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Infrared divergences in plane wave backgrounds

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We show that the emission of soft photons via nonlinear Compton scattering in a pulsed plane wave (laser field) is in general infrared divergent. We give examples of both soft and soft-collinear divergences, and we pay particular attention to the case of crossed fields in both classical and quantum theories.

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I. INTRODUCTION

There is growing interest in the use of strong laser fields to probe both physics beyond the standard model [1,2], and the high intensity, low energy regime of QED [3–5]. Much of this interest is focused on quantum effects such as nonperturbative pair creation from the vacuum [6] and vacuum birefringence [7]. As the center-of-mass energies in laser-particle scattering are typically low compared to accelerator experiments (laser frequencies are in the range 1–10^4 eV [8,9]) quantum effects are hard to detect. However, the high intensity, or flux, of the laser can to some extent compensate for the low energy [10].

The high field strengths of modern lasers require a nonperturbative approach, with most progress having been made when the laser fields are described by a plane wave, a model which holds when beams are not too tightly focused [11,12]. In this case, scattering amplitudes can be calculated for arbitrary field strengths [13–17]. Using this model, increasingly complex semiclassical (tree level) processes are being studied; examples include Compton scattering within the background field [18], Møller scattering [19], trident pair production [20] and the multiphoton emission processes

$$e^-(p) \xrightarrow{\text{in laser}} e^-(p') + \gamma(k^1) + \gamma(k^2) + \dots \gamma(k^n). \qquad (1)$$

The $n = 1$ process is well studied [21–26]. It goes by the name *nonlinear Compton scattering* since, for low background field strengths, the scattering amplitude becomes a sum over ordinary Compton amplitudes for each frequency in the background; see below. The case $n = 2$ has been considered in Refs. [27,28]. The low field strength amplitude for this process is proportional to that of two-photon Compton scattering in ordinary QED,

$$e^-(p) + \gamma(k_{\text{laser}}) \to e^-(p') + \gamma(k^1) + \gamma(k^2), \qquad (2)$$

which is infrared (IR) divergent [29]. The divergence is inherited by the full $n = 2$ process.

In Ref. [30], an incoherent-sum approximation was used to study large n, and it was observed that the (tree level) integrated probabilities $\mathbb{P}(n)$ for the processes (1) can exceed unity. Investigating this statement is not entirely straightforward since, while it is conceptually trivial to calculate S-matrix elements for arbitrary n, it is computationally exhausting and multiple numerical integrations are required to obtain the probabilities.[1] Some intuition can however be obtained by studying the classical limit, in which the probabilities can be calculated exactly. One finds [22,32]

$$\lim_{\hbar\to 0}\mathbb{P}(n) = \frac{1}{n!}(N_\gamma)^n, \qquad (3)$$

where N_γ is the classically obtained *number of photons* emitted by a particle passing through a plane wave. There is no *a priori* reason why N_γ should be smaller than 1. It is not hard to identify the origin of the problem and the higher order corrections which will resolve it; (3) is the archetypal relation associated to the infrared problem. Let us therefore recall how IR divergences arise, and are dealt with, in QED. (See Chap. 6 of Ref. [33] for a particularly lucid introduction.)

One first encounters the IR problem in the process of bremsstrahlung, which is the emission of a photon from an electron as it passes a nucleus. The emitted photon can be arbitrarily soft and this leads to an IR divergence, already at tree level, when the emitted photon frequency goes to zero. However, any detector has a finite resolution and therefore cannot distinguish between sufficiently soft bremsstrahlung and scattering without emission; the Bloch-Nordsieck result is that, when one considers the physically measurable sum of the probabilities of these two IR divergent processes, the IR divergence cancels between them [34]. IR divergences are not solely the domain of bremsstrahlung, though. Ordinary Compton scattering is infrared divergent at one-loop level. To cancel this divergence one must account for both ordinary Compton scattering and double-Compton scattering of

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[1]See Ref. [31] for recent progress in stimulated pair production, obtained from nonlinear Compton by crossing symmetry.

one hard and one soft photon [35]. In general, then, physically measurable quantities which account for detector resolutions and experimentally indistinguishable processes are IR finite [34,36,37].

The IR divergences in the $n = 2$ case of (1) have been dealt with to date by inserting a cutoff or damping factor. Nor does it seem to have been noticed that nonlinear Compton scattering, $n = 1$, can be IR divergent depending on the properties of the chosen background field: as we will show, a divergence arises when the plane wave's physical fields (not its potential) contain a Fourier zero mode. Such pulses are called *unipolar* [38] and, physically, the presence of the zero mode means the pulse can transfer a *net acceleration* to a particle. Vacuum acceleration in a real laser field is possible and has been observed [39–41]. (The *Lawson-Woodard theorem* [42–44] does not apply here; see Ref. [45]). The IR divergences we discuss here will appear in realistic beam geometries, but it is a good idea to understand the simplest model first; we therefore allow for plane waves which model an accelerating structure.

The purpose of this paper is to examine the origin and severity of IR divergences in plane wave backgrounds, starting with the classical theory and then considering tree level quantum calculations, with the background field treated exactly. In a sequel paper, the divergences will be removed by calculating the appropriate higher order corrections. We begin in Sec. II with a general discussion of classical radiation and the *IR catastrophe*. In Sec. III we apply this to the particular case of a plane wave background and show how the IR sector of the emission spectrum is related to the net energy transferred by the plane wave. In Sec. IV, we turn to QED. Lehmann-Symanzik-Zimmermann (LSZ) reduction formulas are derived for unipolar plane waves, and this produces Volkov solutions with the correct boundary conditions. Using these, nonlinear Compton scattering is addressed in Sec. V and shown to be IR divergent. We investigate the soft and perturbative limits, and compare with both bremsstrahlung and ordinary Compton scattering. We also discuss the seemingly contradictory example of crossed fields (constant plane waves), which can accelerate but do not lead to an IR divergence. We conclude in Sec. VI.

II. CLASSICAL RADIATION: IR BEHAVIOR

Consider a particle moving, and radiating, in an arbitrary background field. The four-momentum P^μ of the emitted radiation field is [46,47]

$$P^\mu = -\frac{1}{2}\int \frac{d^4k'}{(2\pi)^3}\,\mathrm{sign}(k'^0)\delta(k'^2)k'^\mu j(k')\cdot j^*(k'), \quad (4)$$

which depends on the particle's trajectory $x^\mu(\tau)$ and velocity $u^\mu(\tau) \equiv \dot{x}^\mu(\tau)$ (both functions of proper time τ) through the Fourier transformed current j_μ,

$$j^\mu(k') = e\int d\tau u^\mu(\tau)e^{ik'\cdot x(\tau)}. \quad (5)$$

This integral does not behave well at large distances and, as a result, does not obey current conservation,

$$k'\cdot j(k') = -ie\int d\tau \frac{d}{d\tau}e^{ik'\cdot x(\tau)} \neq 0, \quad (6)$$

since the boundary terms do not in general cancel each other. To see why, we return to (5) and assume that the background field turns on and off at finite times. We parametrize the path such that the particle enters the field at proper time $\tau = 0$ and position $x^\mu = 0$, with momentum p_μ. (Our final result will be independent of these initial conditions). The particle then exits the pulse at some proper time $\tau = \tau_f$, at some position x_f^μ and with some momentum p'_μ, all determined by the classical equations of motion. With this, the current becomes

$$j^\mu(k') = \frac{e}{m}p_\mu\int_{-\infty}^0 d\tau e^{ik'\cdot p\tau/m} + e\int_0^{\tau_f} d\tau u^\mu(\tau)e^{ik'\cdot x(\tau)} + \frac{e}{m}p'_\mu\int_{\tau_f}^\infty d\tau e^{ik'\cdot[x_f+p'(\tau-\tau_f)/m]}. \quad (7)$$

