Global poles of the two-loop six-point $\mathcal{N} = 4$ SYM integrand

Kasper J. Larsen*

Department of Physics and Astronomy, Uppsala University, SE-75108 Uppsala, Sweden, and Institut de Physique Théorique, CEA-Saclay, F-91191 Gif-sur-Yvette cedex, France

(Received 15 May 2012; published 19 October 2012)

Recently, a recursion relation has been developed, generating the four-dimensional integrand of the amplitudes of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory for any number of loops and legs. In this paper, I provide a comparison of the prediction for the two-loop six-point maximally helicity-violating integrand against the result obtained by use of the leading singularity method. The comparison is performed numerically for a large number of randomly selected momenta and in all cases finds agreement between the two results to high numerical accuracy.

DOI: 10.1103/PhysRevD.86.085032 PACS numbers: 11.15.−q, 11.15.Bt, 11.55.Bq

I. INTRODUCTION

Scattering amplitudes in gauge theories are fascinating quantities partly because they provide a direct link between theory and experiment and partly because their investigation continues to uncover rich structures in quantum field theory. Our understanding of computing amplitudes has undergone a revolution over the past decade and a half, owing in large part to the development of on-shell recursion relations for tree-level amplitudes [1,2] and a purely on-shell formalism for loop-level amplitudes, the modern unitarity method [3–34]. These powerful modern methods have to a large extent made the more traditional approach of Feynman diagrams obsolete for tree-level and one-loop amplitudes. Thus, the current frontier is the development of systematic approaches for computing two-loop amplitudes.

The maximally supersymmetric gauge theory in four dimensions, the $\mathcal{N} = 4$ super Yang-Mills theory, has attracted a great deal of effort over the past several years, serving as a laboratory for testing new ideas. This has lead to the discovery of a surprising new symmetry in the planar limit, the so-called dual conformal symmetry, which, although not inherited from the theory's Lagrangian in any obvious way, is nonetheless a property of its amplitudes. This symmetry combines with the standard superconformal symmetry, dictated by the Lagrangian, to form an infinite-dimensional symmetry algebra, the Yangian of the superconformal group. This integrable structure has been intensively studied [35–50] and has recently been exploited successfully to determine the theory's scattering amplitudes recursively in the number of loops [51]. Another spectacular recent advance, following earlier insights based on a reformulation of the S matrix of $\mathcal{N} = 4$ SYM as a contour integral on a Grassmannian manifold [52–59]—but this time extendable to any planar supersymmetric gauge theory—has been the development of recursion relations *à la* that of Britto, Cachazo, Feng, and Witten (BCFW) for the (strictly four-dimensional) loop integrand [60–63].

Ensuring the soundness of new ideas requires performing careful tests of the results they produce. In this spirit, the question we wish to address in this paper is whether the result in Refs. [60,63] for the two-loop six-point maximally helicity-violating (MHV) integrand of $\mathcal{N} = 4$ SYM theory can be reproduced by more traditional methods. In the generalized-unitarity-based approach, an L-loop amplitude is expressed as a linear combination of known basis integrals, plus terms that are rational functions of the external momenta (and which will not be discussed in this paper),

$$\text{Amplitude} = \sum_{j\in\text{Basis}} \text{coefficient}_j \text{Integral}_j + \text{Rational}. \quad (1.1)$$

The coefficients of the integrals (evaluated in dimensional regularization) are obtained by changing the integration range from $(\mathbb{R}^{4-2\epsilon})^{\otimes L}$ into specific contours σ of real dimension $4L$, embedded in $\mathbb{C}^{4L}$ and encircling the points where the maximum number of denominators of the integrand become zero. Unlike the path followed in the leading singularity method [64–68] in which one allows any choice of contour σ (by virtue of first having expanded the amplitude in an artfully chosen, typically overcomplete, basis), in generalized unitarity the contours σ are subject to the constraint that any function which integrates to zero on $(\mathbb{R}^{4-2\epsilon})^{\otimes L}$ must also integrate to zero on σ [34]. As argued in this paper, multidimensional contours σ satisfying this consistency condition are guaranteed to produce correct results for scattering amplitudes in any gauge theory, not only $\mathcal{N} =$ 4 SYM theory. The change of integration contour has the effect of transforming the integrals in Eq. (1.1) into contour integrals (which are easily evaluated by taking residues). By making the various allowed choices of contours σ one produces a set of linear equations satisfied by the integral coefficients which can then be solved to determine them uniquely.

The set of linear relations between two-loop integrals includes the set of all integration-by-parts (IBP) identities [69–87] between the various tensor integrals arising from the Feynman rules of gauge theory; however, at present a complete knowledge of such relations involving six-point two-loop integrals is not available [88]. For this reason, in this paper we will carry out the analysis following the leading

*Kasper.Larsen@cea.fr

singularity method, allowing any contour σ. This approach was shown to reproduce the result of a unitarity-based calculation [89] for the parity-even part of the two-loop six-point MHV amplitude in Ref. [67]. By expressing the full two-loop amplitude (i.e., including both parity-even and -odd parts) in terms of the same basis as used in Ref. [67], we therefore expect the leading singularity method to also produce the correct result for the parity-odd part of this amplitude.

The results of Refs. [60,63] are expressed in terms of a different basis from that of Ref. [67], and thus it is not meaningful to check agreement between individual integral coefficients in either representation. A quantity that can be meaningfully compared is the two-loop integrand: in general, in the planar limit of any field theory, the loop integrand is a well-defined rational function of the external momenta (which for example can be thought of as being produced by the Feynman rules). We have evaluated the integrand for a large number of randomly selected momenta and in all cases find agreement with the recent literature to high numerical accuracy [90].

An interesting spinoff of the calculation in this paper are the potential applications of the intermediate results (in particular, the enumeration of the global poles of the integrand and the expressions for the cut integrals). Once all the necessary IBP relations do become available, we expect these results to greatly facilitate the task of determining the maximal-cut contours that allow the extraction of integral coefficients in any two-loop six-point gauge theory amplitude.

A complementary approach is that of Refs. [91–95], in which the heptacut integrand is reconstructed by polynomial matching in similarity with the approach of Ossola, Papadopoulos, and Pittau (OPP) [21]. A recent paper by Zhang [96] adds tools required in such an approach for reducing integrands to a basis of monomials.

This paper is organized as follows. In Sec. II, we explain conventions and introduce notation used throughout the paper. In Sec. III, we compute the heptacuts of the general double-box integral and of the factorized double box with six massless legs. In Sec. IV, we apply these heptacuts to the ansatz for the two-loop six-point MHV amplitude of $\mathcal{N} = 4$ SYM and employ the leading singularity method to obtain linear equations satisfied by the integral coefficients. We discuss how these equations can be used to directly obtain the parity-even part of the integrand. We then explain how to evaluate numerically the full $\mathcal{N} = 4$ SYM integrand (i.e., including both parity-even and -odd parts) as obtained by the leading singularity method and report agreement with the result in Refs. [60,63]. Finally, in Sec. V we provide conclusions and suggest directions for future investigation. In Appendix A, we provide full details of all the heptacuts of the two-loop six-point MHV amplitude.

II. NOTATION AND CONVENTIONS

In this section we explain conventions and introduce notation used throughout the paper.

All external momenta in an amplitude are outgoing and will be denoted by k_i. We will make use of the spinor helicity formalism [97–104] in which a given massless four-dimensional momentum is written as a tensor product of two massless Weyl spinors,

$$k_i^\mu = \overline{u_+(k_i)}\sigma^\mu u_+(k_i) = \overline{u_-(k_i)}\sigma^\mu u_-(k_i). \tag{2.1}$$

We define the spinors

$$\lambda_i = u_+(k_i), \qquad \tilde{\lambda}_i = u_-(k_i), \tag{2.2}$$

and the Lorentz invariant inner products formed out of the spinors,

$$\begin{aligned} \langle ij\rangle &= \langle i^-|j^+\rangle = \overline{u_-(k_i)}u_+(k_j), \\ [ij] &= \langle i^+|j^-\rangle = \overline{u_+(k_i)}u_-(k_j), \end{aligned} \tag{2.3}$$

which satisfy

$$\langle ij\rangle[ji] = 2k_i \cdot k_j. \tag{2.4}$$

Spinor strings are defined as follows

$$\langle K_i^-|\not{P} + \not{Q}|K_j^-\rangle = \langle K_i^-|\not{P}|K_j^-\rangle + \langle K_i^-|\not{Q}|K_j^-\rangle \tag{2.5}$$

$$\langle K_i^-|\not{P}|K_j^-\rangle = \langle K_i P\rangle[PK_j] \quad \text{if } P^2 = 0. \tag{2.6}$$

We will use the following notation for sums and invariant masses of external momenta,

$$k_{i_1\cdots i_n} \equiv k_{i_1} + \cdots + k_{i_n} \tag{2.7}$$

$$s_{i_1\cdots i_n} \equiv (k_{i_1} + \cdots + k_{i_n})^2 \tag{2.8}$$

$$S_i \equiv K_i^2. \tag{2.9}$$

Throughout we will make use of the "flattened" momenta introduced in Refs. [21,23]: for a pair of momenta K_1, K_2, define the quantity

$$\gamma_{1\pm} = (K_1 \cdot K_2) \pm \sqrt{\Delta_1}, \qquad \Delta_1 = (K_1 \cdot K_2)^2 - K_1^2 K_2^2, \tag{2.10}$$

which can take two different values if both momenta are massive (i.e., if $S_1S_2 \neq 0$). For a given value of γ_1 one defines a pair of massless "flattened" momenta as follows

$$\begin{aligned} K_{1\pm}^\flat &= \frac{K_1 - (S_1/\gamma_{1\pm})K_2}{1 - S_1S_2/\gamma_{1\pm}^2}, \\ K_{2\pm}^\flat &= \frac{K_2 - (S_2/\gamma_{1\pm})K_1}{1 - S_1S_2/\gamma_{1\pm}^2}. \end{aligned} \tag{2.11}$$

If one of the momenta K_1 or K_2 is massless, $\gamma_{1\pm}$ can only take one value, and we will use the following abbreviated notation:

$$S_1S_2 = 0 \quad \Rightarrow \quad \begin{cases} \gamma_1 = 2K_1 \cdot K_2 \\ K_1^\flat = K_1 - (S_1/\gamma_1)K_2 \\ K_2^\flat = K_2 - (S_2/\gamma_1)K_1 \end{cases}. \tag{2.12}$$

Similarly, we will use the notation

$$\gamma_{2\pm} = (K_4 \cdot K_5) \pm \sqrt{\Delta_2}, \qquad \Delta_2 = (K_4 \cdot K_5)^2 - K_4^2 K_5^2 \tag{2.13}$$

$$K^{\flat}_{4\pm} = \frac{K_4 - (S_4/\gamma_{2\pm})K_5}{1 - S_4 S_5/\gamma_{2\pm}^2}, \qquad K^{\flat}_{5\pm} = \frac{K_5 - (S_5/\gamma_{2\pm})K_4}{1 - S_4 S_5/\gamma_{2\pm}^2}. \tag{2.14}$$

$$S_4 S_5 = 0 \quad \Rightarrow \quad \begin{cases} \gamma_2 = 2K_4 \cdot K_5 \\ K_4^{\flat} = K_4 - (S_4/\gamma_2)K_5 \\ K_5^{\flat} = K_5 - (S_5/\gamma_2)K_4 \end{cases}. \tag{2.15}$$

Finally, we will denote the elements of the dihedral group D_6 as follows

$$\begin{aligned} &\sigma_1 = (1,2,3,4,5,6) && \sigma_2 = (2,3,4,5,6,1) \\ &\sigma_3 = (3,4,5,6,1,2) && \sigma_4 = (4,5,6,1,2,3) \\ &\sigma_5 = (5,6,1,2,3,4) && \sigma_6 = (6,1,2,3,4,5) \\ &\sigma_7 = (6,5,4,3,2,1) && \sigma_8 = (5,4,3,2,1,6) \\ &\sigma_9 = (4,3,2,1,6,5) && \sigma_{10} = (3,2,1,6,5,4) \\ &\sigma_{11} = (2,1,6,5,4,3) && \sigma_{12} = (1,6,5,4,3,2). \end{aligned} \tag{2.16}$$

III. HEPTACUT TWO-LOOP INTEGRALS

As preparation for computing the generalized-unitarity cuts of the two-loop six-point amplitude, in this section we compute the heptacuts of the general double-box integral and the factorized double-box integral, illustrated below in Figs. 1 and 5. These integrals are part of the linear bases in which the two-loop six-point $\mathcal{N} = 4$ SYM helicity amplitudes were expanded in Refs. [67,89,105]. In this paper we shall use the (overcomplete) basis in Ref. [67] (illustrated for convenience in Fig. 7 in Sec. IV). As explained further in Sec. IV A, the heptacuts of the remaining integrals in this basis are easily obtained from the heptacuts of the double box integrals by multiplying

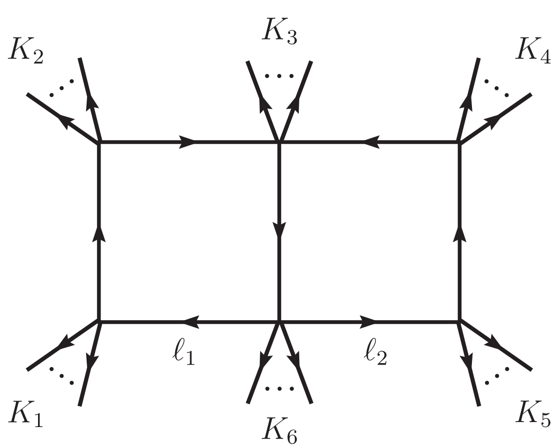


FIG. 1. The general double-box integral.

additional factors (involving propagators and possibly numerator insertions) onto the integrand of the latter. From the knowledge of the generalized cuts of all basis integrals, the cuts of the amplitude are easily obtained, as will be explained further in Sec. IV.

We emphasize that the generalized cuts considered throughout this paper are strictly four-dimensional (as opposed to $(4-2\epsilon)$-dimensional). Moreover, we will suppress the Feynman $i\varepsilon$-prescription when writing propagators. Finally, the internal lines in all diagrams are taken to be massless.

A. The maximal cut of the general double box

In this section we will compute the maximal cut of the double box integral where the vertex momenta K_i shown in Fig. 1 are typically sums of several lightlike momenta, $K_i = k_{i_1} + \cdots + k_{i_n}$ with $k_{i_j}^2 = 0$. Setting $D = 4 - 2\epsilon$, this integral is defined by

$$\int \frac{d^D\ell_1}{(2\pi)^D}\frac{d^D\ell_2}{(2\pi)^D}\bigg(\frac{1}{\ell_1^2}\frac{1}{(\ell_1-K_1)^2}\frac{1}{(\ell_1-K_1-K_2)^2} \times \frac{1}{(\ell_1+\ell_2+K_6)^2}\frac{1}{\ell_2^2}\frac{1}{(\ell_2-K_5)^2}\frac{1}{(\ell_2-K_4-K_5)^2}\bigg), \tag{3.1}$$

where the integration is over real Minkowski space for both loop momenta. By considering the heptacut of the general double-box integral we can easily obtain the heptacuts of all the double boxes appearing in the basis in Fig. 7 (in Sec. IV) by taking appropriate vertex momenta to be massless. As will be explained in Sec. III A 5, in the case of six massless external momenta, there are three qualitatively distinct ways of distributing the momenta at the vertices of the double box. In order to streamline the presentation we will first compute the heptacut of the completely general double-box integral, making no assumptions about masslessness of any vertex momenta. Subsequently, we will describe each of the three cases in turn.

Formally speaking, the four-dimensional heptacut of (3.1) is obtained by replacing each of the seven propagators by a δ-function whose argument is the denominator of the propagator in question, and replacing the D-dimensional integration measure by the corresponding four-dimensional measure (up to factors of $2\pi i$, depending on conventions). Thus, the heptacut replaces the double-box integral in (3.1) by the integral

$$J_{\text{formal}} = \int d^4\ell_1 d^4\ell_2 \delta(\ell_1^2)\delta((\ell_1-K_1)^2) \times \delta((\ell_1-K_1-K_2)^2)\delta((\ell_1+\ell_2+K_6)^2) \times \delta(\ell_2^2)\delta((\ell_2-K_5)^2)\delta((\ell_2-K_4-K_5)^2). \tag{3.2}$$

However, this integral only receives contributions from regions of integration space where the loop momenta solve the joint on-shell constraints

$$\ell_1^2 = 0 \tag{3.3}$$

$$(\ell_1 - K_1)^2 = 0 \tag{3.4}$$

$$(\ell_1 - K_1 - K_2)^2 = 0 \tag{3.5}$$

$$\ell_2^2 = 0 \tag{3.6}$$

$$(\ell_2 - K_5)^2 = 0 \tag{3.7}$$

$$(\ell_2 - K_4 - K_5)^2 = 0 \tag{3.8}$$

$$(\ell_1 + \ell_2 + K_6)^2 = 0, \tag{3.9}$$

which in general only have solutions for complex loop momenta $(\ell_1, \ell_2) \in \mathbb{C}^4 \times \mathbb{C}^4$.

The natural definition of δ-functions with complex arguments involves contour integrals—integrating out a variable q in an integrand involving δ-functions will fix q to some value q_0; in the language of contour integrals, this corresponds to integrating in the complex q-plane along a small circle centered at q_0. Indeed, as observed in Refs. [106,107], Cauchy's residue theorem implies that the localization property

$$\int dq \delta(q - q_0) f(q) = f(q_0), \tag{3.10}$$

remains to hold if we define $\delta(q - q_0) \equiv \frac{1}{2\pi i}\frac{1}{q-q_0}$ and take the integral to be a contour integral along a small circle in the complex q-plane centered at q_0.

By analogy, taking the four-dimensional heptacut of the double-box integral (3.1) should really be understood as a change of integration range from $\mathbb{R}^D \times \mathbb{R}^D$ to a surface (of real dimension 8) embedded in $\mathbb{C}^4 \times \mathbb{C}^4$ while leaving the integrand in Eq. (3.1) unchanged. The maximal-cut integral is thus a multidimensional contour integral whose contour is in general a linear combination of tori encircling the so-called *global poles* of the integrand. These are points $(\ell_1, \ell_2) \in \mathbb{C}^4 \times \mathbb{C}^4$ where all seven propagators in (3.1) become on-shell [108]. The change of contour away from real Minkowski space renders the double-box integral IR and UV finite, and one can therefore disregard the dimensional regulator part of the measure $d^{-2\epsilon}\ell_1 d^{-2\epsilon}\ell_2$ and the (-2ϵ)-dimensional components of the loop momenta.

In the following we will continue to write multidimensional contours symbolically in terms of δ-functions, as in Eq. (3.2), as we find the latter notation more suggestive. As it turns out, in all cases considered in this paper, the only respect in which the multidimensional contour integrals do not behave like integrals of δ-functions is the transformation formula for changing variables: Given a holomorphic function $f = (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ with an isolated zero [109] at $a \in \mathbb{C}^n$, the residue at a is computed by the integral over the contour $\Gamma_\epsilon(a) = \{z \in \mathbb{C}^n : |f_i(z)| = \epsilon_i, i = 1, \ldots, n\}$. This contour integral satisfies the transformation formula

$$\frac{1}{(2\pi i)^n}\int_{\Gamma_\epsilon(a)} \frac{h(z) dz_1 \wedge \cdots \wedge dz_n}{f_1(z)\cdots f_n(z)} = \frac{h(a)}{\det_{i,j}\frac{\partial f_i}{\partial z_j}}, \tag{3.11}$$

which, crucially, does not involve taking the absolute value of the inverse Jacobian. This ensures that this factor is analytic in any variables on which it depends, so that further contour integrations can be carried out.

In order to visualize the multidimensional tori in question, it turns out to be convenient to use the following parametrization of the loop momenta:

$$\ell_1^\mu = \alpha_1 K_1^{\flat\mu} + \alpha_2 K_2^{\flat\mu} + \alpha_3 \langle K_1^{\flat -}|\gamma^\mu|K_2^{\flat -}\rangle + \alpha_4 \langle K_2^{\flat -}|\gamma^\mu|K_1^{\flat -}\rangle \tag{3.12}$$

$$\ell_2^\mu = \beta_1 K_4^{\flat\mu} + \beta_2 K_5^{\flat\mu} + \beta_3 \langle K_4^{\flat -}|\gamma^\mu|K_5^{\flat -}\rangle + \beta_4 \langle K_5^{\flat -}|\gamma^\mu|K_4^{\flat -}\rangle. \tag{3.13}$$

The virtue of this parametrization is that it linearizes as many of the cut constraints as possible, and in turn it becomes easy to locate the positions of the global poles of the integrand in the coordinates $\alpha_1, \ldots, \alpha_4, \beta_1, \ldots, \beta_4$. The multidimensional tori discussed above are then easily obtained as products of small circles each encircling one of the entries of a given global pole.

After changing variables from the components of the loop momenta ℓ_1^μ and ℓ_2^ν to the parameters α_i and β_j, the heptacut of the double-box integral becomes

$$\begin{aligned} J = {}& \frac{1}{\gamma_1^3\gamma_2^3}\int\prod_{i=1}^4 d\alpha_i d\beta_i \left(\det_{\mu,i}\frac{\partial \ell_1^\mu}{\partial\alpha_i}\right)\left(\det_{\nu,j}\frac{\partial \ell_2^\nu}{\partial\beta_j}\right)\delta(\alpha_1\alpha_2 - 4\alpha_3\alpha_4)\delta\left((\alpha_1 - 1)\left(\alpha_2 - \frac{S_1}{\gamma_1}\right) - 4\alpha_3\alpha_4\right)\delta\left(\left(\alpha_1 - \frac{S_2}{\gamma_1} - 1\right)\right. \\ & \times \left.\left(\alpha_2 - \frac{S_1}{\gamma_1} - 1\right) - 4\alpha_3\alpha_4\right)\delta(\beta_1\beta_2 - 4\beta_3\beta_4)\delta\left(\left(\beta_1 - \frac{S_5}{\gamma_2}\right)(\beta_2 - 1) - 4\beta_3\beta_4\right)\delta\left(\left(\beta_1 - \frac{S_5}{\gamma_2} - 1\right)\right. \\ & \times \left.\left(\beta_2 - \frac{S_4}{\gamma_2} - 1\right) - 4\beta_3\beta_4\right)\delta((\ell_1 + \ell_2 + K_6)^2|_{\text{param}}), \end{aligned} \tag{3.14}$$

where the subscript "param" on the argument of the δ-function in the last line indicates that it is to be evaluated in the parametrization (3.12) and (3.13) and where the Jacobians associated with the change of variables are, respectively [110]

$$\det{}_{\mu,i}\frac{\partial \ell_1^\mu}{\partial \alpha_i} = i\gamma_1^2, \qquad \det{}_{\nu,j}\frac{\partial \ell_2^\nu}{\partial \beta_j} = i\gamma_2^2. \tag{3.15}$$

In the parametrization (3.12) and (3.13), the six on-shell constraints (3.3), (3.4), (3.5), (3.6), (3.7), and (3.8) which only involve a single loop momentum are solved by setting

$$\alpha_1 = \frac{\gamma_1(S_2+\gamma_1)}{\gamma_1^2 - S_1S_2}, \qquad \alpha_2 = \frac{S_1S_2(S_1+\gamma_1)}{\gamma_1(S_1S_2-\gamma_1^2)},$$
$$\alpha_3\alpha_4 = -\frac{S_1S_2(S_1+\gamma_1)(S_2+\gamma_1)}{4(\gamma_1^2-S_1S_2)^2} \tag{3.16}$$

$$\beta_1 = \frac{S_4S_5(S_5+\gamma_2)}{\gamma_2(S_4S_5-\gamma_2^2)}, \qquad \beta_2 = \frac{\gamma_2(S_4+\gamma_2)}{\gamma_2^2-S_4S_5},$$
$$\beta_3\beta_4 = -\frac{S_4S_5(S_4+\gamma_2)(S_5+\gamma_2)}{4(\gamma_2^2-S_4S_5)^2}. \tag{3.17}$$

We observe that the integrations of the δ-functions in (3.14) unambiguously fix the values of α_1, α_2, β_1, β_2. After imposing the last on-shell constraint (3.9), the four variables α_3, α_4, β_3, β_4 are subject to three relations, and one is free to choose either one of them to be an unconstrained free complex parameter z. Each of these four choices give rise to different contributions to the heptacut of the double box. As will be explained further in Sec. III A 5, these contributions correspond to the various existing classes of solutions to the on-shell constraints (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9), and the heptacut (3.14) is an appropriately weighted sum of these contributions.

