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QED at a finite chemical potential

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We consider multiflavor QED on a finite lattice at a finite chemical potential and show that the partition function only depends on the variables, $(\frac{\mu_i}{q_i} - \frac{\mu_1}{q_1})$, for $i = 2, \cdots$, where q_i, $i = 1 \cdots$ are integer valued charges of the various flavors and μ_i, $i = 1 \cdots$ are the dimensionless chemical potentials of the various flavors.

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Consider multiflavor QED regularized on the lattice. We will assume that the continuum theory is defined by first defining the theory on a finite periodic lattice and then taking the limit where the extent in all directions goes off to infinity. We will not be concerned in this paper as to whether such a theory has a well-defined continuum limit. For definiteness, we could think about two-dimensional multiflavor QED which has a well-defined continuum limit.

Let $U_\nu(\mathbf{n})$ be the link variable connecting the sites $\mathbf{n}$ and $\mathbf{n} + \hat{\nu}$ on a d dimensional periodic lattice, $L_1 \times \cdots L_d$. Consider the class of gauge fields given by $U_\nu(\mathbf{n}) e^{i\frac{2\pi h_\nu}{L_\nu}}$ with $0 \le h_\nu < 1$ for $\nu = 1, \cdots d$. Gauge fields with different choices of the dimensionless variables, h_ν, within this class are not gauge equivalent but have the same gauge action. The fermion determinant, on the other hand, depends on the variables, h_ν, which we will refer to as the toron variables. Consider a fermion with integer charge q_i and a dimensionless chemical potential μ_i. The chemical potential is introduced [1] by multiplying the parallel transporter in the forward $\nu = d$ direction by $e^{\frac{2\pi\mu_i}{L_d}}$ and in the backward $\nu = d$ direction by $e^{-\frac{2\pi\mu_i}{L_d}}$. The factor of 2π is introduced for convenience and we keep the dimensionless chemical potential, μ_i, fixed as we take $L_d \to \infty$.

The fermionic determinant is a function of $U_\nu(\mathbf{n})$, $\nu = 1, \cdots, d$; h_ν for $\nu = 1, \cdots, d-1$; and $z_i = h_d - i\frac{\mu_i}{q_i}$. Since the gauge action does not depend on the toron variables, we can integrate the fermion determinant over these variables using the uniform measure. Consider, for simplicity, a lattice fermion operator of the naïve, Wilson or staggered type. We will show later that our arguments will also apply to the overlap Dirac operator under certain mild assumptions. The lattice fermion operator is a finite matrix on a finite lattice. Focusing on the dependence on h_d alone, we see that the fermion determinant is a finite polynomial in $e^{i\frac{2\pi q_i z_i}{L_d}}$ and $e^{-i\frac{2\pi q_i z_i}{L_d}}$. The fermion determinant will be an analytic function of z_i in the complex plane. Since the fermion determinant is gauge invariant, it will be periodic under $z_i \to z_i + 1$. A contour integral in the complex plane results in the integral over h_d in the range $[0, 1]$ to be independent of μ_i. If we consider a theory with many flavors, that all have the same value for $\frac{\mu_i}{q_i}$, then again the integral over h_d will yield a result that is independent of all the μ_i. In other words, a multiflavor theory of QED at a finite chemical potential can only depend on the variables, $(\frac{\mu_i}{q_i} - \frac{\mu_1}{q_1})$, for $i = 2, \cdots$. This result will also hold for an anomaly free chiral QED as long as the lattice formulation is gauge invariant, at least in the continuum limit. This is the main observation in this paper.

Massless Schwinger model in the presence of a chemical potential was first studied in Ref. [2]. The problem was treated in the Hamiltonian formalism. In order to deal with a finite problem, a uniform charge background was introduced in a finite region of space. This causes an explicit breaking of translational invariance. The ground state is a classical Wigner crystal which is not destroyed by quantum fluctuations. An explicit chemical potential term was introduced in the Hamiltonian formalism in Ref. [3]. Their formalism also had to break translational invariance and they conclude that there is an inhomogeneous chiral condensate in the Schwinger model at finite density. A path integral formulation of the problem, again with the introduction of a chemical potential that breaks translational invariance results in an inhomogeneous chiral condensate [4]. References to a possible inhomogeneous chiral condensate in the Schwinger model are still being discussed in the literature [5–7]. The problem is discussed in Ref. [8] where the author argues that the inhomogeneous chiral condensate in the Schwinger model at finite density is an artifact of the explicit breaking of translational invariance in the formalism. The generalized Thirring model was analyzed in Ref. [9] and it has been explicitly shown that the chiral condensate does not depend on the chemical potential. Our observation in this paper clearly shows that physics does not depend on the chemical potential in the Schwinger model.