The first and third terms do not behave well at large distances, where the phases diverge.[2] Regulating the integrals using an $i\epsilon$ prescription in the exponents yields

$$j^\mu = ie\frac{e^{ik'\cdot x_f}p'_\mu}{k'\cdot p'} - ie\frac{p_\mu}{k'\cdot p} + e\int_0^{\tau_f} d\tau u^\mu(\tau)e^{ik'\cdot x(\tau)}. \quad (8)$$

The first and second terms now give the (boosted) Coulomb fields of the particle before and after interaction with the background, as follows from inserting (8) into (10) and carrying out the k'_0 integral; see Chap. 6 of Ref. [33]. The current (8) is easily checked to be conserved, $k'.j(k') = 0$. Integrating (8) by parts, the boundary terms cancel the Coulomb terms and we obtain our final, compact result

$$j^\mu(k') = -e\int d\tau e^{ik'\cdot x(\tau)}\frac{d}{d\tau}\left(\frac{u^\mu(\tau)}{ik'\cdot u}\right), \quad (9)$$

where the integral is *automatically* restricted to the pulse duration since the integrand goes like the acceleration $\dot{u}$ (as can be seen by expanding the derivative). This shows manifestly that only accelerated charges radiate (see Chap. 14 of Ref. [50]). Since $u_\mu(\tau)$ is timelike and k'_μ is lightlike, the denominators in (8) and (9) are nonzero unless $\omega' = 0$, which we address below.

Returning now to the radiation spectrum and integrating over k'^0, the emitted energy P^0 may be written

$$P^0 = \int d\omega' d\Omega\, \omega'\rho(k'), \quad (10)$$

where we have introduced the frequency ω' and the spectral density ρ measuring the number of photons radiated per unit frequency per unit solid angle,

[2] This is, as we will see, related to soft IR divergences and not the *phase divergences* which occur in, say, pair creation [48,49].

$$\rho(k') = -\frac{\omega'}{2(2\pi)^3} j(k') \cdot j^*(k'). \tag{11}$$

The "total number of emitted photons" is therefore

$$N_\gamma := \int \mathrm{d}\omega' d\Omega \rho(k'). \tag{12}$$

Consider now the emission of low frequency radiation. We write $k'_\mu = \omega' n'_\mu$, and expand (9) for small ω', finding

$$j^\mu(k') = \frac{ie}{\omega'}\left(\frac{p'^\mu}{n'\cdot p'} - \frac{p^\mu}{n'\cdot p}\right) + \mathcal{O}(\omega'^0). \tag{13}$$

[The same result follows immediately from (8), since the bounded integral term can be dropped in the soft limit]. Again, $p'_\mu (p_\mu)$ is the particle's momentum when it leaves (enters) the pulse. The classical IR problem can now be stated in terms of the total number of emitted photons N_γ; this is logarithmically divergent at low frequencies:

$$N_\gamma = -\frac{\alpha}{(2\pi)^2}\left[\int \mathrm{d}\Omega \left(\frac{p'}{n'\cdot p'} - \frac{p}{n'\cdot p}\right)^2\right]\int_0 \mathrm{d}\omega' \frac{1}{\omega'}, \tag{14}$$

where $\alpha = e^2/4\pi$ and $\omega' \simeq 0$ since we focus on low frequency emission. (The angular integral can be performed exactly and is nonzero). Since the number of photons is not a classical concept, we rephrase (14) in terms of the energy (10); an equivalent statement is that the energy emitted at low frequency *is independent of frequency*, i.e.,

$$P^0 = -\frac{\alpha}{(2\pi)^2}\left[\int \mathrm{d}\Omega \left(\frac{p'}{n'\cdot p'} - \frac{p}{n'\cdot p}\right)^2\right]\int_0 \mathrm{d}\omega' 1. \tag{15}$$

There is therefore no divergence in any measurable classical object, but it is the behavior (14) or (15) which signals a corresponding IR divergence in the quantum theory. In the limit that the background field provides a *sudden kick*, instantaneously changing the particle's momentum, the expressions (14) and (15) become exact. We have said nothing about the properties of the background field, though, which is the statement that "the precise form of the trajectory … does not affect the low-frequency radiation" (see Chap. 6 of Ref. [33]).

III. PLANE WAVES AND THE INFRARED

We now apply the general results above to the case of plane waves. A plane wave traveling in the negative z direction is characterized by the lightlike vector $n_\mu = (1, 0, 0, 1)$ and some scale ω which is usually the dominant frequency of the wave. We write $k_\mu := \omega n_\mu$. The transverse electric fields E_j ($j = 1, 2$) depend arbitrarily on the dimensionless, Lorentz invariant variable $\phi := k.x$, which can be identified with lightfront time. Lightfront variables are defined via $x^\pm = x^0 \pm x^3$, $x_\pm = (x_0 \pm x_3)/2$ and $x^\perp = \{x^1, x^2\}$, so that $\phi = k\cdot x = k_+ x^+$. The plane wave field strength is

$$F_{\mu\nu}(k\cdot x) = f'_j(k\cdot x)(k_\mu a^j_\nu - a^j_\mu k_\nu), \tag{16}$$

where the f'_j are profile functions describing the shape of the electromagnetic fields and, for our choice of k_μ, the polarization vectors become $a^j_\mu = (a_0 m/e)\delta^j_\mu$. We normalize such that $(f'_j f'_j)_{\mathrm{rms}} = 1$, sum over j, rms taken over the whole pulse, so that the parameter a_0 is always equal to $a_0 \equiv eE_{\mathrm{rms}}/m\omega$ [51].

A particle in a plane wave, neglecting radiation reaction, obeys the Lorentz force equation. $k.u$ is then conserved and proper time $\tau \propto \phi$ [51,52]. We assume that the particle is free until some lightfront time ϕ_i, with kinetic momentum p_μ, when it first encounters the field. Its subsequent kinetic momentum $\pi_\mu \equiv m u_\mu$ can be expressed as the following function of ϕ:

$$\pi_\mu(p;\phi) := p_\mu - eC_\mu(\phi) + \frac{2eC(\phi)\cdot p - e^2C^2(\phi)}{2k\cdot p}k_\mu. \tag{17}$$

Here, C_μ is the integral of the field strength,

$$C_\mu(\phi) := a^j_\mu \int_{\phi_i}^{\phi} f'_j(\varphi) =: a^j_\mu f_j(\phi). \tag{18}$$

It is easy to check that (17) obeys the correct initial condition, $\pi_\mu(p;\phi_i) = p_\mu$, and that $\pi^2 \equiv m^2$. When the pulse turns off at, say, $\phi = \phi_f$ the particle again becomes free. By definition, the function C_μ then becomes constant,

$$C_\mu(\phi_f) = C_\mu(\infty) =: C^\infty_\mu. \tag{19}$$

Note that C^∞_μ is a vector of *Fourier zero modes* of the electromagnetic field strengths, i.e.,

$$C^\infty_\mu \equiv a^j_\mu \tilde{f}'_j(0). \tag{20}$$

C^∞_μ will play a crucial role in what follows. Mathematically, it is the Fourier zero mode of the electromagnetic fields and, physically, it neatly encodes the ability of the electromagnetic fields to do net work on a particle.[3] Fields for which the Fourier zero mode is nonvanishing are called unipolar, an example being a subcycle pulse, and can be produced from *ordinary* fields with a vanishing zero mode by interaction with a nonlinear optical medium; see Ref. [38].

For our purposes, all plane waves fall into one of two categories, defined by whether or not the Fourier zero mode vanishes:

$$\int_{\phi_i}^{\phi_f} \mathrm{d}\phi F_{\mu\nu}(\phi)\begin{cases} = 0 \Leftrightarrow C^\infty_\mu = 0, & \text{“whole cycle,”} \\ \neq 0 \Leftrightarrow C^\infty_\mu \neq 0, & \text{unipolar} \end{cases}. \tag{21}$$

[3] We note that the zero mode can also be obtained from the gauge invariant phase of a lightlike Wilson loop [53], and further that acceleration, through sudden kicks, relates IR divergences to cusp singularities in Wilson loops; see Ref. [49] and references therein.

