experimental error, the difference with the LHCb experiment does not look so significant.

APPENDIX: THE COEFFICIENTS $F^{(i)}$ ENTERING THE DIFFERENTIAL CROSS SECTION (22)

$$F^{(0)} = \frac{16384}{9M^4 s^6(M^2 - t)^4(M^2 - s - t)^4}[7776M^{24} - 432M^{22}(73s + 216t) + 6M^{20} \times (9085s^2 + 60336st + 85536t^2)$$
$$- 16M^{18}(3629s^3 + 37686s^2t + 117855st^2 + 106920t^3) + 2t^4(s + t)^4(349s^4 + 2304s^3t + 6192s^2t^2$$
$$+ 7776st^3 + 3888t^4) + 4M^{16}(11927s^4 + 151588s^3t + 745674s^2t^2 + 1470960st^3 + 962280t^4)$$
$$- 4M^{14}(7761s^5 + 109608s^4t + 699467s^3t^2 + 2173908s^2t^3 + 3055320st^4 + 1539648t^5)$$
$$+ 2M^{12}(6952s^6 + 117893s^5t + 897043s^4t^2 + 3741980s^3t^3 + 8278410s^2t^4 + 8872416st^5 + 3592512t^6)$$
$$- 4M^2t^2(s + t)^2(9s^7 + 649s^6t + 6460s^5t^2 + 29630s^4t^3 + 74435s^3t^4 + 105156s^2t^5 + 77868st^6 + 23328t^7)$$
$$- 2M^{10}(1899s^7 + 43398s^6t + 405618s^5t^2 + 2113568s^4t^3 + 6394090s^3t^4 + 10762584s^2t^5$$
$$+ 9189936st^6 + 3079296t^7) + M^8(587s^8 + 19710s^7t + 244772s^6t^2 + 1603468s^5t^3 + 6229962s^4t^4$$
$$+ 14478304s^3t^5 + 19359816s^2t^6 + 13582080st^7 + 3849120t^8) - 2M^6(20s^9 + 1185s^8t + 22153s^7t^2$$
$$+ 193780s^6t^3 + 965358s^5t^4 + 2928368s^4t^5 + 5431786s^3t^6 + 5949528s^2t^7 + 3508920st^8 + 855360t^9)$$
$$+ M^4(s^{10} + 76s^9t + 3756s^8t^2 + 52062s^7t^3 + 353472s^6t^4 + 1398834s^5t^5 + 3421754s^4t^6 + 5210968s^3t^7$$
$$+ 4784622s^2t^8 + 2414880st^9 + 513216t^{10})], \tag{A1}$$

$$F^{(1)} = -\frac{16384}{27M^4 s^8 (M^2-t)^5 (M^2-s-t)^5}[497664M^{32} - 221184M^{30}(13s+36t) + 5760M^{28}(1285s^2+7680st+10368t^2)$$

$$-48M^{26}(243089s^3+2289552s^2t+6612480st^2+5806080t^3) + 12M^{24}(1090607s^4+13899232s^3t$$

$$+62988960s^2t^2+117411840st^3+75479040t^4) - 8M^{22}(1392130s^5+22255745s^4t+136976040s^3t^2$$

$$+399063744s^2t^3+540933120st^4+271724544t^5) + 8t^5(s+t)^5(1867s^6+18256s^5t+77728s^4t^2$$

$$+181152s^3t^3+246096s^2t^4+186624st^5+62208t^6) + 4M^{20}(1800338s^6+35626541s^5t+276305481s^4t^2$$

$$+1095702384s^3t^3+2313080352s^2t^4+2435457024st^5+996323328t^6) - M^{18}(3514643s^7+86600280s^6t$$

$$+825013064s^5t^2+4139468480s^4t^3+11909795760s^3t^4+19466599680s^2t^5+16605388800st^6+5693276160t^7)$$

$$+2M^{16}(659715s^8+19899554s^7t+231809132s^6t^2+1426201784s^5t^3+5209721940s^4t^4+11612866752s^3t^5$$

$$+15333506496s^2t^6+10912112640st^7+3202467840t^8) - M^2t^3(s+t)^3(664s^9+73857s^8t+969897s^7t^2$$

$$+6006640s^6t^3+21622120s^5t^4+49033392s^4t^5+71696784s^3t^6+65938176s^2t^7+34725888st^8+7962624t^9)$$

$$-M^{14}(380999s^9+13782634s^8t+195250554s^7t^2+1459312784s^6t^3+6543590240s^5t^4+18571278768s^4t^5$$

$$+33449165568s^3t^6+36752348160s^2t^7+22298664960st^8+5693276160t^9) + 2M^{12}(39553s^{10}$$