We now proceed to discuss the case of overlap fermions in some detail when the chemical potential is introduced as in Ref. [10]. Since the definition involves the sign function of a complex matrix, it is not *a priori* clear if the arguments presented above will apply. This problem is

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addressed using a specific numerical example, namely, the Schwinger model with a finite chemical potential, μ. Gauge fields are generated at a fixed coupling using the gauge action described in Ref. [11] with zero and unit topological charge. The determinant of both the Wilson-Dirac operator and the overlap-Dirac operator was computed in a fixed gauge field background that has zero topological charge. The determinant of both the Wilson-Dirac operator and the overlap-Dirac operator (we exclude the zero mode in this case) was also computed in a fixed gauge field background that has unit topological charge. Averages of these two quantities over all gauge fields for the overlap-Dirac operator enter the computation of the chiral condensate in the massless Schwinger model.

We first consider two sample gauge fields, one with zero topological charge and one with unit topological charge as set by the gauge fields, on a 7×7 lattice. The same gauge field is used for several different values of the chemical potential. Plots of the real part of the determinants [12] for three sample values of the chemical potential, namely, $\mu = 0, 0.5, 1.0$ are shown in Figs. 1–3, respectively. In all three cases, the determinant of the Wilson Dirac operator with a specific choice for the mass term used in the definition of the overlap Dirac operator kernel behaves smoothly as a function of h_2. The determinants do depend on h_2 and the variation increases as one increases the chemical potential. Furthermore, the function even changes sign for $\mu = 1.0$. In spite of this, we explicitly verified that the integral over h_2 (which remains complex for a fixed gauge field background) is independent of μ as expected by the analytical argument presented in the beginning of the paper.

The sign function of the Wilson Dirac operator need not be a smooth function of h_2. This is due to the fact it depends on the sign of the real part of the eigenvalue of the Wilson Dirac operator and this can change as a function of h_2. The trace of the sign function of the Wilson Dirac operator is defined as twice the topological charge of the gauge field. Whereas, the results are consistent with