A particle entering a whole-cycle field (which, in a loose sense, contains a whole number of cycles) with momentum p_μ leaves with the same momentum, i.e., experiences no net acceleration, since $C^\infty_\mu = 0$ in (17) and therefore $\pi_\mu(p;\phi) = p_\mu$ when $\phi \geq \phi_f$. In this case, the leading order terms in (14) vanish,

$$\frac{p'^\mu}{n'\cdot p'} - \frac{p^\mu}{n'\cdot p} \equiv 0, \tag{22}$$

so that the classical number of photons N_γ becomes IR finite and the low energy spectrum is frequency *dependent*. The implication is that the corresponding quantum processes are IR finite, and this is born out: nonlinear Compton scattering contains no IR divergence provided the pulse contains a whole number of cycles [22–24]. The typical situation for whole-cycle pulses is sketched in Fig. 1, upper panel.

Now consider a unipolar pulse. A particle entering such a pulse with momentum p_μ leaves with a different momentum $\pi_\mu(p;\infty)$,

$$\pi_\mu(p;\infty) = p_\mu - eC^\infty_\mu + \frac{2eC^\infty\cdot p - e^2C^\infty\cdot C^\infty}{2k\cdot p}k_\mu, \tag{23}$$

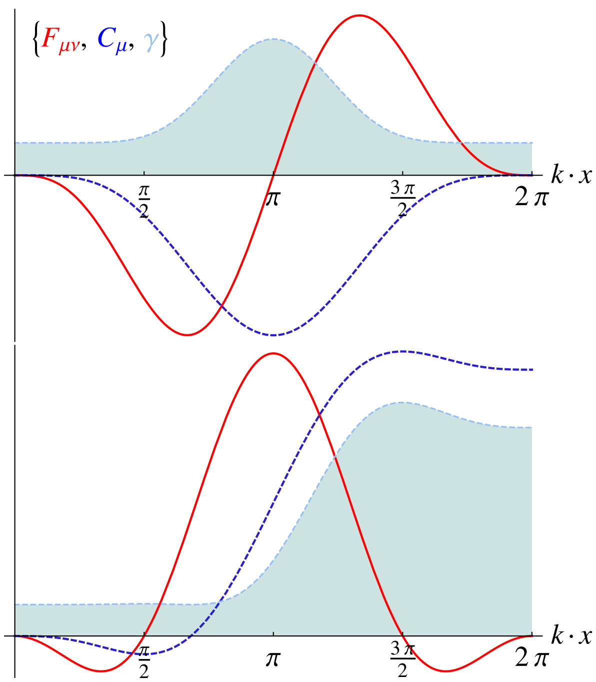


FIG. 1 (color online). *Not to scale.* The γ factor (filled) of an electron in a plane wave with field strength $F_{\mu\nu}$ (red/solid) and integrated field strength C_μ (blue/dashed). Upper panel: In a whole-cycle pulse, the γ factor returns to its initial value when the electron leaves the field. Lower panel: In a unipolar pulse, the electron gains a net acceleration, signaled by the nonzero C^∞_μ. This potential term yields the nonzero boundary term of (13) which signals the soft IR divergence.

which differs from p_μ in both the transverse ($\perp$) and lightfront energy (lower $+$) components because of the nonvanishing Fourier zero mode C^∞_μ. This is the precise form of the Lawson-Woodward theorem for plane waves and holds independently of both the pulse duration and details of its field structure. The typical situation is sketched in the lower panel of Fig. 1; the electric field will clearly push the particle more in one direction than the other, giving a net acceleration. A simple way to model such pulse shapes is to employ a carrier phase; see the Appendix A. Since $C^\infty \neq 0$ in unipolar pulses, the boundary term of (13) is nonzero, and this gives a divergent photon number in (14). We therefore expect nonlinear Compton scattering to exhibit the usual IR divergence of QED when the background field has unipolar structure. We confirm this below. We note that even an *infinitesimal* deviation from whole-cycle structure in the field strength is enough to cause an IR divergence, so it is really unipolar rather than whole-cycle pulses which are the general case. There is also a special case, which we consider before turning to the quantum theory.

A. Soft and collinear divergences

Crossed fields are constant, homogeneous and orthogonal electric and magnetic fields of equal magnitude, i.e., constant plane waves. They provide one of the most common models of intense laser fields and form, for example, the basis of cascade codes [54,55].

The infinite extent of crossed fields is somewhat unphysical. To study their infrared properties in a controlled manner we therefore consider a plane wave which is constant for $-\frac{T}{2} < \phi < \frac{T}{2}$ and otherwise zero. The definition (18) then gives

$$C_\mu = a^1_\mu \begin{cases} 0 & \phi < -\frac{T}{2} \\ \phi + T/2 & -\frac{T}{2} \leq \phi < \frac{T}{2}. \\ T & \phi \geq \frac{T}{2} \end{cases} \tag{24}$$

This field accelerates since $C^\infty_\mu = Ta^1_\mu$ and the photon number is log divergent. If we focus on the soft sector, evaluating (13) in the limit that $T \to \infty$ yields

$$j^\mu(k') = -ie\left(\frac{k^\mu}{k'\cdot k} - \frac{p^\mu}{k'\cdot p}\right) + \mathcal{O}(\omega'^0), \tag{25}$$

independently of the chosen field strength E. As well as the soft divergence, we also have here a *soft and collinear* divergence when $k'_\mu \propto k_\mu$; see also Ref. [56]. Collinear divergences are known to appear only in association with massless particles (for their removal see Refs. [36,37,57]). The reason they can appear here is that any constant electric field, when allowed to persist for an infinite time, accelerates all incoming particles to the speed of light. In this sense, the final state particles are effectively *massless* (as in high energy approximations, for example, in which

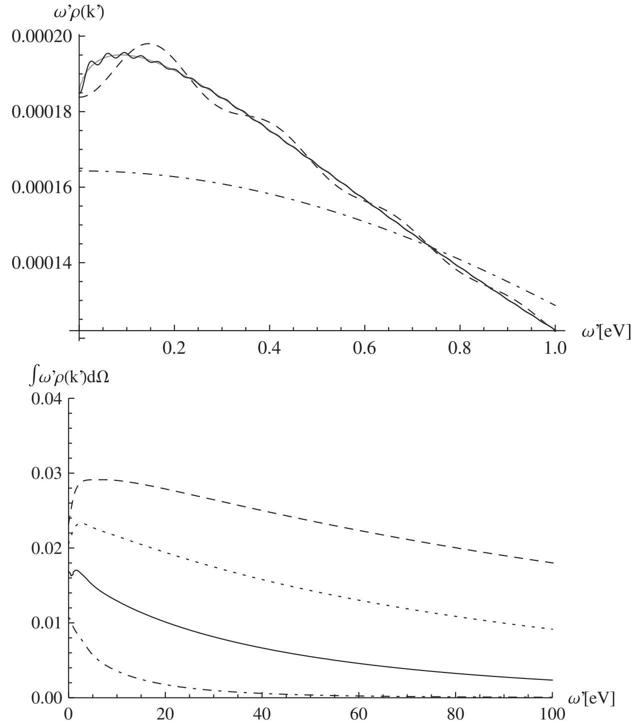


FIG. 2. The energy spectrum for a crossed field (with $E/E_S = 2\times 10^{-6}$). Upper panel: Fixed emission angles, $\theta = \varphi = \pi/2$, for $T = 2, 5, 10, 20$ (dot-dashed, solid, dotted, dashed). At $\omega' = 0$, the spectrum converges to (25) as T increases. Lower panel: $T = 5, 10, 15, 20$. When integrated over emission angles, the soft limit includes the developing soft-collinear divergence (25) and increases like $\log T$ as $T \to \infty$.

one neglects mass terms compared to momentum terms). Indeed, the dominant term in the particle's final momentum for large T is $\pi_\mu(p;\infty) \sim T^2 k_\mu$, which is lightlike, and the replacement of $\pi_\mu(p;\infty)$ with k_μ is manifest in (25). These results are summarized in Fig. 2, where we plot the energy density $\omega'\rho(\omega')$. At fixed emission angles, the value at $\omega' = 0$ is nonzero, illustrating the soft divergence, and converges to (25) as the duration increases. When the emission angle is integrated out, the low frequency value grows logarithmically with T because of the developing collinear divergence. So, crossed fields lead to both soft and soft-collinear divergences: their IR structure is worse than the generic case. Surprisingly, the literature results for the quantum case, i.e., for nonlinear Compton scattering in crossed fields, are IR *finite*. This contradiction will be resolved in Sec. V.