$$+1754120s^9t+30305432s^8t^2+275173836s^7t^3+1497456520s^6t^4+5228976572s^5t^5+12018927060s^4t^6$$

$$+18013988160s^3t^7+16836887232s^2t^8+8856207360st^9+1992646656t^{10}) - 4M^{10}\times(2658s^{11}$$

$$+152991s^{10}t+3330028s^9t^2+37242554s^8t^3+246717845s^7t^4+1047413960s^6t^5+2970379604s^5t^6$$

$$+5692479912s^4t^7+7256996388s^3t^8+5868903744s^2t^9+2712213504st^{10}+543449088t^{11})$$

$$+2M^4t(s+t)(8s^{12}+943s^{11}t+73235s^{10}t^2+1243892s^9t^3+10501734s^8t^4+53251684s^7t^5$$

$$+176208052s^6t^6+396112960s^5t^7+612104570s^4t^8+641021472s^3t^9+434301120s^2t^{10}$$

$$+171417600st^{11}+29859840t^{12}) + 2M^8(267s^{12}+31089s^{11}t+948124s^{10}t^2+13678946s^9t^3$$

$$+112700707s^8t^4+584919678s^7t^5+2023301960s^6t^6+4798139192s^5t^7+7834428690s^4t^8$$

$$+8641058880s^3t^9+6129541440s^2t^{10}+2515968000st^{11}+452874240t^{12}) - M^6(8s^{13}$$

$$+1772s^{12}t+139013s^{11}t^2+3024912s^{10}t^3+33019131s^9t^4+215628282s^8t^5+917347710s^7t^6$$

$$+2664532144s^6t^7+5399964368s^5t^8+7640313480s^4t^9+7391197632s^3t^{10}+4650333696s^2t^{11}$$

$$+1710858240st^{12}+278691840t^{13})], \tag{A2}$$

$$F^{(2)} = -4F^{(0)},$$

$$F^{(3)} = \frac{4096}{81M^4 s^{10}(M^2-t)^6(M^2-s-t)^6}[31850496M^{40}-995328M^{38}(219s+640t)$$

$$+27648M^{36}(33541s^2+153432st+218880t^2)$$

$$-9216M^{34}(313031s^3+1832460s^2t+4240836st^2+3939840t^3)$$

$$+768M^{32}(8244959s^4+63274344s^3t+189539568s^2t^2+295052544st^3+200931840t^4)$$