In Sec. III A 1 below we compute in detail the contribution to the heptacut obtained by letting $z = \alpha_3$ be the unconstrained parameter. The results for the remaining contributions are quoted in Secs. III A 2–III A 4 and are obtained in an entirely analogous way. Finally, in Sec. III A 5 we explain how to assemble the heptacut of the double-box integral from its various contributions.

1. Leaving $z = \alpha_3$ as the free parameter

In this example, we aim to leave $z = \alpha_3$ as the unconstrained parameter, and we will therefore integrate out α_1, α_2, α_4 and β_1, β_2, β_3, β_4. This will proceed in three stages: first we integrate out the three δ-functions involving only the α-variables; then we integrate out the three δ-functions involving only the β-variables; finally, we integrate out the remaining β-variable.

Thus, we start by considering the following integral whose integrand consists of all the δ-functions in (3.14) that only involve the α-variables,

$$J_\alpha = \int d\alpha_1 d\alpha_2 d\alpha_4 \delta(\alpha_1\alpha_2 - 4\alpha_3\alpha_4)$$
$$\times \delta\Big((\alpha_1-1)\Big(\alpha_2 - \frac{S_1}{\gamma_1}\Big) - 4\alpha_3\alpha_4\Big)$$
$$\times \delta\Big(\Big(\alpha_1 - \frac{S_2}{\gamma_1} - 1\Big)\Big(\alpha_2 - \frac{S_1}{\gamma_1} - 1\Big) - 4\alpha_3\alpha_4\Big). \tag{3.18}$$

This integral is the inverse Jacobian associated with the change of integration variables from the α-parameters to the arguments of the δ-functions. It is straightforwardly evaluated to yield

$$J_\alpha = \frac{1}{4(1-\frac{S_1S_2}{\gamma_1^2})\alpha_3}, \tag{3.19}$$

where we recall that the δ-functions in Eqs. (3.14) and (3.18) are a short-hand notation for multidimensional contour integrations. As these are subject to the transformation formula in (3.11), which involves the Jacobian of the transformation rather than its absolute value, there is no absolute value in Eq. (3.19).

Second, we consider the following integral whose integrand consists of all the δ-functions in (3.14) that only involve the β-variables,

$$J_\beta = \int d\beta_1 d\beta_2 d\beta_4 \delta(\beta_1\beta_2 - 4\beta_3\beta_4)$$
$$\times \delta\Big(\Big(\beta_1 - \frac{S_5}{\gamma_2}\Big)(\beta_2 - 1) - 4\beta_3\beta_4\Big)$$
$$\times \delta\Big(\Big(\beta_1 - \frac{S_5}{\gamma_2} - 1\Big)\Big(\beta_2 - \frac{S_4}{\gamma_2} - 1\Big) - 4\beta_3\beta_4\Big). \tag{3.20}$$

In addition to integrating out β_1, β_2, we have here chosen to integrate out β_4. Alternatively, we could have imagined integrating out β_3; but this would ultimately lead to the same final result. Once again, the integral J_β is a Jacobian and is straightforwardly evaluated to yield

$$J_\beta = -\frac{1}{4(1-\frac{S_4S_5}{\gamma_2^2})\beta_3}. \tag{3.21}$$

Putting together the partial results in Eqs. (3.15), (3.19), and (3.21) and applying the loop momentum parametrization to the argument of the remaining factor $\delta((\ell_1+\ell_2+K_6)^2|_{\text{param}})$ in Eq. (3.14) one finds the expression

$$\frac{\gamma_1\gamma_2}{32(\gamma_1^2-S_1S_2)(\gamma_2^2-S_4S_5)}\int\frac{d\alpha_3 d\beta_3}{\alpha_3\beta_3}$$
$$\times\,\delta(B_1\beta_3 + B_0 + B_{-1}\beta_3^{-1}), \tag{3.22}$$

where

$$B_1 = \langle K_4^{\flat-}|\gamma_\mu|K_5^{\flat-}\rangle(\alpha_1 K_1^{\flat\mu} + \alpha_2 K_2^{\flat\mu}$$
$$+ \alpha_3\langle K_1^{\flat-}|\gamma^\mu|K_2^{\flat-}\rangle + \alpha_4\langle K_2^{\flat-}|\gamma^\mu|K_1^{\flat-}\rangle + K_6^\mu) \tag{3.23}$$

$$B_0 = (\beta_1 K^\flat_{4\mu} + \beta_2 K^\flat_{5\mu} + K_{6\mu}) \times (\alpha_1 K_1^{\flat\mu} + \alpha_2 K_2^{\flat\mu} + \alpha_3 \langle K_1^{\flat -}|\gamma^\mu|K_2^{\flat -}\rangle + \alpha_4 \langle K_2^{\flat -}|\gamma^\mu|K_1^{\flat -}\rangle + K_6^\mu) - \frac{1}{2}S_6 \tag{3.24}$$

$$B_{-1} = -\frac{S_4 S_5 (S_4+\gamma_2)(S_5+\gamma_2)\langle K_5^{\flat -}|\gamma_\mu|K_4^{\flat -}\rangle}{4(\gamma_2^2 - S_4 S_5)^2} \times (\alpha_1 K_1^{\flat\mu} + \alpha_2 K_2^{\flat\mu} + \alpha_3 \langle K_1^{\flat -}|\gamma^\mu|K_2^{\flat -}\rangle + \alpha_4 \langle K_2^{\flat -}|\gamma^\mu|K_1^{\flat -}\rangle + K_6^\mu). \tag{3.25}$$

We can integrate out β_3 using

$$\int \frac{d\beta_3}{\beta_3}\delta(B_1\beta_3 + B_0 + B_{-1}\beta_3^{-1}) = (B_0^2 - 4B_1 B_{-1})^{-1/2}. \tag{3.26}$$

In conclusion, leaving $z = \alpha_3$ unconstrained and integrating out the remaining loop momentum parameters from (3.14) produces the following contribution to the heptacut of the double box,

$$J|_{z=\alpha_3} = \frac{\gamma_1\gamma_2}{32(\gamma_1^2 - S_1S_2)(\gamma_2^2 - S_4S_5)} \times \oint \frac{dz}{z}(B_0(z)^2 - 4B_1(z)B_{-1}(z))^{-1/2}, \tag{3.27}$$

where we have relabeled $\alpha_3 \equiv z$ and made the dependence of B_1, B_0, B_{-1} on z explicit. Although the factor $(\cdots)^{-1/2}$ appears to have a branch cut, the radicand turns out to be a perfect square in all cases considered in this paper, and the integrand in Eq. (3.27) always takes the form $\frac{1}{z(z-P)}$. As we will discuss further in Sec. IV, we will allow the integration contour in Eq. (3.27) to encircle any individual singularity of the integrand.

Note that the form of the final result (3.27) does not depend on the order of integration: if one instead chooses to integrate out α_1, α_2, α_4 and β_1, β_2, β_3 from (3.14) and subsequently integrates out β_4, one finds (3.27) as may easily be checked.

2. Leaving $z = \alpha_4$ as the free parameter

Integrating out α_1, α_2, α_3 and β_1, β_2, β_3, β_4 from (3.14) produces

$$J|_{z=\alpha_4} = \frac{\gamma_1\gamma_2}{32(\gamma_1^2 - S_1S_2)(\gamma_2^2 - S_4S_5)} \times \oint \frac{dz}{z}(B_0^\bullet(z)^2 - 4B_1^\bullet(z)B_{-1}^\bullet(z))^{-1/2}, \tag{3.28}$$

where

$$B_1^\bullet(z) = \langle K_4^{\flat -}|\gamma_\mu|K_5^{\flat -}\rangle(\alpha_1 K_1^{\flat\mu} + \alpha_2 K_2^{\flat\mu} + \alpha_3 \langle K_1^{\flat -}|\gamma^\mu|K_2^{\flat -}\rangle + z\langle K_2^{\flat -}|\gamma^\mu|K_1^{\flat -}\rangle + K_6^\mu) \tag{3.29}$$

$$B_0^\bullet(z) = (\beta_1 K^\flat_{4\mu} + \beta_2 K^\flat_{5\mu} + K_{6\mu}) \times (\alpha_1 K_1^{\flat\mu} + \alpha_2 K_2^{\flat\mu} + \alpha_3 \langle K_1^{\flat -}|\gamma^\mu|K_2^{\flat -}\rangle + z \langle K_2^{\flat -}|\gamma^\mu|K_1^{\flat -}\rangle + K_6^\mu) - \frac{1}{2}S_6 \tag{3.30}$$

$$B_{-1}^\bullet(z) = -\frac{S_4 S_5 (S_4+\gamma_2)(S_5+\gamma_2)\langle K_5^{\flat -}|\gamma_\mu|K_4^{\flat -}\rangle}{4(\gamma_2^2 - S_4 S_5)^2} \times (\alpha_1 K_1^{\flat\mu} + \alpha_2 K_2^{\flat\mu} + \alpha_3 \langle K_1^{\flat -}|\gamma^\mu|K_2^{\flat -}\rangle + z \langle K_2^{\flat -}|\gamma^\mu|K_1^{\flat -}\rangle + K_6^\mu). \tag{3.31}$$

Again this result is independent of the order of the integrations.

3. Leaving $z = \beta_3$ as the free parameter

Integrating out α_1, α_2, α_3, α_4 and β_1, β_2, β_4 from (3.14) produces

$$J|_{z=\beta_3} = \frac{\gamma_1\gamma_2}{32(\gamma_1^2 - S_1S_2)(\gamma_2^2 - S_4S_5)} \times \oint \frac{dz}{z}(A_0(z)^2 - 4A_1(z)A_{-1}(z))^{-1/2}, \tag{3.32}$$

where

$$A_1(z) = \langle K_1^{\flat -}|\gamma^\mu|K_2^{\flat -}\rangle(\beta_1 K^\flat_{4\mu} + \beta_2 K^\flat_{5\mu} + z\langle K_4^{\flat -}|\gamma_\mu|K_5^{\flat -}\rangle + \beta_4 \langle K_5^{\flat -}|\gamma_\mu|K_4^{\flat -}\rangle + K_{6\mu}) \tag{3.33}$$

$$A_0(z) = (\alpha_1 K_1^{\flat\mu} + \alpha_2 K_2^{\flat\mu} + K_6^\mu) \times (\beta_1 K^\flat_{4\mu} + \beta_2 K^\flat_{5\mu} + z\langle K_4^{\flat -}|\gamma_\mu|K_5^{\flat -}\rangle + \beta_4 \langle K_5^{\flat -}|\gamma_\mu|K_4^{\flat -}\rangle + K_{6\mu}) - \frac{1}{2}S_6 \tag{3.34}$$

$$A_{-1}(z) = -\frac{S_1 S_2 (S_1+\gamma_1)(S_2+\gamma_1)\langle K_2^{\flat -}|\gamma^\mu|K_1^{\flat -}\rangle}{4(\gamma_1^2 - S_1 S_2)^2} \times (\beta_1 K^\flat_{4\mu} + \beta_2 K^\flat_{5\mu} + z\langle K_4^{\flat -}|\gamma_\mu|K_5^{\flat -}\rangle + \beta_4 \langle K_5^{\flat -}|\gamma_\mu|K_4^{\flat -}\rangle + K_{6\mu}). \tag{3.35}$$

Again this result is independent of the order of the integrations.

4. Leaving $z = \beta_4$ as the free parameter

Integrating out α_1, α_2, α_3, α_4 and β_1, β_2, β_3 from (3.14) produces

$$J|_{z=\beta_4} = \frac{\gamma_1\gamma_2}{32(\gamma_1^2 - S_1S_2)(\gamma_2^2 - S_4S_5)} \times \oint \frac{dz}{z}(A_0^\bullet(z)^2 - 4A_1^\bullet(z)A_{-1}^\bullet(z))^{-1/2}, \tag{3.36}$$

where

$$A_1^\bullet(z) = \langle K_1^{\flat -}|\gamma^\mu|K_2^{\flat -}\rangle(\beta_1 K_{4\mu}^\flat + \beta_2 K_{5\mu}^\flat + \beta_3\langle K_4^{\flat -}|\gamma_\mu|K_5^{\flat -}\rangle + z\langle K_5^{\flat -}|\gamma_\mu|K_4^{\flat -}\rangle + K_{6\mu}) \tag{3.37}$$

$$A_0^\bullet(z) = (\alpha_1 K_1^{\flat\mu} + \alpha_2 K_2^{\flat\mu} + K_6^\mu) \times (\beta_1 K_{4\mu}^\flat + \beta_2 K_{5\mu}^\flat + \beta_3\langle K_4^{\flat -}|\gamma_\mu|K_5^{\flat -}\rangle + z\langle K_5^{\flat -}|\gamma_\mu|K_4^{\flat -}\rangle + K_{6\mu}) - \frac{1}{2}S_6 \tag{3.38}$$

$$A_{-1}^\bullet(z) = -\frac{S_1 S_2(S_1+\gamma_1)(S_2+\gamma_1)\langle K_2^{\flat -}|\gamma^\mu|K_1^{\flat -}\rangle}{4(\gamma_1^2 - S_1 S_2)^2} \times (\beta_1 K_{4\mu}^\flat + \beta_2 K_{5\mu}^\flat + \beta_3\langle K_4^{\flat -}|\gamma_\mu|K_5^{\flat -}\rangle + z\langle K_5^{\flat -}|\gamma_\mu|K_4^{\flat -}\rangle + K_{6\mu}). \tag{3.39}$$

Again this result is independent of the order of the integrations.

In the section below we will explain how to assemble the heptacut double-box integral from the individual contributions in Eqs. (3.27), (3.28), (3.32), and (3.36).

5. Assembling the heptacut double box from its contributions

As alluded to in the beginning of this section, there are three qualitatively distinct ways of distributing six massless external momenta at the vertices of the double box. In this section we shall describe the classification of these cases, and how to assemble the heptacut double-box integral from the individual contributions in Eqs. (3.27), (3.28), (3.32), and (3.36) in each of these cases. Before proceeding to describe the classification of the three cases, it is useful to appreciate that the individual contributions to the heptacut arise from the different existing classes of solutions to the joint on-shell constraints (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9). Indeed, these seven constraints leave one unfixed degree of freedom z in the two loop momenta ℓ_1, ℓ_2, and in general there are several classes of solutions. Each class is parametrized by a free complex variable z and has the remaining seven loop momentum parameters α_i, β_j fixed to specific values. Two classes $\mathcal{S}$ and $\mathcal{S}'$ are identical if and only if there exists an invertible holomorphic function $\varphi(z)$ that maps one class into the other; that is, $\varphi(\alpha_i|_{\mathcal{S}}) = \alpha_i|_{\mathcal{S}'}$ and $\varphi(\beta_i|_{\mathcal{S}}) = \beta_i|_{\mathcal{S}'}$ for $i = 1, \ldots, 4$.

The classification into the three cases is given as stated below.

(i) Case I: all three vertical propagators in Fig. 1 are part of some three-particle vertex. There are six classes of kinematical solutions to the on-shell constraints (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9), illustrated in Fig. 2. The heptacut double-box integral is $J = \sum_{i=1}^6 J_i$ where

$$\begin{aligned} J_1 &= J|_{z=\beta_3} && \text{[given in Eq. (3.32)]} \\ J_2 = J_6 &= J|_{z=\alpha_3} && \text{[given in Eq. (3.27)]} \\ J_3 &= J|_{z=\beta_4} && \text{[given in Eq. (3.36)]} \\ J_4 = J_5 &= J|_{z=\alpha_4} && \text{[given in Eq. (3.28)].} \end{aligned} \tag{3.40}$$

This case encompasses heptacuts #1–5 and #8, discussed in detail below in Secs. IV A, A 2–A 5 and A 8 where one can also find explicit results for the on-shell values P_1, Q_1 etc. quoted in Fig. 2. To determine, for example, the function $\beta_3(z)$ in solution $\mathcal{S}_5$, one expresses the loop momenta appearing in the on-shell constraint $(\ell_1 + \ell_2 + K_6)^2 = 0$ in their parametrized form (3.12) and (3.13), sets the values of the parameters α_i, β_j equal to those quoted in Eqs. (3.16) and (3.17) and below solution $\mathcal{S}_5$ in Fig. 2 and then solves the on-shell constraint for β_3.

(ii) Case II: the left- and rightmost vertical propagators in Fig. 1 are part of some three-particle vertex, but the middle one is not. There are four classes of kinematical solutions to the on-shell constraints (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9), illustrated in Fig. 3. The heptacut double-box integral is $J = \sum_{i=1}^4 J_i$ where

$$\begin{aligned} J_1 &= J|_{z=\beta_3} && \text{[given in Eq. (3.32)]} \\ J_2 &= J|_{z=\beta_4} && \text{[given in Eq. (3.36)]} \\ J_3 &= J|_{z=\alpha_4} && \text{[given in Eq. (3.28)]} \\ J_4 &= J|_{z=\alpha_3} && \text{[given in Eq. (3.27)].} \end{aligned} \tag{3.41}$$

This case encompasses heptacut #6, discussed in detail below in Sec. A 6 where one can also find explicit results for the on-shell values P_1, Q_1 etc. quoted in Fig. 3.

(iii) Case III: the two rightmost vertical propagators in Fig. 1 are part of some three-particle vertex, but the leftmost is not. There are four classes of kinematical solutions to the on-shell constraints (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9), illustrated in Fig. 4. The heptacut double-box integral is $J = \sum_{i=1}^4 J_i$ where

$$\begin{aligned} J_1 = J_2 &= J|_{z=\beta_3} && \text{[given in Eq. (3.32)]} \\ J_3 = J_4 &= J|_{z=\beta_4} && \text{[given in Eq. (3.36)].} \end{aligned} \tag{3.42}$$

This case encompasses heptacut #7, discussed in detail below in Sec. A 7 where one can also find explicit results for the on-shell values $P_1^\pm$, $Q_2^\pm$ etc. quoted in Fig. 4. In this case, because both K_1 and K_2 are massive, there are two solutions for $\gamma_1^\pm$ [given in Eq. (2.10)] and therefore two pairs of flattened momenta $(K_{1\pm}^\flat, K_{2\pm}^\flat)$ [see Eq. (2.11)]. The on-shell values of the loop momenta are independent of which sign is chosen,

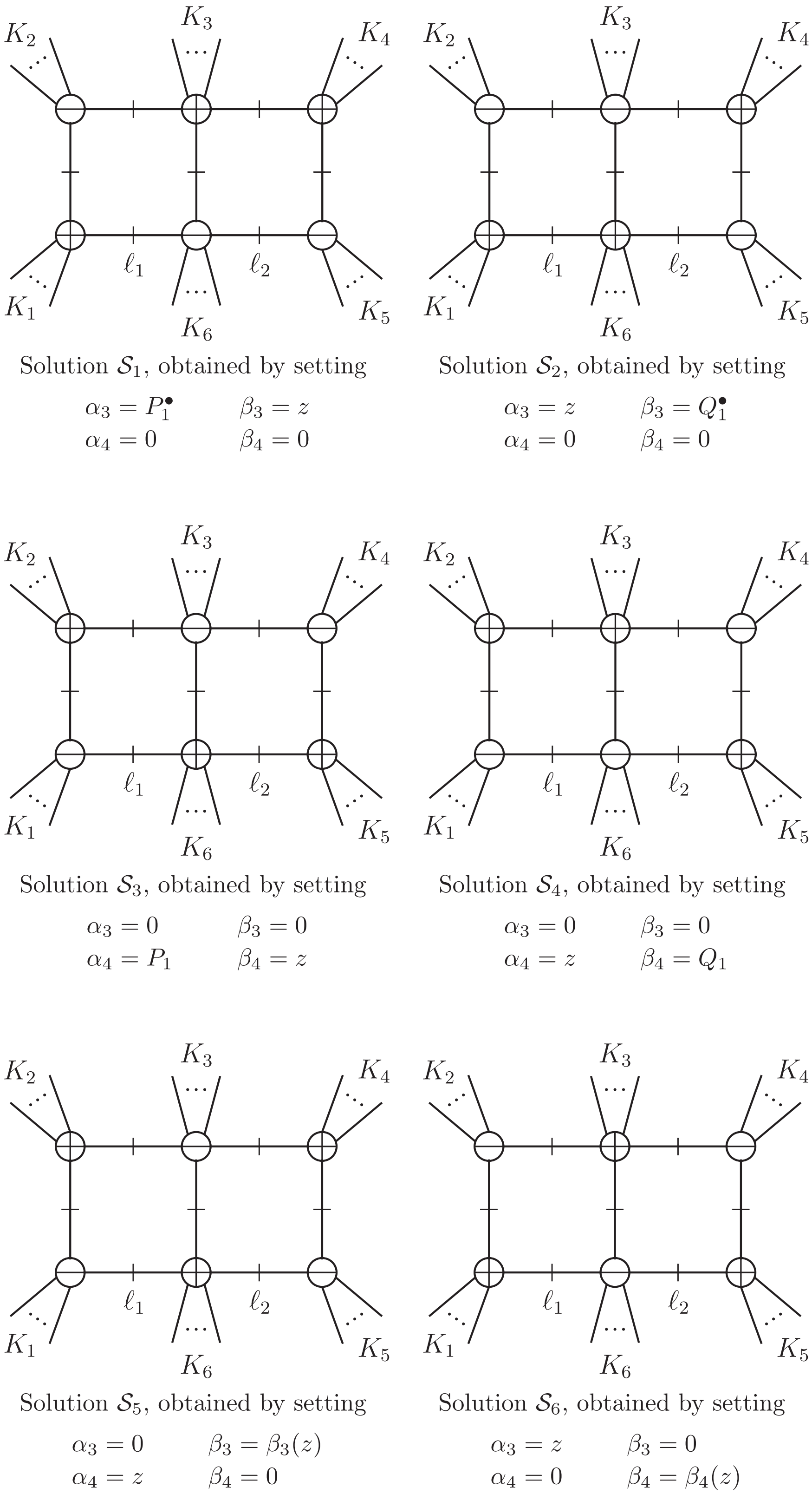


FIG. 2. The six kinematical solutions to the heptacut constraints for the double-box topology in case I. For all solutions, the loop momentum parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are set equal to the values given in Eqs. (3.16) and (3.17). Any blob connecting more than three legs does not have a well-defined chirality and its sign should be ignored. For $\mathcal{S}_5$ and $\mathcal{S}_6$, the parameters β_3 and β_4 are determined by solving the on-shell constraint $(\ell_1 + \ell_2 + K_6)^2 = 0$ for the respective parameter. The on-shell values P_1, Q_1 etc. are functions of the external momenta; examples may be found in Secs. IVA, A 2–A 5 and A 8.