IV. ASYMPTOTIC STATES AND VOLKOV SOLUTIONS

Consider QED coupled to an additional, external gauge field C_μ. (The doubling of notation is deliberate, but for now C_μ is arbitrary). We briefly outline how one can calculate in the theory when the background is treated nonperturbatively. The action is

$$S = \int \mathrm{d}^4x - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar\psi[i\gamma^\mu(\partial_\mu + ieC_\mu) - m]\psi + \text{gauge fixing} + \text{sources} \leftarrow | \rightarrow - e\bar\psi \not{A}\psi, \tag{26}$$

where A_μ is the dynamical (quantized) photon field and $F_{\mu\nu}$ is its field strength. Everything to the left of the bar is considered to be *free*, and everything to the right is *interacting*. With regards to the quantum fields, this is the same split as is made in perturbation theory, except that the free part now contains a background field. Hence, Feynman rules are unchanged from QED except that the fermion propagator is the inverse of $i\not\partial - e\not{C} - m$ rather than $i\not\partial - m$. If this propagator can be calculated exactly, the background field will be accounted for nonperturbatively; plane wave backgrounds, to which we now return, provide an example of such a theory [58]. This holds (at least) when we choose the gauge potential C_μ to be equal to the classical $C_\mu(k.x)$ we encountered in (18). This is the method used in the literature, often implicitly, and we use it here.

In order to convert Feynman diagrams into S-matrix elements one appeals to LSZ reduction. Noting that our chosen potential vanishes in the past, for all plane waves, our asymptotic theory is free as usual. The LSZ reduction formula, applied to the Volkov propagator, yields the incoming wave function

$$\Psi^{\text{in}}_{p,\sigma}(x) := \left[\mathbb{1} + \frac{e}{2k\cdot p}\not{k}\not{C}(k\cdot x)\right]u_p^\sigma \exp\left[-ip\cdot x - \frac{i}{2k\cdot p}\int_{-\infty}^{k.x} 2eC\cdot p - e^2C^2\right]. \tag{27}$$

Neglecting spin contributions, this wave function obeys $iD_\mu = \pi_\mu(\phi;p)$; see (17) [15]. If we use the same wave functions for outgoing particles (as is normally done), the following odd result is found. Consider the S-matrix element for scattering without emission. Since (27) depends, aside from the usual $p.x$ factors, *only* on the lightfront variable $\phi = k.x$, we have overall momentum conservation in $p_\perp$ and p_-. The scattering amplitude will then have support on the conservation law $p'_\mu = p_\mu + sk_\mu$, where s arises as the Fourier transform of the ϕ dependence introduced by the background. For all momenta on-shell, this equation has only one solution, $s = 0$, so that $p' = p$. This is not the correct result for unipolar pulses where we expect p_μ to become $\pi_\mu(p;\infty)$ (classically). In fact, since there is no dependence on, say, $C.x$, it seems that no S-matrix element can recover the transverse push proportional to eC_∞ seen in (17). The resolution of this problem requires a brief, but straightforward, analysis of our theory's asymptotic behavior.

Physically, there is a sense in which the coupling to a unipolar pulse does not switch off; particles accelerated by such fields retain the *information* eC^∞_μ after the pulse itself has switched off, as seen in (27) via

$iD_\mu = \pi_\mu \not\to p_\mu$. This effect is also reflected by the fact that the potential for unipolar pulses switches off in the past but not in the asymptotic future. The label p in the Volkov solutions (27) can therefore only be associated with a momentum in the far past, not the future. Consequently, the result $p' = p$ in scattering without emission does not manifestly relate incoming and outgoing momenta. To account for this, we observe that, in the asymptotic future, $C \to C^\infty$, constant, and so our theory will consist of fermions minimally coupled to a *constant* gauge field C^∞_μ. We will therefore rederive the LSZ reduction formula to account for this. We stress that this theory is an ordinary free theory, since constant gauge fields are pure gauge; we are simply going to describe the free theory using a particular basis of states which is suited to the problem at hand. See Ref. [59] for related discussions in the context of pair creation.

We write $D^\infty_\mu \equiv \partial_\mu + ieC^\infty_\mu$. The electron solutions of the Dirac equation $i\not{D}^\infty - m = 0$ are

$$e^{-i(p'+eC^\infty)\cdot x} u_{p'}. \tag{28}$$

The kinetic momentum is $iD^\infty_\mu = p'_\mu$ with $p'^2 = m^2$: these solutions describe ordinary, free electrons. The usual steps leading to the LSZ reduction formulas then yield the following amputation instruction for outgoing electrons:

$$-i\int \mathrm{d}^4x e^{i(p'+eC^\infty)\cdot x}\bar{u}_{p'}(i\not{D}^\infty - m)_x\langle 0|T\psi(x)\ldots. \tag{29}$$

This differs from the usual result only in the presence of C^∞_μ. Applying (29) to the Volkov propagator, one obtains the following expressions for the outgoing electron wave function:

$$\bar{\Psi}^{\text{out}}_{p',\sigma}(x) := \bar{u}^\sigma_{p'}\bigg[\mathbb{1} + \frac{e}{2k\cdot p'}\delta\not{C}(k\cdot x)\not{k}\bigg] \exp\bigg[i(p'+eC_\infty)\cdot x - \frac{i}{2k\cdot p'}\int_{k\cdot x}^\infty 2e\delta C\cdot p' - e^2\delta C^2\bigg], \tag{30}$$

where $\delta C_\mu(k\cdot x) := C_\mu(k\cdot x) - C^\infty_\mu$. The limits on the integrals follow as part of LSZ. This wave function manifestly recovers (28) in the far future, and in this limit its kinetic momentum becomes p'. So, incoming electrons are described by ordinary Volkov solutions while outgoing electrons are described by (30); both satisfy the Dirac equation in the background $C_\mu(k.x)$. Positron solutions are obtained by sending $u \to v$ and $e \to -e$. Complemented with the usual propagator, the use of (27) and (30) completes the Feynman rules for the theory. Note that these wave functions describe asymptotically free particles; we are simply using different bases so that all wave functions are labeled by the appropriate incoming or outgoing momenta. Nevertheless, we show in the Appendix B how one can *change basis* and calculate with, if one wishes, (27) for both incoming and outgoing particles, at the expense of obscuring the physics.

The corresponding wave functions for scalar particles have appeared in Ref. [60]. They were suggested as an alternative outgoing basis which would remove infinite phase factors from S-matrix elements. However, while $\bar{\Psi}_{\text{out}}(\Psi_{\text{in}})$ behaves well in the far future (past), it does not behave well in the far past (future), and the S-matrix element contains an integral over *all* times. Rather, the use of (30) makes the correct physics manifest, and the divergent phases are only removed by regulating the S-matrix elements in analogy to the procedure used for the classical theory, as we will see.

