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Functional renormalization group flow of the effective Hamiltonian action

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After a brief review of the definition and properties of the quantum effective Hamiltonian action, we describe its renormalization flow by a functional renormalization group equation. This equation can be used for a nonperturbative quantization and study of theories with bare Hamiltonians that are not quadratic in the momenta. As an example, the vacuum energy and gap of quantum mechanical models are computed. Extensions of this framework to quantum field theories are discussed. In particular, one possible Lorentz-covariant approach for simple scalar field theories is developed. Fermionic degrees of freedom, being naturally described by a first order formulation, can easily be accommodated in this approach.

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I. INTRODUCTION

Quantum mechanical systems can be studied with a variety of methods such as, for example, the canonical operatorial approach or the functional methods. The latter are usually employed to construct generating functionals of different kinds of correlators from which the physical observables of interest can be derived. Among them a very useful object is the so-called quantum effective action, a functional mostly used in quantum field theory (QFT), in both perturbative and nonperturbative approaches. This is, in general, a highly nonlocal object that encodes all the quantum properties of the system; for instance, it generates the proper vertices of the theory.

The effective action most commonly discussed in the literature is of the Lagrangian type, since it is derived from the second order Lagrangian formulation of the bare theory. There is a very good reason to do that, namely, that people usually consider bare Hamiltonians that are quadratic in the momenta such that one can easily move to a Lagrangian description. The rationale for this is obtaining a manifestly Lorentz-covariant formulation in d space-time dimensions. Another advantage of passing to a second order formulation is that the number of fields in configuration space is half the one in phase space, since in the functional formulation the conjugated momenta have been integrated out.

On the other hand, one may also consider the reasons to choose a first order Hamiltonian description on the phase space of a theory. Clearly this is unavoidable when dealing with the quantization of theories with bare Hamiltonians nonquadratic in the momenta. In such a case the full phase-space variables are needed for a quantum description of the system. Traditionally the main advantage attributed to the Hamiltonian formulation is that it makes unitarity manifest [1]. This is because of the strict relationship established by canonical quantization between the classical symplectic structure on phase space and the inner product on the Hilbert space. The Hamiltonian approach may be useful also when configuration space is not a vector space, since phase space can usually be interpreted as a cotangent bundle and it could be easier to deal with. In the functional integral representation this is translated with the possibility that the measure in phase space is field independent while the one in configuration space is not. This happens, for instance, in the case of nonlinear sigma models. Of course, even in this case whenever the bare theory is quadratic in the momenta, the Lagrangian and the Hamiltonian formulations lead to the same results (Matthews theorem), as proved by perturbative studies [2,3]. In a functional integral representation, the Hamiltonian approach is based on quantum generating functionals obtained introducing sources in the phase-space path integral [4]. From them, one can define a quantum effective Hamiltonian action that generates the proper vertices. This was recently studied in [5], on the wake of a renewed interest in Hamiltonian gauge theories such as QCD, in particular, in the Coulomb gauge (see [6] and references therein).

The purpose of the present work is to present a nonperturbative framework that allows one to compute, within specific approximation schemes, the quantum effective Hamiltonian action. This approach is based on the definition of a one-parameter family of deformed effective actions, which were introduced in the literature a long time ago under the name of average effective (Lagrangian) actions [7]. This is only one of the many formulations of the functional renormalization group (RG) [8–10], which is a Wilsonian representation of QFT based on a coarse-graining procedure allowing one to interpolate, by moving along an RG flow trajectory in theory space, between the bare theory at the ultraviolet (UV) scale and the quantum effective theory at the infrared (IR) scale. Providing a bare action in the UV and solving the RG flow equation with an appropriate set of boundary conditions, one could in principle obtain the quantum effective action. Since this

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equation is a functional differential equation, generally one is forced to employ specific approximation schemes, essentially strongly constraining the space of functionals the solution belongs to with the help of physical arguments. For almost two decades such a theoretical framework has been applied to investigate several aspects of QFT's and condensed matter systems [11] and has been used to probe the possibility that, for example, Einstein gravity, as a QFT, can be nonperturbatively renormalizable [12] within the paradigm of asymptotic safety [13].

In a previous work [14] we proposed the use of cutoff operators affecting the symplectic form of phase space and implementing a more balanced coarse graining and regularization, with respect to the cases where the coarse graining is performed on the fluctuations in configuration space only, but after this choice of regularization, we restricted our discussion to bare Hamiltonians quadratic in the momenta, and we fully integrated out the momenta, obtaining a cutoff dependent functional measure in the Lagrangian path integral, which was leading to a subtraction term in the RG flow equation. Here instead we are interested in retaining the full dynamics in phase space, building a flow that realizes the idea of shell-by-shell simultaneous integration on both phase-space variables. As a disclaimer, let us add that other nonperturbative RG flows called ''Hamiltonian flows'' already appeared in the literature, but they largely differ from our formulation. Examples are the similarity RG [15], which is generated by iterated unitary transformations within the operatorial representation, and the flows based on a variational solution of the Schrödinger equation [16].

In this paper we start our discussion from quantum mechanical systems [$(0+1)$-dimensional QFT's] with scalar degrees of freedom, for which we review some of the properties of the Hamiltonian effective action in the first part of Sec. II, and we prove some formulas useful for the subsequent developments. In the second part of Sec. II we derive the main equations satisfied by the average effective Hamiltonian action (AEHA) of a quantum mechanical system. They depend on a cutoff operator that suppresses part of the functional integration generating a one-parameter flow from the UV to the IR. In particular, we give the simpler equations associated with the so-called local Hamiltonian approximation (LHA), which is the lowest order term of the derivative expansion of the full functional, for some specific cutoff operators. These are then used (Sec. II C) to study a family of exactly solvable Hamiltonians that are not quadratic in the momenta, and indeed we show that one can easily extract information like the ground state and the first energy gap of such systems. The same approach can be used to study general systems with arbitrary bare Hamiltonians. We conclude Sec. II discussing the extension of the formalism to quantum mechanical theories with fermionic degrees of freedom.

In Sec. III we start to address quantum field theories. The extension to the noncovariant version of QFT is straightforward, and we first discuss it briefly for the case of scalar QFT. Since in the traditional Hamiltonian formulation of QFT one pays explicit unitarity with the disguising of Lorentz invariance, we discuss one possible way around this drawback; that is, we spend the last part of the paper discussing a manifestly Lorentz symmetric (but maybe not manifestly unitary) extension of the previous framework inside the realm of the covariant Hamiltonian formalism.

This is a subject that has a long history in classical physics [17–19], but whose applications to quantum dynamics are pretty rare to be found in the literature. Even if under different names, the covariant Hamiltonian formulation of the Yang-Mills theory is one of the oldest examples. In 1977 Halpern addressed such a formalism for QCD, generically naming it ''first order formalism'' [20], but he immediately abandoned the full phase-space formulation integrating out the gauge vector fields, thus being left with a theory, containing only conjugate momenta, that he called ''field strength formulation,'' which was studied in the following years (see [21] and references therein). More recently a slight variant of the first order formalism (still covariant) for the Yang-Mills theory has received fresh attention from the perspective of topological BF theories [22]. In particular, the reader can find in [23] an explicit one-loop computation of what we call the effective covariant Hamiltonian action of pure Yang-Mills theory, reproducing the expected asymptotic freedom result. Despite these successful examples, the main open question about covariant Hamiltonian QFT is still about its foundations, even if these have begun to be studied recently by some authors [24,25]. These investigations can shed light on the issue of unitarity of this covariant formulation. Without a sound Lorentz covariant quantization prescription, covariant Hamiltonian formalism seems but a game, legitimate only in the special case of Hamiltonians quadratic in the momenta. On the other hand, only by studying this approach in more general cases and by looking for its applications to real physical systems can one hope to find a legitimation for the search of foundations.

In this work, for what concerns a covariant Hamiltonian formulation of QFT's, we restrict ourselves to defining the average effective covariant Hamiltonian action of a scalar field theory in a particularly simple case. This consists in assuming that the nontrivial dependence on the covariant momenta is in the longitudinal [with respect to (w.r.t.) the Fourier variable] subspace of the space of conjugate momenta. This definition is compatible with both quantum mechanics (QM) in $0+1$ dimensions and with QFT's whose bare Hamiltonians are quadratic in the momenta, and it provides a particular dynamical extension outside this domain. For this simple case we present a framework for studying such a model by a nonperturbative RG flow equation. For completeness we also comment on the

corresponding covariant Hamiltonian formulation for theories with Dirac fermions.

In the conclusions the reader will find a discussion about the physical motivations for the introduction of this formalism, as well as a proposal of some possible developments, extensions, and future applications of this method. Several appendices follow, where some technical issues are described in more detail.

II. THE EFFECTIVE HAMILTONIAN ACTION IN QUANTUM MECHANICS

In this section we shall work within QM, i.e., a $(0+1)$-dimensional QFT. As an example we will quantize a classical system with one bosonic degree of freedom governed by the following Hamiltonian action:

$$S[p,q] = \int dt[p(t)\partial_t q(t) - H(p(t),q(t))], \tag{1}$$

where the (bare) Hamiltonian can have an arbitrary dependence in the momenta, departing therefore from the usual quadratic form

$$H(p,q) = \frac{1}{2}p^2 + V(q). \tag{2}$$

Here and in the following p and q denote canonically conjugate variables. The quantization of such a system is performed via the following phase-space path integral:

$$e^{\frac{i}{\hbar}W[I,J]} = \int [dpdq]\mu[p,q]e^{\frac{i}{\hbar}\{S[p,q]+I\cdot p+J\cdot q\}}, \tag{3}$$

where the dots stand for ordinary integrations. The functional measure on the physical phase space is usually assumed to be $\mu[p,q] = \mathrm{Det}\frac{1}{2\pi\hbar}$. Also one can easily extend all the formalism to a Euclidean description. Since we want to keep our discussion as general as possible, we will not specify the precise space of functions on which the functional integral is defined.

It is possible to study the system by a functional that may be called the quantum effective Hamiltonian action, which is a trivial generalization of the more widely known effective Lagrangian action. The latter Γ^L is defined by introducing in the configuration-space path integral external sources J coupled to the Lagrangian variables and by taking the Legendre transform of the generating functional of the connected Green's functions $W[J]$ w.r.t. J. Similarly, to define the effective Hamiltonian action Γ^H, one starts from the phase-space path integral (3) and performs a Legendre transform:

$$\Gamma^H[\bar{p},\bar{q}] = \underset{I,J}{\mathrm{ext}}(W[I,J] - I\cdot\bar{p} - J\cdot\bar{q}), \tag{4}$$

where

$$\bar{p} = \frac{\delta W}{\delta I}, \qquad \bar{q} = \frac{\delta W}{\delta J}.$$

The introduction of such a functional is not a novelty, as we have discussed in the Introduction. There are several ways to convince ourselves that from this functional one can get all the information about the quantum system.

First, by taking functional derivatives w.r.t. $\bar{q}(t)$ and $\bar{p}(t)$, one immediately gets

$$I = -\frac{\delta\Gamma^H}{\delta\bar{p}}, \qquad J = -\frac{\delta\Gamma^H}{\delta\bar{q}}. \tag{5}$$

For zero sources one has the equations for the vacuum configuration $(\bar{q},\bar{p})$. They appear as the classical equations of motion obtained from the quantum effective Hamiltonian action.

Second, Γ^H satisfies the following integro-differential equation:

$$e^{\frac{i}{\hbar}\Gamma^H[\bar{p},\bar{q}]} = \int [dpdq]\mu[p,q]e^{\frac{i}{\hbar}\{S[p,q]-(q-\bar{q})\cdot\frac{\delta\Gamma^H}{\delta\bar{q}}-(p-\bar{p})\cdot\frac{\delta\Gamma^H}{\delta\bar{p}}\}}. \tag{6}$$

This is a central identity, and it could also be promoted to the definition of Γ^H.

Third, from this equation one can get a different proof that the classical equations satisfied by the effective Hamiltonian action encode the full quantum dynamics, because they are equivalent to the Hamiltonian Dyson-Schwinger equations. In fact, the identities

$$\begin{aligned} 0 &= \int [dpdq]\frac{\delta}{\delta p}(\mu[p,q]e^{\frac{i}{\hbar}\{S[p,q]-(q-\bar{q})\cdot\frac{\delta\Gamma^H}{\delta\bar{q}}-(p-\bar{p})\cdot\frac{\delta\Gamma^H}{\delta\bar{p}}\}}) \\ &= \int [dpdq]\frac{\delta}{\delta q}(\mu[p,q]e^{\frac{i}{\hbar}\{S[p,q]-(q-\bar{q})\cdot\frac{\delta\Gamma^H}{\delta\bar{q}}-(p-\bar{p})\cdot\frac{\delta\Gamma^H}{\delta\bar{p}}\}}) \end{aligned}$$

lead to

$$\left\langle -i\hbar\frac{\delta}{\delta p}\log\mu[p,q] + \frac{\delta S}{\delta p}\right\rangle = \frac{\delta\Gamma^H}{\delta\bar{p}},$$
$$\left\langle -i\hbar\frac{\delta}{\delta q}\log\mu[p,q] + \frac{\delta S}{\delta q}\right\rangle = \frac{\delta\Gamma^H}{\delta\bar{q}}.$$

Fourth, just as for the effective action, the effective Hamiltonian action has a similar interpretation as the generator of the one-particle-irreducible (1PI) proper vertices. For more details and a proof of this statement see Appendix A.

Fifth, by evaluating the effective Hamiltonian action on its stationarity $\bar{p}$ values, one gets the effective Lagrangian action. In fact, defining

$$\Gamma^L[\bar{q}] = \underset{\bar{p}}{\mathrm{ext}}\Gamma^H[\bar{p},\bar{q}]$$

and calling $\bar{p}_{\bar{q}}$ the extremal point, it is straightforward to show that

$$I = -\frac{\delta\Gamma^H}{\delta\bar{p}}[\bar{p}_{\bar{q}}, \bar{q}] = 0,$$
$$J = -\frac{\delta\Gamma^H}{\delta\bar{q}}[\bar{p}_{\bar{q}}, \bar{q}] = -\frac{\delta\Gamma^L}{\delta\bar{q}}[\bar{q}].$$

Therefore $\Gamma^L[\bar{q}] = W[0, -\frac{\delta\Gamma^L}{\delta\bar{q}}] + \bar{q}\cdot\frac{\delta\Gamma^L}{\delta\bar{q}}$, from which we learn that Γ^L satisfies the integro-differential equation

$$e^{\frac{i}{\hbar}\Gamma^L[\bar{q}]} = \int [dpdq]\mu[p, q]e^{\frac{i}{\hbar}\{S[p,q]-(q-\bar{q})\cdot\frac{\delta\Gamma^L}{\delta\bar{q}}\}},$$

which is a generalization of the usual configuration-space integro-differential equation satisfied by the effective action, since it does not require S to be quadratic in the momenta. Because of this simple relation between the two effective actions, for the rest of this paper we will use the same letter Γ for both, dropping the superscripts, since the reader will be able to distinguish them by their arguments ($\bar{p}$, $\bar{q}$ for the Hamiltonian one and $\bar{q}$ only for the Lagrangian one).