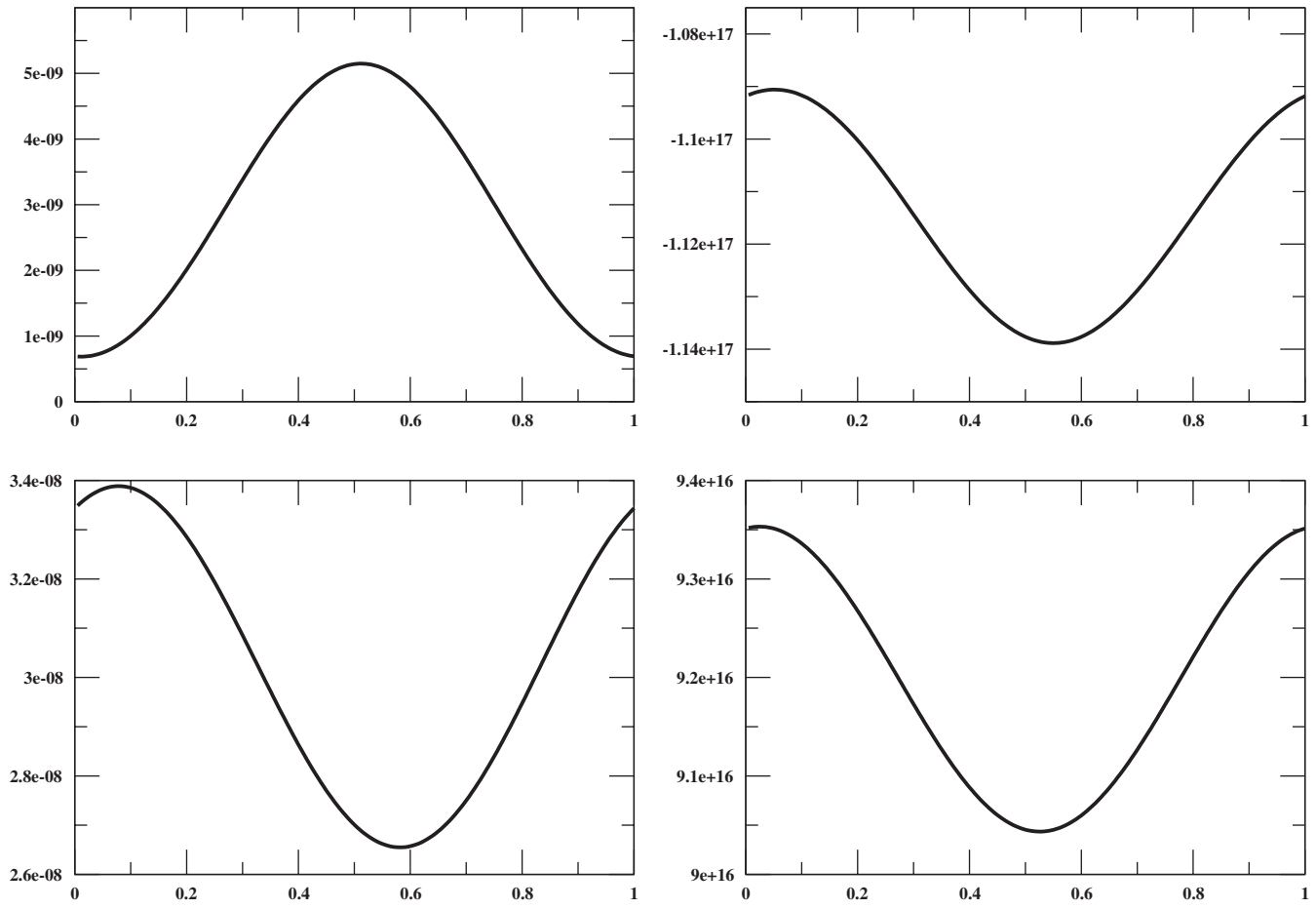

FIG. 1. The left top and bottom panels show the behavior of the determinant of the overlap Dirac operator in the zero and unit topological sectors. The right top and bottom panels show the behavior of the determinant of the Wilson Dirac operator in the zero and unit topological sectors. The results are on a 7×7 lattice and the dimensionless chemical potential is set to 0 and the x axis shows the value of h_2.