Let us briefly check that these *new* LSZ rules describe the correct large distance behavior of the theory. We return to scattering without emission. Using (27) and (30), and again trading $k.x$ for dimensionless s, the S-matrix element for this process now takes the form (for some F which we do not need explicitly)

$$S_{\text{no emission}} = \int \mathrm{d}s\,\delta^4(p' + eC_\infty - p - sk)F. \tag{31}$$

There is again only one point of support for the delta function, as one finds by squaring the conservation law:

$$p'^2 = (p - eC_\infty + sk)^2 \Rightarrow s = \frac{2eC_\infty\cdot p - e^2C^2_\infty}{2k\cdot p}. \tag{32}$$

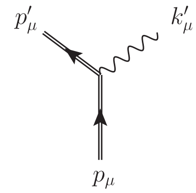
Inserting this into (31) we see that the S-matrix element for scattering without emission has support when $p'^\mu = \pi^\mu(p;\infty)$, where we recognize the asymptotic kinematic momentum π^μ from (23). In other words, the scattering amplitude now tells us that an electron experiences both the longitudinal and transverse pushes implied by the Lorentz force as it passes through a plane wave, as it should. This resolves the puzzle introduced above regarding the transverse terms in the momenta. Our LSZ analysis therefore yields the correct physics, and we can now construct the S-matrix element for nonlinear Compton scattering and examine its IR structure.

V. NONLINEAR COMPTON SCATTERING: IR DIVERGENCE

A. *S*-matrix element: regularization

Nonlinear Compton scattering, $e^-(p) \xrightarrow{\text{in laser}} e^-(p') + \gamma(k')$, has the following S-matrix element to lowest order in the interaction between quantized fields (i.e., to tree level, with the background accounted for to all orders),

$$
\begin{aligned}
S_{fi} &= -ie \int \mathrm{d}^4x \; \bar{\Psi}^{\mathrm{out}}_{p',\sigma'}(x) \not{\epsilon} e^{ik'.x} \Psi^{\mathrm{in}}_{p,\sigma}(x) \\
&= -\frac{ie}{2k_+}(2\pi)^3 \delta^3_{\perp,-}(p' + eC_\infty + k' - p) \int \mathrm{d}\phi \; e^{i\Phi(s_+,\phi)} \; \mathrm{Spin}(\phi)\,.
\end{aligned} \tag{33}
$$



The Ψ's are as in (27) and (30). To reach the second line, the integrals over $x^\perp$ and x^- are performed to yield delta functions. *Spin* contains the photon polarization and all the spin structure coming from the Volkov solutions, while Φ contains all the ϕ-dependent phases coming from the same. These, together with s_+, are given explicitly below. The ϕ integral in (33) needs to be regulated. Proceeding just as in the classical theory, we split the integral into three parts corresponding to before, during and after the pulse. Regulating with a damping factor essentially *cuts out* (in a gauge invariant way, as we confirm shortly) the before and after pieces of the S-matrix element in which no scattering can occur. The resulting expression is

$$
S_{\mathrm{fi}} = \frac{ie}{2k_+}(2\pi)^3 \delta^3_{\perp,-}(p' + eC_\infty + k' - p) \times \int \mathrm{d}\phi e^{i\Phi(s_+,\phi)} \frac{\mathrm{d}}{\mathrm{d}\phi}\left[\frac{\mathrm{Spin}(\phi)}{i\Phi'(s_+,\phi)}\right]. \tag{34}
$$

The dash on Φ is a derivative w.r.t. ϕ. It is conceptually clearer to again Fourier transform, trading ϕ for a dimensionless variable s which represents the lightfront energy taken from the background. The derivative of the term in square brackets is proportional to the background field strength, and hence the integrand vanishes outside the pulse. This means firstly that the Fourier transform is well defined and secondly that there are no infinite phase factors to worry about, as promised. The S-matrix element becomes

$$
S_{\mathrm{fi}} = ie \int \frac{\mathrm{d}s}{2\pi}(2\pi)^4 \delta^4(p' + eC_\infty + k' - p - sk)\Gamma(s),
$$

$$
\text{with} \quad \Gamma(s) := \int \mathrm{d}\phi e^{i\Phi(s,\phi)} \frac{\mathrm{d}}{\mathrm{d}\phi}\left[\frac{\mathrm{Spin}(\phi)}{i\Phi'(s,\phi)}\right]. \tag{35}
$$

Explicitly, the spin and phase parts are

$$
\begin{aligned}
\Phi(s,\phi) &= s\phi - \int_\phi^{\phi_f} \frac{2e\delta C \cdot p' - e^2 \delta C^2}{2k \cdot p'} - \int_{\phi_i}^{\phi} \frac{2eC \cdot p - e^2 C^2}{2k \cdot p}, \\
\mathrm{Spin}(\phi) &= \bar{u}^{\sigma'}_{p'}\left(1 + \frac{e\delta \not{C} \not{k}}{2k \cdot p'}\right) \not{\epsilon} \left(1 + \frac{e\not{k}\not{C}}{2k \cdot p}\right) u^\sigma_p.
\end{aligned} \tag{36}
$$

(Undoing the Fourier transform sets s to a particular value s_+, but working in Fourier space allows us to maintain covariance, and the resulting expressions are clearer). Before proceeding to the emission probability itself we should check that our regularization is gauge invariant with respect to transformations of the quantum fields. This can be confirmed by showing that (35) vanishes when $\varepsilon \to \varepsilon + \xi k'$: one finds that the resulting change in $\Gamma(s)$ is

$$
\delta\Gamma(s) = \xi \bar{u}_{p'} \not{k} u_p \int \mathrm{d}\phi e^{i\Phi(s,\phi)} \frac{\mathrm{d}}{\mathrm{d}\phi}\left[\frac{i\Phi'(s,\phi)}{i\Phi'(s,\phi)}\right] = 0, \tag{37}
$$

as required. We can now wrap the incoming state into a wave packet, normalized per unit lightfront volume (so the incoming particle carries a normalization of $1/\sqrt{2p_-}$ rather than $1/\sqrt{2p_0}$), square up the S-matrix element and obtain the total probability of emitting a photon, averaged over initial spins, summed over final spins and polarizations, as

$$
\mathbb{P} = \frac{e^2}{2k.p} \int \mathrm{d}f \int \frac{\mathrm{d}s}{2\pi}(2\pi)^4 \delta^4(p' + eC_\infty + k' - p - sk) \times \frac{1}{2} \sum_{\sigma,\sigma',\varepsilon} |\Gamma(s)|^2. \tag{38}
$$

As usual, the wave packet drops out of the final expression, and the integral over final states is

$$
\mathrm{d}f = \frac{\mathrm{d}^3 p'}{(2\pi)^3 2p'_0} \frac{\mathrm{d}^3 k'}{(2\pi)^3 2k'_0}. \tag{39}
$$

Of the seven integrals in $\mathbb{P}$, four can be performed using the delta functions. Methods for evaluating the remaining three integrals are discussed in Ref. [61]; see also Ref. [31].

B. Probability of emission: IR divergence

We can now investigate the IR contribution to the probability (38). Using the kinematics implied by the delta function in (38) one finds that the phase Φ has a single stationary point, corresponding to the point of soft emission, $\omega' = 0$. The function $\Gamma(s)$ therefore diverges at this point. The classical analogue of this statement was that $k'.u \neq 0$ unless $\omega' = 0$; see (9) and the discussion following. At the point of soft emission, the argument of the delta function in (38) becomes

$$
p' + eC_\infty - p - sk \to 0, \tag{40}
$$

which is just the inelastic scattering condition we found in (31). In order to study the IR limit we therefore expand around (32), writing

$$
s = t + \frac{2eC_\infty \cdot p - e^2 C_\infty^2}{2k \cdot p}, \tag{41}
$$

and look at the limit of small t. We eliminate the p' integrals in (38) using the delta functions. The remaining calculation is straightforward; the denominator Φ' becomes, for example,

$$\Phi'(\phi) = t + \frac{1}{k\cdot p}[k'\cdot \pi^{\mu}(p;\phi) - k'\cdot \pi(p;\infty)] + \ldots \tag{42}$$

and in the soft limit one has $\omega'_t = tk\cdot p/n'\cdot\pi(p;\infty) + \ldots$ and so $\omega' \propto t$. Hence, the remaining delta function in (38) may be used to perform either the ω' integral or, equivalently, the t integral. After performing the spin sums, the soft-photon contribution (38) is found to be

$$\mathbb{P} = -\frac{\alpha}{(2\pi)^2}\int \mathrm{d}\Omega'\bigg(\frac{\pi(p;\infty)}{n'\cdot\pi(p;\infty)} - \frac{p}{n'\cdot p}\bigg)^2 \int_0 \mathrm{d}t\frac{1}{t} + \ldots \tag{43}$$

This diverges when $t \to 0$, i.e., at the point of soft emission. The probability is therefore IR divergent whenever the term in large brackets is nonzero, i.e., whenever the field is able to accelerate the particle, such that $\pi(p;\infty) \neq p$, which requires a unipolar field with nonvanishing C^{∞}_{μ}. The singularity is logarithmic, as in bremsstrahlung, and depends only on whether $C^{\infty}_{\mu} = 0$ or not. The *probability* (43) matches the classically expected number of photons (14) with $p' = \pi(p;\infty)$, also as for bremsstrahlung. The removal of this divergence is discussed in the conclusions. To understand the physical differences and similarities between nonlinear Compton, bremsstrahlung and Compton scattering, it is helpful to consider the perturbative limit of our results.