Sixth, the effective Hamiltonian action can be defined from the operatorial representation by means of a time-dependent variational principle, in a way that is the direct generalization of the usual construction in configuration space [26]. Let $\hat{H}$ be the Hamiltonian operator of the quantum system and $|0\rangle$ be its time-independent ground state, and let the boundary conditions of the path integral in (3) be chosen such that

$$e^{\frac{i}{\hbar}W[I,J]} = \langle 0|\hat{U}_{I,J}(+\infty, -\infty)|0\rangle = \langle 0|T\exp\left\{-\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt[\hat{H} - J(t)\hat{q} - I(t)\hat{p}]\right\}|0\rangle. \tag{7}$$

Then the effective Hamiltonian action defined in (4) is related in the following way:

$$\Gamma[\bar{p}, \bar{q}] = \underset{|\psi_\pm, t\rangle}{\text{ext}}\left(\int_{-\infty}^{+\infty} dt\langle\psi_-, t|i\hbar\partial_t - \hat{H}|\psi_+, t\rangle\right) \tag{8}$$

to an extremum with respect to variations of the two states $|\psi_\pm, t\rangle$ preserving the constraints

$$\begin{aligned}\langle\psi_-, t|\psi_+, t\rangle &= 1,\\ \langle\psi_-, t|\hat{q}|\psi_+, t\rangle &= \bar{q}(t),\\ \langle\psi_-, t|\hat{p}|\psi_+, t\rangle &= \bar{p}(t)\end{aligned} \tag{9}$$

for any t, and the boundary conditions

$$\lim_{t\to\mp\infty}|\psi_\pm, t\rangle = |0\rangle. \tag{10}$$

A sketch of the proof of this statement is given in Appendix B. A special role is played by time-independent $\bar{p}$ and $\bar{q}$, because the previous proposition reduces to $\Gamma[\bar{p}, \bar{q}] = -\mathcal{E}(\bar{p}, \bar{q})\int dt$ where $\mathcal{E}$ is the usual energy density functional defined by the minimum

$$\mathcal{E}(\bar{p}, \bar{q}) = \min_{|\psi\rangle}\langle\psi|\hat{H}|\psi\rangle \tag{11}$$

with respect to variations of the time-independent state $|\psi\rangle$ preserving the time-independent version of the constraints in (9).

This clearly provides an energy interpretation for the effective Hamiltonian action. In particular, if one evaluates this action on the constant $(\bar{p}, \bar{q})$ values that make it stationary, the resulting number is just minus the "time volume" times the ground state energy. In principle, it is possible to compute all the energy levels by means of Γ, but higher levels require more work. One possible way is through the two point functions. In a Hamiltonian framework the propagator splits in the entries of the matrix:

$$i\left\langle\mathcal{T}\begin{pmatrix}(p-\bar{p})_{t'}(p-\bar{p})_t & (q-\bar{q})_{t'}(p-\bar{p})_t\\ (p-\bar{p})_{t'}(q-\bar{q})_t & (q-\bar{q})_{t'}(q-\bar{q})_t\end{pmatrix}\right\rangle = W^{(2)}_{tt'}[I,J] = \begin{pmatrix}\frac{\delta^2 W}{\delta I_{t'}\delta I_t} & \frac{\delta^2 W}{\delta J_{t'}\delta I_t}\\ \frac{\delta^2 W}{\delta I_{t'}\delta J_t} & \frac{\delta^2 W}{\delta J_{t'}\delta J_t}\end{pmatrix} = \begin{pmatrix}\frac{\delta\bar{p}_t}{\delta I_{t'}} & \frac{\delta\bar{p}_t}{\delta J_{t'}}\\ \frac{\delta\bar{q}_t}{\delta I_{t'}} & \frac{\delta\bar{q}_t}{\delta J_{t'}}\end{pmatrix} \tag{12}$$

(where $\mathcal{T}$ is the time ordering operator) so that one could try to think about p and q as different "fields" but should also remember the existence of an unusual mixed propagator connecting p legs to q legs or vice versa. Thanks to (5) one can write this matrix in terms of Γ as follows:

$$W^{(2)}_{tt'}[I,J] = \begin{pmatrix}\frac{\delta\bar{p}}{\delta I} & \frac{\delta\bar{p}}{\delta J}\\ \frac{\delta\bar{q}}{\delta I} & \frac{\delta\bar{q}}{\delta J}\end{pmatrix}_{tt'} = \begin{pmatrix}\frac{\delta I}{\delta\bar{p}} & \frac{\delta I}{\delta\bar{q}}\\ \frac{\delta J}{\delta\bar{p}} & \frac{\delta J}{\delta\bar{q}}\end{pmatrix}^{-1}_{tt'} = -\begin{pmatrix}\frac{\delta^2\Gamma}{\delta\bar{p}\delta\bar{p}} & \frac{\delta^2\Gamma}{\delta\bar{q}\delta\bar{p}}\\ \frac{\delta^2\Gamma}{\delta\bar{p}\delta\bar{q}} & \frac{\delta^2\Gamma}{\delta\bar{q}\delta\bar{q}}\end{pmatrix}^{-1}_{tt'} = -(\Gamma^{(2)}[\bar{p}, \bar{q}])^{-1}_{tt'}. \tag{13}$$

To make the last expression for the two point function more explicit, one needs to invert a matrix whose elements are operators. In the particular case in which all block entries of the original matrix are nonsingular, its inverse is given by

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}^{-1} = \begin{pmatrix}(A - BD^{-1}C)^{-1} & (C - DB^{-1}A)^{-1}\\ (B - AC^{-1}D)^{-1} & (D - CA^{-1}B)^{-1}\end{pmatrix}. \tag{14}$$

In our case the operator $W^{(2)}_k$ is symmetric, and one can use the formula in Eq. (14) setting $C = B^T$. Let us stress that in order to put the off-diagonal blocks of this inverse in the form of Eq. (14) with $C = B^T$ it is only necessary to assume that B is nonsingular, a condition that is met by $\frac{\delta^2\Gamma}{\delta\bar{p}\delta\bar{q}}$ unless Γ is extremely pathological. Once we know how to compute the two point functions by means of Γ, we could have access to all the energy gaps $\Delta E_n = E_n - E_0$ through the Källen-Lehmann representation of the propagators

$$\frac{\delta^2 W}{\delta I(\tau)\delta I(0)} = i\sum_{n\neq 0} e^{-i\Delta E_n \tau}|\langle 0|\hat{p}|n\rangle|^2 = -\sum_{n\neq 0}\int\frac{dE}{2\pi}e^{-iE\tau}\frac{2\Delta E_n|\langle 0|\hat{p}|n\rangle|^2}{E^2-\Delta E_n^2+i\epsilon},$$

$$\frac{\delta^2 W}{\delta J(\tau)\delta J(0)} = i\sum_{n\neq 0} e^{-i\Delta E_n \tau}|\langle 0|\hat{q}|n\rangle|^2 = -\sum_{n\neq 0}\int\frac{dE}{2\pi}e^{-iE\tau}\frac{2\Delta E_n|\langle 0|\hat{q}|n\rangle|^2}{E^2-\Delta E_n^2+i\epsilon}.$$

Similar expressions hold for mixed derivatives of W. This tells us that, in principle, by studying the pole structure of the Fourier transformed two point functions, we could compute all the ΔE_n. As Eq. (13) shows, this requires the knowledge of the exact $\Gamma^{(2)}$. In most cases this is not available, and only approximations are possible. In certain contexts one popular approximation scheme for the computation of the effective action is the derivative expansion. The zeroth order of such an expansion in the present Hamiltonian framework is the LHA and consists of the ansatz $\Gamma = \int dt(\bar{p}\partial_t\bar{q} - H_{\rm eff}(\bar{p},\bar{q}))$ where the effective Hamiltonian $H_{\rm eff}$, which is an ultralocal function of its arguments (i.e., it does not depend on their derivatives), can be computed by setting the fields $\bar{p}$ and $\bar{q}$ to constant values. For this choice, since the second derivatives of Γ on constant field configurations commute with each other, the inversion rule (14) leads to a simple expression

$$\frac{\delta^2 W}{\delta I(\tau)\delta I(0)} = -\left[\frac{\delta^2\Gamma}{\delta\bar{p}\delta\bar{p}} - \frac{\delta^2\Gamma}{\delta\bar{q}\delta\bar{p}}\left(\frac{\delta^2\Gamma}{\delta\bar{q}\delta\bar{q}}\right)^{-1}\frac{\delta^2\Gamma}{\delta\bar{p}\delta\bar{q}}\right]^{-1}_{0\tau} \overset{LHA}{=} -\int\frac{dE}{2\pi}e^{-iE\tau}\frac{\frac{\partial^2 H_{\rm eff}}{\partial\bar{q}\partial\bar{q}}}{E^2-\det H^{(2)}_{\rm eff}+i\epsilon}$$

$$\frac{\delta^2 W}{\delta J(\tau)\delta J(0)} = -\left[\frac{\delta^2\Gamma}{\delta\bar{q}\delta\bar{q}} - \frac{\delta^2\Gamma}{\delta\bar{p}\delta\bar{q}}\left(\frac{\delta^2\Gamma}{\delta\bar{p}\delta\bar{p}}\right)^{-1}\frac{\delta^2\Gamma}{\delta\bar{q}\delta\bar{p}}\right]^{-1}_{0\tau} \overset{LHA}{=} -\int\frac{dE}{2\pi}e^{-iE\tau}\frac{\frac{\partial^2 H_{\rm eff}}{\partial\bar{p}\partial\bar{p}}}{E^2-\det H^{(2)}_{\rm eff}+i\epsilon}, \qquad (15)$$

and similar formulas hold for mixed derivatives of W. Here $\det H^{(2)} = \partial^2_{\bar{q}\,\bar{q}}H\partial^2_{\bar{p}\,\bar{p}}H - (\partial^2_{\bar{q}\,\bar{p}}H)^2$ is the determinant of the Hessian matrix of H. Therefore we see that in the LHA, whenever the second derivatives of $H_{\rm eff}$ commute (as in the case they are single numbers and not matrices), only one pole appears in the propagators at the value $(\det H^{(2)}_{\rm eff})^{1/2}$. Since we are performing a derivative (low energy) expansion, in general this pole is the one closer to $E=0$, that is to say, the first gap ΔE_1, unless the matrix elements $\langle 0|\hat{p}|1\rangle$ and $\langle 0|\hat{q}|1\rangle$ vanish. Therefore we shall use in the LHA the relations

$$E_0 = H_{\rm eff}|_{\rm min}, \qquad \Delta E_1 = \sqrt{\det H^{(2)}_{\rm eff}}|_{\rm min}. \qquad (16)$$

So far we have discussed how many properties of a quantum system can be deduced from the effective Hamiltonian action, but how can we compute this action? One way is to use perturbation theory. First of all, one needs to define propagators and vertex functions. We already know that in a Hamiltonian framework the propagators of a theory with Hamiltonian action Γ are given by Eq. (13). The vertex functions generated by Γ are simply given by

$$\frac{\delta^m}{\delta\bar{p}^m}\frac{\delta^n\Gamma}{\delta\bar{q}^n}\bigg|_{\bar{q}=\bar{p}=0}, \qquad m+n>2 \qquad (17)$$

and therefore generically comprehend m p legs and n q legs. Since perturbation theory in phase space is built on tree level propagators and vertices, one can read these ingredients from (13) and (17) by substituting Γ with the bare action S. For instance, to get the one-loop result one changes variables of integration in (6) according to $p = \bar{p} + \hbar^{\frac{1}{2}}p'$, $q = \bar{q} + \hbar^{\frac{1}{2}}q'$, and Taylor expands both S and Γ around $\hbar = 0$ up to linear terms

$$S(p,q) = S(\bar{p},\bar{q}) + \frac{\hbar}{2}(p',q')S^{(2)}(\bar{p},\bar{q})(p',q')^T + o(\hbar^2),$$

$$\Gamma[\bar{p},\bar{q}] = \Gamma_0[\bar{p},\bar{q}] + \hbar\Gamma_1[\bar{p},\bar{q}] + o(\hbar^2).$$

The change of variable goes along with a change of measure, owing to the Jacobian determinant $\mathrm{Det}\,\hbar$, such that the new measure becomes $\mu[p',q'] = \mathrm{Det}\frac{1}{2\pi}$. The Gaussian path integral over p' and q' combined with such a measure gives $\Gamma_1[\bar{p},\bar{q}] = \frac{i}{2}\,\mathrm{logDet}(-iS^{(2)}[\bar{p},\bar{q}])$, where S is the bare Hamiltonian action (together with the obvious result $\Gamma_0[\bar{p},\bar{q}] = S[\bar{p},\bar{q}]$). The block determinant can be written in a more explicit form by means of the general formula

$$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \det A\,\det(D - CA^{-1}B) = \det D\,\det(A - BD^{-1}C), \qquad (18)$$

where the first expression is true if $\det A \neq 0$ and the second if $\det D \neq 0$. Therefore, if $\frac{\delta^2 S}{\delta\bar{p}\delta\bar{p}}$ is nonvanishing,

$$\Gamma_1[\bar{p},\bar{q}] = \frac{i}{2}\,\mathrm{logDet}\bigg[-\frac{\delta^2 S}{\delta\bar{p}\delta\bar{p}}\left(\frac{\delta^2 S}{\delta\bar{q}\delta\bar{q}} - \frac{\delta^2 S}{\delta\bar{p}\delta\bar{q}}\times\left(\frac{\delta^2 S}{\delta\bar{p}\delta\bar{p}}\right)^{-1}\frac{\delta^2 S}{\delta\bar{q}\delta\bar{p}}\right)\bigg]$$

$$= \frac{i}{2}\,\mathrm{logDet}\bigg[\left(-\partial_t^2 - \det H^{(2)} + \left(\partial_t\frac{\partial^2 H}{\partial\bar{p}\partial\bar{q}}\right) + \left(\partial_t\log\frac{\partial^2 H}{\partial\bar{p}\partial\bar{p}}\right)\left(\partial_t - \frac{\partial^2 H}{\partial\bar{p}\partial\bar{q}}\right)\right)\delta\bigg], \qquad (19)$$

which reduces to the usual one-loop formula for the effective action in the case of a bare Hamiltonian as the one in (2). In the formula above we have used the symbol δ for $\delta(t-t')$, which defines the operator. If instead $\frac{\delta^2 S}{\delta\bar{p}\delta\bar{p}}$ vanishes while $\frac{\delta^2 S}{\delta\bar{q}\delta\bar{q}}$ is nonvanishing, the result can be obtained from (19) by replacing $\delta_{\bar{p}}$ with $\delta_{\bar{q}}$ and vice versa.