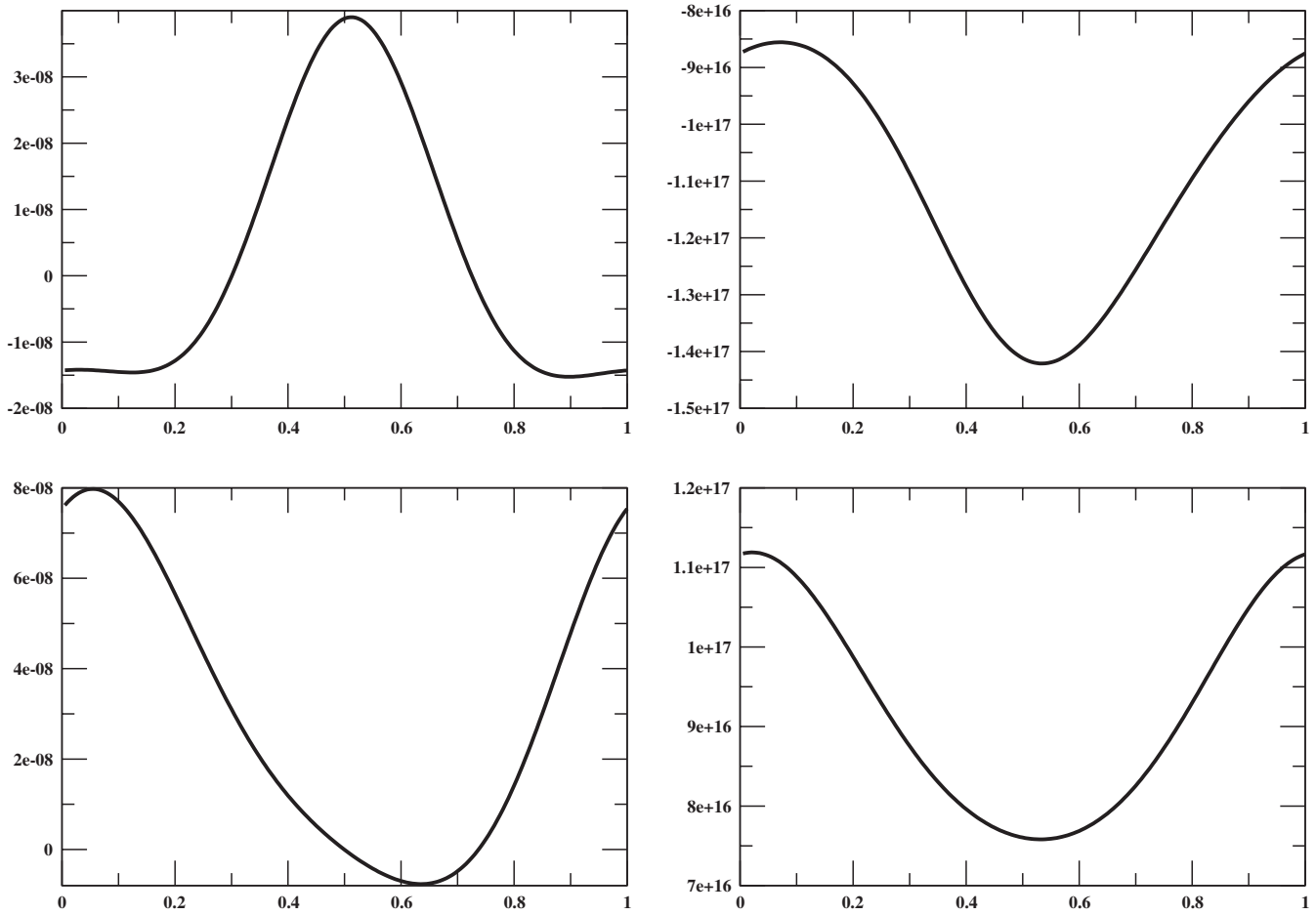

FIG. 2. The left top and bottom panels show the behavior of the determinant of the overlap Dirac operator in the zero and unit topological sectors. The right top and bottom panels show the behavior of the determinant of the Wilson Dirac operator in the zero and unit topological sectors. The results are on a 7×7 lattice and the dimensionless chemical potential is set to 0.5 and the x axis shows the value of h_2.

the topological charge of the constructed gauge field background for small values of the chemical potential, it does not remain consistent for large values of the chemical potential as seen in the right panel of Fig. 3. The top panel should have been consistent with zero and the bottom panel should have been consistent with unity. Since this is not the case for $\mu = 1$, the determinant of the overlap Dirac operator in the two different topological sectors is not a smooth function of h_2 as seen in the left panels of Fig. 3. As a consequence the independence of the integral of the determinant of the overlap Dirac operator on the chemical potential breaks down beyond a certain value of the chemical potential. This is due to the discretization of the dimensional chemical potential, μ, over a finite number of slices, L_d, taken to be 7 in Fig. 3. If we increase the number of slices, L_d, keeping the physical gauge coupling fixed, we found that the value of the chemical potential where the independence breakdown increases. For example, we could only go up to $\mu = 0.7$ on 7×7 lattice at a given coupling but we could go up to $\mu = 1.0$ on a 9×9 lattice at the same physical coupling. This is illustrated in Fig. 4 for the case of $\mu = 1.0$ on a 9×9 lattice where the behavior of the determinants of the overlap Dirac operator are smooth as a function of h_2. The problem with the overlap Dirac operator in the presence of a chemical potential was anticipated in Ref. [10] but is expected not to affect the continuum limit.

We have presented an analytical argument in this paper that there is no dependence on the chemical potential in the path integral formalism of QED. This result is valid within the lattice formulation and in the continuum limit. Whereas, there is some justification to the argument presented in Ref. [8] toward the problems with breaking translation invariance in the Hamiltonian formalism, we have shown that the main reason for recovering the correct behavior in the presence of a chemical potential is the integration over the toron variable as emphasized in Ref. [9] and further emphasized in