$$-192M^{30}(52201757s^5+519109072s^4t+2012650416s^3t^2+4109435136s^2t^3$$

$$+4827651840st^4+2571927552t^5) + 16M^{28}(760108913s^6+9260081144s^5t$$

$$+46005056832s^4t^2+120457679616s^3t^3+188334270720s^2t^4+177640667136st^5$$

$$+77157826560t^6) - 8M^{26}(1475952353s^7+20899343744s^6t+127598524184s^5t^2$$

$$+423537421056s^4t^3+844308839808s^3t^4+1076247502848s^2t^5$$

$$+846934050816st^6+308631306240t^7)$$

$$
\begin{aligned}
&+ 16t^6(s+t)^6(117307s^8 + 841072s^7t + 2960704s^6t^2 + 7010976s^5t^3 + 12425424s^4t^4 + 16277760s^3t^5 \\
&+ 14715648s^2t^6 + 7962624st^7 + 1990656t^8) + 4M^{24} \times (2269174169s^8 + 37035895336s^7t + 266823310648s^6t^2 \\
&+ 1087787332416s^5t^3 + 2722434981120s^4t^4 + 4414063320576s^3t^5 + 4767985797120s^2t^6 + 3212405194752st^7 \\
&+ 1003051745280t^8) - M^{22}(5273945643s^9 + 102674924584s^8t + 863595364448s^7t^2 + 4191387926848s^6t^3 \\
&+ 12844048277696s^5t^4 + 25933611095040s^4t^5 + 35654616990720s^3t^6 + 33483324506112s^2t^7 \\
&+ 19674332061696st^8 + 5349609308160t^9) + 2M^{20}(1108300275s^{10} + 26970163113s^9t + 270274383346s^8t^2 \\
&+ 1544514084352s^7t^3 + 5656684280280s^6t^4 + 13921443451712s^5t^5 + 23690831678976s^4t^6 + 28410755383296s^3t^7 \\
&+ 23612613792768s^2t^8 + 12267586805760st^9 + 2942285119488t^{10}) - M^2t^4(s+t)^4(82320s^{11} + 10526659s^{10}t \\
&+ 122742683s^9t^2 + 663347888s^8t^3 + 2199787352s^7t^4 + 5137832384s^6t^5 + 9050135872s^5t^6 + 12183171072s^4t^7 \\
&+ 12114432000s^3t^8 + 8258236416s^2t^9 + 3403026432st^{10} + 637009920t^{11}) - M^{18}(658318562s^{11} \\
&+ 20774136911s^{10}t + 256196780079s^9t^2 + 1742958116720s^8t^3 + 7550194930440s^7t^4 + 22215242834368s^6t^5 \\
&+ 45850626752448s^5t^6 + 67807235088384s^4t^7 + 72355739968512s^3t^8 + 53902515363840s^2t^9 \\
&+ 25023941369856st^{10} + 5349609308160t^{11}) + 4M^{16}(34194446s^{12} + 1432273039s^{11}t + 22383564112s^{10}t^2 \\
&+ 185308892030s^9t^3 + 954476169886s^8t^4 + 3325907679280s^7t^5 + 8190596683920s^6t^6 + 14624279688576s^5t^7 \\
&+ 19220073992064s^4t^8 + 18510299039232s^3t^9 + 12475169980416s^2t^{10} + 5218508685312st^{11} \\
&+ 1003051745280t^{12}) - M^{14}(19246515s^{13} + 1103736798s^{12}t + 22478276084s^{11}t^2 + 232094334116s^{10}t^3 \\
&+ 1443234697510s^9t^4 + 5968097322672s^8t^5 + 17386794613376s^7t^6 + 36925721101440s^6t^7 \\
&+ 58395873784128s^5t^8 + 69341740385280s^4t^9 + 60881635077120s^3t^{10} + 37374891540480s^2t^{11} \\
&+ 14182623756288st^{12} + 2469050449920t^{13}) + 2M^4t^2(s+t)^2 \times (1024s^{14} + 151570s^{13}t + 12969265s^{12}t^2 \\
&+ 212727487s^{11}t^3 + 1644135147s^{10}t^4 + 7651899125s^9t^5 + 24229924530s^8t^6 + 56242255280s^7t^7 \\
&+ 100001584488s^6t^8 + 138359684800s^5t^9 + 146996284416s^4t^{10} + 115115443200s^3t^{11} + 61947293184s^2t^{12} \\
&+ 20275826688st^{13} + 3025797120t^{14}) + 2M^{12}(824111s^{14} + 70166611s^{13}t + 1952942482s^{12}t^2 \\
&+ 25962244078s^{11}t^3 + 199207601182s^{10}t^4 + 988657795894s^9t^5 + 3408779823892s^8t^6 + 8542312029248s^7t^7 \\
&+ 16016481697464s^6t^8 + 22847237069376s^5t^9 + 24808543913472s^4t^{10} + 19996345055232s^3t^{11} \\
&+ 11238532171776s^2t^{12} + 3890175787008st^{13} + 617262612480t^{14}) - M^6t(s+t)(2176s^{15} + 414368s^{14}t \\
&+ 34529090s^{13}t^2 + 736175475s^{12}t^3 + 7430265423s^{11}t^4 + 44335186685s^{10}t^5 + 175863201255s^9t^6 \\
&+ 499176650440s^8t^7 + 1064569592864s^7t^8 + 1756640077312s^6t^9 + 2262689749056s^5t^{10} + 2242313957376s^4t^{11} \\
&+ 1643405663232s^3t^{12} + 831421919232s^2t^{13} + 256940937216st^{14} + 36309565440t^{15}) \\
&- M^{10}(48840s^{15} + 10192723s^{14}t + 435518065s^{13}t^2 + 7912226162s^{12}t^3 + 77545065352s^{11}t^4 \\
&+ 471337098042s^{10}t^5 + 1940776757542s^9t^6 + 5736471274304s^8t^7 + 12651167529752s^7t^8 \\
&+ 21342766010240s^6t^9 + 27846873221696s^5t^{10} + 27881390733312s^4t^{11} + 20726379168768s^3t^{12} \\
&+ 10705439195136s^2t^{13} + 3395959603200st^{14} + 493810089984t^{15}) + 4M^8(24s^{16} + 60117s^{15}t + 6480104s^{14}t^2 \\
&+ 187031341s^{13}t^3 + 2509040584s^{12}t^4 + 19370940805s^{11}t^5 + 97105255536s^{10}t^6 + 341150600006s^9t^7 \\
&+ 884070584925s^8t^8 + 1749157202600s^7t^9 + 2695987058680s^6t^{10} + 3249848355136s^5t^{11} \\
&+ 3016059149568s^4t^{12} + 2073684017664s^3t^{13} + 987205294080s^2t^{14} + 288123568128st^{15} + 38578913280t^{16})].
\end{aligned}
$$

$$(A3)$$

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