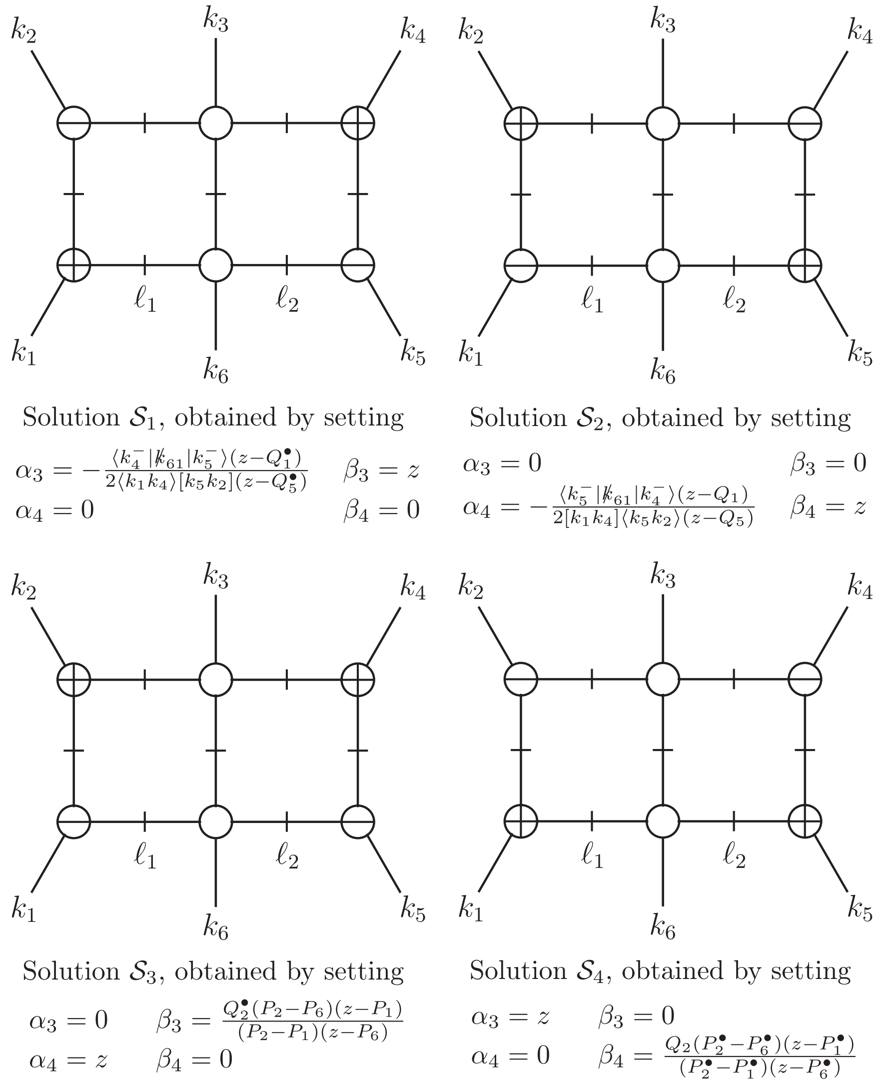


FIG. 3. The four kinematical solutions to the heptacut constraints for the double-box topology in case II. For all solutions, the loop momentum parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are set equal to the values given in Eqs. (3.16) and (3.17). The on-shell values P_1, Q_1 etc. are functions of the external momenta and may be found in Sec. A 6.

$$\ell_1(\alpha_1^+, \alpha_2^+, \alpha_3^+, \alpha_4^+) = \ell_1(\alpha_1^-, \alpha_2^-, \alpha_3^-, \alpha_4^-) \tag{3.43}$$

$$\ell_2(\beta_1^+, \beta_2^+, \beta_3^+, \beta_4^+) = \ell_2(\beta_1^-, \beta_2^-, \beta_3^-, \beta_4^-). \tag{3.44}$$

$$\xi^\pm = -\frac{S_1 S_2 (S_1 + \gamma_1^\pm)(S_2 + \gamma_1^\pm)}{4(\gamma_1^{\pm 2} - S_1 S_2)^2}. \tag{3.45}$$

B. Heptacut of the factorized double box

For the double-box integral considered in Sec. III A there are various ways of distributing six external momenta at the vertices, and we therefore computed the heptacut with an arbitrary number of external legs. In contrast, the factorized double-box integral

$$\left(\int \frac{d^D\ell_1}{(2\pi)^D} \frac{1}{\ell_1^2} \frac{1}{(\ell_1 - k_1)^2} \frac{1}{(\ell_1 - k_{12})^2} \frac{1}{(\ell_1 - k_{123})^2}\right) \times \left(\int \frac{d^D\ell_2}{(2\pi)^D} \frac{1}{\ell_2^2} \frac{1}{(\ell_2 - k_6)^2} \frac{1}{(\ell_2 - k_{56})^2} \frac{1}{(\ell_2 - k_{456})^2}\right), \tag{3.46}$$

with which we shall be concerned in this section, admits a unique way of distributing six (cyclically ordered) external momenta at its vertices, and we therefore restrict ourselves to this case. The factorized double box with six massless legs is illustrated in Fig. 5 below.

We consider the heptacut defined by imposing the following joint on-shell constraints

$$\ell_1^2 = 0 \tag{3.47}$$

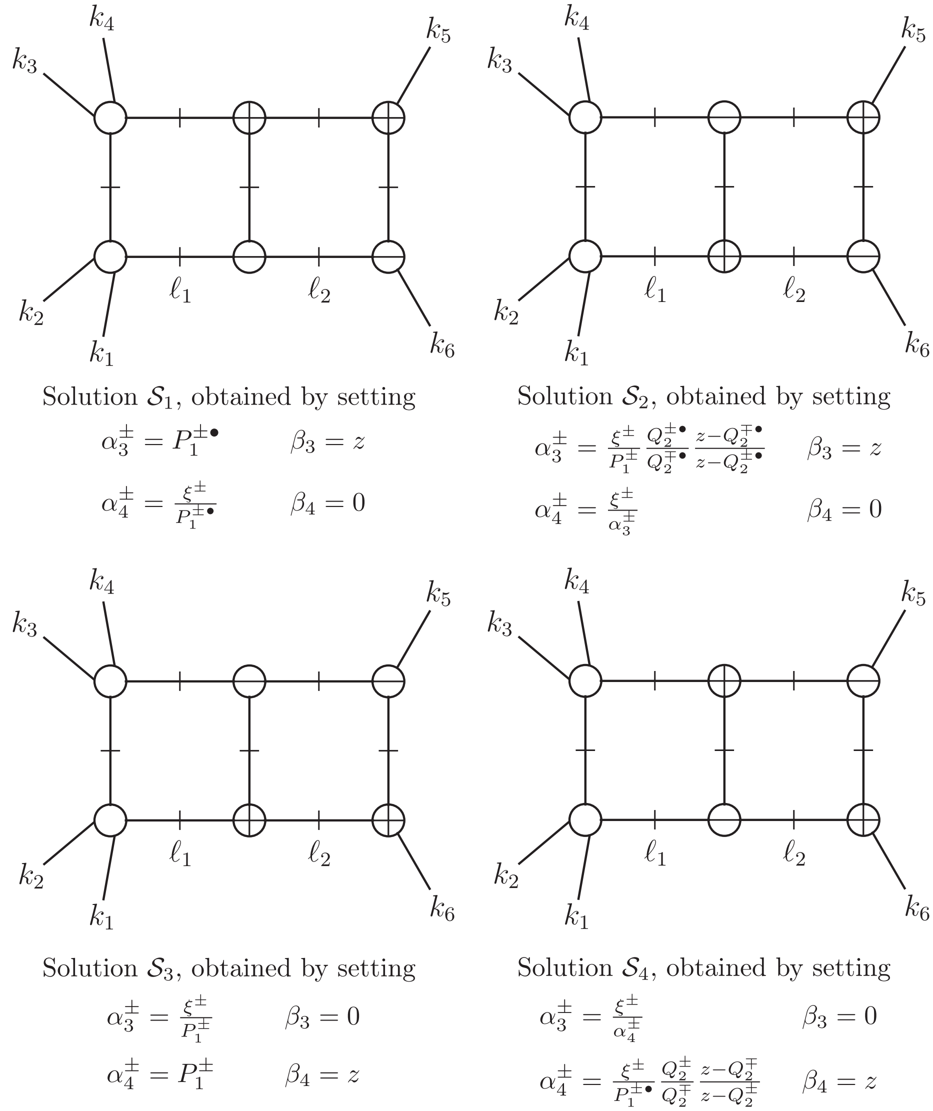


FIG. 4. The four kinematical solutions to the heptacut constraints for the double-box topology in case III. In this case, because both K_1 and K_2 are massive, there are two solutions for $\gamma_1^\pm$ and therefore two pairs of flattened momenta $(K_{1\pm}^\flat, K_{2\pm}^\flat)$. For all solutions, the loop momentum parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are set equal to the values given in Eqs. (3.16) and (3.17). The on-shell values $P_1^\mp$, $Q_2^\pm$ etc. are functions of the external momenta and may be found in Sec. A 7. The quantity $\xi^\pm$ is defined in Eq. (3.45).

$$(\ell_1 - k_1)^2 = 0 \tag{3.48}$$

$$(\ell_1 - k_{12})^2 = 0 \tag{3.49}$$

$$\ell_2^2 = 0 \tag{3.50}$$

$$(\ell_2 - k_6)^2 = 0 \tag{3.51}$$

$$(\ell_2 - k_{56})^2 = 0 \tag{3.52}$$

$$(\ell_2 - k_{456})^2 = 0. \tag{3.53}$$

This is not a maximal cut: we deliberately leave one degree of freedom z in the loop momenta unfrozen to make manifest the singularities in the Jacobians arising from changes of variables. This in turn makes the global poles of the integrand easy to identify.

We will use the following parametrization of the loop momenta

$$\ell_1^\mu = \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \alpha_3 \langle k_1^- | \gamma^\mu | k_2^- \rangle + \alpha_4 \langle k_2^- | \gamma^\mu | k_1^- \rangle \tag{3.54}$$

$$\ell_2^\mu = \beta_1 k_5^\mu + \beta_2 k_6^\mu + \beta_3 \langle k_5^- | \gamma^\mu | k_6^- \rangle + \beta_4 \langle k_6^- | \gamma^\mu | k_5^- \rangle. \tag{3.55}$$

The constraints (3.47), (3.48), (3.49), (3.50), (3.51), and (3.52) form a special case of Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), and (3.8) after appropriately relabeling the external momenta. From Eqs. (3.16) and (3.17) we then find that the constraints are satisfied by setting

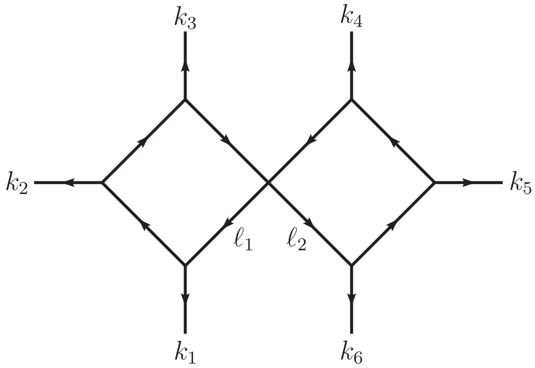


FIG. 5. The factorized double-box integral with six massless external momenta.

$$\begin{aligned} &\alpha_1 = 1, \qquad \alpha_2 = 0, \qquad \alpha_3\alpha_4 = 0 \\ &\beta_1 = 0, \qquad \beta_2 = 1, \qquad \beta_3\beta_4 = 0. \end{aligned} \tag{3.56}$$

The final on-shell constraint (3.53) combined with Eq. (3.56) turns out to have four classes of solutions which can be conveniently expressed by defining the spinor ratios

$$Q_1 = \frac{\langle 45\rangle}{2\langle 46\rangle}, \qquad Q_1^\bullet = \frac{[45]}{2[46]}. \tag{3.57}$$

Thus, the four classes of solutions to the joint on-shell constraints (3.47), (3.48), (3.49), (3.50), (3.51), (3.52), and (3.53) are

$$\begin{aligned} &\mathcal{S}_1: \begin{cases} \alpha_3 = z, & \beta_3 = Q_1^\bullet \\ \alpha_4 = 0, & \beta_4 = 0 \end{cases}; \\ &\mathcal{S}_2: \begin{cases} \alpha_3 = z, & \beta_3 = 0 \\ \alpha_4 = 0, & \beta_4 = Q_1 \end{cases}; \\ &\mathcal{S}_3: \begin{cases} \alpha_3 = 0, & \beta_3 = Q_1^\bullet \\ \alpha_4 = z, & \beta_4 = 0 \end{cases}; \\ &\mathcal{S}_4: \begin{cases} \alpha_3 = 0, & \beta_3 = 0 \\ \alpha_4 = z, & \beta_4 = Q_1 \end{cases} \end{aligned} \tag{3.58}$$

where the parameters α_1, α_2, β_1, β_2 are put equal to the values quoted in Eq. (3.56). The solutions are illustrated below in Fig. 6.

Since we are not cutting all propagators, the relevant quantity to compute is not the heptacut of the factorized double-box integral itself, but rather the Jacobian

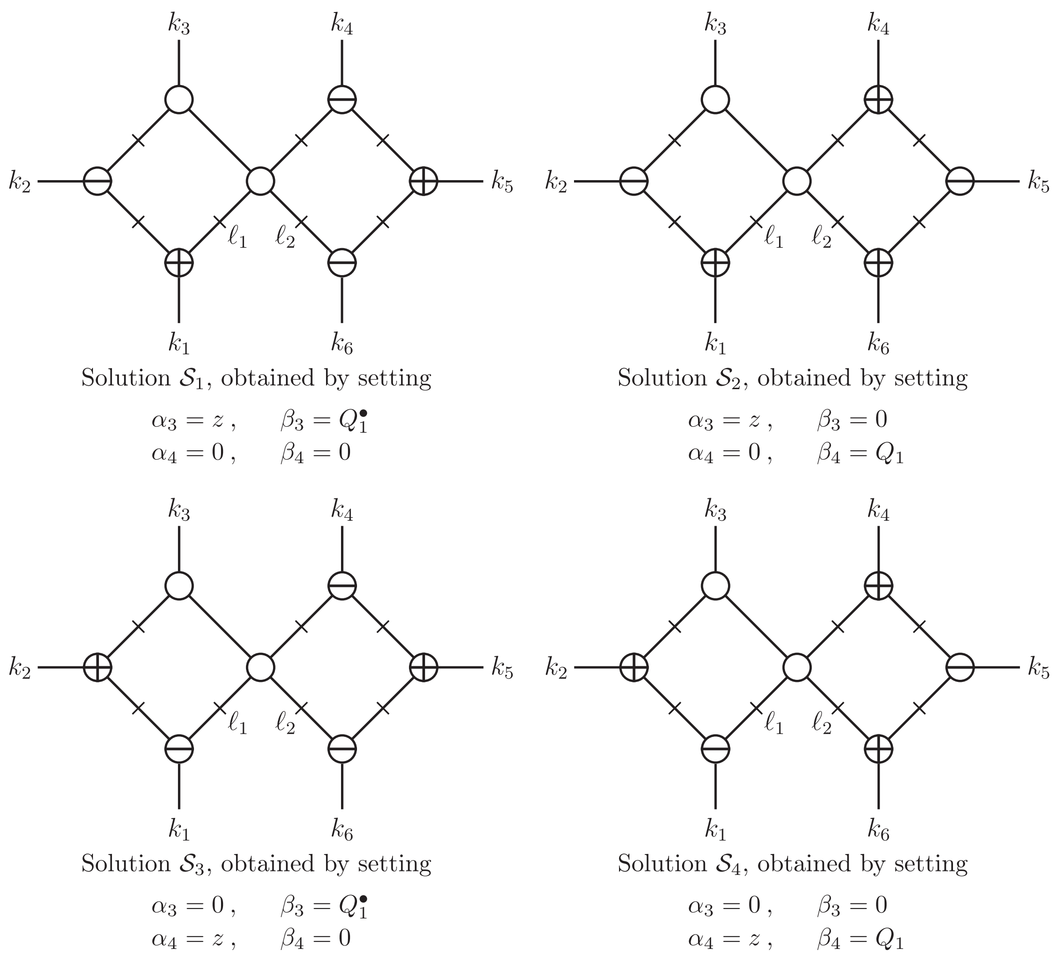


FIG. 6. The four kinematical solutions to the heptacut constraints given in Eqs. (3.47), (3.48), (3.49), (3.50), (3.51), (3.52), and (3.53) for the factorized double box with six massless external momenta. The on-shell values Q_1 and $Q_1^\bullet$ are defined in Eq. (3.57).

$$J = \left(\int d^4\ell_1 \delta(\ell_1^2)\delta((\ell_1 - k_1)^2)\delta((\ell_1 - k_{12})^2)\right) \times \left(\int d^4\ell_2 \delta(\ell_2^2)\delta((\ell_2 - k_6)^2)\delta((\ell_2 - k_{56})^2) \times \delta((\ell_2 - k_{456})^2)\right), \quad (3.59)$$

since from this one can easily obtain the heptacut of any given object, as we will see below. The δ-functions in Eq. (3.59) may be integrated out along the lines described in Sec. III A 1, and one finds that for all four kinematical solutions $\mathcal{S}_i$ the contribution to the Jacobian J is given by

$$J|_{\mathcal{S}_i} = -\frac{1}{16 s_{12} s_{45} s_{56}} \oint_{\Gamma_i} \frac{dz}{z} \quad \text{for } i = 1, \ldots, 4. \quad (3.60)$$

Then the heptacut factorized double-box integral is

$$-\frac{1}{16 s_{12} s_{45} s_{56}} \sum_{i=1}^{4} \oint_{\Gamma_i} \frac{dz}{z} \frac{1}{(\ell_1 - k_{123})^2}\bigg|_{\mathcal{S}_i}, \quad (3.61)$$

where the subscript $(\cdots)|_{\mathcal{S}_i}$ indicates that the propagator is to be evaluated in the parametrization (3.54) and (3.55) with the parameters set equal to the values in Eqs. (3.56) and (3.58). Likewise, the heptacut (3.47), (3.48), (3.49), (3.50), (3.51), (3.52), and (3.53) of the two-loop amplitude is

$$-\frac{1}{16 s_{12} s_{45} s_{56}} \sum_{i=1}^{4} \oint_{\Gamma_i} \frac{dz}{z} \prod_{j=1}^{6} A_j^{\text{tree}}(z)\bigg|_{\mathcal{S}_i}. \quad (3.62)$$

IV. GLOBAL POLES OF THE TWO-LOOP SIX-POINT INTEGRAND

In this section we apply the heptacuts discussed in Sec. III to compute the two-loop six-point MHV integrand in $\mathcal{N} = 4$ SYM theory. The parity-even part of this integrand was first calculated in Ref. [89] and was later reexamined using the leading singularity method in Ref. [67]. In Sec. IV A we give a pedagogical review of the use of the leading singularity method to determine the two-loop six-point integrand. In particular, we discuss how one can set up linear equations to determine the parity-even part of the integrand directly. The approach here is similar to that of Ref. [67], but differs in the use of the loop-momentum parametrization (3.12) and (3.13) as well as (3.54) and (3.55), which has the virtue of making the multidimensional contours associated with the heptacuts completely explicit. It is also worth remarking on the similarity between the approach followed here and that of Forde in Ref. [23]: the heptacut at two loops, in analogy with the triple cut at one loop, leaves one *a priori* undetermined contour integration. The associated contour may be chosen to encircle the various poles coming from the measure or from additional propagators, allowing the determination of the integral coefficients by use of the residue theorem.

The main result of this paper is contained in Sec. IV B in which we report on a numerical check that the result produced by the leading singularity method for the full (i.e., parity-even and -odd) two-loop six-point MHV integrand is in agreement with recent predictions in the literature [60,63].

In general one can apply integral reductions to any two-loop six-point amplitude to express it as a linear combination of integrals in some sufficiently large basis. Due to the special symmetries of $\mathcal{N} = 4$ SYM theory amplitudes, it is natural to include integrals in the basis that reflect these symmetries (in particular, dual pseudoconformal integrals), and in this paper we shall use the basis in Ref. [67], for convenience illustrated in Fig. 7 below. Note that in order to express the parity-odd part of the amplitude, this basis contains additional (non–dual conformal invariant) integrals beyond those included in the bases of Refs. [89,105] and as such is overcomplete. We shall discuss the consequences of the overcompleteness in detail in Sec. IV A.

Thus, we will use the following ansatz for the planar two-loop six-point MHV amplitude of $\mathcal{N} = 4$ SYM theory

$$A_{6,\text{MHV}}^{(2)} = \frac{1}{4} \sum_{\substack{i=1,\ldots,24\\ j=1,\ldots,12}} r_i c_{i,\sigma_j} I_{i,\sigma_j}, \quad (4.1)$$

in which it is expressed as a linear combination of the basis integrals illustrated in Fig. 7 whose symmetry factors r_i are given as follows,

$$\begin{aligned}
(r_1, r_2, r_3, r_4, r_5, r_6) &= \left(\frac{1}{4}, 1, \frac{1}{2}, \frac{1}{2}, 1, 1\right)\\
(r_7, r_8, r_9, r_{10}, r_{11}, r_{12}) &= \left(\frac{1}{4}, \frac{1}{2}, 1, 1, 1, \frac{1}{2}\right)\\
(r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}) &= \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right)\\
(r_{19}, r_{20}, r_{21}, r_{22}, r_{23}, r_{24}) &= \left(\frac{1}{2}, 1, 1, 1, \frac{1}{4}, 1\right).
\end{aligned} \quad (4.2)$$

Thus, the task of computing the two-loop amplitude reduces to determining the coefficients c_{i,σ_j} as functions of the external momenta. This is achieved by applying generalized cuts to both sides of Eq. (4.1) which has the effect of turning either side into a contour integral in the complex plane. More specifically, the left-hand side of Eq. (4.1) is turned into a product of tree amplitudes whose external states coincide with either the external states of the two-loop amplitude or with the states traveling along the propagators that are becoming on-shell. On the right-hand side of Eq. (4.1), the generalized cut has the effect of removing all the integrals that do not contain the propagators being cut, and turning the remaining integrals into contour integrals.