In the rest of this paper we will work on a nonperturbative setting for the computation of the effective Hamiltonian action, and we will choose $\hbar$ as our unit of action.

A. The average effective Hamiltonian action

A nonperturbative definition of the path integral (6) can be given by a functional RG flow equation. The starting point of this construction is the introduction of an external parameter in the theory. This allows one to reduce the task of computing the functional integral in the simpler task of computing its infinitesimal variation under changes of such a parameter. In quantum mechanics the external parameter can be dimensionless, since the number of degrees of freedom is finite and no regularization is needed. Instead, the generalization of the construction to field theories requires the introduction of a dimensionful parameter k, such that its variation corresponds to a *coarse-graining* operation (otherwise we meet infinities in the computation of the infinitesimal variation). An alternative way is to assume that the theory has already been regularized, as, for example, by the introduction of a UV cutoff Λ, in which case it is possible to deal with a dimensionless parameter also in field theories (related to the ratio between the dimensionful k and Λ). Since by varying k we will get a one-parameter flow of theories, we will need initial conditions in order to integrate it. A convenient way to deal with this issue is to choose the dependence on k in such a way that the flow interpolates between full functional integration (conventionally at $k=0$) and no integration at all (conventionally at $k=\Lambda$, even if Λ might in some cases be displaced at $+\infty$). The no integration limit can also be realized considering k as a mathematical parameter unrelated to a physical sounding coarse-graining procedure, and, in the presence of the physical UV cutoff Λ, taking the limit $k\to\infty$. Sticking to this framework we introduce such a parameter, by means of a modification of the bare action and of the functional measure

$$e^{iW_k[I,J]}=\int[dpdq]\mu_k[p,q]e^{i\{S[p,q]+\Delta S_k[p,q]+I\cdot p+J\cdot q\}}, \tag{20}$$

and ask for $\mu_k\exp\{i\Delta S_k\}$ to become μ as $k\to 0$ and to provide a rising delta functional as $k\to\Lambda$. As traditional, to keep the framework as simple as possible, we choose ΔS_k to be quadratic in the fields

$$\Delta S_k[p,q]=\frac{1}{2}(p,q)\cdot R_k\cdot(p,q)^T \tag{21}$$

such that we need $R_k\to 0$ and $\mu_k\to\mu$ when $k\to 0$, as well as $R_k\to\infty$ and $\mu_k\to(\mathrm{Det}\frac{R_k}{2\pi})^{\frac{1}{2}}$ when $k\to\Lambda$. These constraints can be satisfied by several choices for the symmetric matrix R_k and for the measure μ_k. In this paper we will consider only two simple cases in which the only nonvanishing entries of R_k are either off-diagonal and built out of an odd differential operator or diagonal and built out of even differential operators. These, respectively, read

$$R_k(t,t')=\begin{pmatrix}0 & r_k(-\partial_t^2)\partial_t\delta(t-t')\\ -r_k(-\partial_t^2)\partial_t\delta(t-t') & 0\end{pmatrix}, \tag{22}$$

$$R_k(t,t')=\begin{pmatrix}\mathcal{R}_k^p(-\partial_t^2)\delta(t-t') & 0\\ 0 & \mathcal{R}_k^q(-\partial_t^2)\delta(t-t')\end{pmatrix}. \tag{23}$$

The first choice can be interpreted as a k-dependent deformation of the symplectic potential $\lambda=pdq$, by means of an operator $(1+r_k)$ that, after the pullback by a section defining the specific path, might become a differential operator. This interpretation suggests the appropriate k-dependent deformation of the functional measure: if the new symplectic potential is $\lambda_k=p(1+r_k)dq$, the new nontrivial Liouville measure would become $\mu_k=(\mathrm{Det}\frac{\sigma_k}{2\pi})^{\frac{1}{2}}$, where $\sigma_k=d\lambda_k$ is the regularized symplectic form. This choice for the measure indeed provides the correct normalization of the Gaussian rising delta functional [14]. Following this line of thought, we can guess a convenient choice for the regularized measure also in the second case of a diagonal regulator. The straightforward adaptation of the previous argument is insisting on adding to the fundamental symplectic matrix our regulator matrix, and then taking its determinant. To summarize, the regularized functional measures we will use together with the regulators (22) and (23), respectively, are

$$\mu_k=\left[\mathrm{Det}\frac{1}{2\pi}\begin{pmatrix}0 & (1+r_k(-\partial_t^2))\partial_t\delta(t-t')\\ -(1+r_k(-\partial_t^2))\partial_t\delta(t-t') & 0\end{pmatrix}\right]^{\frac{1}{2}}, \tag{24}$$

$$\mu_k = \left[\mathrm{Det} \frac{1}{2\pi} \begin{pmatrix} \mathcal{R}_k^p(-\partial_t^2)\delta(t-t') & \partial_t \delta(t-t') \\ -\partial_t \delta(t-t') & \mathcal{R}_k^q(-\partial_t^2)\delta(t-t') \end{pmatrix} \right]^{\frac{1}{2}}. \tag{25}$$

The definition of the AEHA $\Gamma_k[\bar{p}, \bar{q}]$ is

$$\Gamma_k[\bar{p}, \bar{q}] + \Delta S_k[\bar{p}, \bar{q}] = \underset{I,J}{\mathrm{ext}} (W_k[I, J] - I \cdot \bar{p} - J \cdot \bar{q}).$$

Note that the sources minimizing the right-hand side (r.h.s.) will in general depend on k. Again it is easy to write an integro-differential equation for the AEHA:

$$e^{i\Gamma_k[\bar{p},\bar{q}]} = \int [dpdq] \mu_k[p, q] \times e^{i\{S[p,q] + \Delta S_k[p-\bar{p}, q-\bar{q}] - (p-\bar{p})\frac{\delta \Gamma_k}{\delta \bar{p}} - (q-\bar{q})\frac{\delta \Gamma_k}{\delta \bar{q}}\}}. \tag{26}$$

When $k \to 0$ Eq. (26) trivially reduces to Eq. (6) and the AEHA becomes the full effective Hamiltonian action. It is not hard to check that when $k \to \Lambda$ the r.h.s. of Eq. (26) reduces to $\exp\{iS[\bar{p}, \bar{q}]\}$ and the AEHA coincides with the bare Hamiltonian action. A sketch of the proof can be found in Appendix C.

The relation between the average effective Hamiltonian and Lagrangian actions is the same as for the full effective actions:

$$\Gamma_k[\bar{q}] = \underset{\bar{p}}{\mathrm{ext}} \Gamma_k[\bar{p}, \bar{q}]. \tag{27}$$

We observe that this is evident in the simplest possible case, i.e., when the bare action is quadratic in the momenta, as in (2), since $\frac{\partial^2 H}{\partial p^2}$ and $\frac{\partial^2 H}{\partial p \partial q}$ are constant (the latter is actually zero). Indeed, the integration over p in (26) can be performed exactly, and in such a case one discovers that the AEHA must also be quadratic in the momenta and that for any k the canonical momentum that extremizes it is $\bar{p} = \partial_t \bar{q}$. As a result, plugging this field configuration in (26), using the definition (27), and integrating out the momenta, one obtains

$$e^{i\Gamma_k[\bar{q}]} = \int [dq] \mu_k[q] e^{i\{S[q] + \Delta S_k[q-\bar{q}] - (q-\bar{q})\frac{\delta \Gamma_k}{\delta \bar{q}}\}}, \tag{28}$$

where now $\mu_k[q] \equiv \int [dp] \mu_k[p, q] e^{-i\frac{p^2}{2}}$ and $\Delta S_k[q]$ arises from the chosen $\Delta S_k[p, q]$. For example, if one adopts the scheme of Eqs. (22) and (24), then

$$\mu_k[q] = \left[\mathrm{Det} \frac{1}{2\pi} (1 + r_k(-\partial_t^2))^2 (-\partial_t^2) \delta \right]^{\frac{1}{2}},$$
$$\Delta S_k[q] = \frac{1}{2} \partial_t q \cdot (r_k^2 + 2r_k) \partial_t q.$$

As usual, the $k \to \Lambda$ limit of the average effective Lagrangian action coincides with the bare Lagrangian action while the $k \to 0$ limit gives the full quantum effective Lagrangian action.

In this work we are interested in the cases that depart from such a simple situation.

B. RG flow equation for the AEHA

In this section we discuss the translation of the functional integro-differential equation (26) in a functional differential equation describing a flow parametrized by k.

Denoting by "$\cdot$" the operation $k\partial_k$, and acting with it on Eq. (26), one obtains

$$i\dot{\Gamma}_k = \frac{\dot{\mu}_k}{\mu_k} + i\langle \dot{\Delta S}_k[p - \bar{p}, q - \bar{q}] \rangle_k.$$

Since ΔS_k has been chosen quadratic in the fields, the expectation value can be rewritten by means of the k-dependent version of Eqs. (12) and (13). Denoting $\tilde{\Gamma}_k[\bar{p}, \bar{q}] \equiv \Gamma_k[\bar{p}, \bar{q}] + \Delta S_k[\bar{p}, \bar{q}]$, these read

$$\begin{aligned} & i\langle \mathcal{T} \begin{pmatrix} (p-\bar{p})_{t'}(p-\bar{p})_t & (q-\bar{q})_{t'}(p-\bar{p})_t \\ (p-\bar{p})_{t'}(q-\bar{q})_t & (q-\bar{q})_{t'}(q-\bar{q})_t \end{pmatrix} \rangle_k \\ & = W^{(2)}_{ktt'}[I, J] = \begin{pmatrix} \frac{\delta^2 W_k}{\delta I_{t'} \delta I_t} & \frac{\delta^2 W_k}{\delta J_{t'} \delta I_t} \\ \frac{\delta^2 W_k}{\delta I_{t'} \delta J_t} & \frac{\delta^2 W_k}{\delta J_{t'} \delta J_t} \end{pmatrix} = \begin{pmatrix} \frac{\delta \bar{p}_t}{\delta I_{t'}} & \frac{\delta \bar{p}_t}{\delta J_{t'}} \\ \frac{\delta \bar{q}_t}{\delta I_{t'}} & \frac{\delta \bar{q}_t}{\delta J_{t'}} \end{pmatrix} \\ & = \begin{pmatrix} \frac{\delta I}{\delta \bar{p}} & \frac{\delta I}{\delta \bar{q}} \\ \frac{\delta J}{\delta \bar{p}} & \frac{\delta J}{\delta \bar{q}} \end{pmatrix}^{-1}_{tt'} = - \begin{pmatrix} \frac{\delta^2 \tilde{\Gamma}_k}{\delta \bar{p} \delta \bar{p}} & \frac{\delta^2 \tilde{\Gamma}_k}{\delta \bar{q} \delta \bar{p}} \\ \frac{\delta^2 \tilde{\Gamma}_k}{\delta \bar{p} \delta \bar{q}} & \frac{\delta^2 \tilde{\Gamma}_k}{\delta \bar{q} \delta \bar{q}} \end{pmatrix}^{-1}_{tt'} \\ & = -(\tilde{\Gamma}^{(2)}_k[\bar{p}, \bar{q}])^{-1}_{tt'}. \end{aligned}$$

Therefore, for any quadratic regulator, the flow equation can be written as

$$i\dot{\Gamma}_k = \frac{\dot{\mu}_k}{\mu_k} - \frac{1}{2} \mathrm{Tr}[(\Gamma^{(2)}_k + R_k \delta)^{-1} \dot{R}_k \delta], \tag{29}$$

where $R_k \delta = \Delta S^{(2)}_k$. Here one still has freedom for the choice of μ_k as a functional of R_k. By using the inversion formula (14), one can find a more explicit form for the flow equation. Adopting the regulator (22) affecting only the Legendre transform term of the bare action (i.e., the symplectic potential) and the corresponding minimally deformed Liouville functional measure (24), Eq. (29) becomes

$$\begin{aligned} i\dot{\Gamma}_k = & \mathrm{Tr}[\dot{r}_k (1 + r_k)^{-1} \delta] - \mathrm{Tr}\Bigg[(\dot{r}_k \partial \delta) \Bigg(\Bigg(r_k \partial \delta + \frac{\delta^2 \Gamma_k}{\delta \bar{q} \delta \bar{p}} \Bigg) \\ & - \frac{\delta^2 \Gamma_k}{\delta \bar{p} \delta \bar{p}} \Bigg(-r_k \partial \delta + \frac{\delta^2 \Gamma_k}{\delta \bar{p} \delta \bar{q}} \Bigg)^{-1} \frac{\delta^2 \Gamma_k}{\delta \bar{q} \delta \bar{q}} \Bigg)^{-1} \Bigg], \end{aligned} \tag{30}$$

where we denote $(\partial \delta)_{t_1 t_2} = \partial_{t_1} \delta(t_1 - t_2)$. Instead, the choice of a diagonal regulator (23) and of the corresponding measure (25) leads to the flow equation

$$
\begin{aligned}
i\dot{\Gamma}_k = {} & \frac{1}{2}\,\mathrm{Tr}[(\dot{\mathcal{R}}_k^p\delta)(\mathcal{R}_k^p\delta - (\partial\delta)(\mathcal{R}_k^q\delta)^{-1}(-\partial\delta))^{-1}] + \frac{1}{2}\,\mathrm{Tr}[(\dot{\mathcal{R}}_k^q\delta)(\mathcal{R}_k^q\delta - (-\partial\delta)(\mathcal{R}_k^p\delta)^{-1}(\partial\delta))^{-1}] \\
& - \frac{1}{2}\,\mathrm{Tr}\left[(\dot{\mathcal{R}}_k^p\delta)\left(\left(\mathcal{R}_k^p\delta + \frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{p}}\right) - \frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{p}}\left(\mathcal{R}_k^q\delta + \frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{q}}\right)^{-1}\frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{q}}\right)^{-1}\right] \\
& - \frac{1}{2}\,\mathrm{Tr}\left[(\dot{\mathcal{R}}_k^q\delta)\left(\left(\mathcal{R}_k^q\delta + \frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{q}}\right) - \frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{q}}\left(\mathcal{R}_k^p\delta + \frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{p}}\right)^{-1}\frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{p}}\right)^{-1}\right].
\end{aligned}
\tag{31}
$$

Notice that, thanks to the regularization of the functional measure, these equations correctly reproduce the nonrenormalization of H_k in the trivial cases in which the bare Hamiltonian either vanishes or depends on just one field out of p and q. As far as the reality properties of this equation are concerned, there is no difference with the standard Lagrangian formalism in real time; that is to say, the imaginary unit on the left-hand side (l.h.s.) is needed in order to ensure reality of Γ_k. This is because in real time the traces on the r.h.s. usually are integrals of functions with poles on the real axis, which thus lead to imaginary values. An appropriate prescription should be given in order to displace these poles off the real axis. As usual in QFT, one adopts the prescription that relates the Minkowskian theory to the Euclidean theory by a continuous Wick rotation. The same can be done in QM. The reader can find more details about this translation to imaginary time in Appendix D.