C. Perturbative expansion

We assume the incoming electron is at rest in order to give the clearest results. To lowest order in the background, the probability (38) then becomes

$$\mathbb{P}_{\text{pert}} = a_0^2\int_0^{\infty}\frac{\mathrm{d}s}{2\pi}\frac{|\tilde{f}'_j(s)|^2}{s} \times \frac{\alpha}{2}\int_{-1}^{1}\mathrm{d}(\cos\theta)\bigg(\frac{\omega'_s}{s\omega}\bigg)^2 \times\bigg[\frac{\omega'_s}{s\omega} + \frac{s\omega}{\omega'_s} - \sin^2\theta\bigg]. \tag{44}$$

This is a sum over ordinary Klein-Nishima probabilities for Compton scattering of incoming photons of all frequencies $s\omega$ (second line), modulated by the strength of the electromagnetic fields (first line). The corresponding Feynman diagrams are shown in Fig. 3. The photon frequencies which can be produced by each s are

$$\omega'_s = \frac{s\omega}{1 + \frac{s\omega}{m}(1-\cos\theta)}. \tag{45}$$

The essential difference between Compton and nonlinear Compton is the range of produced photon frequencies,

$$\frac{s\omega}{1 + 2\frac{s\omega}{m}} < \omega' < s\omega. \tag{46}$$

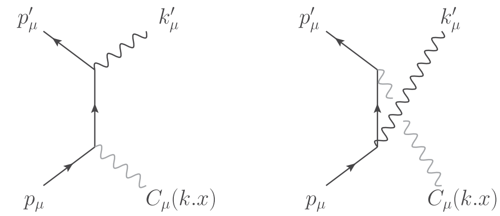


FIG. 3. Nonlinear Compton scattering at tree level, to lowest order in the background field.

In Compton scattering one obtains only the second line of (44) with $s = 1$. The fixed and nonzero incoming photon frequency ω then acts as an *IR cutoff*, since it forbids the outgoing photon from having zero frequency; one obtains (46) with $s = 1$. In nonlinear Compton, though, the background field contains a range of frequencies, and each can lead to photon production in the range (46). Even though the range for each s is bounded, s is continuous with $s \geq 0$ and so the emitted photons can in principle be arbitrarily soft; this is just as in bremsstrahlung, but *not* as in Compton scattering.

Whether or not the point $s = 0$ can contribute depends on the low frequency composition of the beam. For any compactly supported field, i.e., a pulse, we can expand $\tilde{f}'(s) = \tilde{f}'(0) + \mathcal{O}(s)$ for small s. From (45) we have $\omega'_s/s\omega = 1 + \mathcal{O}(s)$, and we find that the soft contribution to the probability (44) is

$$\mathbb{P}_{\text{pert}} = \frac{8\alpha a_0^2}{6}\int_0 \frac{\mathrm{d}s}{2\pi}\frac{|\tilde{f}'_j(0)|^2}{s} + \mathcal{O}(s). \tag{47}$$

We again obtain the result that the probability is log divergent at $s = 0$, corresponding to the emission of a zero frequency photon, when $\tilde{f}'(0) \neq 0$. Hence, we confirm that the IR divergence can be attributed to the Fourier zero (frequency) mode of the background field strength; this mode permits the production of a zero frequency photon in a kind of *forward scattering*. This coincides exactly with the ability of the field to accelerate the particle following (21).

D. Example: crossed fields

Crossed fields can accelerate particles and one therefore expects a soft IR divergence; recall also (25). Despite this, the literature results state that the nonlinear Compton probability in crossed fields is IR finite [15,16,54]. From a QED perspective, the lack of an IR divergence when the coupling does not turn off is potentially interesting, since it is the assumption of adiabatic switching which is responsible for IR divergences [62,63]. However, homogeneous fields are often rather special cases when it comes to radiation (see Ref. [47] and Chap. 37 of Ref. [64]), and crossed fields are no exception. The nonlinear Compton probability is given in the Appendix C. The conclusion therein is that the literature results are indeed IR finite but cannot be obtained from the limiting case of a homogeneous field of large duration, which does contain an IR divergence. We do not need the details however: we can show using only classical arguments that the reason for the difference between the literature results and our own is a different choice of boundary conditions.

The literature results describe the crossed field, from the outset, as persisting for all time, and with a gauge potential $C_\mu(\phi) = a^1_\mu \phi \equiv a_\mu \phi$ [16]. The Volkov solutions for this potential, used in the quantum calculation of Nikishov and Ritus [16], carry the kinematic momentum

$$\pi^c_\mu(\phi) = p_\mu - ea_\mu\phi + \frac{2ea\cdot p\phi - e^2 a\cdot a\phi^2}{2k\cdot p}k_\mu, \quad (48)$$

and the path x^c is the indefinite integral of π^c_μ, where c stands for crossed. The (unregulated) classical current corresponding to the S-matrix element of Nikishov and Ritus [15,16] is

$$j^c_\mu = \frac{e}{k\cdot p}\int_{-\infty}^{\infty} \mathrm{d}\varphi\, \pi^c_\mu(\varphi) e^{ik'\cdot x^c(\varphi)}. \quad (49)$$

It follows that p_μ is the kinematic momentum at $\phi = 0$; for a particle to have finite momentum after spending an infinite time in the crossed field means that it must have *started* with infinite momentum. This is clear from (48). There are two possible interpretations.

First, one can protest that the field should be considered to turn on at finite time, say $\phi = 0$ so that p_μ is the incoming momentum. In this case, (49) and the corresponding literature results contain unphysical contributions from before the particle entered the pulse. Removing them reveals the soft divergence as in Sec. II.

Second, one can take (48) and (49) at face value. In principle there is no need to regulate the current since the particle never enters or leaves the crossed field. If we did regulate as above, the difference between the two expressions would be a boundary term as in (13), which we have seen is key to understanding the IR, so let us calculate it. The particle described by (48) has infinite kinetic momentum in both the asymptotic past and future, with the leading term being

$$\pi^c_\mu(\phi) = -e^2\frac{a\cdot a\phi^2}{2k\cdot p}k_\mu + \dots \quad (50)$$

For low frequencies, the boundary term which would cause a soft divergence is therefore

$$\lim_{\phi\to\infty}\left(\frac{\pi^c_\mu(\phi)}{k'\cdot\pi^c(\phi)} - \frac{\pi^c_\mu(-\phi)}{k'\cdot\pi^c(-\phi)}\right) = \frac{k_\mu}{k'\cdot k} - \frac{k_\mu}{k'\cdot k} = 0. \quad (51)$$

This means that, from the point of the view of the radiation formulas, the momenta in the asymptotic past and future are not only lightlike but *equal*: since the particle is decelerated from and reaccelerated to the speed of light, there is effectively no net acceleration, and hence no IR divergence.[4]

In summary, the literature results for nonlinear Compton scattering in crossed fields are indeed IR finite, but *only* on the assumptions that (1) the electron begins with an infinite momentum in the past, and (2) it is decelerated from and then reaccelerated to the speed of light over an infinite time. Dropping these assumptions amounts to allowing the particle to enter and exit the pulse at finite times, and the IR divergence reappears. We leave it to the reader to decide which scenario is more physical.