To limit the number of terms remaining on the right-hand of Eq. (4.1) as much as possible, it would therefore be

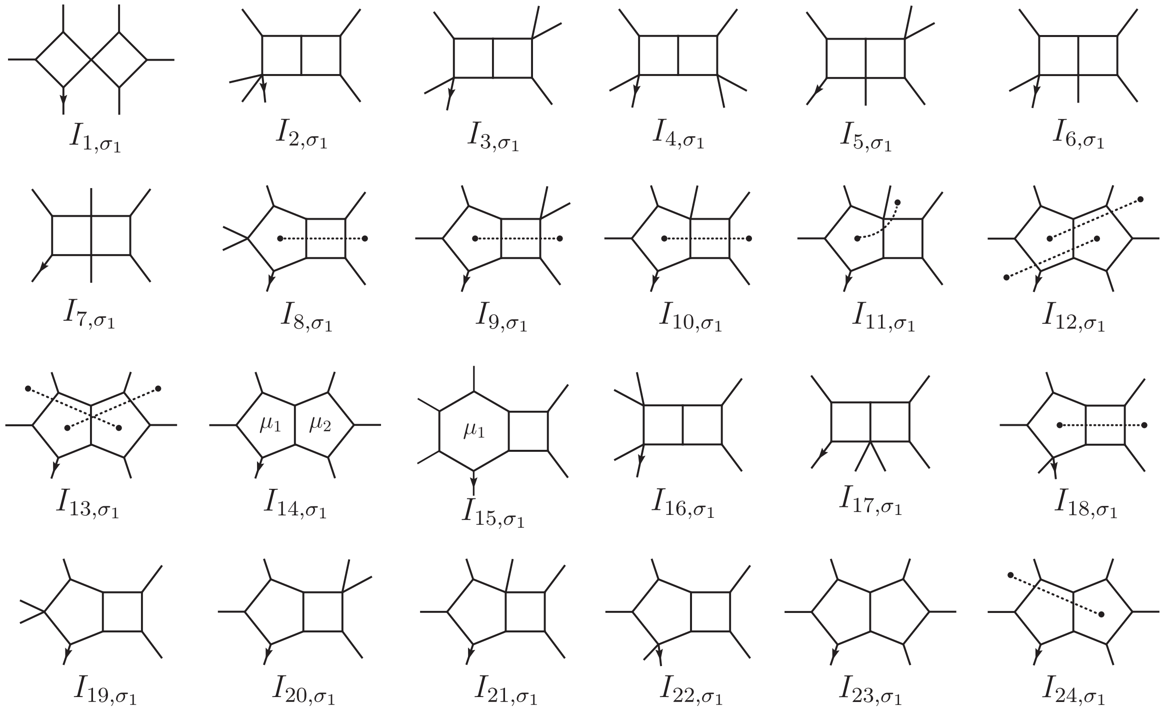


FIG. 7. The integral basis in terms of which the two-loop six-point amplitude is expressed. The integrals shown here are recorded in the σ_1 permutation where the external momenta are labeled clockwise starting with k_1 at the position of the arrow. The basis is overcomplete, giving rise to nonuniqueness of the integral coefficients in Eq. (4.1).

natural to start by imposing as many simultaneous four-dimensional cut constraints as possible, which at two loops would lead us to consider octacuts. Thus having obtained the coefficients of integrals that admit an octacut, we could then proceed to relax one cut constraint to allow the contributions of the double-box integrals in Fig. 7 and in turn determine their coefficients. This, however, is not the route we will be following: instead, we will focus our attention on the heptacuts of Eq. (4.1). Indeed, imposing seven on-shell constraints on two four-dimensional loop momenta leaves an unconstrained parameter z, and this makes it easy to survey the global poles of the integrand (to be defined properly in Sec. IVA 3) on the right-hand side of Eq. (4.1). This in turn will make it straightforward to determine the tori encircling the global poles, which we shall refer to as leading singularity contours.

It is important to distinguish between these leading singularity contours and the maximal-cut contours. Indeed, as explained in Ref. [34], the maximal-cut contours are particular linear combinations σ of these tori whose coefficients are determined by the requirement that any function that integrates to zero on $\mathbb{R}^D \times \mathbb{R}^D$ should also integrate to zero on σ. This constraint ensures that two Feynman integrals which are equal, possibly through some nontrivial relations, will also have equal maximal cuts. As argued in Ref. [34], multidimensional contours σ satisfying this consistency condition are guaranteed to produce correct results for scattering amplitudes in any gauge theory, not only $\mathcal{N} = 4$ SYM theory. It would therefore be very interesting to determine these maximal-cut contours.

Such an analysis is, however, not currently possible: the set of linear relations between two-loop integrals includes the set of all integration-by-parts (IBP) identities between the various tensor integrals [111] arising from the Feynman rules of gauge theory, and at present a complete knowledge of these relations is not available. Nevertheless, all the details needed to carry out this analysis (in particular, the enumeration of the global poles of the integrand and the expressions for the cut integrals) are provided in the intermediate results in Sec. IVA and in Appendix A, and the correct heptacut contours can therefore be determined immediately once the IBP relations do become available.

With the cyclic ordering of the external momenta shown in Fig. 7, the two-loop six-point helicity amplitudes in $\mathcal{N} = 4$ SYM theory admit nine different heptacuts, dictated by the eight double-box integrals and the factorized double-box integral in Fig. 7, respectively I_{2,σ_1}, I_{3,σ_1}, I_{4,σ_1}, I_{5,σ_1}, I_{6,σ_1}, I_{7,σ_1}, I_{16,σ_1}, I_{17,σ_1}, and I_{1,σ_1}. We label these heptacuts respectively #1, ..., #9 and study them in turn in Secs. IVA, A 2–A 9.

In attempting to obtain the integral coefficients c_{i,σ_j} in Eq. (4.1) from generalized four-dimensional cuts, one encounters two technical issues. The first point is that the basis is overcomplete, and the integral coefficients are therefore not uniquely defined. This feature will manifest itself as the appearance of free parameters in the solutions of the linear equations satisfied by the integral coefficients. This in turn means that one has to set a subset of the integral coefficients equal to specific values in order to obtain unique solutions for the remaining coefficients. The nonuniqueness of the integral coefficients is accounted for by the existence of various linear relations between the integrals in Fig. 7, as was explained carefully in Ref. [67].

The second point is that the coefficients of the μ-integrals I_{14,σ_1} and I_{15,σ_1} (thus called because their integrands contain factors involving the (-2ϵ)-dimensional part of the loop momenta) are of $\mathcal{O}(\epsilon)$ in the dimensional regulator and hence are not obtainable from four-dimensional cuts [112]. As we restrict ourselves to taking four-dimensional generalized cuts in this paper, we shall therefore not be concerned with these integral coefficients.

A. Example: evaluation of heptacut #1

This section is intended as a pedagogical example of the use of the leading singularity method to obtain integral coefficients of two-loop scattering amplitudes in $\mathcal{N}=4$ SYM theory. In this example, we evaluate both sides of the two-loop Eq. (4.1) resulting after imposing the simplest of the nine heptacuts that the six-point amplitude admits, labeled #1. This heptacut is defined by the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) with the vertex momenta

$$\begin{aligned} K_1 &= k_{123} & K_2 &= k_4 & K_3 &= 0 \\ K_4 &= k_5 & K_5 &= k_6 & K_6 &= 0. \end{aligned} \tag{4.3}$$

After having evaluated this heptacut of Eq. (4.1), we will set up equations satisfied by the full integral coefficients (i.e., including both parity-even and -odd parts) and solve these equations explicitly. Along the way, we also present the 22 tori encircling the global poles of the integrand associated with this heptacut. Finally, we comment briefly on how the linear equations obtained from the remaining heptacuts #2, ..., #9 (details of which are provided in Appendix A) can be used to obtain the parity-even part of the two-loop six-point integral coefficients directly and report agreement with the results originally found in Ref. [89].

1. Heptacut #1 of the right-hand side of Eq. (4.1)

Applying heptacut #1 to the right-hand side of Eq. (4.1) will leave the linear combination of cut integrals shown in Fig. 8 below.

We will use the loop momentum parametrizations in Eqs. (3.12) and (3.13). Furthermore, it will be convenient to define the spinor ratios

$$\begin{aligned} P_1 &= -\frac{\langle K_1^\flat k_6\rangle}{2\langle K_2^\flat k_6\rangle}, & P_2 &= -\frac{K_1^\flat \cdot k_{12} - \frac{1}{2}s_{12}}{\langle K_2^{\flat -}|\not{k}_{12}|K_1^{\flat -}\rangle}, \\ P_3 &= -\frac{\langle K_1^\flat k_1\rangle}{2\langle K_2^\flat k_1\rangle} & P_4 &= -\frac{\langle K_1^\flat K_4^\flat\rangle}{2\langle K_2^\flat K_4^\flat\rangle}, \\ Q_1 &= -\frac{[K_1^\flat K_5^\flat]}{2[K_1^\flat K_4^\flat]}, \end{aligned} \tag{4.4}$$

and their parity conjugates

$$\begin{aligned} P_1^\bullet &= -\frac{[K_1^\flat k_6]}{2[K_2^\flat k_6]}, & P_2^\bullet &= -\frac{K_1^\flat \cdot k_{12} - \frac{1}{2}s_{12}}{\langle K_1^{\flat -}|\not{k}_{12}|K_2^{\flat -}\rangle}, \\ P_3^\bullet &= -\frac{[K_1^\flat k_1]}{2[K_2^\flat k_1]} & P_4^\bullet &= -\frac{[K_1^\flat K_4^\flat]}{2[K_2^\flat K_4^\flat]}, \\ Q_1^\bullet &= -\frac{\langle K_1^\flat K_5^\flat\rangle}{2\langle K_1^\flat K_4^\flat\rangle}. \end{aligned} \tag{4.5}$$

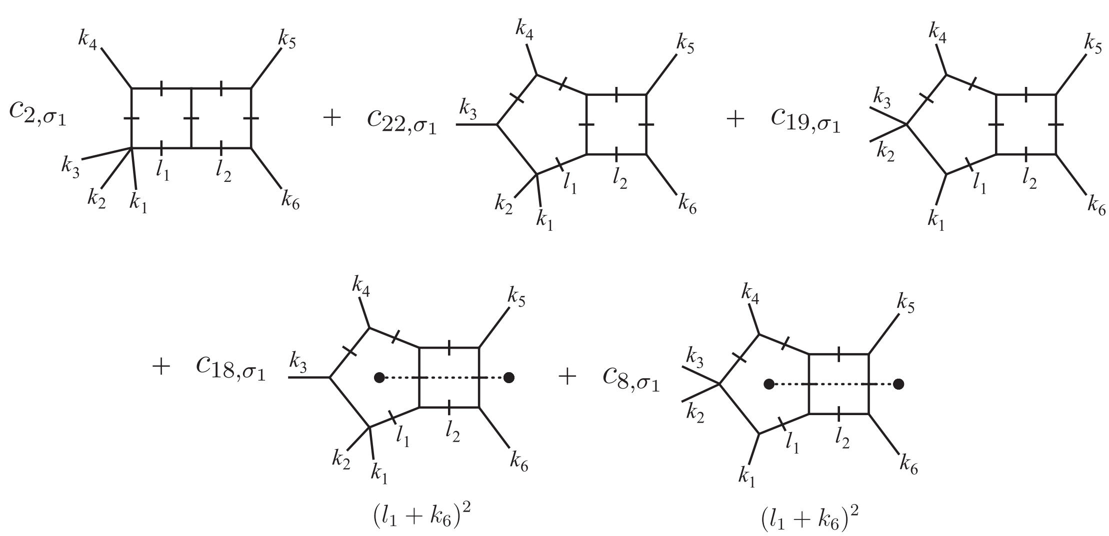


FIG. 8. The integrals remaining on the right-hand side of Eq. (4.1) after applying the heptacut labeled #1. The cut propagators are illustrated by the inclusion of an additional orthogonal line. This heptacut is defined by the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) with the vertex momenta given by Eq. (4.3).

This heptacut belongs to case I treated in Sec. III: each of the vertical propagators in the double-box integral I_{2,σ_1} is part of some three-particle vertex. There are thus six kinematical solutions (shown in Fig. 2) to the on-shell constraints. The heptacut of the double-box integral receives contributions from each of the kinematical solutions. These contributions were found in Sec. III [see Eq. (3.40)] and take the form

$$\oint_{\Gamma_i} dz J_i(z), \tag{4.6}$$

where, using the notation in Eqs. (4.3), (4.4), and (4.5), the Jacobians are

$$J_i(z) = \frac{1}{32\gamma_1\gamma_2} \times \begin{cases} (\langle K_1^{\flat -}|\not{K}_5^{\flat}|K_2^{\flat -}\rangle z(z-P_1^\bullet))^{-1} & \text{for } i=2,6 \\ (\langle K_2^{\flat -}|\not{K}_5^{\flat}|K_1^{\flat -}\rangle z(z-P_1))^{-1} & \text{for } i=4,5 \\ (\langle K_4^{\flat -}|\not{K}_1^{\flat}|K_5^{\flat -}\rangle z(z-Q_1^\bullet))^{-1} & \text{for } i=1 \\ (\langle K_5^{\flat -}|\not{K}_1^{\flat}|K_4^{\flat -}\rangle z(z-Q_1))^{-1} & \text{for } i=3 \end{cases}. \tag{4.7}$$

Accordingly, the heptacut right-hand side of Eq. (4.1), displayed in Fig. 8, receives contributions from each of the six kinematical solutions of the form

$$\oint_{\Gamma_i} dz J_i(z)\bigg(c_{2,\sigma_1} + \frac{c_{22,\sigma_1}}{(\ell_1-k_{12})^2} + \frac{c_{19,\sigma_1}}{(\ell_1-k_1)^2} + \frac{c_{18,\sigma_1}(\ell_1+k_6)^2}{(\ell_1-k_{12})^2} + \frac{c_{8,\sigma_1}(\ell_1+k_6)^2}{(\ell_1-k_1)^2}\bigg)\bigg|_{\mathcal{S}_i}, \tag{4.8}$$

where the subscript in $(\cdots)|_{\mathcal{S}_i}$ indicates that the function is to be evaluated in the parametrization (3.12) and (3.13) with the parameters set equal to the values

$$\alpha_1 = 1, \qquad \alpha_2 = 0, \qquad \beta_1 = 0, \qquad \beta_2 = 1 \tag{4.9}$$

and those given in Fig. 2 with the functions $\beta_3(z)$ and $\beta_4(z)$ quoted below solutions $\mathcal{S}_5$ and $\mathcal{S}_6$ being given by

$$\beta_3(z) = -\frac{\langle k_6 k_4\rangle(z-P_1)}{2\langle k_5 k_4\rangle(z-P_4)} \tag{4.10}$$

$$\beta_4(z) = -\frac{[k_6 k_4](z-P_1^\bullet)}{2[k_5 k_4](z-P_4^\bullet)}. \tag{4.11}$$

Instead of displaying all six contributions individually, we can make use of the fact that the kinematical solutions come in three parity-conjugate pairs. Namely, the contributions coming from two parity-conjugate solutions are obtainable from each other by the replacements

$$\langle ij\rangle \leftrightarrow [ij] \tag{4.12}$$

$$\langle K_i^-|\not{P}|K_j^-\rangle \leftrightarrow \langle K_j^-|\not{P}|K_i^-\rangle \tag{4.13}$$

$$P_i \leftrightarrow P_i^\bullet \tag{4.14}$$

$$Q_i \leftrightarrow Q_i^\bullet \tag{4.15}$$

$$\alpha_3 \leftrightarrow \alpha_4 \tag{4.16}$$

$$\beta_3 \leftrightarrow \beta_4, \tag{4.17}$$

where the replacement rules (4.16) and (4.17) specify that if one solution has, e.g., $z = \alpha_3$ as the free parameter, then the parity-conjugate solution should be understood as having $z = \alpha_4$ as the free parameter.

Putting everything together, the result of applying heptacut #1 to the right-hand side of Eq. (4.1) is

$$\frac{1}{4}\sum_{i=1}^{6}\oint_{\Gamma_i} dz J_i(z) K_i(z), \tag{4.18}$$

where the Jacobians are given in Eq. (4.7) and the kernels evaluated on the six kinematical solutions (illustrated in Fig. 2) are

$$K_1(z) = c_{2,\sigma_1} - \frac{1}{2}\frac{c_{22,\sigma_1}}{\langle K_1^{\flat -}|\not{k}_{12}|K_2^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)} - \frac{1}{2}\frac{c_{19,\sigma_1}}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle(P_1^\bullet - P_3^\bullet)} \tag{4.19}$$

$$K_2(z) = c_{2,\sigma_1} - \frac{1}{2}\frac{c_{22,\sigma_1}}{\langle K_1^{\flat -}|\not{k}_{12}|K_2^{\flat -}\rangle(z - P_2^\bullet)} - \frac{1}{2}\frac{c_{19,\sigma_1}}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle(z - P_3^\bullet)} - \frac{c_{18,\sigma_1}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(z-P_1^\bullet)}{\langle K_1^{\flat -}|\not{k}_{12}|K_2^{\flat -}\rangle(z-P_2^\bullet)} - \frac{c_{8,\sigma_1}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(z-P_1^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle(z-P_3^\bullet)} \tag{4.20}$$

$$K_3(z) = \text{parity conjugate of } K_1(z) \quad [\text{obtained by applying Eqs.(4.12)–(4.17)}] \tag{4.21}$$

$$K_4(z) = \text{parity conjugate of } K_2(z) \quad [\text{obtained by applying Eqs.(4.12)–(4.17)}] \tag{4.22}$$

$$K_5(z) = c_{2,\sigma_1} - \frac{1}{2}\frac{c_{22,\sigma_1}}{\langle K_2^{\flat -}|\not{k}_{12}|K_1^{\flat -}\rangle(z - P_2)} - \frac{1}{2}\frac{c_{19,\sigma_1}}{\langle K_2^{\flat -}|\not{k}_1|K_1^{\flat -}\rangle(z - P_3)} - \frac{c_{18,\sigma_1}\langle K_2^{\flat -}|\not{k}_6|K_1^{\flat -}\rangle(z-P_1)}{\langle K_2^{\flat -}|\not{k}_{12}|K_1^{\flat -}\rangle(z-P_2)} - \frac{c_{8,\sigma_1}\langle K_2^{\flat -}|\not{k}_6|K_1^{\flat -}\rangle(z-P_1)}{\langle K_2^{\flat -}|\not{k}_1|K_1^{\flat -}\rangle(z-P_3)} \tag{4.23}$$

$$K_6(z) = \text{parity conjugate of } K_5(z)$$
$$[\text{obtained by applying Eqs. (4.12)–(4.17)}]. \qquad (4.24)$$

2. Heptacut #1 of the left-hand side of Eq. (4.1)

For MHV configurations in $\mathcal{N} = 4$ SYM theory, the ratio of the two-loop amplitude to the corresponding tree-level amplitude is independent of the distribution of the helicities of the external states [113,114]. Without loss of generality, we will therefore assume throughout the paper that the helicities of the external states are $(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$. In this case, the result of applying heptacut #1 to the left-hand side of Eq. (4.1) is

$$i\sum_{i=1}^{6}\oint_{\Gamma_i} dz J_i(z)\prod_{j=1}^{6} A_j^{\text{tree}}(z)\Big|_{\mathcal{S}_i}, \qquad (4.25)$$

where the cut amplitude evaluated on the six different kinematical solutions yields

$$\prod_{j=1}^{6} A_j^{\text{tree}}(z)\Big|_{\mathcal{S}_i} = -\frac{i}{16}A^{\text{tree}}_{--++++} \times \begin{cases} \frac{1}{J_3(z)}\left(\frac{1}{z} - \frac{1}{z-Q_1}\right) & \text{for } i=3 \\ -\frac{1}{J_4(z)}\left(\frac{1}{z-P_1} - \frac{1}{z-P_3}\right) & \text{for } i=4 \\ -\frac{1}{J_5(z)}\left(\frac{1}{z-P_1} - \frac{1}{z-P_3}\right) & \text{for } i=5 \\ 0 & \text{for } i=1,2,6, \end{cases} \qquad (4.26)$$

with the tree-level amplitude given by

$$A^{\text{tree}}_{--++++} = \frac{i\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}. \qquad (4.27)$$

The expressions in Eq. (4.26) can be obtained by first exploiting momentum conservation to simplify the heptacut amplitude $\prod_{j=1}^{6} A_j^{\text{tree}}|_{\mathcal{S}_i}$ as much as possible and then substituting the parametrization of the loop momenta in Eqs. (3.12) and (3.13) to obtain the heptacut amplitude as a function of z. To further simplify, one can make use of the fact that, for $\mathcal{N} = 4$ SYM theory, the function

$$\varphi_i(z) = \frac{16i}{A^{\text{tree}}_{--++++}} J_i(z)\prod_{j=1}^{6} A_j^{\text{tree}}(z)\Big|_{\mathcal{S}_i}, \qquad (4.28)$$

only has simple poles in z and has residues ± 1 or 0 at finite poles, and 0 at infinity [52,65], [115].

Denoting all finite poles of $\varphi_i(z)$ by X_j, these facts combined imply that

$$\varphi_i(z) = \sum_j \frac{\text{Res}_{z=X_j}\varphi_i(z)}{z - X_j}. \qquad (4.29)$$

It is from this latter form of $\varphi_i(z)$ that the expressions in Eq. (4.26) were extracted.

3. Extraction of integral coefficients

To summarize the results of Secs. IVA 1 and IVA 2, we find that applying heptacut #1 to both sides of Eq. (4.1) produces the equation

$$i\sum_{i=1}^{6}\oint_{\Gamma_i} dz J_i(z)\prod_{j=1}^{6} A_j^{\text{tree}}(z)\Big|_{\mathcal{S}_i} = \frac{1}{4}\sum_{i=1}^{6}\oint_{\Gamma_i} dz J_i(z) K_i(z), \qquad (4.30)$$

where the Jacobians $J_i(z)$, the cut amplitude $\prod_{j=1}^{6} A_j^{\text{tree}}(z)$ and the kernels $K_i(z)$ are given in Eqs. (4.7), (4.26), (4.19), (4.20), (4.21), (4.22), (4.23), and (4.24), respectively. The kernels $K_i(z)$ contain the integral coefficients, and by making various appropriate choices of the contours Γ_i, Eq. (4.30) produces a system of linear equations which can be solved to obtain the integral coefficients.

Before proceeding to discussing how Eq. (4.30) may be used to determine the integral coefficients of an amplitude, let us first remind ourselves of the relation of this equation to the original two-loop Eq. (4.1). As we found in Secs. IVA 1 and IVA 2, the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), (3.9), and (4.3) for heptacut #1 are solved by setting the parameters α_i, β_j equal to the values quoted in Eq. (4.9) and in Fig. 2, with the spinor ratios P_1, Q_1 etc. and the functions $\beta_3(z)$, $\beta_4(z)$ being given in Eqs. (4.4) and (4.5) as well as (4.10) and (4.11), respectively. For any of the six solutions to the on-shell constraints shown in Fig. 2, one of the loop momentum parameters α_i, β_j is set equal to an unconstrained complex parameter z.

The leading singularity contour is, by definition, a torus consisting of circle factors centered around the seven on-shell values of the parameters left fixed and where, in addition, one makes a choice of contour for the unconstrained degree of freedom z. The contour in z can be chosen to encircle the poles of the Jacobians (4.7) or the poles of the loop momentum parametrization (3.12) and (3.13) at which one of the loop momenta becomes infinite [see Eqs. (4.10) and (4.11)]. The point $(\alpha_1, \ldots, \alpha_4, \beta_1, \ldots, \beta_4) \in \mathbb{C}^4 \times \mathbb{C}^4$ encircled by the torus is referred to as a *global pole* of the integrand of the right-hand side of Eq. (4.1).