The previous flow equations are still too general for a first approach to their meaning and application; therefore let us give more specific and simple forms of the first one of them, Eq. (30). As a first example let us consider the truncation $\Gamma_k = \int dt(\bar{p}\partial_t\bar{q} - \frac{1}{2}\bar{p}^2 - V_k(\bar{q}))$. Introducing the notation $P_k(-\partial_t^2) = -\partial_t^2(1 + r_k)^2$, one finds the RG flow equation

$$
-i\int dt\dot{V}_k(\bar{q}) = \frac{1}{2}\,\mathrm{Tr}[\dot{P}_k P_k^{-1}] - \frac{1}{2}\,\mathrm{Tr}[\dot{P}_k(P_k - V_k^{(2)}(\bar{q}))^{-1}],
\tag{32}
$$

which is what one gets by the effective average Lagrangian action approach [14] in the local potential approximation (LPA). A more general example is the LHA, i.e., the case in which the flow equation for the truncation $\Gamma_k = \int dt(\bar{p}\partial_t\bar{q} - H_k(\bar{p},\bar{q}))$ is evaluated on constant $\bar{q}$ and $\bar{p}$ configurations. For this choice, if the second derivatives of Γ_k commute with each other as in the present case where they are 1-by-1 bosonic matrices, the operators in the trace can be simplified and one obtains

$$
\begin{aligned}
& -i\int dt\dot{H}_k(\bar{p},\bar{q}) \\
& \quad = -\mathrm{Tr}\left[\left(\frac{\dot{r}_k}{1 + r_k}\delta\right)\frac{\det H_k^{(2)}(\bar{p},\bar{q})}{-\partial^2(1 + r_k)^2\delta - \det H_k^{(2)}(\bar{p},\bar{q})}\right] \\
& \qquad + \mathrm{Tr}\left[\frac{(\dot{r}_k\partial\delta)\frac{\delta^2 H_k}{\delta\bar{p}\delta\bar{q}}(\bar{p},\bar{q})}{-\partial^2(1 + r_k)^2\delta - \det H_k^{(2)}(\bar{p},\bar{q})}\right],
\end{aligned}
\tag{33}
$$

where $\det H_k^{(2)} = \partial_{\bar{q}}^2{}_{\bar{q}} H_k \partial_{\bar{p}}^2{}_{\bar{p}} H_k - (\partial_{\bar{q}}^2{}_{\bar{p}} H_k)^2$ is the determinant of the Hessian matrix of H_k. Notice that the second trace vanishes whenever it is possible to evaluate it in Fourier space and when the domain in such space is symmetric around the origin. If this is the case, we are left with

$$
i\int dt\dot{H}_k(\bar{p},\bar{q}) = \frac{1}{2}\,\mathrm{Tr}\left[\frac{\dot{P}_k}{P_k}\delta\frac{\det H_k^{(2)}(\bar{p},\bar{q})}{P_k\delta - \det H_k^{(2)}(\bar{p},\bar{q})}\right].
\tag{34}
$$

Here one could adopt any of the regulators $\mathcal{R}_k$ developed in the vast literature about the average effective Lagrangian action [11,27] and plug it in the last formula by $P_k(-\partial_t^2) = -\partial_t^2 + \mathcal{R}_k(-\partial_t^2)$. One of the simplest choices for the regulator is a constant r_k, that is to say, an operator that is multiplicative in both time and frequency representations; in other words, a function of k and Λ only. If no UV cutoff is present, this choice is possible only in quantum mechanics because it does not produce any coarse graining, and therefore it does not regularize the functional traces. Assuming $\dot{r}_k > 0$, $\forall\ k \in (0,\Lambda)$, one can trade k for the dimensionless parameter r_k. Thus, in LHA and if the second derivatives of H_k commute with each other, assuming that the traces can be written as $\int dt\int\frac{dE}{2\pi}$ (after Fourier transform) and that there is no UV cutoff in the theory, then by Wick rotating the trace ($E \longrightarrow iE$) one gets

$$
\frac{dH_r}{dr} = -\frac{1}{2(1 + r)^2}(\det H_r^{(2)})^{\frac{1}{2}}.
\tag{35}
$$

A different choice that makes the computation of the traces even simpler than for a constant r_k is the square root of the Litim regulator [27]. Denoting by $r_k(E^2)E$ the Fourier transform of $r_k(-\partial_t^2)i\partial_t$, and with θ the Heaviside step function, after Wick rotation such a regulator reads

$$
\begin{aligned}
r_k(E^2)E = {} & -(k + E)\theta(k + E)\theta(-E) \\
& + (k - E)\theta(k - E)\theta(E).
\end{aligned}
$$

In the LHA and if the second derivatives of H_k commute with each other, this gives the same result as (34) for $P_k(E^2) = k^2\theta(k^2 - E^2) + E^2\theta(E^2 - k^2)$, that is,

$$
\dot{H}_k = -\frac{k}{\pi}\frac{\det H_k^{(2)}}{k^2 + \det H_k^{(2)}}.
\tag{36}
$$

Of course, if one considers $H_k(\bar{p},\bar{q}) = T_k(\bar{p}) + V_k(\bar{q})$ as an initial condition for the flow, whenever both T_k and V_k

are polynomials of degree higher than two, the determinant becomes a function of both $\bar{q}$ and $\bar{p}$ so that the flow generates also mixed $\bar{p}$ and $\bar{q}$ dependence in the effective Hamiltonian. Therefore one should consider a larger truncation in order to track such terms. Also a structure of a σ-model kind, quadratic in the momenta, generates a dependence in the momenta that is more than quadratic. We stress that in general the flow will also generate a dependence on time derivatives of q and p variables. This goes beyond the LHA, but it is still compatible with the standard Hamiltonian approach as long as one starts the flow at the UV with a derivatives-free bare Hamiltonian.

C. Exercise: The ground state energy and gap of models that are more than quadratic in the momenta

As an example of the application of the framework discussed in the previous subsections to specific problems, we will present the computation of the first two energy levels of some exactly solvable systems for which no simple Lagrangian description is available, due to the fact that the functional integral over the conjugate momenta is not Gaussian. This will serve as a check of the soundness of the formalism, but the reader is invited to remember that the very same simple computations explained in the following would also work for much more complicated models. Let us recall that the functional RG has already been successfully applied to the computation of the spectrum of quantum mechanical models in the configuration-space formulation [28,29].

The systems we are going to address have the following classical Hamiltonian:

$$H_n(p,q) = \left(\frac{p^2+\omega^2 q^2}{2}\right)^n. \tag{37}$$

They are easy to solve because of the $O(2)$ symmetry that forces the Hamiltonian to depend only on the "action" and not on the "angle" coordinate in phase space. Even without performing a canonical transformation to such coordinates, the energy spectrum can be built by ladder operators. Rescaling the variables $q = q'/\sqrt{\omega}$ and $p = \sqrt{\omega}p'$ as well as the Hamiltonian $H = \omega^n H'$, we can reduce the problem to the one with $\omega = 1$; therefore in the following we will restrict to such a case. The operator algebra of these quantum models is completely described by

$$\hat{a} = \frac{\hat{q}+i\hat{p}}{\sqrt{2}}, \qquad \hat{a}^\dagger = \frac{\hat{q}-i\hat{p}}{\sqrt{2}}, \qquad \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1. \tag{38}$$

The Hamiltonian operator is just the nth power of $(\hat{N}+\frac{1}{2})$ where $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator. This is enough to deduce the whole energy spectrum for any positive integer n.

To reproduce such a spectrum by means of the RG flow equation, the first step is to specify the initial condition for the integration of the flow. From the discussion of the previous subsections follows that the most suitable initial condition is $\Gamma_{k=\Lambda} = S$, where S is the bare action to be inserted in a path integral, as the input specifying which system is being studied. At this point it is necessary to recall that such a bare action is in one-to-one correspondence with the Hamiltonian of the operator representation: the bare Hamiltonian is just the Weyl symbol of the Hamiltonian operator. Let us remember that an operator $\hat{O}(\hat{p},\hat{q})$ can always be written as a sum of symmetrized (in $\hat{p}$ and $\hat{q}$) operators

$$\hat{O} = \hat{O}_S + \sum_i \hat{O}_{iS} = \hat{O}_W, \tag{39}$$

which is what one calls the Weyl-ordered version of $\hat{O}$. Also, its average on coordinate $\hat{q}$ eigenstates with eigenvalues x and y is conveniently given by

$$\langle x|\hat{O}|y\rangle = \int dp \langle x|p\rangle O_W\left(p, \frac{x+y}{2}\right)\langle p|y\rangle. \tag{40}$$

The function O_W in the r.h.s. of Eq. (40) is called the Weyl symbol of $\hat{O}$, and it can be considered as the classical counterpart of $\hat{O}$. There are many ways to compute this function; one is to Weyl order $\hat{O}$ and then to replace the operators in $\hat{O}_W$ with c numbers. Another way is through the relation

$$O_W(p,q) = \int dx e^{ipx}\left\langle q - \frac{x}{2}\middle|\,\hat{O}(\hat{p},\hat{q})\,\middle| q+\frac{x}{2}\right\rangle, \tag{41}$$

where the bra's and ket's are again eigenstates of the $\hat{q}$ operator. For instance, considering the models in Eq. (37), in the $n=2$ and $n=3$ cases such symbols read

$$\begin{aligned} H_{2W}(p,q) &= \left(\frac{p^2+q^2}{2}\right)^2 - \frac{1}{4}, \\ H_{3W}(p,q) &= \left(\frac{p^2+q^2}{2}\right)^3 - \frac{5}{4}\left(\frac{p^2+q^2}{2}\right). \end{aligned} \tag{42}$$

Notice that both subtraction terms above, generated by Weyl ordering, are proportional to $\hbar^2$, but in natural units such a dependence disappears.

Inserting these initial conditions in the flow equation for the LHA, one can compute the full quantum effective Hamiltonian at $k=0$. Such a task can be performed by numerically integrating the flow equation. However, if one is interested in simple quantities as the first two energy levels, this might be unnecessary: it could be enough to truncate the LHA to a polynomial in $z \equiv (p^2+q^2)/2$ of finite order. Indeed, if the bare Hamiltonian depends on p and q only through z, in the LHA approximation H_k can also be shown to respect this symmetry for suitable cutoff operators.

We started by studying these polynomial truncated flows as generated by Eqs. (35) and (36), finding that singularities appear at nonvanishing values of k. This happens

TABLE I. The ground state energy E_0 and the first energy gap ΔE_1 for the bare Hamiltonians of Eqs. (37) and (42), as computed from the flow Eq. (43) by means of two polynomial truncations: up to the same order of the bare Hamiltonian and up to the next order (+ 1 superscript).

Bare Hamiltonian	E_0^{exact}	E_0	E_0^{+1}	$\Delta E_1^{\text{exact}}$	ΔE_1	ΔE_1^{+1}
H_{2W}	1/4	0.249 36	0.249 36	2	1.998 71	1.998 71
H_2	1/2	0.499 89	0.499 94	2	1.998 67	1.999 85
H_{3W}	1/8	0.124 92	0.124 89	13/4	3.247 36	3.249 05
H_3	3/4	0.749 85	0.748 56	9/2	4.4991	4.4939

because at some k the radius of convergence of the necessary expansion of the r.h.s. in powers of z goes to zero, a fact related to the vanishing of the terms quadratic in the fields in the bare Hamiltonian of the $n = 2$ model. If no expansion is performed, as in the numerical integration of the flow equation for H_k, no singularity is met and the ground state and gap can be estimated by the value of H_k and of $(\det H_k^{(2)})^{1/2}$ at the minimum. However, these estimates do not reach a great accuracy either because of spurious dependence on the boundary conditions (which can be controlled by some nonlinear redefinitions of H_k) or because of numerical errors: typically we reached no more than two digit accuracy in the region around the minimum. To get stable predictions with a precision better than 1%, we turned to a different choice of regulators, curing the problem about the polynomial expansion of the flow equation. Such a choice is that of a diagonal regulator, as in Eq. (43). We chose this regulator to be constant, i.e., $\mathcal{R}_k^p = \mathcal{R}_k^q = \mathcal{R}$ a multiplicative operator (recall that we are assuming $\omega^2 = 1$; therefore even if $\mathcal{R}_k^p$ and $\mathcal{R}_k^q$ have different dimensions, we can set them equal if we assume their ratio to be some power of ω). We also introduced a UV cutoff Λ to control the convergence of the flow for $\mathcal{R} \to \infty$. As a result, we observed that, for such a constant regulator, Λ can be removed only after the integration of the flow from $\mathcal{R} = \infty$ to $\mathcal{R} = 0$. The resulting flow equation in the LHA is

$$\partial_{\mathcal{R}} H_{\mathcal{R}} = -\frac{1}{\pi}\arctan\left(\frac{\Lambda}{\mathcal{R}}\right) + \frac{2\mathcal{R} + \partial_{\bar{p}}^2 H_{\mathcal{R}} + \partial_{\bar{q}}^2 H_{\mathcal{R}}}{2\pi \mathcal{D}_{\mathcal{R}}} \times \arctan\left(\frac{\Lambda}{\mathcal{D}_{\mathcal{R}}}\right), \tag{43}$$

where we defined

$$\mathcal{D}_{\mathcal{R}} = \sqrt{\mathcal{R}^2 + \mathcal{R}(\partial_{\bar{p}}^2 H_{\mathcal{R}} + \partial_{\bar{q}}^2 H_{\mathcal{R}}) + \det H_{\mathcal{R}}^{(2)}}.$$

In this scheme good estimates for the ground state energy E_0 and the energy gap $\Delta E_1 = E_1 - E_0$ can be obtained by simple polynomial truncations. For a bare Hamiltonian that is a polynomial of order n we consider two cases: a truncation with a polynomial of the same order n and another of order $n + 1$. In the latter case we add a superscript $+1$ to the corresponding quantities E_0^{+1} and ΔE_1^{+1}. In Table I we give the results obtained by choosing as an initial condition both the Weyl-ordered H_{nW} and the Weyl-uncorrected Hamiltonian H_n.