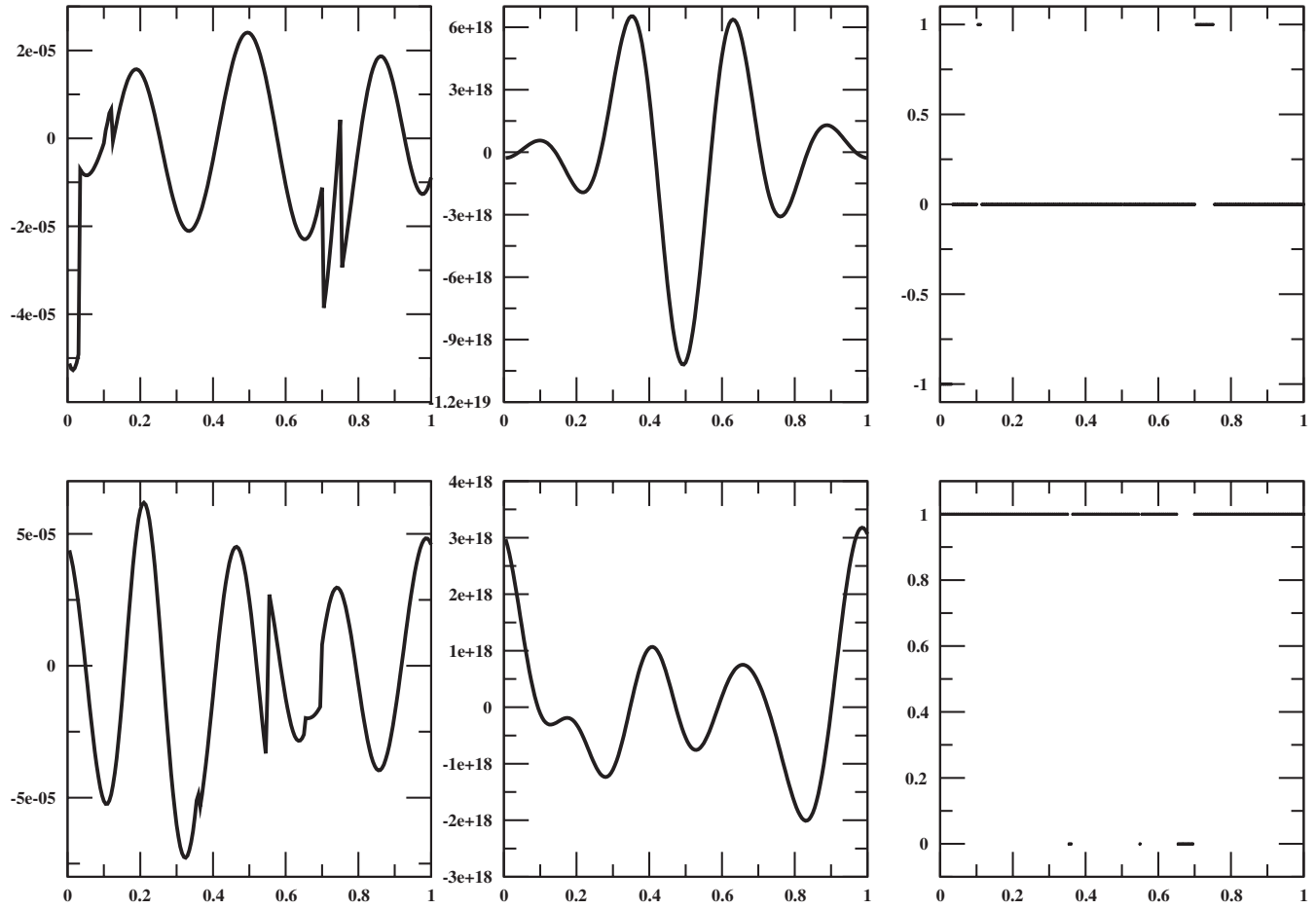

FIG. 3. The left top and bottom panels show the behavior of the determinant of the overlap Dirac operator in the zero and unit topological sectors with zero chemical potential. The middle top and bottom panels show the behavior of the determinant of the Wilson Dirac operator in the zero and unit topological sectors. The right top and bottom panels show the topological charge measured by the overlap Dirac operator in the presence of a chemical potential for the presumed zero and unit topological charge configuration. The results are on a 7×7 lattice and the dimensionless chemical potential is set to 1.0 and the x axis shows the value of h_2.

Ref. [13]. One does not see the toron variable, h_d, in the Hamiltonian formalism since one starts in the Coloumb gauge. The realization of the toron variable in the Hamiltonian formalism would be to integrate over all boundary conditions for fermions in the Euclidean time (temperature) direction [14].

We have not performed an analytic continuation in the chemical potential in this paper. One could reproduce the central argument in this paper by working with an imaginary chemical potential [15]. Since there is no periodicity when the chemical potential is real, we refrained from using imaginary chemical potential—our results are valid for all values of the real chemical potential. A nontrivial dependence on the chemical potential will be seen in the two flavor Schwinger model and finite density phase transitions are expected in four-dimensional two-flavor QED.

The continuous toron variable becomes a discrete Z_N variable in a $SU(N)$ gauge theory and our argument of independence on the chemical potential will not go through [16] In the limit of $N \rightarrow \infty$, we have a continuous toron variable. Therefore, we expect the toron variable to play a part in the analysis of the 't Hooft model in the presence of a chemical potential as discussed in Ref. [18]. The Gross-Neveu model [5,6] is different in this aspect since one does not integrate over all possible fermionic boundary conditions in the Euclidean time direction. This will be the case even if we introduce a bosonic variable to convert the four-fermi coupling into a fermion bilinear since we will have a Gaussian term for the bosonic field.