Defining

$$\tau = C_{\alpha_1}(1) \times C_{\alpha_2}(0) \times C_{\beta_1}(0) \times C_{\beta_2}(1), \qquad (4.31)$$

the six sets of on-shell values of the loop momentum parameters quoted in Fig. 2, combined with the various possible choices of contours in z, give rise to the following 22 tori corresponding to heptacut #1,

$$
\begin{aligned}
T_{\mathcal{S}_1,0} &= \tau \times C_{\alpha_3}(P_1^\bullet) \times C_{\alpha_4}(0) \times C_{\beta_3=z}(0) \times C_{\beta_4}(0) \\
T_{\mathcal{S}_1,P_1^\bullet} &= \tau \times C_{\alpha_3}(P_1^\bullet) \times C_{\alpha_4}(0) \times C_{\beta_3=z}(P_1^\bullet) \times C_{\beta_4}(0) \\
T_{\mathcal{S}_2,0} &= \tau \times C_{\alpha_3=z}(0) \times C_{\alpha_4}(0) \times C_{\beta_3}(Q_1^\bullet) \times C_{\beta_4}(0) \\
T_{\mathcal{S}_2,P_i^\bullet} &= \tau \times C_{\alpha_3=z}(P_i^\bullet) \times C_{\alpha_4}(0) \times C_{\beta_3}(Q_1^\bullet) \times C_{\beta_4}(0), \quad i = 1,2,3 \\
T_{\mathcal{S}_3,0} &= \tau \times C_{\alpha_3}(0) \times C_{\alpha_4}(P_1) \times C_{\beta_3}(0) \times C_{\beta_4=z}(0) \\
T_{\mathcal{S}_3,P_1} &= \tau \times C_{\alpha_3}(0) \times C_{\alpha_4}(P_1) \times C_{\beta_3}(0) \times C_{\beta_4=z}(P_1) \\
T_{\mathcal{S}_4,0} &= \tau \times C_{\alpha_3}(0) \times C_{\alpha_4=z}(0) \times C_{\beta_3}(0) \times C_{\beta_4}(Q_1) \\
T_{\mathcal{S}_4,P_i} &= \tau \times C_{\alpha_3}(0) \times C_{\alpha_4=z}(P_i) \times C_{\beta_3}(0) \times C_{\beta_4}(Q_1), \quad i = 1,2,3 \\
T_{\mathcal{S}_5,0} &= \tau \times C_{\alpha_3}(0) \times C_{\alpha_4=z}(0) \times C_{\beta_3}(\beta_3(z)) \times C_{\beta_4}(0) \\
T_{\mathcal{S}_5,P_i} &= \tau \times C_{\alpha_3}(0) \times C_{\alpha_4=z}(P_i) \times C_{\beta_3}(\beta_3(z)) \times C_{\beta_4}(0), \quad i = 1,\ldots,4 \\
T_{\mathcal{S}_6,0} &= \tau \times C_{\alpha_3=z}(0) \times C_{\alpha_4}(0) \times C_{\beta_3}(0) \times C_{\beta_4}(\beta_4(z)) \\
T_{\mathcal{S}_6,P_i^\bullet} &= \tau \times C_{\alpha_3=z}(P_i^\bullet) \times C_{\alpha_4}(0) \times C_{\beta_3}(0) \times C_{\beta_4}(\beta_4(z)), \quad i = 1,\ldots,4
\end{aligned} \tag{4.32}
$$

where the indices in $T_{\mathcal{S}_i,X}$ refer to the kinematical solution $\mathcal{S}_i$ associated with the given torus, and X is the pole around which the z-contour is taken. Furthermore, $C_{\alpha_j}(X)$ (e.g.) denotes a small circle in the α_j-plane centered around $\alpha_j = X$; in addition, we write $C_{\alpha_j=z}(X)$ whenever the α_j-variable in question is left unfixed by the heptacut constraints. Finally, the spinor ratios P_1, Q_1 etc. and the functions $\beta_3(z)$, $\beta_4(z)$ are given in Eqs. (4.4) and (4.5) as well as (4.10) and (4.11), respectively.

The relation of Eq. (4.30) to the original two-loop Eq. (4.1) can now easily be stated: the former equation is obtained from the latter by changing the integration contour from the real slice $\mathbb{R}^D \times \mathbb{R}^D$ to an arbitrary linear combination of the 22 tori given in Eq. (4.32). The contributions from, e.g., the first two terms in this linear combination $a_{\mathcal{S}_1,0}T_{\mathcal{S}_1,0} + a_{\mathcal{S}_1,P_1^\bullet}T_{\mathcal{S}_1,P_1^\bullet} + \cdots$ are found by integrating out all parameters except $z = \beta_3$; this leads to the Γ_1-integral in Eq. (4.30) where $\Gamma_1 = a_{\mathcal{S}_1,0}C_{\beta_3=z}(0) + a_{\mathcal{S}_1,P_1^\bullet}C_{\beta_3=z}(P_1^\bullet)$.

Thus having explicitly stated the relation of Eq. (4.30) to the original two-loop Eq. (4.1), let us now return to the question of how the Eq. (4.30) may be used to determine the integral coefficients of an amplitude. As explained in the beginning of this section, the class of contours $\sum_{i,j} a_{i,j} T_{\mathcal{S}_i,X_j}$ that will produce correct results for the integral coefficients in any gauge theory amplitude are those that annihilate all functions that integrate to zero on the real slice $\mathbb{R}^D \times \mathbb{R}^D$. Determining such contours requires the knowledge of all integration-by-parts identities at six points; however, as a complete knowledge of all such relations is presently not available, we will here proceed as in Ref. [67] and assume that *any* contour is valid. Indeed, as already mentioned above, the purpose of this section is mainly to provide a pedagogical exposition of the use of the leading singularity method to obtain integral coefficients of $\mathcal{N} = 4$ SYM amplitudes.

Below we will use the following notation: $C_\epsilon(X_k)$ denotes a circle centered around $z = X_k$ of some appropriately small radius ϵ (i.e., small enough to not enclose any other poles), and $\Gamma_i = \delta_{i,j}C_\epsilon(X_k)$ denotes a contour which is zero on all six sheets except for the sheet supporting kinematical solution $\mathcal{S}_j$; on this sheet, the contour is a small circle centered around X_k. From Eq. (4.30) we then find, for example, that

(i) setting $\Gamma_i = \delta_{i,5}C_\epsilon(P_1)$ produces the equation

$$
\begin{aligned}
&\frac{1}{4}\bigg(c_{2,\sigma_1} - \frac{1}{2}\frac{c_{22,\sigma_1}}{\langle K_2^{\flat-}|\not{k}_{12}|K_1^{\flat-}\rangle(P_1 - P_2)} \\
&\quad - \frac{1}{2}\frac{c_{19,\sigma_1}}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle(P_1 - P_3)}\bigg) \\
&= -2\gamma_1\gamma_2 P_1\langle K_2^{\flat-}|\not{K}_5|K_1^{\flat-}\rangle A^{\text{tree}}_{--++++}
\end{aligned} \tag{4.33}
$$

(ii) setting $\Gamma_i = \delta_{i,5}C_\epsilon(P_2)$ produces the equation

$$
\begin{aligned}
&\frac{1}{4}\bigg(-\frac{1}{2}\frac{c_{22,\sigma_1}}{\langle K_2^{\flat-}|\not{k}_{12}|K_1^{\flat-}\rangle} \\
&\quad - \frac{c_{18,\sigma_1}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle(P_2 - P_1)}{\langle K_2^{\flat-}|\not{k}_{12}|K_1^{\flat-}\rangle}\bigg) = 0,
\end{aligned} \tag{4.34}
$$

(iii) setting $\Gamma_i = \delta_{i,5}C_\epsilon(P_3)$ produces the equation

$$
\begin{aligned}
&\frac{1}{4}\bigg(-\frac{1}{2}\frac{c_{19,\sigma_1}}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle} \\
&\quad - \frac{c_{8,\sigma_1}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle(P_3 - P_1)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle}\bigg) \\
&= 2\gamma_1\gamma_2 P_3(P_3 - P_1)\langle K_2^{\flat-}|\not{K}_5|K_1^{\flat-}\rangle A^{\text{tree}}_{--++++}
\end{aligned} \tag{4.35}
$$

(iv) setting $\Gamma_i = \delta_{i,2} C_\epsilon(P_1^\bullet)$ produces the equation

$$\frac{1}{4}\left(c_{2,\sigma_1} - \frac{1}{2}\frac{c_{22,\sigma_1}}{\langle K_1^{\flat-}|\not{k}_{12}|K_2^{\flat-}\rangle(P_1^\bullet - P_2^\bullet)} - \frac{1}{2}\frac{c_{19,\sigma_1}}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle(P_1^\bullet - P_3^\bullet)}\right) = 0, \tag{4.36}$$

(v) setting $\Gamma_i = \delta_{i,2} C_\epsilon(P_2^\bullet)$ produces the equation

$$\frac{1}{4}\left(-\frac{1}{2}\frac{c_{22,\sigma_1}}{\langle K_1^{\flat-}|\not{k}_{12}|K_2^{\flat-}\rangle} - \frac{c_{18,\sigma_1}\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle(P_2^\bullet - P_1^\bullet)}{\langle K_1^{\flat-}|\not{k}_{12}|K_2^{\flat-}\rangle}\right) = 0, \tag{4.37}$$

(vi) setting $\Gamma_i = \delta_{i,2} C_\epsilon(P_3^\bullet)$ produces the equation

$$\frac{1}{4}\left(-\frac{1}{2}\frac{c_{19,\sigma_1}}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle} - \frac{c_{8,\sigma_1}\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle(P_3^\bullet - P_1^\bullet)}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle}\right) = 0. \tag{4.38}$$

We have here deliberately chosen an overcomplete system of equations as a consistency check on the method. We in fact find that the six equations (4.33), (4.34), (4.35), (4.36), (4.37), and (4.38) are consistent, and that expressed in terms of the quantity

$$Y = -8\gamma_1\gamma_2 P_1\langle K_2^{\flat-}|\not{k}_5|K_1^{\flat-}\rangle A^{\text{tree}}_{--++++} \times\left(1 - \frac{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle(P_1^\bullet - P_3^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle(P_1 - P_3)}\right)^{-1}, \tag{4.39}$$

the solution of Eqs. (4.33), (4.34), (4.35), (4.36), (4.37), and (4.38) takes the following form

$$c_{2,\sigma_1} = Y \tag{4.40}$$

$$c_{22,\sigma_1} = 0 \tag{4.41}$$

$$c_{19,\sigma_1} = 2\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle(P_1^\bullet - P_3^\bullet)Y \tag{4.42}$$

$$c_{18,\sigma_1} = 0 \tag{4.43}$$

$$c_{8,\sigma_1} = \frac{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle}Y. \tag{4.44}$$

In the following it will be useful to consider the two-loop amplitude normalized with respect to the tree-level amplitude,

$$M^{(2)}_{6,\text{MHV}} = \frac{A^{(2)}_{6,\text{MHV}}}{A^{\text{tree}}_{\text{MHV}}} = \frac{1}{4}\sum_{\substack{i=1,\dots,24\\ j=1,\dots,12}} r_i \bar{c}_{i,\sigma_j} I_{i,\sigma_j}, \tag{4.45}$$

where the normalized coefficients are defined by

$$\bar{c}_{\,i,\sigma_j} \equiv \frac{c_{i,\sigma_j}}{A^{\text{tree}}_{\text{MHV}}}, \tag{4.46}$$

because this object is independent of the distribution of the helicities of the external states [113,114]; that is, for example, $M^{(2)}_{6,--++++} = M^{(2)}_{6,-++++-}$. Moreover, the integrand of $M^{(2)}_{6,\text{MHV}}$ can be decomposed into two terms that are even and odd under parity and which respectively coincide with its real and imaginary parts.

Thus, the parity-even part of the coefficients found in Eqs. (4.40), (4.41), (4.42), (4.43), and (4.44) is obtained by dividing by the tree-level amplitude A^{tree}_{--++++} and taking the real part, yielding

$$\text{Re}\left(\frac{c_{2,\sigma_1}}{A^{\text{tree}}_{--++++}}\right) = 2s_{45}s_{56}^2 \tag{4.47}$$

$$\text{Re}\left(\frac{c_{22,\sigma_1}}{A^{\text{tree}}_{--++++}}\right) = 0 \tag{4.48}$$

$$\text{Re}\left(\frac{c_{19,\sigma_1}}{A^{\text{tree}}_{--++++}}\right) = 0 \tag{4.49}$$

$$\text{Re}\left(\frac{c_{18,\sigma_1}}{A^{\text{tree}}_{--++++}}\right) = 0 \tag{4.50}$$

$$\text{Re}\left(\frac{c_{8,\sigma_1}}{A^{\text{tree}}_{--++++}}\right) = 2s_{56}(s_{123}s_{234} - s_{56}s_{23}), \tag{4.51}$$

where the equalities have been found to hold numerically. The expressions on the right-hand sides of Eqs. (4.47), (4.48), (4.49), (4.50), and (4.51) are in agreement with the results originally found in Ref. [89] and reproduced by the leading singularity method in Ref. [67]. Due to a Ward identity [113,114,116] valid for $\mathcal{N} = 4$ supersymmetry, the coefficients of the cyclically permuted integrals can simply be obtained by cyclic permutation of the results found here.

For the remaining heptacuts #2, ..., #9 (details of which are provided in Appendix A), the leading singularity method provides similar linear equations satisfied by the integral coefficients. However, when solving the linear equations for these more complicated heptacuts, the overcompleteness of the basis in Fig. 7 entails a problem, carefully discussed in Ref. [67]: due to linear relations between various integrals in the basis, the integral coefficients are not unique. This feature will manifest itself as the appearance of free parameters in the solutions of the linear equations. Thus, one has to set some of the integral coefficients equal to specific values in order to obtain unique solutions for the remaining coefficients.

This "gauge fixing" can be easily done for the parity-even part of the integral coefficients as analytic results for these were already obtained in Ref. [89], and one can proceed as follows. For a given heptacut, encircle a pole and its parity conjugate to obtain equations of the schematic form

$$\alpha c_1 + \beta c_2 = \gamma \tag{4.52}$$

$$\alpha^* c_1 + \beta^* c_2 = \delta, \tag{4.53}$$

where, for example, α^* denotes the complex conjugate of α. From this pair of equations immediately follows

$$(\alpha + \alpha^*)\bar{c}_1^{\text{even}} + (\beta + \beta^*)\bar{c}_2^{\text{even}} = \text{Re}\left(\frac{\gamma + \delta}{A^{\text{tree}}_{--++++}}\right) \tag{4.54}$$

$$(\alpha - \alpha^*)\bar{c}_1^{\text{even}} + (\beta - \beta^*)\bar{c}_2^{\text{even}} = i\,\text{Im}\left(\frac{\gamma - \delta}{A^{\text{tree}}_{--++++}}\right) \tag{4.55}$$

where the parity-even part of the coefficients is simply given as the real part of the (normalized) coefficient,

$$\bar{c}^{\,\text{even}}_{i,\sigma_j} \equiv \text{Re}\bar{c}_{i,\sigma_j}. \tag{4.56}$$

The pair of equations (4.54) and (4.55) thus allows one to solve directly for the parity-even part of the integral coefficients. In order to obtain unique answers for these, we make the following "gauge choices": we set

$$\left.\begin{aligned} \bar{c}^{\text{even}}_{11,\sigma_j} &= s_{\sigma_j(61)}s_{\sigma_j(12)}s_{\sigma_j(123)} \\ \bar{c}^{\text{even}}_{12,\sigma_j} &= s_{\sigma_j(456)}(s_{\sigma_j(345)}s_{\sigma_j(456)} - s_{\sigma_j(12)}s_{\sigma_j(45)}) \\ \bar{c}^{\text{even}}_{24,\sigma_j} &= 0 \end{aligned}\right\} \quad \text{for } j = 1, \ldots, 12 \tag{4.57}$$

and also recall that $c_{14,\sigma_j} = c_{15,\sigma_j} = 0$ for the full coefficients of the μ-integrals. With this choice, the equations produced by taking leading singularities of both sides of Eq. (4.1) and then projecting out the parity-odd part of the coefficients in analogy with Eqs. (4.54) and (4.55) have a unique solution which is in agreement with the results for the parity-even coefficients originally found in Ref. [89]; in particular, one finds that

$$\bar{c}^{\,\text{even}}_{19,\sigma_j} = \cdots = \bar{c}^{\text{even}}_{24,\sigma_j} = 0, \tag{4.58}$$

in agreement with the parity-even part of the amplitude being dual conformally invariant.

B. Leading singularities vs loop-level recursion

In this section we report on a comparison between results for the full (i.e., parity-even and -odd) two-loop six-point MHV $\mathcal{N} = 4$ SYM integrand as produced by the leading singularity method on one hand and recent predictions in the literature based on a BCFW-like loop-level recursion relation [60,63] on the other. Because the results of these papers are expressed in terms of a different basis from the one used in this paper, it is obviously not meaningful to check agreement between individual integral coefficients in the two representations.

A quantity that can be meaningfully compared is the two-loop integrand: in general, in the planar limit of any field theory, the loop integrand is a well-defined rational function of the external momenta (which for example can be thought of as being produced by the Feynman rules). In our case, the two-loop integrand is the quantity under the integral sign in Eq. (4.45), obtained as the sum of the integrands of the 24 basis integrals I_{i,σ_j}, weighted by the integral coefficients c_{i,σ_j} and symmetry factors r_i, where the summation is taken over all dihedral permutations σ_j of the external momentum labels. Agreement between the two-loop integrand as computed by either method would imply agreement between the integrated expressions; that is, the results for the amplitude [117].

We have performed the comparison of the two-loop integrands numerically, by verifying agreement to high accuracy for a large number of randomly selected external and internal momenta. Accordingly, the remainder of this section will be devoted to discussing how, given a set of randomly generated momenta, one may evaluate the integrand of Eq. (4.45)—that is, how the integral coefficients are obtained, and how the integrands of the basis integrals I_{i,σ_j} are added in a meaningful way.

The integral coefficients are determined in analogy with the procedure explained in Sec. IVA 3. However, in that context we were concerned with obtaining analytical results for the coefficients and could find the coefficients of all dihedral permutations of, for example, I_{2,σ_1} by applying the appropriate permutation to the algebraic expression for c_{2,σ_1}. This is obviously not possible when one is aiming to find numerical results for the coefficients. Instead, one must act on the external momentum labels implicit in Eq. (4.30) with each of the dihedral permutations $\sigma_j \in D_6$ in turn so as to produce distinct linear equations for c_{2,σ_j}. Again, because of linear relations between the basis integrals, the solutions of the linear equations will, for the more complicated heptacuts #2, …, #9, contain free parameters. Accordingly, one must set some of the coefficients equal to specific values in order to obtain unique solutions for the remaining coefficients. In analogy with Eq. (4.57), we choose the "gauge fixing"

$$\left.\begin{aligned} \bar{c}_{11,\sigma_j} &= s_{\sigma_j(61)}s_{\sigma_j(12)}s_{\sigma_j(123)} \\ \bar{c}_{12,\sigma_j} &= s_{\sigma_j(456)}(s_{\sigma_j(345)}s_{\sigma_j(456)} - s_{\sigma_j(12)}s_{\sigma_j(45)}) \\ \bar{c}_{24,\sigma_j} &= 0 \end{aligned}\right\} \quad \text{for } j = 1, \ldots, 12 \tag{4.59}$$

whereby $\bar{c}^{\text{odd}}_{11,\sigma_j} = \bar{c}^{\text{odd}}_{12,\sigma_j} = \bar{c}^{\text{odd}}_{24,\sigma_j} = 0$. We then find unique results for the remaining coefficients with the property that $\text{Re}\bar{c}_{i,\sigma_j} = \bar{c}^{\text{even}}_{i,\sigma_j}$.

Expressed as functions of internal and external momenta, the integrands of the basis integrals cannot be added in any meaningful way as the value of any term would depend on the labeling of the internal lines of the corresponding graph (i.e., which propagators are labeled ℓ_1

and ℓ_2). To remedy this, the integrand must be expressed in terms of dual x-space coordinates, defined by

$$\begin{aligned} x_i - x_{i+1} &= k_i & & i = 1, \ldots, 6 \pmod 6 \\ x_{\sigma_j(1)} - x_7 = \ell_1 \quad & x_{\sigma_j(1)} - x_8 = -\ell_2 & & j = 1, \ldots, 6 \\ x_{\sigma_j(6)} - x_7 = -\ell_1 \quad & x_{\sigma_j(6)} - x_8 = \ell_2 & & j = 7, \ldots, 12 \\ x_{ij} &\equiv x_i - x_j & & i, j = 1, \ldots, 8, \end{aligned} \tag{4.60}$$

with the additional requirement that, for any given graph, ℓ_1 and ℓ_2 be offset by appropriate translations by external momenta so that all propagators take the form $\frac{1}{x_{ij}^2}$. Finally, the integrand must be symmetrized in x_7 and x_8. Namely, any assignment of these points to a given graph will fail to be invariant under vertical reflections of the graph; to ensure that the value of the integrand is not dependent on how its contributing graphs happen to be drawn, one must therefore average over the two possible assignments of these points. The integrands of the basis integrals in Fig. 7 have been presented in Sec. A 1 for convenience.

In summary, given a set of random internal and external momenta, the evaluation of the integrand of Eq. (4.45) proceeds in three steps. First, the integral coefficients are obtained by solving the linear equations that follow from taking the leading singularities of Eq. (4.45). Second, the integrands of the basis integrals are computed after converting the momenta into the dual x-space (which is achieved by solving Eq. (4.60) and choosing, e.g., the base point $x_6 = 0$). Finally, the intermediate results are combined, weighting the contributions by the appropriate symmetry factors (4.2) of the integrals. This is essentially the procedure followed by our code [118] which is available online. As a simple consistency check of the code, we remark that the results produced for the integrand indeed satisfy crossing symmetry; that is,

$$\begin{aligned} &\text{integrand of } M^{(2)}_{6,\text{MHV}}(x_1, \ldots, x_6) \\ &\quad = \text{integrand of } M^{(2)}_{6,\text{MHV}}(x_{\sigma_j(1)}, \ldots, x_{\sigma_j(6)}) \;\; \text{for } \sigma_j \in D_6. \end{aligned} \tag{4.61}$$

We have evaluated the two-loop six-point MHV integrand for a large number of randomly selected rational momenta [119] and in all cases find agreement with Refs. [60,63] to high numerical accuracy [90]. In Table I below we have provided a few sample points to allow the curious reader to reproduce our results. Further data points can be generated by the Mathematica notebook [118] available online.

Finally, let us observe that we only made use of the assumption that the two-loop amplitude is MHV when evaluating the heptacuts of the left-hand side of Eq. (4.1). The form of the heptacuts of the right-hand side of Eq. (4.1) is independent of the external helicities, and the results presented in this paper can therefore straightforwardly be extended to obtain the NMHV integrand as well.

The two-loop six-point integrand as computed in this paper was expressed in terms of the basis in Fig. 7 which does not include integrals containing subloops with less than four propagators. The exclusion of such integrals from the basis owes to the observation that cutting two propagators would factor out a subtriangle or sub-bubble—but the latter integrals are known not to contribute to one-loop amplitudes in $\mathcal{N} = 4$ SYM. This strongly suggests that the uncut two-loop integral would appear with zero coefficient if included in the basis expansion.

One possibility which cannot be rigorously ruled out by this argument, however, occurs when the two-particle cut is shared between several integrals containing subtriangles or sub-bubbles: in principle, the coefficients of the respective cut two-loop integrals could be nonzero, but such that the contributions cancel.