We note that the quantities E_0 and ΔE_1 depend on the local properties of the effective Hamiltonian at the minimum ($\bar{p} = \bar{q} = 0$) and therefore can be extracted with a good approximation adopting simple polynomial truncations. From Table I we see that there is no clear pattern on the change of the precision of the results when increasing the order of the truncation. In the worst case we find a relative error of order 10^{-3}. To achieve a better accuracy, going to next-to-leading order in the derivative expansion would probably do the job.

We remark that for the first time in the functional RG approach one faces the ordering problem in the choice of the bare Hamiltonian function, which corresponds to the initial condition for the flow. This feature generally extends to QFT, and therefore one needs to keep it in mind before interpreting the results obtained by choosing an initial condition that is nonseparable in p and q.

D. The average effective Hamiltonian action in fermionic quantum mechanics

Since fermions usually have a first order dynamics, the Hamiltonian formulation of it is identical to the Lagrangian one. Therefore the AEHA formalism in this case is identical to the traditional Lagrangian approach. For completeness we will briefly review it in this subsection.

Let us consider as an example a free system whose Lagrangian variables are n real Grassmann-valued functions of time: $\{\theta^i(t)\}_{i=1,\dots,n}$, evolving according to the following Lagrangian:

$$L(\theta(t), \partial_t \theta(t)) = \frac{1}{2}\theta^i(t) i \partial_t \theta^j(t) \delta_{ij} - V(\theta^i(t)). \tag{44}$$

Defining the momenta π_i as the right partial derivatives of L with respect to $\partial_t \theta^i$, we find n second class primary constraints

$$\chi_i(t) = \pi_i(t) - \frac{i}{2}\delta_{ij}\theta^j(t) = 0, \tag{45}$$

which cause the canonical Hamiltonian $H = \pi_i \partial_t \theta^i - L = V(\theta^i)$ to be independent of π_i. The relevant phase space is the surface $\mathcal{S}$ defined by (45), a complete set of independent coordinates on it is given by θ^i and the functional integral is to be taken over all paths $\theta^i(t)$ lying on

this surface. In the presence of second class constraints and assuming that the whole phase space is endowed with a symplectic structure $\sigma = d\lambda$, we can define a nondegenerate symplectic form $\tilde{\sigma} = \tilde{d}\,\tilde{\lambda}$ on the reduced phase space, simply by restricting σ to $\mathcal{S}$. As the inverse of σ is the Poisson bracket $[\,,\,]$, the inverse of $\tilde{\sigma}$ is the Dirac bracket $[\,,\,]_\sim$, which in the reduced phase-space coordinates θ^i has components $[\theta^i, \theta^j]_\sim = -i\delta^{ij} = [\chi_i, \chi_j]$. The kinetic term in (44) can be interpreted as the Legendre transform term on $\mathcal{S}$, i.e., as the pullback of the symplectic potential $\tilde{\lambda}$ by a section $\theta^i(t)$. The appropriate measure for functional integration over $\mathcal{S}$ is again the square root of the superdeterminant of the symplectic form $\tilde{\sigma}$ [30]. In conclusion, the functional integral over the reduced phase space reads

$$Z = \int [d\theta]\mu[\theta] e^{iS[\theta]},$$
$$S[\theta] = \int dt \left[\frac{1}{2}\theta^i i\partial_t \theta^j \delta_{ij} - V(\theta^i)\right]. \tag{46}$$

Following the same coarse-graining scheme explained in the previous subsections we modify the symplectic structure of the reduced phase space by replacing $\tilde{\sigma}$ with $\tilde{\sigma}_k = (1 + r_k)\tilde{\sigma}$. This is tantamount to adding the term $\Delta S_k[\theta] = \int dt[\frac{1}{2}\theta^i r_k(-\partial_t^2) i\partial_t \theta^j \delta_{ij}]$ to the bare action. Correspondingly the functional measure becomes $\mu_k = (\mathrm{SDet}\frac{\tilde{\sigma}_k}{2\pi})^{1/2} = \mu(\mathrm{SDet}(1 + r_k)\delta)^{1/2}$, where δ stands for a product of Dirac and Kronecker deltas. Then the modified path integral reads

$$Z_k[J_i] = \int [d\theta]\mu_k[\theta] e^{i\{S[\theta] + \Delta S_k[\theta] + J_i \cdot \theta^i\}}. \tag{47}$$

Starting from it, one defines the AEHA

$$\Delta S_k[\bar{\theta}] + \Gamma_k[\bar{\theta}^i] = \operatorname*{ext}_{J_i}(W_k[J^i] - J_i \cdot \bar{\theta}^i), \tag{48}$$

which satisfy the following integro-differential equation:

$$e^{i\Gamma_k[\bar{\theta}^i]} = \int [d\theta]\mu_k[\theta] e^{i\{S[\theta] + \Delta S_k[\theta - \bar{\theta}] - \Gamma_k \overleftarrow{\frac{\delta}{\delta\bar{\theta}^i}} \cdot (\theta - \bar{\theta})^i\}}, \tag{49}$$

and therefore the $k \longrightarrow \Lambda$ limit of Γ_k is just the bare action. The flow equation for Γ_k reads

$$i\dot{\Gamma}_k = -\frac{1}{2}\mathrm{Tr}[\dot{r}_k(1 + r_k)^{-1}\delta] + \frac{1}{2}\mathrm{Tr}\left[(\dot{r}_k i\partial_t \delta)\left(r_k i\partial_t \delta + \frac{\vec{\delta}}{\delta\bar{\theta}}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\bar{\theta}}\right)^{-1}\right], \tag{50}$$

where the trace is over $\{i, j\}$ indices as well and, as in the bosonic case, in the matrix $r_k i\partial\delta$ the derivatives act on the first index.

III. THE EFFECTIVE HAMILTONIAN ACTION IN QUANTUM FIELD THEORY

There are at least two possible generalizations of the previous formalism to QFT.

The simplest can be obtained by embracing the traditional Hamiltonian formulation of field theory, where one associates a canonically conjugate field (momentum) with the time derivative of each Lagrangian coordinate. This choice leads to a noncovariant formulation. The translation of all previous formulas to this framework can be obtained by replacing the bare Hamiltonian with the spatial integral of a Hamiltonian density and by promoting the integrals and functional traces to sums over spatial positions as well as time instants. In this way one can obtain a formal definition of the noncovariant effective Hamiltonian action and extend all the previous discussions developed in Sec. II A.

However, in so doing, willing to construct the corresponding coarse-graining procedure for the flow of the average effective Hamiltonian action, one faces the necessity to regularize the spatial part of these summations, which are otherwise ill defined. In other words, the regulator matrix R_k, appearing in ΔS_k and μ_k, must now contain operators depending on spatial derivatives too. For instance, choosing an off-diagonal R_k one could consider

$$R_k(x, x') = \begin{pmatrix} 0 & r_k(-\Box)\partial_0\delta(x - x') \\ -r_k(-\Box)\partial_0\delta(x - x') & 0 \end{pmatrix}\mu_k$$
$$= \left[\mathrm{Det}\frac{1}{2\pi}\begin{pmatrix} 0 & (1 + r_k(-\Box))\partial_0\delta(x - x') \\ -(1 + r_k(-\Box))\partial_0\delta(x - x') & 0 \end{pmatrix}\right]^{\frac{1}{2}},$$

but this choice would explicitly break Lorentz symmetry. Instead, it would be easy to write more general regulators preserving such a symmetry, even if in an implicit form. In both cases one may study the AEHA defined by the integro-differential equation

$$e^{i\Gamma_k[\bar{\pi},\bar{\phi}]} = \int [d\pi d\phi]\mu_k[\pi, \phi] e^{i\{S[\pi,\phi] + \Delta S_k[\pi - \bar{\pi}, \phi - \bar{\phi}] - (\pi - \bar{\pi})\frac{\delta\Gamma_k}{\delta\bar{\pi}} - (\phi - \bar{\phi})\frac{\delta\Gamma_k}{\delta\bar{\phi}}\}}.$$

This road could be useful if one is interested in nonrelativistic field theories, but for relativistic systems, since Lorentz invariance is not manifest, in this framework it is hard to distinguish truncations for Γ_k that are Lorentz symmetric from those that are not (one would have to deal with Ward-Takahashi-Slavnov-Taylor identities).

Another possibility is to choose a covariant Hamiltonian formalism, in which one introduces a momentum field for each first order partial derivative of the Lagrangian coordinates, thus preserving manifest Lorentz covariance. In the following we will give the two simplest examples of how this could work: spin zero and spin one-half field theories. There are several choices one can do. In this work we shall attempt to use a *reduced* approach, which has the advantage of being the minimal extension that on one side preserves the general results in $0+1$ dimensions (QM) and on the other side leads to the usual QFT results in the case of quadratic bare Hamiltonians. More general formulations as well as specific applications will be considered elsewhere.

A. Covariant Hamiltonian scalar field theory

Let us build the covariant Hamiltonian formulation of a classical unconstrained single scalar field in d space-time dimensions with the standard Lagrangian density

$$\mathcal{L}(\phi, \partial_\nu\phi) = -\frac{1}{2}(\partial_\nu\phi)(\partial^\nu\phi) - V(\phi)$$

(in a Minkowski mostly plus signature). The covariant Hamiltonian density is defined as the extremum

$$\begin{aligned}\mathcal{H}(\pi^\nu, \phi) &= \operatorname*{ext}_{\partial_\nu\phi}(-\pi^\nu\partial_\nu\phi - \mathcal{L}(\phi, \partial_\nu\phi)) \\ &= -\frac{\pi_\nu\pi^\nu}{2} + V(\phi), \end{aligned} \tag{51}$$

and by demanding the stationarity of the Hamiltonian action

$$S = \int d^dx[-\pi^\nu\partial_\nu\phi - \mathcal{H}], \tag{52}$$

one finds the De Donder–Weyl equations

$$\pi^\nu = \partial^\nu\phi, \qquad \partial_\nu\pi^\nu = V'(\phi),$$

i.e., a first order system equivalent to $\Box\phi - V'(\phi) = 0$. Here the dynamics of π^ν and ϕ seem to be completely coupled; however, this is not the case. In fact, the Lorentz vector π^ν can be decomposed into a transverse and a gradient part $\pi^\nu = \pi^\nu_\perp + \pi^\nu_\parallel$, by means of the standard projectors $\Pi^{\mu\nu}_\parallel = \partial^\mu\partial^\nu/\Box$ and $\Pi^{\mu\nu}_\perp = \eta^{\mu\nu} - \Pi^{\mu\nu}_\parallel$. Rewriting the Hamiltonian action density in terms of these reduced degrees of freedom (and assuming that the boundary terms coming from integration by parts do not contribute), one finds $-\pi^\nu_\parallel\partial_\nu\phi - \mathcal{H}$ with

$$\mathcal{H}(\pi^\nu_\perp, \pi^\nu_\parallel, \phi) = -\frac{\pi_{\perp\nu}\pi^\nu_\perp}{2} - \frac{\pi^\nu_\parallel\pi_{\parallel\nu}}{2} + V(\phi)$$

and the corresponding Hamiltonian equations

$$\pi^\nu_\parallel = \partial^\nu\phi, \qquad \partial_\nu\pi^\nu_\parallel = V'(\phi), \qquad \pi^\nu_\perp = 0.$$

Hence the transverse momenta are classically irrelevant if the Hamiltonian is quadratically depending on them. This translates into the following quantum property: if the bare Hamiltonian is separable in π and ϕ and quadratically depending on π, the functional integration over transverse momenta factorizes from those on the other two fields.

Now let us address the possibility to extend this formalism to covariant Hamiltonian densities that are more than quadratic in the momenta. The classical decoupling of the transverse momenta, i.e., their factorization in the functional integral, can happen also for nonquadratic Hamiltonians, such as $\mathcal{H} = T(\pi^\mu\pi_\mu) + V(\phi)$. Insisting on the validity of the classical variational principle for the action (52), the classical equations read

$$\partial_\nu\pi^\nu_\parallel = \frac{\delta}{\delta\phi}\int d^dx\mathcal{H}, \qquad -\partial_\nu\phi = \frac{\delta}{\delta\pi^\nu_\parallel}\int d^dx\mathcal{H},$$
$$\frac{\delta}{\delta\pi^\nu_\perp}\int d^dx\mathcal{H} = 0.$$

The interesting question now is whether the third equation is a constraint or it gives a dynamics to the transverse momenta. If $\mathcal{H}$ does not contain derivatives of $\pi^\nu_\perp$, and if one can perform some sort of Fourier transform such that $\pi^\nu_\perp$ can be considered orthogonal to ∂^ν with respect to the metric in Minkowski space-time, then the third equation cannot contain derivatives of $\pi^\nu_\perp$. Therefore, under these assumptions, one can always solve the third equation by writing $\pi^\nu_\perp$ as a local (if $\mathcal{H}$ is local) function of $\pi^\nu_\parallel$, ϕ and their derivatives. By substituting this solution in the first two equations, one gets a coupled dynamics for the unconstrained variables $\pi^\nu_\parallel$ and ϕ only. That is, under these assumptions the transverse momenta do not have their own independent dynamics and behave only as redundant variables that can be eliminated without losing the locality of the action. However, even in this case the quantization of the theory containing the $\pi^\nu_\perp$ fields is not equivalent to the quantization of the theory in which one got rid of them by means of the classical equations, since in the first case one has a full functional integral over $\pi^\nu_\perp$, whose stationary phase approximation gives the second quantum theory. Nevertheless, considering a Hamiltonian action depending on parallel momenta only, although it is not the most general case, is already a consistent and covariant generalization of the standard noncovariant Hamiltonian approach, reproducing the known results for quadratic Hamiltonians. Therefore in this paper we will restrict ourselves to such a case.