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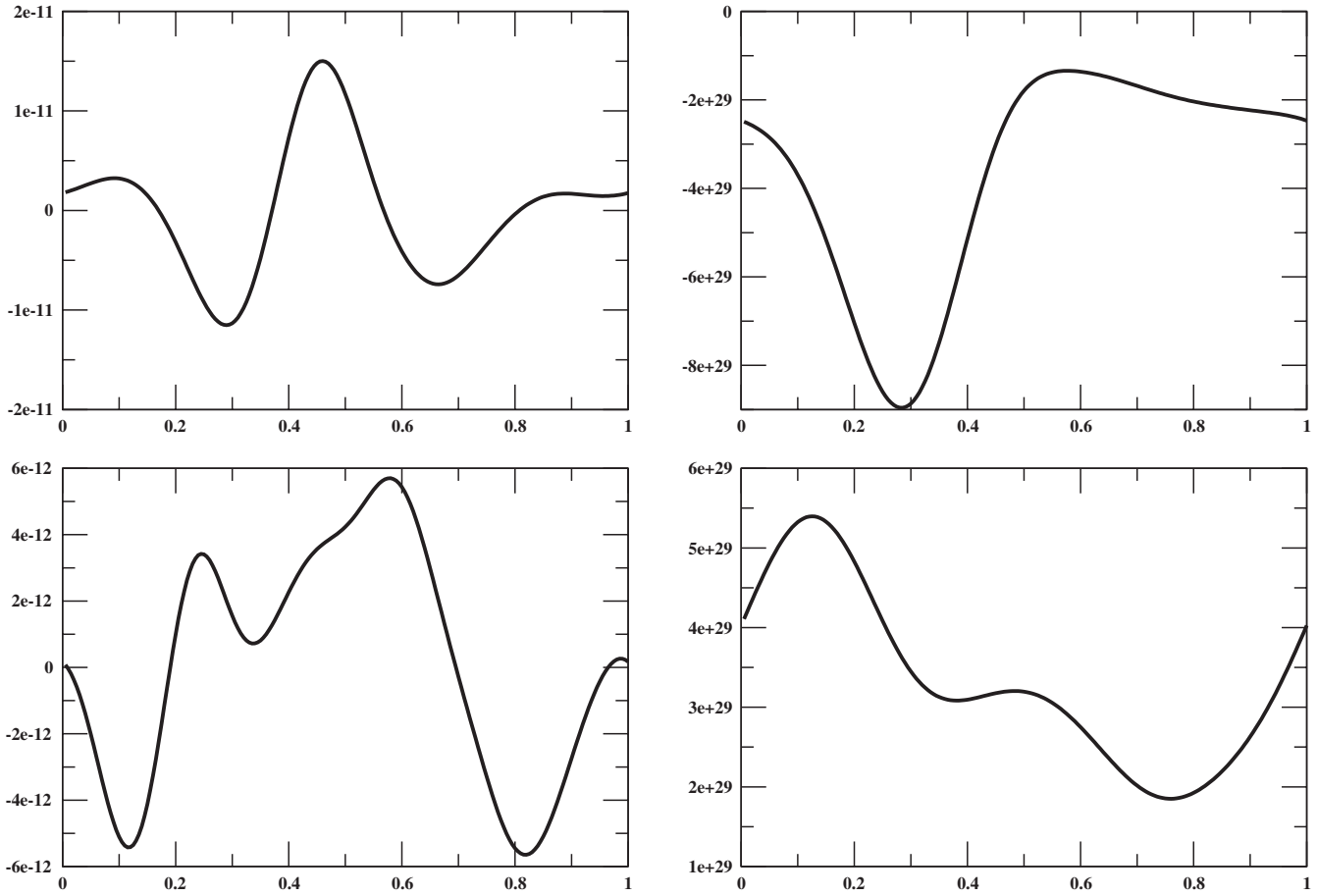

FIG. 4. The left top and bottom panels show the behavior of the determinant of the overlap Dirac operator in the zero and unit topological sectors with zero chemical potential. The right top and bottom panels show the behavior of the determinant of the Wilson Dirac operator in the zero and unit topological sectors. The results are on a 9×9 lattice and the dimensionless chemical potential is set to 1.0 and the x axis shows the value of h_2.

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