Ruling out such a scenario completely would require extending the analysis of this paper to consider hexacuts, pentacuts etc. Two-loop integrals whose subloops contain at least three propagators all admit T^8-integration contours [analogous to those in Eq. (4.32)] that appropriately define such cuts. Moreover, the cut integrand is a holomorphic function, and the contour integrations can thus be performed directly by means of the global residue theorem [52]. In contrast, two-loop integrals with bubble-subloops do not admit T^8-contours: for example, the bubble-box integral

TABLE I. Values of the two-loop six-point MHV integrand (normalized with respect to the tree-level amplitude) at three randomly chosen sets of points $(x_1, \ldots, x_6)$ and (x_7, x_8) in dual x-space, respectively encoding external and internal momenta as prescribed by Eq. (4.60). The parity-even and -odd parts of the integrand of $M^{(2)}_{6,\text{MHV}}$ respectively coincide with its real and imaginary parts. Further data points can be generated by the Mathematica notebook [118] available online.

$(x_1, \ldots, x_6)$	(x_7, x_8)	integrand of $M^{(2)}_{6,\text{MHV}}$
$((-\frac{1}{2}, \frac{1}{2}, 0, 0), (-\frac{11}{12}, \frac{1}{6}, \frac{1}{4}, 0), (-\frac{4}{3}, \frac{5}{12}, \frac{7}{12}, 0), (-\frac{7}{4}, \frac{29}{36}, \frac{13}{18}, -\frac{1}{18}), (-\frac{23}{18}, \frac{35}{54}, \frac{28}{27}, -\frac{10}{27}), (0, 0, 0, 0))$	$((\frac{1}{4}, \frac{1}{4}, 2, 1), (0, 0, 2, 0))$	$-\frac{31\,230\,748\,253}{22\,094\,130\,240} - \frac{994\,276\,085}{981\,961\,344} i$
$((\frac{3}{8}, \frac{7}{24}, \frac{1}{6}, -\frac{1}{6}), (\frac{7}{8}, -\frac{5}{24}, \frac{1}{6}, -\frac{1}{6}), (\frac{157}{120}, -\frac{3}{8}, -\frac{7}{30}, -\frac{1}{6}), (\frac{69}{40}, -\frac{1}{8}, -\frac{17}{30}, -\frac{1}{6}), (\frac{73}{90}, \frac{5}{54}, -\frac{53}{135}, \frac{19}{27}), (0, 0, 0, 0))$	$((1, \frac{1}{3}, 0, 2), (0, 0, 1, 1))$	$\frac{4\,777\,009\,838\,357}{201\,230\,662\,913\,280} + \frac{1\,802\,603\,853\,899}{259\,652\,468\,275\,200} i$
$((\frac{5}{12}, \frac{1}{3}, 0, \frac{1}{4}), (\frac{79}{96}, -\frac{1}{24}, \frac{1}{8}, \frac{5}{32}), (\frac{59}{48}, -\frac{1}{6}, \frac{7}{32}, -\frac{7}{32}), (\frac{157}{96}, -\frac{13}{24}, \frac{1}{8}, -\frac{3}{32}), (\frac{125}{224}, \frac{15}{28}, \frac{1}{8}, -\frac{3}{32}), (0, 0, 0, 0))$	$((\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, 0), (\frac{1}{2}, 0, 0, 0))$	$\frac{3\,393\,545\,258\,977\,272}{16\,669\,297\,265} - \frac{43\,045\,877\,862\,533\,664}{183\,362\,269\,915} i$

rather admits a $T^6 \times S^2$ contour. Several approaches are available to deal with bubble-inherited S^2 contours, among the more elegant ones is that of Mastrolia [120], exploiting Stokes's theorem; a related approach is that of Arkani-Hamed *et al.* in Ref. [121]. However, we leave such extensions for future work.

V. CONCLUSIONS

In this paper we have provided a check that recent results in the literature [60,63] for the full (i.e., parity-even and -odd) two-loop six-point MHV integrand of $\mathcal{N} = 4$ SYM theory can be reproduced by the leading singularity method. Equivalently, assuming the validity of Refs. [60,63], one can view the analysis carried out in this paper as a check that the leading singularities of the $\mathcal{N} = 4$ SYM integrand evaluated in strictly four dimensions (as opposed to in $D = 4 - 2\epsilon$ dimensions) are sufficient to detect the parity-odd part. This has already been shown to be the case for the two-loop five-gluon $\mathcal{N} = 4$ SYM amplitude in Ref. [66], but the six-gluon MHV amplitude provides a much richer testing ground owing to the larger number of global poles of the integrand and of integrals in terms of which the amplitude is expressed. As the main part of the results presented in this paper are independent of the helicities of the six external states, the NMHV integrand can be straightforwardly obtained by supplementing the requisite helicity-dependent data. We leave this as an open problem.

For any two-loop integral, the leading singularities are obtained by changing the integration range from $\mathbb{R}^D \times \mathbb{R}^D$ into tori of real dimension 8 (embedded in $\mathbb{C}^4 \times \mathbb{C}^4$) that encircle the global poles of the integrand. As explained in Ref. [34], the maximal cuts are particular linear combinations σ of these tori whose coefficients are determined by the requirement that any function that integrates to zero on $\mathbb{R}^D \times \mathbb{R}^D$ should also integrate to zero on σ. This constraint ensures that two Feynman integrals which are equal, possibly through some nontrivial relations, will also have identical maximal cuts. As argued in Ref. [34], multidimensional contours σ satisfying this consistency condition are guaranteed to produce correct results for scattering amplitudes in any gauge theory, not only $\mathcal{N} = 4$ SYM theory.

The set of linear relations between two-loop integrals includes the set of all integration-by-parts (IBP) identities between the various tensor integrals arising from the Feynman rules of gauge theory; however, at present a complete knowledge of such relations is not available. Thus, an interesting open problem to be pursued once all necessary IBP relations do become available is to determine the maximal-cut contours that allow the extraction of integral coefficients in any two-loop six-point gauge theory amplitude. We expect the intermediate results provided in this paper to greatly facilitate this task.

ACKNOWLEDGMENTS

I am indebted to Simon Caron-Huot for assistance in comparing the $\mathcal{N} = 4$ SYM integrand and to Henrik Johansson for innumerable useful discussions. I would like to thank Gregory Korchemsky and David Kosower for comments on the manuscript and for discussions. I would also like to thank Ruth Britto for discussions. I thank the Niels Bohr Institute for hospitality where part of this work was carried out. This work is supported by the European Research Council under Advanced Investigator Grant ERC-AdG-228301.

APPENDIX A: DETAILS OF HEPTACUTS #2, ..., #9

1. Integrands of basis integrals in dual coordinates

The expressions below are the four-dimensional integrands of the basis integrals in Fig. 7, expressed in dual x-space coordinates (related to the internal and external momenta through Eq. (4.60)). The results are recorded in the σ_1 permutation of the external momentum labels; the integrands for the remaining dihedral permutations may be obtained by applying Eq. (A2) below.

$$\text{integrand}_{1,\sigma_1} = \frac{1}{2}((x_{17}^2 x_{27}^2 x_{37}^2 x_{47}^2 x_{18}^2 x_{68}^2 x_{58}^2 x_{48}^2)^{-1} + (x_7 \leftrightarrow x_8))$$

$$\text{integrand}_{2,\sigma_1} = \frac{1}{2}((x_{17}^2 x_{47}^2 x_{57}^2 x_{78}^2 x_{18}^2 x_{68}^2 x_{58}^2)^{-1} + (x_7 \leftrightarrow x_8))$$

$$\text{integrand}_{3,\sigma_1} = \frac{1}{2}((x_{17}^2 x_{37}^2 x_{47}^2 x_{78}^2 x_{18}^2 x_{68}^2 x_{48}^2)^{-1} + (x_7 \leftrightarrow x_8))$$

$$\text{integrand}_{4,\sigma_1} = \frac{1}{2}((x_{17}^2 x_{37}^2 x_{47}^2 x_{78}^2 x_{18}^2 x_{58}^2 x_{48}^2)^{-1} + (x_7 \leftrightarrow x_8))$$

$$\text{integrand}_{5,\sigma_1} = \frac{1}{2}((x_{17}^2 x_{27}^2 x_{37}^2 x_{78}^2 x_{68}^2 x_{58}^2 x_{38}^2)^{-1} + (x_7 \leftrightarrow x_8))$$

$$\text{integrand}_{6,\sigma_1} = \frac{1}{2}((x_{17}^2 x_{37}^2 x_{47}^2 x_{78}^2 x_{68}^2 x_{58}^2 x_{48}^2)^{-1} + (x_7 \leftrightarrow x_8))$$

$$\text{integrand}_{7,\sigma_1} = \frac{1}{2}((x_{17}^2 x_{27}^2 x_{37}^2 x_{78}^2 x_{68}^2 x_{58}^2 x_{48}^2)^{-1} + (x_7 \leftrightarrow x_8))$$

$$
\begin{aligned}
\text{integrand}_{8,\sigma_1} &= \frac{1}{2}\left(\frac{x_{67}^2}{x_{17}^2x_{27}^2x_{47}^2x_{57}^2x_{78}^2x_{18}^2x_{68}^2x_{58}^2} + (x_7 \leftrightarrow x_8)\right)\\
\text{integrand}_{9,\sigma_1} &= \frac{1}{2}\left(\frac{x_{67}^2}{x_{17}^2x_{27}^2x_{37}^2x_{47}^2x_{78}^2x_{18}^2x_{68}^2x_{48}^2} + (x_7 \leftrightarrow x_8)\right)\\
\text{integrand}_{10,\sigma_1} &= \frac{1}{2}\left(\frac{x_{67}^2}{x_{17}^2x_{27}^2x_{37}^2x_{47}^2x_{78}^2x_{18}^2x_{68}^2x_{58}^2} + (x_7 \leftrightarrow x_8)\right)\\
\text{integrand}_{11,\sigma_1} &= \frac{1}{2}\left(\frac{x_{57}^2}{x_{17}^2x_{27}^2x_{37}^2x_{47}^2x_{78}^2x_{18}^2x_{68}^2x_{58}^2} + (x_7 \leftrightarrow x_8)\right)\\
\text{integrand}_{12,\sigma_1} &= \frac{1}{2}\left(\frac{x_{57}^2x_{28}^2}{x_{17}^2x_{27}^2x_{37}^2x_{47}^2x_{78}^2x_{18}^2x_{68}^2x_{58}^2x_{48}^2} + (x_7 \leftrightarrow x_8)\right)\\
\text{integrand}_{13,\sigma_1} &= \frac{1}{2}\left(\frac{x_{57}^2x_{38}^2}{x_{17}^2x_{27}^2x_{37}^2x_{47}^2x_{78}^2x_{18}^2x_{68}^2x_{58}^2x_{48}^2} + (x_7 \leftrightarrow x_8)\right)\\
\text{integrand}_{14,\sigma_1} &= 0\\
\text{integrand}_{15,\sigma_1} &= 0\\
\text{integrand}_{16,\sigma_1} &= \frac{1}{2}((x_{17}^2x_{37}^2x_{57}^2x_{78}^2x_{18}^2x_{68}^2x_{58}^2)^{-1} + (x_7 \leftrightarrow x_8))\\
\text{integrand}_{17,\sigma_1} &= \frac{1}{2}((x_{17}^2x_{27}^2x_{37}^2x_{78}^2x_{58}^2x_{48}^2x_{38}^2)^{-1} + (x_7 \leftrightarrow x_8))\\
\text{integrand}_{18,\sigma_1} &= \frac{1}{2}\left(\frac{x_{67}^2}{x_{17}^2x_{37}^2x_{47}^2x_{57}^2x_{78}^2x_{18}^2x_{68}^2x_{58}^2} + (x_7 \leftrightarrow x_8)\right)\\
\text{integrand}_{19,\sigma_1} &= \frac{1}{2}((x_{17}^2x_{27}^2x_{47}^2x_{57}^2x_{78}^2x_{18}^2x_{68}^2x_{58}^2)^{-1} + (x_7 \leftrightarrow x_8))\\
\text{integrand}_{20,\sigma_1} &= \frac{1}{2}((x_{17}^2x_{27}^2x_{37}^2x_{47}^2x_{78}^2x_{18}^2x_{68}^2x_{48}^2)^{-1} + (x_7 \leftrightarrow x_8))\\
\text{integrand}_{21,\sigma_1} &= \frac{1}{2}((x_{17}^2x_{27}^2x_{37}^2x_{47}^2x_{78}^2x_{18}^2x_{68}^2x_{58}^2)^{-1} + (x_7 \leftrightarrow x_8))\\
\text{integrand}_{22,\sigma_1} &= \frac{1}{2}((x_{17}^2x_{37}^2x_{47}^2x_{57}^2x_{78}^2x_{18}^2x_{68}^2x_{58}^2)^{-1} + (x_7 \leftrightarrow x_8))\\
\text{integrand}_{23,\sigma_1} &= \frac{1}{2}((x_{17}^2x_{27}^2x_{37}^2x_{47}^2x_{78}^2x_{18}^2x_{68}^2x_{58}^2x_{48}^2)^{-1} + (x_7 \leftrightarrow x_8))\\
\text{integrand}_{24,\sigma_1} &= \frac{1}{2}\left(\frac{x_{38}^2}{x_{17}^2x_{27}^2x_{37}^2x_{47}^2x_{78}^2x_{18}^2x_{68}^2x_{58}^2x_{48}^2} + (x_7 \leftrightarrow x_8)\right).
\end{aligned}
\tag{A1}
$$

The integrands for the remaining dihedral permutations $\sigma_j \in D_6$ can be obtained from Eq. (A1) by applying

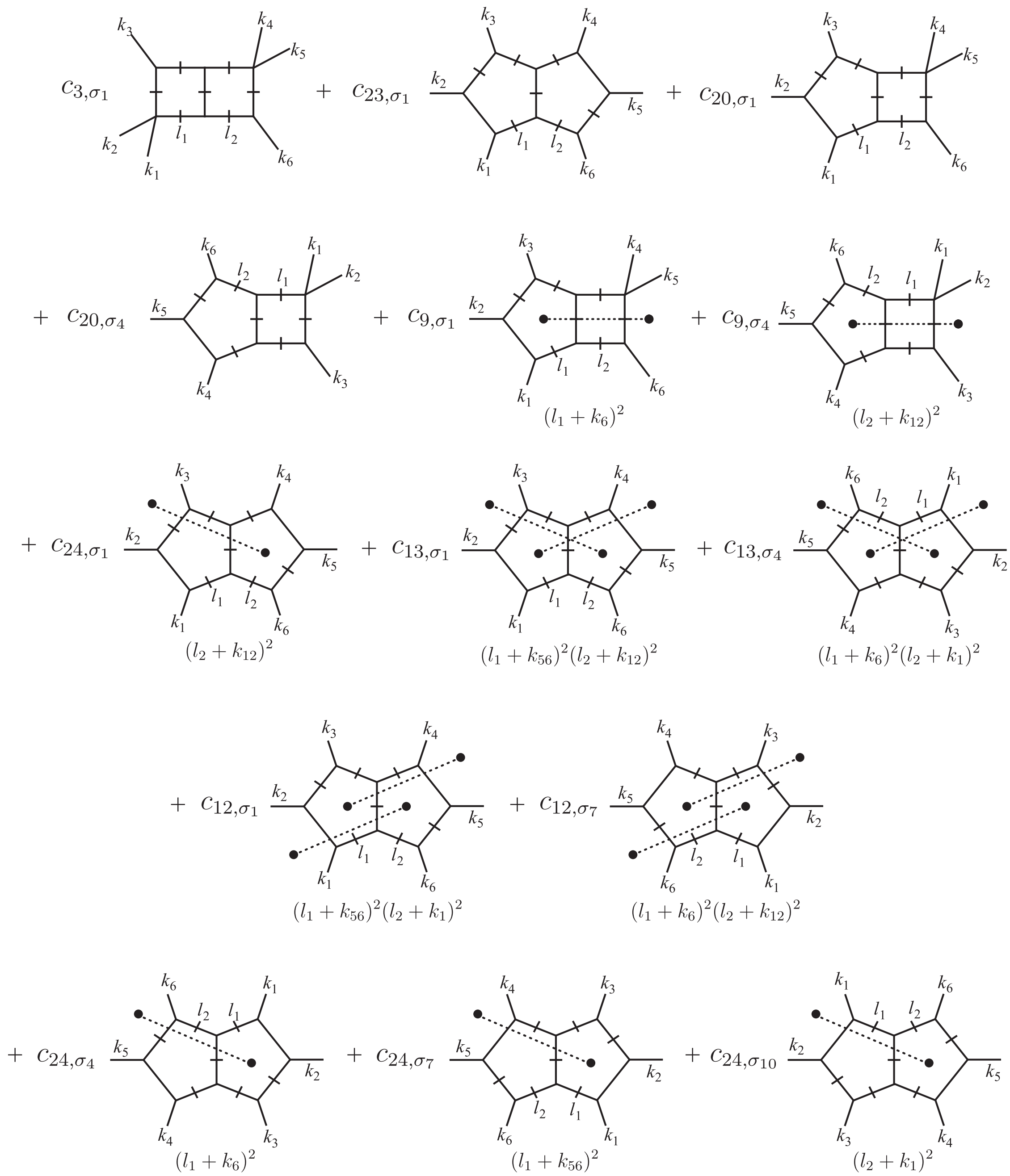
$$
\text{integrand}_{i,\sigma_j}(x_1, x_2, \ldots, x_6; x_7, x_8) = \begin{cases} \text{integrand}_{i,\sigma_1}(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \ldots, x_{\sigma_j(6)}; x_7, x_8) & \text{for } j = 1, \ldots, 6 \\ \text{integrand}_{i,\sigma_1}(x_{\sigma_j(6)}, x_{\sigma_j(1)}, \ldots, x_{\sigma_j(5)}; x_7, x_8) & \text{for } j = 7, \ldots, 12 \end{cases}. \tag{A2}
$$

2. Heptacut #2

This heptacut is defined by the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) with the vertex momenta

$$
K_1 = k_{12} \qquad K_2 = k_3 \qquad K_3 = 0 \qquad K_4 = k_{45} \qquad K_5 = k_6 \qquad K_6 = 0. \tag{A3}
$$

Applying this heptacut to the right-hand side of Eq. (4.1) leaves the following linear combination of cut integrals



We define the spinor ratios

$$P_1 = -\frac{\langle K_1^\flat k_6\rangle}{2\langle K_2^\flat k_6\rangle}, \qquad P_2 = -\frac{\langle K_1^\flat k_1\rangle}{2\langle K_2^\flat k_1\rangle}, \qquad P_3 = -\frac{K_1^\flat \cdot k_{56} + \frac{1}{2}s_{56}}{\langle K_2^{\flat -}|\not{k}_{56}|K_1^{\flat -}\rangle}, \qquad P_4 = -\frac{\langle K_1^\flat K_4^\flat\rangle}{2\langle K_2^\flat K_4^\flat\rangle}$$

$$Q_1 = -\frac{(1+\frac{S_4}{\gamma_2})[K_5^\flat K_1^\flat]}{2[K_4^\flat K_1^\flat]}, \qquad Q_2 = -\frac{(1+\frac{S_4}{\gamma_2})K_5^\flat \cdot k_{56} - \frac{1}{2}s_{56}}{\langle K_5^{\flat -}|\not{k}_{56}|K_4^{\flat -}\rangle}, \qquad Q_3 = -\frac{(1+\frac{S_4}{\gamma_2})[K_5^\flat k_1]}{2[K_4^\flat k_1]}, \tag{A4}$$

and their parity conjugates

$$P_1^\bullet = -\frac{[K_1^\flat k_6]}{2[K_2^\flat k_6]}, \qquad P_2^\bullet = -\frac{[K_1^\flat k_1]}{2[K_2^\flat k_1]}, \qquad P_3^\bullet = -\frac{K_1^\flat \cdot k_{56} + \frac{1}{2}s_{56}}{\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle}, \qquad P_4^\bullet = -\frac{[K_1^\flat K_4^\flat]}{2[K_2^\flat K_4^\flat]}$$
$$Q_1^\bullet = -\frac{(1+\frac{S_4}{\gamma_2})\langle K_5^\flat K_1^\flat\rangle}{2\langle K_4^\flat K_1^\flat\rangle}, \qquad Q_2^\bullet = -\frac{(1+\frac{S_4}{\gamma_2})K_5^\flat \cdot k_{56} - \frac{1}{2}s_{56}}{\langle K_4^{\flat -}|\not{k}_{56}|K_5^{\flat -}\rangle}, \qquad Q_3^\bullet = -\frac{(1+\frac{S_4}{\gamma_2})\langle K_5^\flat k_1\rangle}{2\langle K_4^\flat k_1\rangle}. \tag{A5}$$

This heptacut belongs to case I treated in Sec. III A, and there are thus six kinematical solutions (shown in Fig. 2). Parametrizing the loop momenta according to Eqs. (3.12) and (3.13), the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) are solved by setting the parameters equal to the values

$$\alpha_1 = 1, \qquad \alpha_2 = 0, \qquad \beta_1 = 0, \qquad \beta_2 = 1 + \frac{S_4}{\gamma_2}, \tag{A6}$$

and those given in Fig. 2 with

$$\beta_3(z) = -\left(1 + \frac{S_4}{\gamma_2}\right)\frac{\langle K_2^\flat K_5^\flat\rangle(z - P_1)}{2\langle K_2^\flat K_4^\flat\rangle(z - P_4)}, \tag{A7}$$

for kinematical solution $\mathcal{S}_5$. The heptacut double-box integral I_{3,σ_1} is $\sum_{i=1}^6 \oint_{\Gamma_i} dz J_i(z)$ where

$$J_i(z) = \frac{1}{32\gamma_1\gamma_2} \times \begin{cases} \left(\left(1+\frac{S_4}{\gamma_2}\right)\langle K_1^{\flat -}|\not{K}_5^\flat|K_2^{\flat -}\rangle z(z - P_1^\bullet)\right)^{-1} & \text{for } i = 2, 6 \\ \left(\left(1+\frac{S_4}{\gamma_2}\right)\langle K_2^{\flat -}|\not{K}_5^\flat|K_1^{\flat -}\rangle z(z - P_1)\right)^{-1} & \text{for } i = 4, 5 \\ (\langle K_4^{\flat -}|\not{K}_1^\flat|K_5^{\flat -}\rangle z(z - Q_1^\bullet))^{-1} & \text{for } i = 1 \\ (\langle K_5^{\flat -}|\not{K}_1^\flat|K_4^{\flat -}\rangle z(z - Q_1))^{-1} & \text{for } i = 3. \end{cases} \tag{A8}$$

a. Heptacut #2 of the right-hand side of Eq. (4.1)