The aim of the rest of this section is to give meaning to the quantization of the classical theory with the bare action (52) under the assumption that $\mathcal{H}$ depends on $\pi^\nu_\parallel$ only. Since in this case the bare action S does not depend on $\pi^\nu_\perp$, we are in the presence of a gauge symmetry: by introducing projectors where needed, S can be rewritten in a form that is manifestly invariant under the infinitesimal

transformation: $\delta\pi^\nu(x) = \Pi_\perp^{\nu\rho}\epsilon_\rho(x)$, for any infinitesimal vector field ϵ. In this paper we will discuss the functional integral quantization of the theory by means of the introduction of the constraint $\Pi_\perp^{\nu\rho}\pi_\rho = 0$ in the functional measure (something like a sharp gauge fixing[1]). Thus, the generating functional of the theory will be

$$Z[I_\nu, J] = e^{iW[I_\nu,J]} = \int [d\pi^\nu d\phi]\delta[\Pi_\perp^{\nu\rho}\pi_\rho]\mu e^{i\{S[\pi^\nu,\phi]+I_\nu\cdot\pi^\nu+J\cdot\phi\}}. \tag{53}$$

Notice that, depending on which regularization and precise definition of the functional integral is chosen, the functional integration over $[d\pi_\parallel^\nu]$ and the constrained integration $[d\pi^\nu]\delta[\Pi_\perp^{\nu\rho}\pi_\rho]$ could differ by a field-independent Jacobian determinant. A skeletonized definition in Fourier space, i.e., the use of a discretization of Fourier space, would make this Jacobian to be equal to one. Whenever such a Jacobian is unity, since the constraint kills all but one of the integrals over the π's, the usual functional measure $\mu = \mathrm{Det}\frac{1}{2\pi}$ provides the normalization needed to reproduce the known results for bare Hamiltonian actions quadratic in the momenta. Otherwise μ needs to be different (but still field independent) to balance the Jacobian determinant. Starting from Eq. (53) the definition of the effective Hamiltonian action is again

$$\Gamma[\bar\pi^\nu, \bar\phi] = \operatorname*{ext}_{I_\nu,J}(W[I_\nu, J] - I_\nu\cdot\bar\pi^\nu - J\cdot\bar\phi), \tag{54}$$

which is equivalent to state that Γ is the solution of the following integro-differential equation with suitable boundary conditions:

$$e^{i\Gamma[\bar\pi^\nu,\bar\phi]} = \int [d\pi^\nu d\phi]\delta[\Pi_\perp^{\nu\rho}\pi_\rho] \times \mu e^{i\{S[\pi^\nu,\phi]-(\pi-\bar\pi)^\nu\frac{\delta\Gamma}{\delta\bar\pi^\nu}-(\phi-\bar\phi)\frac{\delta\Gamma}{\delta\bar\phi}\}}. \tag{55}$$

In the following we shall try to give a definition of the integrals (53) and (55) based on an RG flow equation for the average version of the effective action. First of all, one has to introduce k-dependent operators that disappear in the $k \to 0$ limit and that provide a rising delta functional in the $k \to \Lambda$ limit. As before let us denote this regularization as follows:

$$Z_k[I_\nu, J] = \int [d\pi^\nu d\phi]\delta[\Pi_\perp^{\nu\rho}\pi_\rho] \times \mu_k e^{i\{S[\pi^\nu,\phi]+\Delta S_k[\pi^\nu,\phi]+I_\nu\cdot\pi^\nu+J\cdot\phi\}}.$$

We will choose a regularization corresponding to a k-dependent deformation of the term whose one-dimensional version is the Legendre transform term, i.e., $-\pi^\mu\partial_\mu\phi$. In other words, we will restrict to an off-diagonal R_k, or more explicitly

$$\Delta S_k[\pi^\nu, \phi] = \int d^dx[-\pi^\nu r_k(-\Box)\partial_\nu\phi],$$
$$\mu_k = \mu\left[\mathrm{Det}\begin{pmatrix} 0 & -(1 + r_k(-\Box))\partial_\nu\delta(x - x') \\ (1 + r_k(-\Box))\partial_\nu\delta(x - x') & 0 \end{pmatrix}\right]^{\frac{1}{2}}. \tag{56}$$

The definition of the AEHA is the same as in quantum mechanics

$$\Gamma_k[\bar\pi^\nu, \bar\phi] + \Delta S_k[\bar\pi^\nu, \bar\phi] = \operatorname*{ext}_{I_\nu,J}(W_k[I_\nu, J] - I_\nu\cdot\bar\pi^\nu - J\cdot\bar\phi), \tag{57}$$

from which the usual integro-differential equation

$$e^{i\Gamma_k[\bar\pi^\nu,\bar\phi]} = \int [d\pi^\nu d\phi]\delta[\Pi_\perp^{\nu\rho}\pi_\rho]\mu_k e^{i\{S[\pi^\nu,\phi]+\Delta S_k[(\pi-\bar\pi)^\nu,\phi-\bar\phi]-(\pi-\bar\pi)^\nu\frac{\delta\Gamma_k}{\delta\bar\pi^\nu}-(\phi-\bar\phi)\frac{\delta\Gamma_k}{\delta\bar\phi}\}}. \tag{58}$$

By taking the $k\partial_k$ derivative of Eq. (58) one finds

$$i\dot\Gamma_k = \frac{\dot\mu_k}{\mu_k} - i\int d^dx\langle(\pi - \bar\pi)^\nu\dot r_k\partial_\nu(\phi - \bar\phi)\rangle. \tag{59}$$

For the second term, we need to write the two point function in terms of derivatives of Γ_k. Since this theory contains one Lagrangian coordinate and one momentum, $\Gamma^{(2)}$ is a two-dimensional square matrix, as in quantum mechanics. However, our momentum is a vector field bringing a Lorentz index, and even if it lies in a one-dimensional subspace, such a subspace varies from point to point in space-time. Thus, unless we want to choose a frame in the tangent bundle such that at every space-time point x the vector $\pi^\nu(x)$ has only one and the same non-vanishing component, we are forced to deal with it as a generic Lorentz vector. Since we prefer to write formulas in a generic frame, we will treat $\Gamma^{(2)}$ as a generic $(d + 1)$-dimensional square matrix, whose entries can be written as four blocks: a (1,1) tensor (d-by-d square

[1]Dirac's classification of constraints and the consequent quantization schemes for gauge theories are based on the non-covariant Hamiltonian formalism and therefore are not straightforwardly applicable to the present case. However, classical constrained dynamics has been extensively discussed in the literature about the covariant Hamiltonian formalism(s) [18], and some proposals have been provided about the corresponding path integral quantization of gauge theories [25].

matrix), one contravariant (column) vector, one covariant (row) vector, and one Lorentz scalar. Because the momenta enter the theory naturally with high indices (to be contracted with derivatives), we will treat them as column vectors. Therefore the source I will become a row vector. We will denote by $()^t$ the transposition of these objects, that is, the canonical isomorphism defined by the space-time metric. Thus π^t and I^t will denote row and column vectors, respectively. Of course, derivatives with respect to contravariant (covariant) vectors will be considered covariant (contravariant). Going back to the task of computing the two point functions, since

$$i\left\langle \mathcal{T}\begin{pmatrix} (\pi-\bar\pi)_x\otimes(\pi-\bar\pi)^t_{x'} & (\pi-\bar\pi)_x(\phi-\bar\phi)_{x'} \\ (\phi-\bar\phi)_x(\pi-\bar\pi)^t_{x'} & (\phi-\bar\phi)_x(\phi-\bar\phi)_{x'}\end{pmatrix}\right\rangle_k = W_k^{(2)}{}_{xx'}[I,J] = \begin{pmatrix} \frac{\delta W_k}{\delta I_x}\otimes\left(\frac{\overleftarrow{\delta}}{\delta I_{x'}}\right)^t & \frac{\delta^2 W_k}{\delta J_{x'}\delta I_x} \\ \left(\frac{\delta^2 W_k}{\delta I_{x'}\delta J_x}\right)^t & \frac{\delta^2 W_k}{\delta J_{x'}\delta J_x}\end{pmatrix}$$

one needs an explicit expression for the vector $\frac{\delta^2 W_k}{\delta J\delta I}$ in terms of Γ_k. This can be found by using

$$I_\mu = r_k\partial_\mu\bar\phi - \frac{\delta\Gamma_k}{\delta\bar\pi^\mu}, \qquad J = -r_k\partial_\nu\bar\pi^\nu - \frac{\delta\Gamma_k}{\delta\bar\phi},$$

and thus getting

$$\begin{aligned} W_k^{(2)}{}_{xx'}[I,J] &= \begin{pmatrix} \bar\pi\otimes\left(\frac{\overleftarrow{\delta}}{\delta I}\right)^t & \frac{\delta\bar\pi}{\delta J} \\ \left(\frac{\delta\bar\phi}{\delta I}\right)^t & \frac{\delta\bar\phi}{\delta J}\end{pmatrix}_{xx'} = \begin{pmatrix} I^t\otimes\frac{\overleftarrow{\delta}}{\delta\bar\pi} & (\frac{\delta I}{\delta\bar\phi})^t \\ \frac{\delta J}{\delta\bar\pi} & \frac{\delta J}{\delta\bar\phi}\end{pmatrix}^{-1}_{xx'} = -\begin{pmatrix} \left(\frac{\delta\Gamma_k}{\delta\bar\pi}\right)^t\otimes\frac{\overleftarrow{\delta}}{\delta\bar\pi} & \left(-r_k\partial\delta+\frac{\delta^2\Gamma_k}{\delta\bar\phi\delta\bar\pi}\right)^t \\ r_k\partial\delta+\frac{\delta^2\Gamma_k}{\delta\bar\pi\delta\bar\phi} & \frac{\delta^2\Gamma_k}{\delta\bar\phi\delta\bar\phi}\end{pmatrix}^{-1}_{xx'} \\ &\equiv -\begin{pmatrix} A & B \\ B^T & D\end{pmatrix}^{-1}_{xx'}, \end{aligned}$$

where $(r_k\partial\delta)_{x_1x_2} = r_k(-\partial^2_{x_1})\partial_{x_1}\delta(x_1-x_2)$ is a Lorentz-covariant (row) vector. This matrix is manifestly symmetric with respect to full transposition T of both Lorentz and space-time-position indices. Since the building blocks B and B^T are not square matrices, we cannot use formula (14). Anyway, if A and $(D-B^TA^{-1}B)$ are nonsingular, this becomes

$$W_k^{(2)}[I,J] = -\begin{pmatrix} A^{-1}+A^{-1}B(D-B^TA^{-1}B)^{-1}B^TA^{-1} & -A^{-1}B(D-B^TA^{-1}B)^{-1} \\ -(D-B^TA^{-1}B)^{-1}B^TA^{-1} & (D-B^TA^{-1}B)^{-1}\end{pmatrix}. \tag{60}$$

If instead D and $(A-BD^{-1}B^T)$ are nonsingular, then we can write

$$W_k^{(2)}[I,J] = -\begin{pmatrix} (A-BD^{-1}B^T)^{-1} & -(A-BD^{-1}B^T)^{-1}BD^{-1} \\ -D^{-1}B^T(A-BD^{-1}B^T)^{-1} & D^{-1}+D^{-1}B^T(A-BD^{-1}B^T)^{-1}BD^{-1}\end{pmatrix}. \tag{61}$$

The off-diagonal entries of these matrices can finally be plugged into Eq. (59). Thus, if, for instance, A and $(D-B^TA^{-1}B)$ are nonsingular, the final flow equation is

$$\begin{aligned} i\dot\Gamma_k = {}& \mathrm{Tr}[\dot r_k(1+r_k)^{-1}\delta] - \mathrm{Tr}\Bigg[\left(r_k\partial\delta+\frac{\delta^2\Gamma_k}{\delta\bar\pi\delta\bar\phi}\right)\left(\frac{\delta^2\Gamma_k}{\delta\bar\pi\delta\bar\pi}\right)^{-1} \\ &\times(\dot r_k\partial\delta)\left[\frac{\delta^2\Gamma_k}{\delta\bar\phi\delta\bar\phi}-\left(r_k\partial\delta+\frac{\delta^2\Gamma_k}{\delta\bar\pi\delta\bar\phi}\right)\left(\frac{\delta^2\Gamma_k}{\delta\bar\pi\delta\bar\pi}\right)^{-1}\left(r_k\partial\delta+\frac{\delta^2\Gamma_k}{\delta\bar\pi\delta\bar\phi}\right)^T\right]^{-1}\Bigg]. \end{aligned} \tag{62}$$

Here for the sake of notational simplicity we dropped the symbols for tensor products and Lorentz transpositions. By means of Eq. (61) the reader can write down a similar flow equation for the case in which D and $(A-BD^{-1}B^T)$ are nonsingular.

As an example let us discuss the LHA for a scalar theory enjoying Z_2 symmetry under simultaneous reflections: $\pi^\nu\to-\pi^\nu$, $\phi\to-\phi$. In other words, we are going to insert the approximation $\Gamma_k = \int d^dx(-\bar\pi^\nu\partial_\nu\bar\phi - \mathcal{H}_k(\frac{\bar\pi^2}{2},\frac{\bar\phi^2}{2}))$, where $\bar\pi^2\equiv\bar\pi^\nu\bar\pi_\nu$, in the previous flow equation. To project the r.h.s. of the flow equation inside such an ansatz for Γ_k, one usually evaluates it on constant field configurations. This can be done also in the present

case, without contradicting the assumption that the momenta $\bar{\pi}^\nu$ be longitudinal, by choosing the Fourier transform of $\bar{\pi}^\nu$ pointing in the same direction of the Fourier variable and being proportional to a delta function. We will denote by $\mathcal{H}_k^{(i,j)}$ the result of differentiating $\mathcal{H}_k$ i times w.r.t. $\frac{\bar{\pi}^2}{2}$ and j times w.r.t. $\frac{\bar{\phi}^2}{2}$. Let us recall the notation already used in quantum mechanics [see Eq. (34)] for the regulator in the LHA, i.e., $P_k(-\Box) = (1 + r_k(-\Box))^2 \times (-\Box)$. Let us also introduce for convenience the function

$$\sigma_d(\alpha) = {}_2F_1\left(\frac{1}{2}, 1; \frac{d}{2}; \alpha\right) \tag{63}$$

and the following threshold functional

$$l_0^d[\alpha, \beta] = \frac{1}{4} v_d^{-1} k^{-d} \int \frac{d^d p}{(2\pi)^d} \times \frac{\dot{P}_k(p^2)}{P_k(p^2) + k^2 \beta(p^2)} \sigma_d(\alpha(p^2)), \tag{64}$$

where $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(\frac{d}{2})$. Then the flow equation for the dimensionful average effective Hamiltonian density can be written

$$i\dot{\mathcal{H}}_k = 2 v_d k^d (l_0^d[\alpha_{\mathcal{H}}, \beta_{\mathcal{H}}] - l_0^d[\alpha_{\mathcal{H}}, 0]), \tag{65}$$

where we further defined the dimensionless quantities

$$\alpha_{\mathcal{H}}(p^2) = \frac{P_k(p^2)}{P_k(p^2) + k^2 \beta_{\mathcal{H}}} \frac{\bar{\pi}^2 \mathcal{H}_k^{(2,0)}}{\mathcal{H}_k^{(1,0)} + \bar{\pi}^2 \mathcal{H}_k^{(2,0)}}, \tag{66}$$