VI. DISCUSSION AND CONCLUSIONS

The IR problem in QED is first encountered in perturbation theory, at tree level, in the bremsstrahlung (breaking radiation) amplitude for a particle decelerated by an external Coulomb potential. Replacing this potential with a plane wave, we have seen that the same IR divergence is found in nonlinear Compton scattering when the plane wave can give a net acceleration to a particle passing through it.

Our results hold for arbitrary pulses. The only case for which we have allowed a nonvanishing asymptotic field strength is crossed fields. The literature results for nonlinear Compton scattering in crossed fields are, surprisingly, IR finite. We have shown that this results from assuming somewhat unphysical boundary conditions for the scattered particles.

In order to obtain finite and measurable results for nonlinear Compton scattering, soft emission and higher loop effects must be accounted for. Tree-level results for the production of one hard photon and an *arbitrary* number of soft photons have been calculated and follow the expected IR structure of QED [32]: thus, the cancellation of IR divergences to all orders is expected to go through as normal. (See Ref. [67] for an example of how such structures arise naturally in exactly solvable systems, and also Ref. [30]). To lowest order in perturbation theory, this cancellation requires adding the calculated probability (38) of nonlinear Compton scattering to that of scattering without emission, to one loop. The required diagram is shown in Fig. 4. The loop has never been calculated for general plane waves (for crossed fields, see Ref. [68] and references in Ref. [69]; for monochromatic fields see Ref. [18]), and it will be interesting to investigate both its UV and IR structures when the background is treated nonperturbatively. This will be discussed in a sequel paper.

[4] S-matrix elements in crossed fields can be obtained from the low frequency limit of those in monochromatic waves [15] and are then interpreted as *locally constant* approximations for low frequency lasers. Monochromatic waves can themselves be obtained as the limit of N whole-cycle wave trains when $N \to \infty$ [65,66]: these do not accelerate, which gives a more convoluted explanation for why crossed fields yield IR finite results.

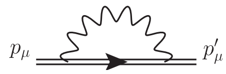


FIG. 4. One-loop contribution to scattering without emission in a background field. One sees by expanding in powers of the background field that the diagram combines both self-energy and vertex corrections.

Let us finally address the impact of our results on nonlinear Compton scattering in whole-cycle pulses. In Sec. IV we saw that even whole-cycle pulses, which give IR finite results, can produce photons with arbitrarily low frequencies when the Fourier spectrum of the pulse extends down to zero frequency. It follows that no detector of finite resolution can distinguish between sufficiently soft emission via nonlinear Compton and scattering without emission (just as soft bremsstrahlung cannot be distinguished from scattering without emission). Experimentally indistinguishable processes must therefore still be accounted for in order to yield measurable probabilities and cross sections for nonlinear Compton experiments, even when the IR divergence is absent.

ACKNOWLEDGMENTS

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APPENDIX A: CARRIER PHASE

The following simple example provides the quantitative results behind Fig. 1. Consider a short pulse with field strength profile

$$f'(\phi) = -n(c)\sin\left(\frac{\phi}{2}\right)^2 \sin(\phi + c), \tag{A1}$$

for $0 \leq \phi \leq 2\pi$ and zero otherwise. The parameter c can be thought of as a *carrier phase* [70,71], and $n(c)$ is a normalization which ensures the pulse energy is carrier-phase independent. The field strength and potential C_μ are plotted in Fig. 5. For $c = 0$ the field describes a sinusoidal-like cycle. As $c \to \pi/2$ the pulse acquires a typical *half cycle* shape; see Ref. [72] for experimental applications of such pulses. For nonzero c, C_μ becomes constant and nonzero when the fields turn off, as is also shown. In Fig. 6 we display the classical energy density in these pulses as a function of ω' for small ω'. The energy density goes to zero only when $c = 0$; from (14), it is therefore only when $c = 0$ (no net acceleration) that the number of photons N_γ is finite.

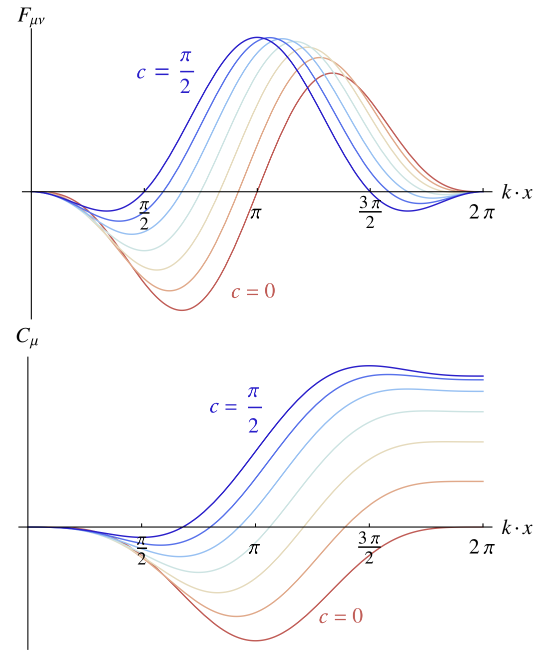

FIG. 5 (color online). The profiles for field strength, $f'(\phi)$ (upper panel), and potential, $f(\phi)$ (lower panel), of a single-cycle pulse with carrier phase $c \in \{0, \pi/2\}$ (red to blue/bottom to top); see Eq. (A1). For all $c \neq 0$, the potential is nonvanishing when the pulse turns off, implying a net acceleration.

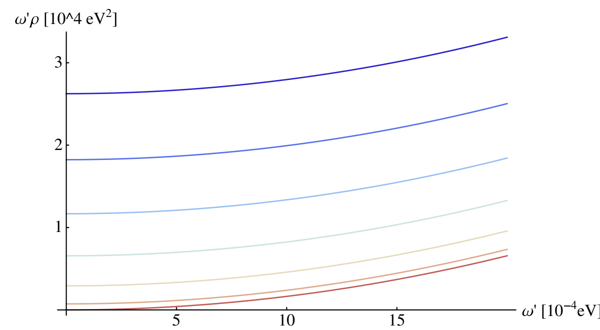

FIG. 6 (color online). The IR behavior of the energy density $\omega'\rho(\omega')$ in the pulse (A1), at fixed emission angles. Electron initially at rest, $a_0 = 1$, backscattered radiation. The carrier phase c is in the range 6×10^{-3} (top/blue) to 0 (bottom/red). The energy density is zero at $\omega' = 0$ only for zero carrier phase.

APPENDIX B: CHOICE OF BASIS

We have allowed for gauge potentials which are constant and nonzero, but still pure gauge, in the asymptotic future. Doing so gives us, via LSZ reduction, Volkov solutions labeled by physical momentum and spin. We will show in this section that if one uses the incoming wave functions (27) also for outgoing electrons, the same results are obtained for integrated probabilities. This confirms that the use of (30) is just a choice of basis. We will establish the equivalence at the level of the probability (38), rather than the amplitude level.