The result of applying heptacut #2 to the right-hand side of Eq. (4.1) is

$$\frac{1}{4}\sum_{i=1}^6 \oint_{\Gamma_i} dz J_i(z) K_i(z), \tag{A9}$$

where the kernels evaluated on the six kinematical solutions are

$$\begin{aligned} K_1(z) = c_{3,\sigma_1} &+ \frac{1}{4}\frac{c_{23,\sigma_1}}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{56}|K_5^{-}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} - \frac{1}{2}\frac{c_{20,\sigma_1}}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)} \\ &+ \frac{1}{2}\frac{c_{24,\sigma_1}\langle K_4^{\flat -}|\not{k}_{12}|K_5^{\flat -}\rangle(z - Q_1^\bullet) + c_{24,\sigma_7}\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle(P_1^\bullet - P_3^\bullet) + c_{24,\sigma_{10}}\langle K_4^{\flat -}|\not{k}_1|K_5^{\flat -}\rangle(z - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{56}|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} \\ &+ \frac{c_{13,\sigma_1}\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{12}|K_5^{\flat -}\rangle(P_1^\bullet - P_3^\bullet)(z - Q_1^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{56}|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} - \frac{1}{2}\frac{c_{20,\sigma_4}}{\langle K_4^{\flat -}|\not{k}_{56}|K_5^{\flat -}\rangle(z - Q_2^\bullet)} \\ &+ \frac{c_{12,\sigma_1}\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_1|K_5^{\flat -}\rangle(P_1^\bullet - P_3^\bullet)(z - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{56}|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} - \frac{c_{9,\sigma_4}\langle K_4^{\flat -}|\not{k}_{12}|K_5^{\flat -}\rangle(z - Q_1^\bullet)}{\langle K_4^{\flat -}|\not{k}_{56}|K_5^{\flat -}\rangle(z - Q_2^\bullet)} \end{aligned} \tag{A10}$$

$$\begin{aligned} K_2(z) = c_{3,\sigma_1} &+ \frac{1}{4}\frac{c_{23,\sigma_1}}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{56}|K_5^{\flat -}\rangle(z - P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)} - \frac{1}{2}\frac{c_{20,\sigma_1}}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle(z - P_2^\bullet)} \\ &+ \frac{1}{2}\frac{c_{24,\sigma_4}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(z - P_1^\bullet) + c_{24,\sigma_7}\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle(z - P_3^\bullet) + c_{24,\sigma_{10}}\langle K_4^{\flat -}|\not{k}_1|K_5^{\flat -}\rangle(Q_1^\bullet - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{56}|K_5^{\flat -}\rangle(z - P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)} \\ &+ \frac{c_{13,\sigma_4}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_1|K_5^{\flat -}\rangle(z - P_1^\bullet)(Q_1^\bullet - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{56}|K_5^{\flat -}\rangle(z - P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)} - \frac{1}{2}\frac{c_{20,\sigma_4}}{\langle K_4^{\flat -}|\not{k}_{56}|K_5^{\flat -}\rangle(Q_1^\bullet - Q_2^\bullet)} \\ &+ \frac{c_{12,\sigma_1}\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_1|K_5^{\flat -}\rangle(z - P_3^\bullet)(Q_1^\bullet - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{56}|K_5^{\flat -}\rangle(z - P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)} - \frac{c_{9,\sigma_1}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(z - P_1^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle(z - P_2^\bullet)} \end{aligned} \tag{A11}$$

$$K_3(z) = \text{parity conjugate of } K_1(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}] \tag{A12}$$

$$K_4(z) = \text{parity conjugate of } K_2(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}] \tag{A13}$$

$$\begin{aligned}
K_5(z) = {} & c_{3,\sigma_1} + \frac{1}{4}\frac{c_{23,\sigma_1}}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} \\
& + \frac{1}{2}\frac{c_{24,\sigma_1}\langle K_4^{\flat-}|\not{k}_{12}|K_5^{\flat-}\rangle(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} \\
& + \frac{1}{2}\frac{c_{24,\sigma_4}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle(z-P_1) + c_{24,\sigma_7}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle(z-P_3) + c_{24,\sigma_{10}}\langle K_4^{\flat-}|\not{k}_1|K_5^{\flat-}\rangle(\beta_3(z)-Q_3^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} \\
& + \frac{c_{13,\sigma_1}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{12}|K_5^{\flat-}\rangle(z-P_3)(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} - \frac{1}{2}\frac{c_{20,\sigma_1}}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle(z-P_2)} \\
& + \frac{c_{13,\sigma_4}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_1|K_5^{\flat-}\rangle(z-P_1)(\beta_3(z)-Q_3^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} - \frac{1}{2}\frac{c_{20,\sigma_4}}{\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle(\beta_3(z)-Q_2^\bullet)} \\
& + \frac{c_{12,\sigma_1}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_1|K_5^{\flat-}\rangle(z-P_3)(\beta_3(z)-Q_3^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} - \frac{c_{9,\sigma_1}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle(z-P_1)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle(z-P_2)} \\
& + \frac{c_{12,\sigma_7}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{12}|K_5^{\flat-}\rangle(z-P_1)(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} - \frac{c_{9,\sigma_4}\langle K_4^{\flat-}|\not{k}_{12}|K_5^{\flat-}\rangle(\beta_3(z)-Q_1^\bullet)}{\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle(\beta_3(z)-Q_2^\bullet)}
\end{aligned} \tag{A14}$$

$$K_6(z) = \text{parity conjugate of } K_5(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}], \tag{A15}$$

where $\beta_3(z)$ is given in Eq. (A7).

b. Heptacut #2 of the left-hand side of Eq. (4.1)

The result of applying heptacut #2 to the left-hand side of Eq. (4.1) is

$$i\sum_{i=1}^{6}\oint_{\Gamma_i} dz J_i(z)\prod_{j=1}^{6}A_j^{\text{tree}}(z)\bigg|_{\mathcal{S}_i}, \tag{A16}$$

where, assuming without loss of generality the external helicities are $(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$, the cut amplitude evaluated on the six different kinematical solutions yields

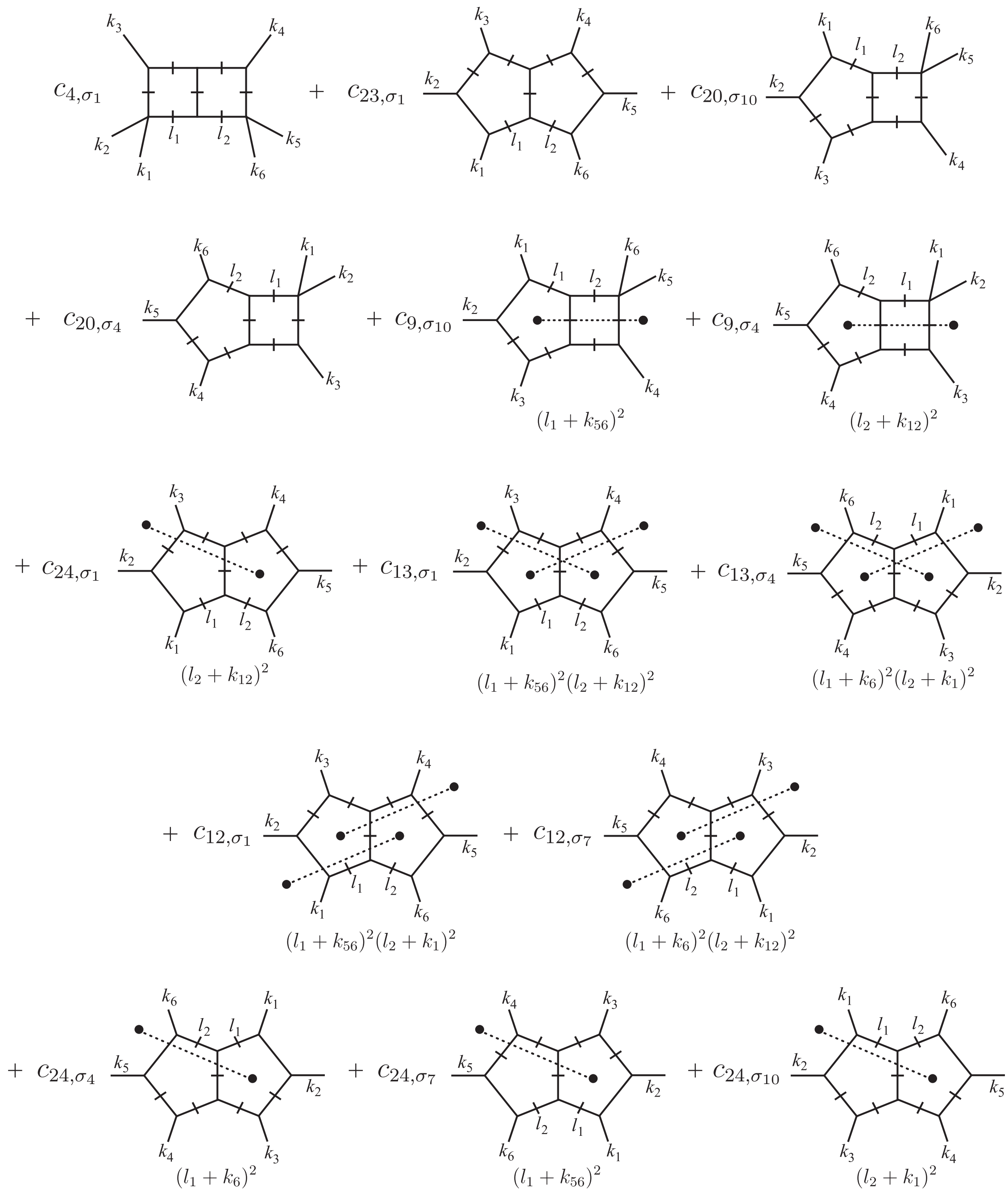
$$\prod_{j=1}^{6}A_j^{\text{tree}}(z)\bigg|_{\mathcal{S}_i} = \frac{i}{16}A^{\text{tree}}_{--++++}\times\begin{cases}\frac{1}{J_3(z)}\left(\frac{1}{z-Q_1}-\frac{1}{z-Q_2}\right) & \text{for } i=3\\ \frac{1}{J_4(z)}\left(\frac{1}{z-P_1}-\frac{1}{z-P_2}\right) & \text{for } i=4\\ 0 & \text{for } i=1,2,5,6.\end{cases} \tag{A17}$$

3. Heptacut #3

This heptacut is defined by the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) with the vertex momenta

$$K_1 = k_{12} \qquad K_2 = k_3 \qquad K_3 = 0 \qquad K_4 = k_4 \qquad K_5 = k_{56} \qquad K_6 = 0. \tag{A18}$$

Applying this heptacut to the right-hand side of Eq. (4.1) leaves the following linear combination of cut integrals



We define the spinor ratios

$$P_1 = -\frac{\langle K_1^\flat K_5^\flat\rangle}{2\langle K_2^\flat K_5^\flat\rangle}, \qquad P_2 = -\frac{\langle K_1^\flat k_1\rangle}{2\langle K_2^\flat k_1\rangle}, \qquad P_3 = -\frac{\langle K_1^\flat K_4^\flat\rangle}{2\langle K_2^\flat K_4^\flat\rangle}, \qquad P_4 = -\frac{\langle K_1^\flat k_6\rangle}{2\langle K_2^\flat k_6\rangle}$$
$$Q_1 = -\frac{[K_1^\flat K_5^\flat]}{2[K_1^\flat K_4^\flat]}, \qquad Q_2 = -\frac{[K_5^\flat k_6]}{2[K_4^\flat k_6]}, \qquad Q_3 = -\frac{[K_5^\flat k_1]}{2[K_4^\flat k_1]}, \tag{A19}$$

and their parity conjugates

$$P_1^\bullet = -\frac{[K_1^\flat K_5^\flat]}{2[K_2^\flat K_5^\flat]}, \qquad P_2^\bullet = -\frac{[K_1^\flat k_1]}{2[K_2^\flat k_1]}, \qquad P_3^\bullet = -\frac{[K_1^\flat K_4^\flat]}{2[K_2^\flat K_4^\flat]} \qquad P_4^\bullet = -\frac{[K_1^\flat k_6]}{2[K_2^\flat k_6]}$$
$$Q_1^\bullet = -\frac{\langle K_1^\flat K_5^\flat\rangle}{2\langle K_1^\flat K_4^\flat\rangle}, \qquad Q_2^\bullet = -\frac{\langle K_5^\flat k_6\rangle}{2\langle K_4^\flat k_6\rangle}, \qquad Q_3^\bullet = -\frac{\langle K_5^\flat k_1\rangle}{2\langle K_4^\flat k_1\rangle}. \tag{A20}$$

This heptacut belongs to case I treated in Sec. III A, and there are thus six kinematical solutions (shown in Fig. 2). Parametrizing the loop momenta according to Eqs. (3.12) and (3.13), the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) are solved by setting the parameters equal to the values

$$\alpha_1 = 1, \qquad \alpha_2 = 0, \qquad \beta_1 = 0, \qquad \beta_2 = 1, \tag{A21}$$

and those given in Fig. 2 with

$$\beta_3(z) = -\frac{\langle K_2^\flat K_5^\flat\rangle(z - P_1)}{2\langle K_2^\flat K_4^\flat\rangle(z - P_3)}, \tag{A22}$$

for kinematical solution $\mathcal{S}_5$. The heptacut double-box integral I_{4,σ_1} is $\sum_{i=1}^6 \oint_{\Gamma_i} dz J_i(z)$ where

$$J_i(z) = \frac{1}{32\gamma_1\gamma_2} \times \begin{cases} (\langle K_1^{\flat-}|\not{K}_5^\flat|K_2^{\flat-}\rangle z(z-P_1^\bullet))^{-1} & \text{for } i = 2, 6 \\ (\langle K_2^{\flat-}|\not{K}_5^\flat|K_1^{\flat-}\rangle z(z-P_1))^{-1} & \text{for } i = 4, 5 \\ (\langle K_4^{\flat-}|\not{K}_1^\flat|K_5^{\flat-}\rangle z(z-Q_1^\bullet))^{-1} & \text{for } i = 1 \\ (\langle K_5^{\flat-}|\not{K}_1^\flat|K_4^{\flat-}\rangle z(z-Q_1))^{-1} & \text{for } i = 3. \end{cases} \tag{A23}$$

a. Heptacut #3 of the right-hand side of Eq. (4.1)

The result of applying heptacut #3 to the right-hand side of Eq. (4.1) is

$$\frac{1}{4}\sum_{i=1}^{6}\oint_{\Gamma_i} dz J_i(z) K_i(z), \tag{A24}$$

where the kernels evaluated on the six kinematical solutions are

$$\begin{aligned} K_1(z) = c_{4,\sigma_1} &+ \frac{1}{4}\frac{c_{23,\sigma_1}}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(P_1^\bullet - P_2^\bullet)(z-Q_2^\bullet)} - \frac{c_{9,\sigma_4}\langle K_4^{\flat-}|\not{k}_{12}|K_5^{\flat-}\rangle(z-Q_1^\bullet)}{\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-Q_2^\bullet)} \\ &+ \frac{1}{2}\frac{c_{24,\sigma_1}\langle K_4^{\flat-}|\not{k}_{12}|K_5^{\flat-}\rangle(z-Q_1^\bullet) + c_{24,\sigma_4}\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle(P_1^\bullet - P_4^\bullet) + c_{24,\sigma_{10}}\langle K_4^{\flat-}|\not{k}_1|K_5^{\flat-}\rangle(z-Q_3^\bullet)}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(P_1^\bullet - P_2^\bullet)(z-Q_2^\bullet)} \\ &+ \frac{c_{13,\sigma_4}\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_1|K_5^{\flat-}\rangle(P_1^\bullet - P_4^\bullet)(z-Q_3^\bullet)}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(P_1^\bullet - P_2^\bullet)(z-Q_2^\bullet)} - \frac{1}{2}\frac{c_{20,\sigma_{10}}}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle(P_1^\bullet - P_2^\bullet)} \\ &+ \frac{c_{12,\sigma_7}\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{12}|K_5^{\flat-}\rangle(P_1^\bullet - P_4^\bullet)(z-Q_1^\bullet)}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(P_1^\bullet - P_2^\bullet)(z-Q_2^\bullet)} - \frac{1}{2}\frac{c_{20,\sigma_4}}{\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-Q_2^\bullet)} \end{aligned} \tag{A25}$$

$$\begin{aligned} K_2(z) = c_{4,\sigma_1} &+ \frac{1}{4}\frac{c_{23,\sigma_1}}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)} - \frac{c_{9,\sigma_{10}}\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle(z-P_1^\bullet)}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle(z-P_2^\bullet)} \\ &+ \frac{1}{2}\frac{c_{24,\sigma_4}\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle(z-P_4^\bullet) + c_{24,\sigma_7}\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle(z-P_1^\bullet) + c_{24,\sigma_{10}}\langle K_4^{\flat-}|\not{k}_1|K_5^{\flat-}\rangle(Q_1^\bullet - Q_3^\bullet)}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)} \\ &+ \frac{c_{13,\sigma_4}\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_1|K_5^{\flat-}\rangle(z-P_4^\bullet)(Q_1^\bullet - Q_3^\bullet)}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)} - \frac{1}{2}\frac{c_{20,\sigma_{10}}}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle(z-P_2^\bullet)} \\ &+ \frac{c_{12,\sigma_1}\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_1|K_5^{\flat-}\rangle(z-P_1^\bullet)(Q_1^\bullet - Q_3^\bullet)}{\langle K_1^{\flat-}|\not{k}_1|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)} - \frac{1}{2}\frac{c_{20,\sigma_4}}{\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(Q_1^\bullet - Q_2^\bullet)} \end{aligned} \tag{A26}$$

$$K_3(z) = \text{parity conjugate of } K_1(z) \qquad [\text{obtained by applying Eqs.}(4.12)\text{–}(4.17)] \tag{A27}$$

$$K_4(z) = \text{parity conjugate of } K_2(z) \qquad [\text{obtained by applying Eqs.}(4.12)\text{–}(4.17)] \tag{A28}$$

$$\begin{aligned} K_5(z) = {}& c_{4,\sigma_1} + \frac{1}{4}\frac{c_{23,\sigma_1}}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} \\ & + \frac{1}{2}\frac{c_{24,\sigma_1}\langle K_4^{\flat-}|\not{k}_{12}|K_5^{\flat-}\rangle(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} \\ & + \frac{1}{2}\frac{c_{24,\sigma_4}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle(z-P_4) + c_{24,\sigma_7}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle(z-P_1) + c_{24,\sigma_{10}}\langle K_4^{\flat-}|\not{k}_1|K_5^{\flat-}\rangle(\beta_3(z)-Q_3^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} \\ & + \frac{c_{13,\sigma_1}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{12}|K_5^{\flat-}\rangle(z-P_1)(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} - \frac{1}{2}\frac{c_{20,\sigma_{10}}}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle(z-P_2)} \\ & + \frac{c_{13,\sigma_4}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_1|K_5^{\flat-}\rangle(z-P_4)(\beta_3(z)-Q_3^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} - \frac{1}{2}\frac{c_{20,\sigma_4}}{\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\beta_3(z)-Q_2^\bullet)} \\ & + \frac{c_{12,\sigma_1}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_1|K_5^{\flat-}\rangle(z-P_1)(\beta_3(z)-Q_3^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} - \frac{c_{9,\sigma_{10}}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle(z-P_1)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle(z-P_2)} \\ & + \frac{c_{12,\sigma_7}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{12}|K_5^{\flat-}\rangle(z-P_4)(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat-}|\not{k}_1|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} - \frac{c_{9,\sigma_4}\langle K_4^{\flat-}|\not{k}_{12}|K_5^{\flat-}\rangle(\beta_3(z)-Q_1^\bullet)}{\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\beta_3(z)-Q_2^\bullet)} \end{aligned} \tag{A29}$$

$$K_6(z) = \text{parity conjugate of } K_5(z) \qquad [\text{obtained by applying Eqs.}(4.12)\text{–}(4.17)], \tag{A30}$$

where $\beta_3(z)$ is given in Eq. (A22).

b. Heptacut #3 of the left-hand side of Eq. (4.1)

The result of applying heptacut #3 to the left-hand side of Eq. (4.1) is

$$i\sum_{i=1}^{6}\oint_{\Gamma_i} dz J_i(z)\prod_{j=1}^{6} A_j^{\text{tree}}(z)\Big|_{\mathcal{S}_i}, \tag{A31}$$

where, assuming without loss of generality the external helicities are $(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$, the cut amplitude evaluated on the six different kinematical solutions yields

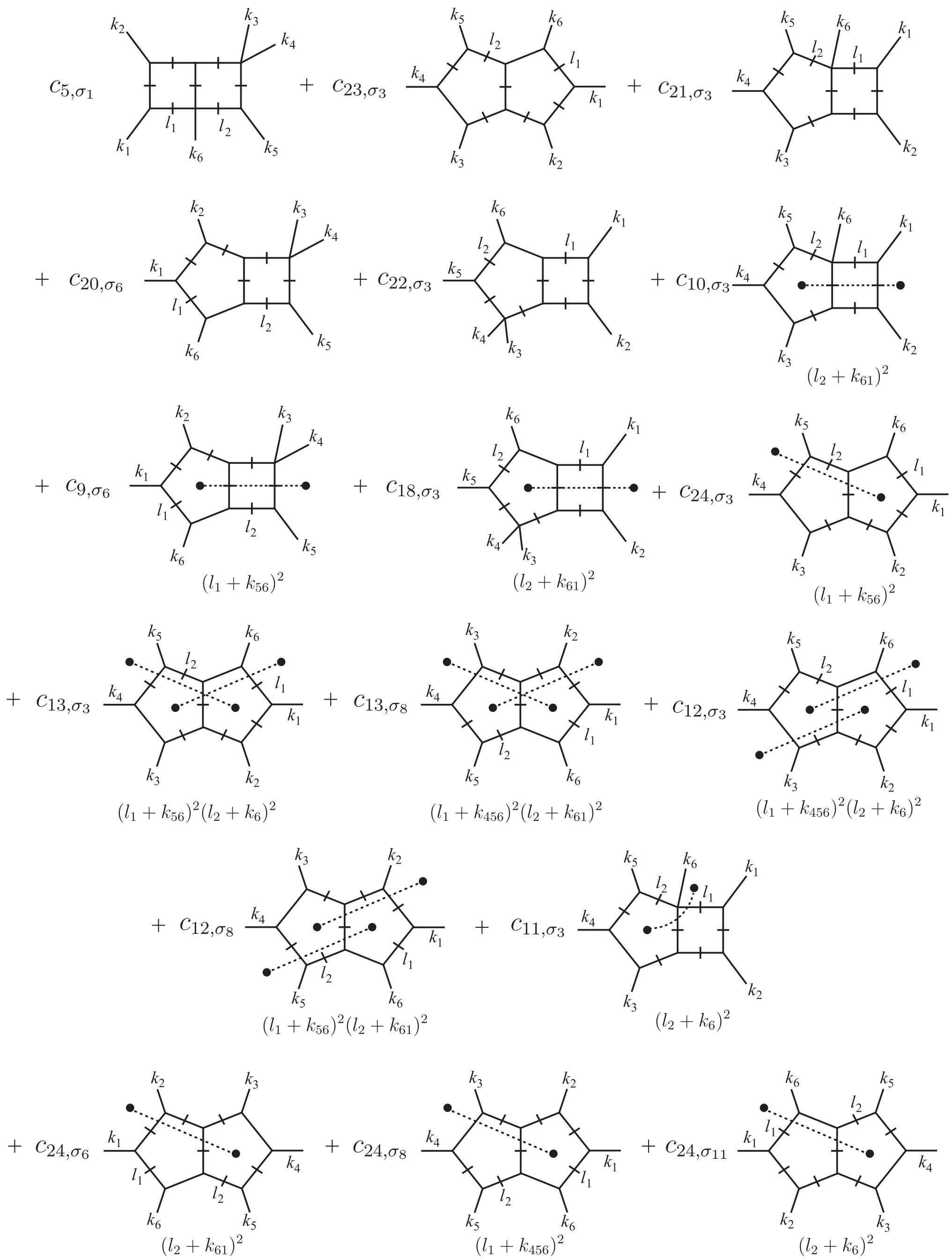
$$\prod_{j=1}^{6} A_j^{\text{tree}}(z)\Big|_{\mathcal{S}_i} = -\frac{i}{16}A^{\text{tree}}_{--++++} \times \begin{cases} \frac{1}{J_5(z)}\left(\frac{1}{z-P_2} - \frac{1}{z-P_4}\right) & \text{for } i=5 \\ 0 & \text{for } i = 1,2,3,4,6. \end{cases} \tag{A32}$$