$$\beta_{\mathcal{H}} = \frac{1}{k^2}\left[\bar{\pi}^2 \bar{\phi}^2 (\mathcal{H}_k^{(1,1)})^2 \frac{\mathcal{H}_k^{(1,0)}}{\mathcal{H}_k^{(1,0)} + \bar{\pi}^2 \mathcal{H}_k^{(2,0)}} - \mathcal{H}_k^{(1,0)}(\mathcal{H}_k^{(0,1)} + \bar{\phi}^2 \mathcal{H}_k^{(0,2)})\right], \tag{67}$$

the second of which is not a function of p^2. First of all, let us notice that if we make the ansatz that the theory is quadratic in the momenta at every scale, then the vanishing of $\mathcal{H}_k^{(2,0)}$ entails the vanishing of $\alpha_{\mathcal{H}}$ and we recover the Lagrangian flow in the LPA. If instead $\alpha_{\mathcal{H}}$ is nonvanishing, the presence of a p-dependent denominator in the argument of the function σ_d in general makes the analytic computation of l_0^d quite hard. For this reason it is wise to choose the regulator in such a way to kill the p dependence of all the denominators. In the LHA this can be accomplished by means of the optimized regulator $r_k(p^2) = (k/\sqrt{p^2} - 1)\theta(k^2 - p^2)$, i.e., $P_k(p^2) = (k^2 - p^2)\theta(k^2 - p^2)$. For such a choice

$$\alpha_{\mathcal{H}}(p^2) = \frac{1}{1 + \beta_{\mathcal{H}}} \frac{\bar{\pi}^2 \mathcal{H}_k^{(2,0)}}{\mathcal{H}_k^{(1,0)} + \bar{\pi}^2 \mathcal{H}_k^{(2,0)}} \tag{68}$$

is p independent, and the threshold function for constant argument becomes

$$l_0^d[\alpha, \beta] = \frac{2}{d} \frac{1}{1 + \beta} \sigma_d(\alpha).$$

To sum up, for the optimized regulator the flow equation of the LHA reads (after Wick rotation)

$$\dot{\mathcal{H}} = -\frac{4}{d} v_d k^d \frac{\beta_{\mathcal{H}}}{1 + \beta_{\mathcal{H}}} \sigma_d(\alpha_{\mathcal{H}}) \tag{69}$$

with $\beta_{\mathcal{H}}$ and $\alpha_{\mathcal{H}}$ given by (67) and (68). The function σ_d takes simpler forms for integer d. For instance, in $d = 2$, $d = 3$, and $d = 4$ it, respectively, reads

$$\sigma_2(\alpha) = (1 - \alpha)^{-\frac{1}{2}}, \qquad \sigma_3(\alpha) = \frac{\operatorname{arctanh}(\sqrt{\alpha})}{\sqrt{\alpha}}, \qquad \sigma_4(\alpha) = \frac{2}{\alpha}[1 - (1 - \alpha)^{-\frac{1}{2}}]. \tag{70}$$

Equation (65) can be taken as a first step toward the nonperturbative study of scalar QFT in the covariant Hamiltonian formalism. In particular, one of the first questions to be addressed is whether such an equation admits non-Gaussian fixed points. In case a positive answer exists, these could provide a possible solution to the triviality problem of scalar QFT in four dimensions. In fact, choosing the engineering dimensions of the fields in such a way that the coefficients of the $\bar{\pi}^2$ and Legendre terms are dimensionless, dimensional analysis tells us that the coupling multiplying the operator $(\bar{\pi}^2)^i(\bar{\phi}^2)^j$ has dimensionality $d_{ij} = (1 - i - j)d + 2j$. Therefore in $d = 4$ the only momentum dependent non IR-irrelevant term is $\bar{\pi}^2$, and all other terms with positive integers (i, j) are IR irrelevant. In other words, scalar theories more than quadratic in the momenta are expected to be highly favored in the UV and to be well approximated by quadratic theories in the IR. From this point of view it seems reasonable to look for the UV completion of four-dimensional scalar QFT in a general Hamiltonian framework. For instance, this could be done according to the paradigm of asymptotic safety [13], i.e., by looking for nontrivial fixed points of the RG flow having a finite dimensional UV critical surface (a finite number of UV attractive directions in theory space). On the other hand, this very same argument in the case of a simpler scalar QFT in configuration space is often used for a qualitative understanding of the absence of Z_2-symmetric non-Gaussian fixed points in $d = 4$: in this case the only IR-relevant monomial-like operator is the mass term, all other monomials being either marginal or IR-irrelevant. Nevertheless, since in the present formulation the theory contains not only a scalar field but also a longitudinal vector field, we believe that the understanding of this issue requires explicit computations in order to reveal the details of the underlying dynamics.

Another interesting question regarding Eq. (65) is whether it can teach us to what extent the covariant Hamiltonian framework adopted in this paper is sound and useful. In particular, it would be interesting to

compare, within a fixed approximation such as the LHA, the RG flow of the traditional noncovariant Hamiltonian formulation with that of the covariant one allowing for longitudinal momenta only (the present case) and with the one allowing also for transverse momenta. These and other questions will be left open by the present work.

B. Spinor field theory

Let us build the covariant Hamiltonian formulation of a classical Lagrangian field theory for a single Dirac field in a number d (allowing Dirac spinors) of space-time dimensions with the standard Lagrangian density

$$\mathcal{L}(\psi, \partial_\nu \psi) = -\bar{\psi} \not{\partial} \psi - V(\bar{\psi}, \psi)$$

(in a Minkowski mostly plus signature) where $\bar{\psi} = i\psi^\dagger \gamma^0$. Defining the momenta π^ν as the right partial derivatives of $-\mathcal{L}$ with respect to $\partial_\nu \psi$, we find d second class primary constraints

$$\chi^\nu(x) = \pi^\nu(x) - \bar{\psi}(x)\gamma^\nu = 0, \tag{71}$$

whose solution is $\bar{\psi} = \frac{1}{d}\pi^\nu \gamma_\nu$. These constraints boil down the momenta to functions of just one field; hence there is no room here for the other $d-1$ conjugate fields that in the bosonic case could be identified with the transverse momenta. The relevant phase space is the surface $\mathcal{S}$ defined by (71), the only independent coordinate on it is ψ, and the functional integral is to be taken over all histories $\psi(x)$. The covariant Hamiltonian density is defined as

$$\begin{aligned}\mathcal{H}(\pi^\nu, \psi) &= \operatorname*{ext}_{\partial_\nu \psi}(-\pi^\nu \partial_\nu \psi - \mathcal{L}(\psi, \partial_\nu \psi)) \\ &= V\left(-\frac{1}{d}\pi^\nu \gamma_\nu, \psi\right),\end{aligned}$$

and on $\mathcal{S}$ it is just $V(\bar{\psi}, \psi)$. Thus the covariant AEHA formalism in this case is equivalent to the usual Lagrangian approach, exactly as was previously described for fermionic QM; one just has to replace time derivatives with $\not{\partial}$ operators.

IV. CONCLUSIONS

In this work we have focused on the description of quantum dynamics by means of the quantum effective Hamiltonian action (EHA). We have first reviewed its properties by a discussion in quantum mechanics, taking advantage of the fact that QM and noncovariant QFT's are very similar in this respect. We have then discussed how to compute the effective action. For instance, we have derived a general one-loop formula, which can be useful to compare the results obtained by other approaches, and we have generalized the variational definition provided a long time ago by Jackiw and Kerman [26] for its Lagrangian counterpart. But the main goal of this work is to provide an alternative nonperturbative tool to compute the EHA. This is a Hamiltonian generalization of the so-called functional renormalization group, in particular, of the formulation by Wetterich based on the average effective (Lagrangian) action [7].

Such a generalization, which is one of the main results of our work, is straightforward in QM, even if the family of one-parameter-dependent cutoff operators is wider and in general the formulas are more cumbersome. Starting from the most general flow equation, we have derived simpler equations as the one associated with the so-called LHA, i.e., the leading order in the derivative expansion. To show that the approach is trustworthy, we have studied, as an example, a family of quantum mechanical systems with bare Hamiltonians nonquadratic in the momenta, we have computed for two cases the ground state energy and the first energy gap, and we have successfully compared them to the exact results, employing different kind of schemes and approximations. We stress that for the models under consideration we needed to take into account, as expected, the issue of Weyl ordering, which turns out to be at the base of the present flow equation quantization as it is well known to be for the functional integral quantization. This fact calls for some care in defining the concept of a bare nonseparable Hamiltonian action.

The application of the formalism developed for QM to the QFT case is straightforward and quickly discussed but, as in all Hamiltonian approaches to QFT, one must pay full generality and manifest unitarity with nonmanifest Lorentz covariance. This is unpleasant and complicates the job of performing approximations without breaking such a symmetry. For this reason, in the second part of the paper, we have discussed the possibility to generalize the EHA formalism to include also covariant Hamiltonian QFT. Functional integral quantizations of such theories have already been addressed in the literature, especially for gauge theories. In the present work we have addressed the simplest cases of scalar and spinor degrees of freedom. Actually, for scalar QFT we further restricted our work to the presence of one conjugate momentum only, namely, a longitudinal vector field. In this specific case we have provided an RG flow equation representation of the corresponding QFT, and we have worked out its explicit form in the LHA.

Let us close this work addressing the issue of the physical motivations for it and of its usefulness. Clearly, the use of this framework is related to Hamiltonian systems nonquadratic in momenta, and therefore we should comment on the question: where are they or why should we look for them?

Quantum mechanical systems more than quadratic in the momenta may be interesting on the base of first principles (think about the action of the free relativistic particle) or arise as effective descriptions of physical systems. Also, they could appear as intermediate technical tools for the description of more complicated systems. For instance, within the worldline formalism, one-loop computations

are reduced to quantum mechanical path integrals with Hamiltonians, which sometimes are nonquadratic in the momenta [31]. In these cases one can hope to use this approach as an alternative or a complementary tool to perturbation theory.

Theories more than quadratic in the momenta, when reduced to the Lagrangian formulation, show a nonlinear dependence on the derivatives of the fields. This dependence, if expanded in powers and truncated, typically generates violations of unitarity. Nevertheless, before truncation nothing prevents such theories from being unitary. That is, there might be some interesting nontrivial extensions of quantum models that are nonquadratic in the momenta and that make perfect sense from a quantum mechanical point of view.

Why should we look for them? As already commented at the end of the section on scalar QFT, the study of the RG flow on the Hamiltonian theory space might show new possibilities for the UV or IR behavior of systems that at some intermediate scale are well approximated by simple Lagrangian theories. Stated in different words, keeping both phase-space variables could make easier the task of parametrizing the quantum dynamics far from that intermediate simple Lagrangian scale. One reason for such an expectation is the following: we know that the effective actions are in general nonlocal and that integrating out non-Gaussian degrees of freedom is responsible for such nonlocalities; therefore avoiding integrating out the momenta should be of help in the hard task of reducing as far as possible the importance of nonlocal interactions. Restated one more time: even by studying the running of approximate local actions on the Hamiltonian theory space one can, just by putting the momenta on-shell, have access to at least part of the running of nonlocal actions in the Lagrangian theory space. For these reasons also the study of theories whose bare actions are quadratic but that flow to AEHA's more than quadratic in the momenta could benefit from this first order formulation. Examples are the covariant Hamiltonian formulation of Yang-Mills theory and generic nonlinear sigma models, which in our opinion deserve future investigations within the present framework.

The analysis of Hamiltonian flows might open the intriguing possibility of finding systems belonging to new universality classes, by looking for fixed points of the flow in the Hamiltonian formulation. We have started to consider this challenging problem within the "reduced" covariant formulation of scalar QFT presented in this paper, and we hope to report on this soon. The results of all these studies will in general depend on the kind of Hamiltonian formulation we choose, a fact that enables one to quantitatively compare different quantization prescriptions as well as to look for physical systems described by each of them. Thus, in our opinion, a vast playground lies open, waiting for future investigations.

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APPENDIX A: THE EFFECTIVE HAMILTONIAN ACTION AS THE GENERATING FUNCTIONAL OF 1PI VERTEX FUNCTIONS

In this appendix we are going to prove that the effective Hamiltonian action is the generating functional of the 1PI proper vertices, in the sense that the tree level amplitudes computed with vertex functions and propagators extracted from it are equal to the full perturbative series generated by the bare Hamiltonian action. For the ease of the explanation we limit this discussion to the QM case, choosing $\hbar = 1$ as a unit of action. The proof works just as for the usual Lagrangian effective action [32].

(1) Write down a path integral based on a Hamiltonian bare action that is $(1/g)$ times the Hamiltonian effective action, with g an external parameter. This rescaling of the action entails a corresponding rescaling of the Liouville form $\lambda_g \equiv \frac{1}{g}\lambda = \frac{1}{g}\bar{p}d\bar{q}$. Thus, to define the new path integral, we must adopt a functional measure $\mu_g = \sqrt{\mathrm{Det}\sigma_g}$ corresponding to the symplectic structure $\sigma_g = d\lambda_g$:

$$e^{iW_g[I,J]} = \int [dpdq]\mu_g[p,q]e^{\frac{i}{g}\{\Gamma[\bar{p},\bar{q}]+I\cdot\bar{p}+J\cdot\bar{q}\}}. \tag{A1}$$

(2) Recognize that the parameter g allows one to distinguish different loop orders in the perturbative evaluation of this path integral. In fact, Eqs. (17) and (13) show that in the perturbation theory generated by $\Gamma_g \equiv \frac{1}{g}\Gamma$ the vertex functions are proportional to $1/g$ while propagators are proportional to g. Thus any graph with i internal lines and v vertices gives a contribution proportional to g^{i-v}. Since the number of loops is $l = i - v + 1$, any loop expansion is an expansion in powers of g of the kind

$$W_g[I,J] = \sum_{l=0}^{\infty} g^{l-1}W_{g,l}[I,J]. \tag{A2}$$

(3) Evaluate the same path integral by a stationary phase method, an approximation that can be made arbitrarily good by tuning g arbitrarily close to zero. Since by definition the exponent at the stationarity point gives the $W[I,J]$ of Eq. (4), one gets

$$e^{iW_g[I,J]} \underset{g\to 0}{\sim} \mu_g[p,q]\left(\mathrm{Det}\frac{1}{2\pi g}\Gamma[\bar{p},\bar{q}]^{(2)}\right)^{-\frac{1}{2}} e^{\frac{i}{g}W[I,J]}. \tag{A3}$$

(4) Expand the logarithm of the last result in powers of g. Because

$$\log\mu_g[p,q] = -\mathrm{Tr}\log g + \log\mu[p,q],$$
$$\log\left(\mathrm{Det}\frac{1}{2\pi g}\Gamma[\bar p,\bar q]^{(2)}\right)^{-\frac12}$$
$$= \mathrm{Tr}\log g + \log\left(\mathrm{Det}\frac{1}{2\pi}\Gamma[\bar p,\bar q]^{(2)}\right)^{-\frac12},$$

the combination of Eqs. (A2) and (A3) gives

$$\sum_{l=0}^{\infty} g^{l-1}W_{g,l}[I,J] \underset{g\to 0}{\sim} \frac{1}{g}W[I,J]$$
$$- i\log\left\{\mu[p,q]\left(\mathrm{Det}\frac{1}{2\pi}\Gamma[\bar p,\bar q]^{(2)}\right)^{-\frac12}\right\},$$

that is, $W_{g,0}[I,J] = W[I,J]$.