We begin by introducing a new variable $\bar{p}$ which obeys

$$\pi_\mu(\bar{p};\infty) = p'_\mu. \tag{B1}$$

In other words, $\bar{p}$ is, in the absence of emission, the momentum a particle had *before it entered the wave*, if it leaves with momentum p'; see Fig. 7. Explicitly, $\bar{p}$ is obtained from (17) by sending $p \to p'$ and $e \to -e$,

$$\bar{p}_\mu = p'_\mu + eC^\infty_\mu + \frac{-2eC^\infty \cdot p' - e^2 C^\infty \cdot C^\infty}{2k \cdot p'} k_\mu. \tag{B2}$$

(This is reminiscent of crossing symmetry and amounts to evolving the particle with p' *back in time*, through the field, to identify the momentum $\bar{p}$ it began with; see Fig. 7). (B2) can also be derived from the momentum conservation law for scattering without emission (31), by squaring up with C^∞ on the left-hand side, so that the support depends on outgoing p' rather than incoming p. Momentum conservation then becomes the requirement that $\bar{p} = p$. We now turn to nonlinear Compton. Starting with (38), three transformations are needed, as follows: *(i) Momentum:* Change variables,

$$s \to s + \frac{2eC^\infty \cdot p' - e^2 C^\infty \cdot C^\infty}{2k \cdot p'} k_+. \tag{B3}$$

This trades p' for $\bar{p}$ in Φ and the delta function, and removes the *explicit* dependence on C^∞ therein. *(ii) Spin:* We consider only probabilities summed over final spins. This spin sum gives

$$\Big(1 + \frac{e\not{k}\delta\not{C}}{2k \cdot p'}\Big)(\not{p}' + m)\Big(1 + \frac{e\delta\not{C}\not{k}}{2k \cdot p'}\Big) = \Big(1 + \frac{e\not{k}\not{C}}{2k \cdot \bar{p}}\Big)(\bar{\not{p}} + m)\Big(1 + \frac{e\not{C}\not{k}}{2k \cdot \bar{p}}\Big), \tag{B4}$$

which is the sum one obtains from incoming Volkov wave functions with momentum $\bar{p}$. *(iii) Final states:* $\bar{p}_\mu$ and p'_μ are two on-shell momenta related by the Lorentz transformation

$$\Lambda^\mu{}_\nu = \exp\Big[\frac{e}{k \cdot p'}(C^\infty k - kC^\infty)^\mu{}_\nu\Big], \tag{B5}$$

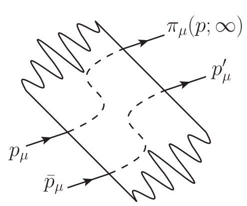

FIG. 7. In the absence of other interactions, a particle entering the wave with momentum p_μ leaves with momentum $\pi_\mu(p;\infty)$. A particle which therefore *leaves* the wave with momentum p'_μ had a momentum $\bar{p}$ when it entered, where $p'_\mu = \pi_\mu(\bar{p},\infty)$.

and, despite the momentum dependence of this transformation, the measure over final states is invariant under $\Lambda^\mu{}_\nu$ [73], so that

$$\int \frac{\mathrm{d}^3\bar{p}}{2\bar{p}_0} = \int \frac{\mathrm{d}^3 p'}{2p'_0}. \tag{B6}$$

Performing these manipulations, (38) reduces to the result one would obtain by using the same Volkov solutions to describe both incoming and outgoing particles, as has appeared in the literature to date. In summary, our *LSZ approach* uses different bases for incoming and outgoing states, so that all wave functions are labeled by physical momenta. There is nothing to stop us, though, from expanding out-states in a basis of in-states and using the same Volkov wave functions for all external particles. Both approaches yield the same results when the outgoing electron's degrees of freedom are integrated out. However, if one is interested in *differential* rates or probabilities with respect to the electron momentum, one should change variables from $\bar{p}$ back to the physical p'_μ.

APPENDIX C: PROBABILITIES AND CROSSED FIELDS

Using the above results, the nonlinear Compton probability is most compactly written

$$\mathbb{P} = \frac{e^2m^2}{k \cdot p}\int \frac{\mathrm{d}^3\bar{p}}{(2\pi)^3 2\bar{p}_0}\int \frac{\mathrm{d}^3 k'}{(2\pi)^3 2\omega'} \times \int \frac{\mathrm{d}s}{2\pi}(2\pi)^4\delta^4(\bar{p} + k' - p - sk)\mathcal{J}. \tag{C1}$$

Let $x \equiv k.k'/k.p'$. All dependence on the pulse profile is contained in

$$\mathcal{J} = -2|B_0|^2 + a^2\Big(1 + \frac{x^2}{2(1+x)}\Big) \times (2|B_1|^2 + 2|B_2|^2 - B_0B_3^* - B_0^*B_3), \tag{C2}$$

through four functions B_μ,

$$B_\mu = \int \mathrm{d}\phi\, e^{i\Phi}\frac{\mathrm{d}}{\mathrm{d}\phi}\Big(\frac{f_\mu(\phi)}{i\Phi'}\Big), \tag{C3}$$

where we define $f_0 \equiv 1$, $f_3 \equiv f_1^2 + f_2^2$ and the phase Φ is now

$$\Phi(x) := sx - \alpha_j\int_{-\infty}^x \mathrm{d}y f_j(y), \tag{C4}$$

with j summed over $\{1, 2, 3\}$. The α parameters are constructed from the incoming Volkov solutions,

$$\alpha_j = e a_j^\mu \cdot \left(\frac{p_\mu}{k\cdot p} - \frac{\bar{p}_\mu}{k\cdot\bar{p}}\right), \quad j = 1, 2,$$
$$\alpha_3 = -\frac{m^2 a_0^2}{2}\frac{k\cdot k'}{k\cdot pk\cdot\bar{p}}, \tag{C5}$$

and the B_μ obey $sB_0 = \alpha_j B_j$ as a consequence of (37).

Our assumption on the behavior of the electromagnetic fields (that they vanish asymptotically) does not allow us to apply the above results to crossed fields directly. Instead we take the limit of the more physical situation in which a particle enters and leaves a patch of constant field strength. We will compare these results with those in the literature which assume a crossed field from the outset. The potential for a field which is constant for a lightfront time T is given in (24). In this case, the functions (C3) become (changing variables $\varphi = \phi + T/2$)

$$B_\mu = -\int_0^T \mathrm{d}\varphi e^{i\Phi_c}\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{b_\mu(\varphi)}{i\Phi'(\varphi)}\right), \tag{C6}$$

with $b_\mu(\varphi) = (1, \varphi, 0, \varphi^2)$ and

$$\Phi_c(\varphi) = \left(s + \frac{\alpha_1^2}{4\alpha_3}\right)\varphi - \frac{\alpha_3}{3}\left(\varphi + \frac{\alpha_1}{2\alpha_3}\right)^3. \tag{C7}$$

In order to compare this result with that in the literature, we integrate by parts—without dropping the boundary term—to find

$$B_\mu = -e^{i\Phi_c}\frac{b_\mu(\varphi)}{i\Phi'(\varphi)}\bigg|_0^T + \int_0^T \mathrm{d}\varphi e^{i\Phi_c} b_\mu(\varphi). \tag{C8}$$

As $\omega' \to 0$, and using momentum conservation, the boundary term survives, reproducing the IR divergence of (25), while the second, *bulk* term vanishes. Now consider the limit as $T \to \infty$. The only T dependence in the bulk term is in the integral limit, so we replace $T \to \infty$ there: this should be compared with the corresponding literature expression for B_μ, which is [15,16]

$$B_\mu \stackrel{!}{=} \int_{-\infty}^{\infty} \mathrm{d}\varphi e^{i\Phi_c} b_\mu(\varphi). \tag{C9}$$

(To convert from the conventions of Nikishov and Ritus [15,16] to ours, use $p'_{\mathrm{N.R.}} \to \bar{p}$, $\alpha_{\mathrm{N.R.}} \to -\alpha_1$, $s_{\mathrm{N.R.}} \to -s$ and $\beta_{\mathrm{N.R.}} \to 4\alpha_3$). The results (C8) and (C9) are not equivalent, even as $T \to \infty$. The former contains a boundary term giving an IR divergence and has *semi-infinite* integration limits, both of which are a consequence of the particle being allowed to enter and leave the background. The literature result (C9) assumes a constant field from the outset, which the particles never enter or leave. This is consistent, but it shows that (C9) cannot be obtained as the large-duration limit of (C8). Further discussion may be found in Sec. IV.

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