APPENDIX B: THE EFFECTIVE HAMILTONIAN ACTION FROM A VARIATIONAL FORMULA ON THE HILBERT SPACE

This appendix is to prove the proposition of Sec. II about the possibility to define the effective Hamiltonian action in the operator representation by means of a variational principle. The following arguments are not original, but just the obvious extension of those presented in [26]. To compute the extremum (8) with the constraints (9), one introduces three Lagrange multipliers $w(t)$, $I(t)$, $J(t)$ and looks for the extremum of $\langle\psi_-,t|i\partial_t - \hat H + J(t)\hat q + I(t)\hat p - w(t)|\psi_+,t\rangle$ with respect to the two states $|\psi_\pm,t\rangle$. Setting the two functional derivatives to zero gives

$$(i\partial_t - \hat H + J(t)\hat q + I(t)\hat p)|\psi_+,t\rangle = w(t)|\psi_+,t\rangle, \quad \text{(B1)}$$

$$(i\partial_t - \hat H + J(t)\hat q + I(t)\hat p)|\psi_-,t\rangle = w^*(t)|\psi_-,t\rangle. \quad \text{(B2)}$$

It is possible to define the states

$$|+,t\rangle = \exp\left\{i\int_{-\infty}^{t} dt' w(t')\right\}|\psi_+,t\rangle,$$
$$|-,t\rangle = \exp\left\{-i\int_{t}^{+\infty} dt' w^*(t')\right\}|\psi_-,t\rangle, \quad \text{(B3)}$$

which solve the following Schrödinger equation:

$$(i\partial_t - \hat H + J(t)\hat q + I(t)\hat p)|\pm,t\rangle = 0 \quad \text{(B4)}$$

and satisfy the boundary conditions $\lim_{t\to\mp\infty}|\pm,t\rangle = |0\rangle$. In other words, $|+,t\rangle = \hat U_{I,J}(t,-\infty)|0\rangle$ and $\langle -,t| = \langle 0|\hat U_{I,J}(+\infty,t)$, such that

$$e^{iW[I,J]} = \langle 0|\hat U_{I,J}(+\infty,-\infty)|0\rangle = \langle -,t|+,t\rangle$$
$$= e^{i\int_{-\infty}^{+\infty} dt' w(t')}, \quad \text{(B5)}$$

that is, $W[I,J] = \int_{-\infty}^{+\infty} dt' w(t')$. On the other hand, by contracting Eq. (B1) with $\langle\psi_-,t|$ and using the previous equation, along with the constraints (9), one finds that for the stationarity states the following relation holds:

$$\int_{-\infty}^{+\infty} dt\langle\psi_-,t|i\hbar\partial_t - \hat H|\psi_+,t\rangle$$
$$= W[I,J] - \int_{-\infty}^{+\infty} dt[J(t)\bar q(t) + I(t)\bar p(t)]. \quad \text{(B6)}$$

To prove that the values of I and J on the r.h.s. are the extremal ones, it is necessary to take derivatives of this equation with respect to the sources and remember that on the l.h.s. the extremal value cannot depend on the Lagrange multipliers, nor can the constraint points $\bar p$ and $\bar q$ on the r.h.s.

APPENDIX C: THE REALIZATION OF THE RISING DELTA FUNCTIONAL WHEN $k \to \Lambda$

To analyze the $k \to \Lambda$ limit of Eq. (26), we first perform a change of variables in the path integral:

$$p' = p - \bar p + (r_k\partial_t)^{-1}\frac{\delta\Gamma_k}{\delta\bar q},$$
$$q' = q - \bar q - (r_k\partial_t)^{-1}\frac{\delta\Gamma_k}{\delta\bar p},$$

and then define the complex variable: $z = (p' - iq')/\sqrt{2}$. The result of these manipulations is

$$e^{i\Gamma_k[\bar p,\bar q]} = \int[dz]\mu_k \exp i\left\{\frac12\int dt(z^* r_k i\partial_t z - z r_k i\partial_t z^*)\right.$$
$$- \frac{\delta\Gamma_k}{\delta\bar q}\cdot(r_k\partial_t)^{-1}\frac{\delta\Gamma_k}{\delta\bar p} + S\left[\bar p - (r_k\partial_t)^{-1}\frac{\delta\Gamma_k}{\delta\bar q}\right.$$
$$\left.\left. + \sqrt{2}\Re(z),\ \bar q + (r_k\partial_t)^{-1}\frac{\delta H_k}{\delta\bar p} - \sqrt{2}\Im(z)\right]\right\}.$$

Under the assumption that Γ_k stays finite for any $k \in [0,\Lambda]$, when $k \to \Lambda$ every Γ_k-dependent term on the r.h.s. gets killed by the divergence of r_k. On the other hand, since $\mu_k = \mathrm{Det}(\frac{1+r_k}{2\pi}\delta)$ (excluding the possible zero eigenvalues), the first term in the exponent together with the regularized functional measure provides a rising delta functional, constraining z, i.e., $(p - \bar p)$ and $(q - \bar q)$, to vanish.[2] Thus in this limit the r.h.s. reduces to $\exp\{iS[\bar p,\bar q]\}$ and the AEHA coincides with the bare Hamiltonian action. To show that a rising delta functional is indeed realized, we need to prove that the quadratic form $(z^* r_k i\partial_t z - z r_k i\partial_t z^*)$ is positive definite. This is not obvious since $i\partial_t$ is a real operator on the spaces of functions one is usually interested in, but whose sign is not fixed. However, if the domain of the functional integral is such that all contributions coming from the time boundaries are

[2]Although the quadratic form $(r_k i\partial_t\delta)$ in the exponent and the operator in the measure $(1 + r_k)\delta$ asymptotically differ for a factor of $i\partial_t$, the path integral is properly normalized [14] in such a way to be finite for a free system ($\forall k \in [0,\Lambda]$) and to show a k-independent divergence in the $H = 0$ case.

vanishing, and if the Fourier transform is allowed, then one can write (the reader should interpret the integrals as generic sums over unspecified domains)

$$\begin{aligned}\frac{i}{2}\int_t (z(t)^* r_k i\partial_t z(t) - z(t) r_k i\partial_t z(t)^*) &= \int_t p'(t) r_k i\partial_t q'(t)\\ &= \frac{1}{2}\int_E r_k(E^2)E(p'(-E)q'(E) - q'(-E)p'(E))\\ &= \int_E \theta(E) r_k(E^2)E(p'(-E)q'(E) - q'(-E)p'(E))\\ &= i\int_E \theta(E) r_k(E^2)E(x_-(E)^* x_-(E) - x_+(E)^* x_+(E)),\end{aligned}$$

where we assumed $q(t)$ and $p(t)$ real, such that for their Fourier transforms satisfy $p(-E) = p(E)^*$ and $q(-E) = q(E)^*$, we defined $x_\pm(E) = (p'(E) \pm iq'(E))/\sqrt{2}$, and we denoted by θ the Heaviside step function. The last equation shows that the diagonalization of the quadratic form gives two complex Gaussians that can be independently rotated to real Gaussians with positive definite inverse variances. In reality they might not be positive definite and allow for zero modes, but we will not discuss this possibility in the present work.

APPENDIX D: THE AVERAGE EFFECTIVE HAMILTONIAN ACTION IN EUCLIDEAN SPACE AND WICK ROTATION

Of course, the Hamiltonian formalism without time makes little sense. However, it could be nice to forget about the evaluation of integrals with poles once and for all by working in Euclidean space from the very beginning. In this appendix the reader will find the translation of some of the main formulas of the present work to Euclidean space and a discussion on the possible equivalence of the theories in Minkowski and Euclidean space, i.e., on the feasibility of a Wick rotation to imaginary time.

Let us start with scalar QM. In this case Wick rotation ($t \to -i\tau$) of Eq. (3) with action (1) is safe and leads to a convergent path integral

$$e^{W[I,J]} = \int [dpdq]\mu[p,q] e^{-\{S[p,q] - Jq - Ip\}}$$

with action

$$S[p,q] = \int d\tau[-p(\tau) i\partial_\tau q(\tau) + H(p(\tau), q(\tau))]. \quad \text{(D1)}$$

The regularization goes as usual

$$e^{W_k[I,J]} = \int [dpdq]\mu_k e^{-\{S[p,q] + \Delta S_k[p,q] - Jq - Ip\}}$$

with ΔS_k and μ_k, which can still be chosen according to Eqs. (21)–(25) if we replace ∂_t with $-i\partial_\tau$ (the minus sign here is because of the global minus factorized in front of the action). The definition of the AEHA is

$$\Gamma_k[\bar{p},\bar{q}] + \Delta S_k[\bar{p},\bar{q}] = \operatorname*{ext}_{I,J}(I\cdot\bar{p} + J\cdot\bar{q} - W_k[I,J]),$$

which is equivalent to

$$\begin{aligned}e^{-\Gamma_k[\bar{p},\bar{q}]} = \int [dpdq]\mu_k[p,q]\\ \times e^{-\{S[p,q] + \Delta S_k[p-\bar{p},q-\bar{q}] - (p-\bar{p})\frac{\delta\Gamma_k}{\delta\bar{p}} - (q-\bar{q})\frac{\delta\Gamma_k}{\delta\bar{q}}\}}.\end{aligned} \quad \text{(D2)}$$

From it the flow equation follows:

$$\dot{\Gamma}_k = \frac{1}{2}\operatorname{Tr}[(\Gamma_k^{(2)} + R_k\delta)^{-1}\dot{R}_k\delta] - \frac{\dot{\mu}_k}{\mu_k}, \quad \text{(D3)}$$

where $R_k\delta = \Delta S_k^{(2)}$. We see that this equation formally differs from the Minkowskian one (29) by the absence of the imaginary factor i on the l.h.s., by a global minus factor on the r.h.s., and by the fact that inside R_k we find the operator $i\partial_\tau$ instead of ∂_t. Thus, for instance, in the particular case of an off-diagonal regulator the explicit form of the flow equation becomes

$$\begin{aligned}\dot{\Gamma}_k = \operatorname{Tr}\Bigg[(-\dot{r}_k i\partial\delta)\Bigg(\Bigg(-r_k i\partial\delta + \frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{p}}\Bigg)\\ - \frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{p}}\Bigg(r_k i\partial\delta + \frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{q}}\Bigg)^{-1}\frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{q}}\Bigg)^{-1}\Bigg]\\ - \operatorname{Tr}[\dot{r}_k(1+r_k)^{-1}\delta].\end{aligned} \quad \text{(D4)}$$

Next let us consider scalar covariant Hamiltonian QFT. Since π^ν is a vector, Wick rotation involves also its zero component, independently of whether we allow for transverse momenta: $x^0 \to -ix^4$ and $\pi^0 \to -i\pi^4$. However, performing such a replacement in the action (52) with Hamiltonian (51), one finds that

$$iS \to \int d^dx\Big[\frac{1}{2}(\pi - \partial\phi)^2 - \frac{1}{2}(\partial\phi)^2 - V(\phi)\Big];$$

therefore the integral over π diverges. In other words, such a Wick rotation cannot be performed. The main difference from the case of QM, or the reason for such a failure, is the fact that the momenta are assumed to rotate along with time. Despite this problem, one possible reason for studying a Euclidean covariant Hamiltonian formulation is that we know that the Euclidean noncovariant Hamiltonian theory makes perfect sense because it is related by a continuos Wick rotation to the corresponding Minkowskian theory. Therefore the Euclidean covariant formulation can be derived from the noncovariant Hamiltonian formulation and studied as a generalization of it. By definition the bare action of such a covariant Hamiltonian Euclidean theory reads

$$S[\pi^\nu,\phi] = \int d^dx[-\pi^\nu i\partial_\nu\phi + \mathcal{H}(\pi^\nu,\phi)]. \quad \text{(D5)}$$

Its Hamiltonian quantization in a scheme where only longitudinal momenta are present is based on the functional integral

$$Z[I_\nu, J] = \int [d\pi^\nu d\phi]\delta[\Pi_\perp^{\nu\rho}\pi_\rho]\mu e^{-\{S[\pi^\nu,\phi]-I_\nu\cdot\pi^\nu-J\cdot\phi\}}.$$

Again, to get a functional RG flow equation representation of this integral, one introduces a k dependence in the bare action and in the measure. In the following we choose an off-diagonal quadratic regularization, i.e., of the kind (56), but with ∂_ν replaced by $i\partial_\nu$. The definition of the AEHA is the same as in Euclidean quantum mechanics

$$\Gamma_k[\bar\pi^\nu, \bar\phi] + \Delta S_k[\bar\pi^\nu, \bar\phi] = \underset{I_\nu, J}{\text{ext}}(I_\nu\cdot\bar\pi^\nu + J\cdot\bar\phi - W_k[I_\nu, J]), \tag{D6}$$

from which the usual integro-differential equation

$$e^{-\Gamma_k[\bar\pi^\nu,\bar\phi]} = \int [d\pi^\nu d\phi]\delta[\Pi_\perp^{\nu\rho}\pi_\rho]\mu_k \times e^{-\{S[\pi^\nu,\phi]+\Delta S_k[(\pi-\bar\pi)^\nu,\phi-\bar\phi]-(\pi-\bar\pi)^\nu\frac{\delta\Gamma_k}{\delta\bar\pi^\nu}-(\phi-\bar\phi)\frac{\delta\Gamma_k}{\delta\bar\phi}\}}. \tag{D7}$$

Again, the Euclidean flow equation can be obtained from the Minkowskian one by stripping the imaginary i on the l.h.s., by changing the global sign on the r.h.s., and by replacing $r_k\partial\delta$ with $r_k i\partial\delta$.

As far as fermions are concerned, no new behavior under Wick rotation shows up, because of the identification of configuration space with the reduced phase space.

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