4. Heptacut #4

This heptacut is defined by the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) with the vertex momenta

$$K_1 = k_1 \qquad K_2 = k_2 \qquad K_3 = 0 \qquad K_4 = k_{34} \qquad K_5 = k_5 \qquad K_6 = k_6. \tag{A33}$$

Applying this heptacut to the right-hand side of Eq. (4.1) leaves the following linear combination of cut integrals



We define the spinor ratios

$$P_1 = -\frac{(1+\frac{S_4}{\gamma_2})K_5^\flat\cdot(K_1^\flat+k_6)+K_1^\flat\cdot k_6}{\langle K_2^{\flat-}|(1+\frac{S_4}{\gamma_2})\not{K}_5^\flat+\not{k}_6|K_1^{\flat-}\rangle}, \qquad P_2 = -\frac{K_1^\flat\cdot k_6}{\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle}, \qquad P_3 = -\frac{K_1^\flat\cdot k_{56}+\frac{1}{2}s_{56}}{\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle}$$

$$P_4 = -\frac{\langle K_4^{\flat-}|\not{K}_1^\flat+\not{k}_6|K_5^{\flat-}\rangle}{2\langle K_2^\flat K_4^\flat\rangle[K_5^\flat K_1^\flat]}, \qquad P_5 = -\frac{K_1^\flat\cdot k_{456}+\frac{1}{2}s_{456}}{\langle K_2^{\flat-}|\not{k}_{456}|K_1^{\flat-}\rangle}$$

$$Q_1 = -\frac{(1+\frac{S_4}{\gamma_2})K_5^\flat\cdot(K_1^\flat+k_6)+K_1^\flat\cdot k_6}{\langle K_5^{\flat-}|\not{K}_1^\flat+\not{k}_6|K_4^{\flat-}\rangle}, \qquad Q_2 = -\frac{(1+\frac{S_4}{\gamma_2})K_5^\flat\cdot k_{45}-\frac{1}{2}s_{45}}{\langle K_5^{\flat-}|\not{k}_{45}|K_4^{\flat-}\rangle},$$

$$Q_3 = -\frac{(1+\frac{S_4}{\gamma_2})K_5^\flat\cdot k_6}{\langle K_5^{\flat-}|\not{k}_6|K_4^{\flat-}\rangle} \qquad Q_4 = -\frac{(1+\frac{S_4}{\gamma_2})K_5^\flat\cdot k_1}{\langle K_5^{\flat-}|\not{k}_1|K_4^{\flat-}\rangle}, \qquad \text{(A34)}$$

and their parity conjugates

$$P_1^\bullet = -\frac{(1+\frac{S_4}{\gamma_2})K_5^\flat\cdot(K_1^\flat+k_6)+K_1^\flat\cdot k_6}{\langle K_1^{\flat-}|(1+\frac{S_4}{\gamma_2})\not{K}_5^\flat+\not{k}_6|K_2^{\flat-}\rangle},\qquad P_2^\bullet = -\frac{K_1^\flat\cdot k_6}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle},\qquad P_3^\bullet = -\frac{K_1^\flat\cdot k_{56}+\frac{1}{2}s_{56}}{\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle}$$
$$P_4^\bullet = -\frac{\langle K_5^{\flat-}|\not{K}_1^\flat+\not{k}_6|K_4^{\flat-}\rangle}{2[K_2^\flat K_4^\flat]\langle K_5^\flat K_1^\flat\rangle},\qquad P_5^\bullet = -\frac{K_1^\flat\cdot k_{456}+\frac{1}{2}s_{456}}{\langle K_1^{\flat-}|\not{k}_{456}|K_2^{\flat-}\rangle}$$
$$Q_1^\bullet = -\frac{(1+\frac{S_4}{\gamma_2})K_5^\flat\cdot(K_1^\flat+k_6)+K_1^\flat\cdot k_6}{\langle K_4^{\flat-}|\not{K}_1^\flat+\not{k}_6|K_5^{\flat-}\rangle},\qquad Q_2^\bullet = -\frac{(1+\frac{S_4}{\gamma_2})K_5^\flat\cdot k_{45}-\frac{1}{2}s_{45}}{\langle K_4^{\flat-}|\not{k}_{45}|K_5^{\flat-}\rangle},$$
$$Q_3^\bullet = -\frac{(1+\frac{S_4}{\gamma_2})K_5^\flat\cdot k_6}{\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle}\qquad Q_4^\bullet = -\frac{(1+\frac{S_4}{\gamma_2})K_5^\flat\cdot k_1}{\langle K_4^{\flat-}|\not{k}_1|K_5^{\flat-}\rangle}. \tag{A35}$$

This heptacut belongs to case I treated in Sec. III A, and there are thus six kinematical solutions (shown in Fig. 2). Parametrizing the loop momenta according to Eqs. (3.12) and (3.13), the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) are solved by setting the parameters equal to the values

$$\alpha_1 = 1,\qquad \alpha_2 = 0,\qquad \beta_1 = 0,\qquad \beta_2 = 1+\frac{S_4}{\gamma_2}, \tag{A36}$$

and those given in Fig. 2 with

$$\beta_3(z) = \frac{Q_3^\bullet(P_2-P_4)(z-P_1)}{(P_2-P_1)(z-P_4)}, \tag{A37}$$

for kinematical solution $\mathcal{S}_5$. The heptacut double-box integral I_{5,σ_1} is $\sum_{i=1}^6 \oint_{\Gamma_i} dz J_i(z)$ where

$$J_i(z) = \frac{1}{32\gamma_1\gamma_2}\times\begin{cases}\left(\left\langle K_1^{\flat-}\middle|\left(1+\frac{S_4}{\gamma_2}\right)\not{K}_5^\flat+\not{k}_6\middle|K_2^{\flat-}\right\rangle z(z-P_1^\bullet)\right)^{-1} & \text{for } i=2,6\\ \left(\left\langle K_2^{\flat-}\middle|\left(1+\frac{S_4}{\gamma_2}\right)\not{K}_5^\flat+\not{k}_6\middle|K_1^{\flat-}\right\rangle z(z-P_1)\right)^{-1} & \text{for } i=4,5\\ (\langle K_4^{\flat-}|\not{K}_1^\flat+\not{k}_6|K_5^{\flat-}\rangle z(z-Q_1^\bullet))^{-1} & \text{for } i=1\\ (\langle K_5^{\flat-}|\not{K}_1^\flat+\not{k}_6|K_4^{\flat-}\rangle z(z-Q_1))^{-1} & \text{for } i=3.\end{cases} \tag{A38}$$

a. Heptacut #4 of the right-hand side of Eq. (4.1)

The result of applying heptacut #4 to the right-hand side of Eq. (4.1) is

$$\frac{1}{4}\sum_{i=1}^6 \oint_{\Gamma_i} dz J_i(z) K_i(z), \tag{A39}$$

where the kernels evaluated on the six kinematical solutions are

$$K_1(z) = c_{5,\sigma_1} - \frac{1}{4}\frac{c_{23,\sigma_3}}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} - \frac{1}{2}\frac{c_{21,\sigma_3}}{\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z - Q_2^\bullet)}$$
$$+ \frac{1}{2}\frac{c_{20,\sigma_6}}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)} + \frac{1}{2}\frac{c_{22,\sigma_3}}{\langle K_4^{\flat -}|\not{K}_6|K_5^{\flat -}\rangle(z - Q_3^\bullet)}$$
$$- \frac{1}{2}\frac{c_{24,\sigma_3}\langle K_1^{\flat -}|\not{K}_{56}|K_2^{\flat -}\rangle(P_1^\bullet - P_3^\bullet)}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)}$$
$$- \frac{1}{2}\frac{c_{24,\sigma_6}\langle K_4^{\flat -}|\not{K}_{61}|K_5^{\flat -}\rangle(z - Q_1^\bullet) + c_{24,\sigma_8}\langle K_1^{\flat -}|\not{K}_{456}|K_2^{\flat -}\rangle(P_1^\bullet - P_5^\bullet) + c_{24,\sigma_{11}}\langle K_4^{\flat -}|\not{K}_6|K_5^{\flat -}\rangle(z - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)}$$
$$- \frac{c_{13,\sigma_3}\langle K_1^{\flat -}|\not{K}_{56}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_6|K_5^{\flat -}\rangle(P_1^\bullet - P_3^\bullet)(z - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} - \frac{c_{10,\sigma_3}\langle K_4^{\flat -}|\not{K}_{61}|K_5^{\flat -}\rangle(z - Q_1^\bullet)}{\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z - Q_2^\bullet)}$$
$$- \frac{c_{13,\sigma_8}\langle K_1^{\flat -}|\not{K}_{456}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{61}|K_5^{\flat -}\rangle(P_1^\bullet - P_5^\bullet)(z - Q_1^\bullet)}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} + \frac{c_{9,\sigma_6}\langle K_1^{\flat -}|\not{K}_{56}|K_2^{\flat -}\rangle(P_1^\bullet - P_3^\bullet)}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)}$$
$$- \frac{c_{12,\sigma_3}\langle K_1^{\flat -}|\not{K}_{456}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_6|K_5^{\flat -}\rangle(P_1^\bullet - P_5^\bullet)(z - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} + \frac{c_{18,\sigma_3}\langle K_4^{\flat -}|\not{K}_{61}|K_5^{\flat -}\rangle(z - Q_1^\bullet)}{\langle K_4^{\flat -}|\not{K}_6|K_5^{\flat -}\rangle(z - Q_3^\bullet)}$$
$$- \frac{c_{12,\sigma_8}\langle K_1^{\flat -}|\not{K}_{56}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{61}|K_5^{\flat -}\rangle(P_1^\bullet - P_3^\bullet)(z - Q_1^\bullet)}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} - \frac{c_{11,\sigma_3}\langle K_4^{\flat -}|\not{K}_6|K_5^{\flat -}\rangle(z - Q_3^\bullet)}{\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z - Q_2^\bullet)} \tag{A40}$$

$$K_2(z) = c_{5,\sigma_1} - \frac{1}{4}\frac{c_{23,\sigma_3}}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z - P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)}$$
$$- \frac{1}{2}\frac{c_{21,\sigma_3}}{\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(Q_1^\bullet - Q_2^\bullet)} + \frac{1}{2}\frac{c_{20,\sigma_6}}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle(z - P_2^\bullet)}$$
$$+ \frac{1}{2}\frac{c_{22,\sigma_3}}{\langle K_4^{\flat -}|\not{K}_6|K_5^{\flat -}\rangle(Q_1^\bullet - Q_3^\bullet)} + \frac{c_{9,\sigma_6}\langle K_1^{\flat -}|\not{K}_{56}|K_2^{\flat -}\rangle(z - P_3^\bullet)}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle(z - P_2^\bullet)}$$
$$- \frac{1}{2}\frac{c_{24,\sigma_3}\langle K_1^{\flat -}|\not{K}_{56}|K_2^{\flat -}\rangle(z - P_3^\bullet) + c_{24,\sigma_8}\langle K_1^{\flat -}|\not{K}_{456}|K_2^{\flat -}\rangle(z - P_5^\bullet) + c_{24,\sigma_{11}}\langle K_4^{\flat -}|\not{K}_6|K_5^{\flat -}\rangle(Q_1^\bullet - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z - P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)}$$
$$- \frac{c_{13,\sigma_3}\langle K_1^{\flat -}|\not{K}_{56}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_6|K_5^{\flat -}\rangle(z - P_3^\bullet)(Q_1^\bullet - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z - P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)}$$
$$- \frac{c_{11,\sigma_3}\langle K_4^{\flat -}|\not{K}_6|K_5^{\flat -}\rangle(Q_1^\bullet - Q_3^\bullet)}{\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(Q_1^\bullet - Q_2^\bullet)}$$
$$- \frac{c_{12,\sigma_3}\langle K_1^{\flat -}|\not{K}_{456}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_6|K_5^{\flat -}\rangle(z - P_5^\bullet)(Q_1^\bullet - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z - P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)} \tag{A41}$$

$$K_3(z) = \text{parity conjugate of } K_1(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}] \tag{A42}$$

$$K_4(z) = \text{parity conjugate of } K_2(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}] \tag{A43}$$

$$
\begin{aligned}
K_5(z) = {} & c_{5,\sigma_1} - \frac{1}{2}\frac{c_{21,\sigma_3}}{\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(\beta_3(z)-Q_2^\bullet)} + \frac{1}{2}\frac{c_{20,\sigma_6}}{\langle K_2^{\flat -}|\not{K}_{6}|K_1^{\flat -}\rangle(z-P_2)} + \frac{1}{2}\frac{c_{22,\sigma_3}}{\langle K_4^{\flat -}|\not{K}_{6}|K_5^{\flat -}\rangle(\beta_3(z)-Q_3^\bullet)} \\
& - \frac{1}{4}\frac{c_{23,\sigma_3}}{\langle K_2^{\flat -}|\not{K}_{6}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} - \frac{c_{10,\sigma_3}\langle K_4^{\flat -}|\not{K}_{61}|K_5^{\flat -}\rangle(\beta_3(z)-Q_1^\bullet)}{\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(\beta_3(z)-Q_2^\bullet)} \\
& + \frac{c_{9,\sigma_6}\langle K_2^{\flat -}|\not{K}_{56}|K_1^{\flat -}\rangle(z-P_3)}{\langle K_2^{\flat -}|\not{K}_{6}|K_1^{\flat -}\rangle(z-P_2)} + \frac{c_{18,\sigma_3}\langle K_4^{\flat -}|\not{K}_{61}|K_5^{\flat -}\rangle(\beta_3(z)-Q_1^\bullet)}{\langle K_4^{\flat -}|\not{K}_{6}|K_5^{\flat -}\rangle(\beta_3(z)-Q_3^\bullet)} \\
& - \frac{1}{2}\frac{c_{24,\sigma_3}\langle K_2^{\flat -}|\not{K}_{56}|K_1^{\flat -}\rangle(z-P_3)}{\langle K_2^{\flat -}|\not{K}_{6}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} - \frac{1}{2}\frac{c_{24,\sigma_6}\langle K_4^{\flat -}|\not{K}_{61}|K_5^{\flat -}\rangle(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat -}|\not{K}_{6}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} \\
& - \frac{c_{11,\sigma_3}\langle K_4^{\flat -}|\not{K}_{6}|K_5^{\flat -}\rangle(\beta_3(z)-Q_3^\bullet)}{\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(\beta_3(z)-Q_2^\bullet)} - \frac{1}{2}\frac{c_{24,\sigma_8}\langle K_2^{\flat -}|\not{K}_{456}|K_1^{\flat -}\rangle(z-P_5)}{\langle K_2^{\flat -}|\not{K}_{6}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} \\
& - \frac{1}{2}\frac{c_{24,\sigma_{11}}\langle K_4^{\flat -}|\not{K}_{6}|K_5^{\flat -}\rangle(\beta_3(z)-Q_3^\bullet)}{\langle K_2^{\flat -}|\not{K}_{6}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} - \frac{c_{13,\sigma_3}\langle K_2^{\flat -}|\not{K}_{56}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{6}|K_5^{\flat -}\rangle(z-P_3)(\beta_3(z)-Q_3^\bullet)}{\langle K_2^{\flat -}|\not{K}_{6}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} \\
& - \frac{c_{13,\sigma_8}\langle K_2^{\flat -}|\not{K}_{456}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{61}|K_5^{\flat -}\rangle(z-P_5)(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat -}|\not{K}_{6}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} \\
& - \frac{c_{12,\sigma_3}\langle K_2^{\flat -}|\not{K}_{456}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{6}|K_5^{\flat -}\rangle(z-P_5)(\beta_3(z)-Q_3^\bullet)}{\langle K_2^{\flat -}|\not{K}_{6}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)} \\
& - \frac{c_{12,\sigma_8}\langle K_2^{\flat -}|\not{K}_{56}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{61}|K_5^{\flat -}\rangle(z-P_3)(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat -}|\not{K}_{6}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{45}|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^\bullet)}
\end{aligned}
\tag{A44}
$$

$$
K_6(z) = \text{parity conjugate of } K_5(z) \qquad [\text{obtained by applying Eqs.(4.12)–(4.17)}], \tag{A45}
$$

where $\beta_3(z)$ is given in Eq. (A37).

b. Heptacut #4 of the left-hand side of Eq. (4.1)

The result of applying heptacut #4 to the left-hand side of Eq. (4.1) is

$$
i\sum_{i=1}^{6}\oint_{\Gamma_i} dz J_i(z)\prod_{j=1}^{6}A_j^{\text{tree}}(z)\bigg|_{\mathcal{S}_i}, \tag{A46}
$$

where, assuming without loss of generality the external helicities are $(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$, the cut amplitude evaluated on the six different kinematical solutions yields

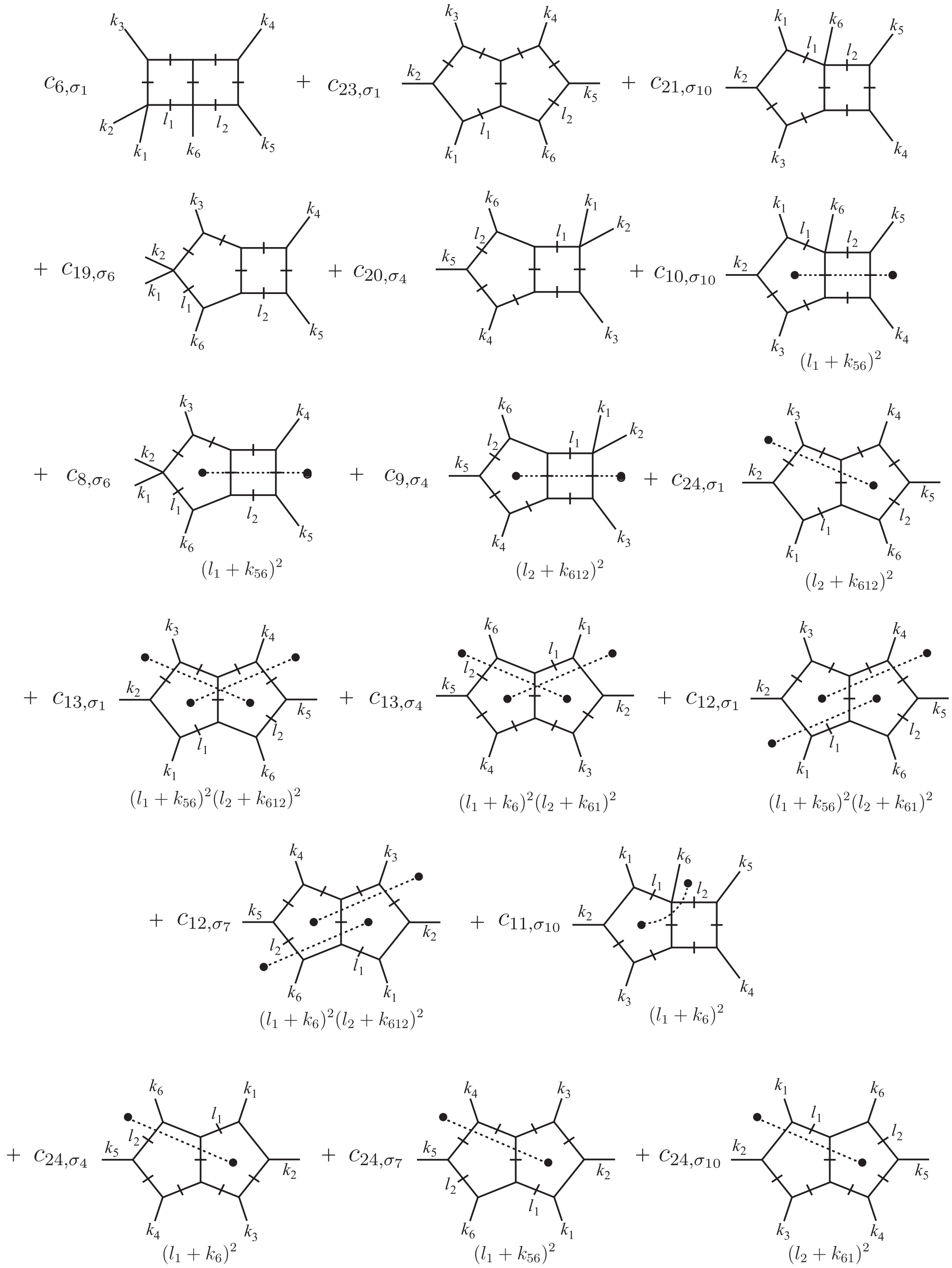
$$
\prod_{j=1}^{6}A_j^{\text{tree}}(z)\bigg|_{\mathcal{S}_i} = -\frac{i}{16}A^{\text{tree}}_{--++++}\times\begin{cases}\frac{1}{J_4(z)}\left(\frac{1}{z}-\frac{1}{z-P_2}\right) & \text{for } i=4\\ \frac{1}{J_6(z)}\left(\frac{1}{z}-\frac{1}{z-P_5^\bullet}\right) & \text{for } i=6\\ 0 & \text{for } i=1,2,3,5.\end{cases} \tag{A47}
$$

5. Heptacut #5

This heptacut is defined by the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) with the vertex momenta

$$
K_1 = k_{12} \qquad K_2 = k_3 \qquad K_3 = 0 \qquad K_4 = k_4 \qquad K_5 = k_5 \qquad K_6 = k_6. \tag{A48}
$$

Applying this heptacut to the right-hand side of Eq. (4.1) leaves the following linear combination of cut integrals



We define the spinor ratios

$$P_1 = -\frac{K_5^\flat \cdot k_6 + (K_5^\flat + k_6)\cdot K_1^\flat}{\langle K_2^{\flat-}|\not{K}_5^\flat + \not{k}_6|K_1^{\flat-}\rangle}, \qquad P_2 = -\frac{\langle K_1^\flat k_1\rangle}{2\langle K_2^\flat k_1\rangle},$$
$$P_3 = -\frac{\langle K_1^\flat k_6\rangle}{2\langle K_2^\flat k_6\rangle} \qquad P_4 = -\frac{\langle K_1^\flat k_5\rangle}{2\langle K_2^\flat k_5\rangle}, \qquad P_5 = -\frac{\langle K_4^{\flat-}|\not{K}_1^\flat + \not{k}_6|K_5^{\flat-}\rangle}{2\langle K_2^\flat K_4^\flat\rangle [K_5^\flat K_1^\flat]}$$
$$Q_1 = -\frac{K_1^\flat \cdot k_6 + K_5^\flat \cdot (K_1^\flat + k_6)}{\langle K_5^{\flat-}|\not{K}_1^\flat + \not{k}_6|K_4^{\flat-}\rangle}, \qquad Q_2 = -\frac{[K_5^\flat k_6]}{2[K_4^\flat k_6]}, \qquad Q_3 = -\frac{K_5^\flat \cdot k_{61} + \frac{1}{2}s_{61}}{\langle K_5^{\flat-}|\not{k}_{61}|K_4^{\flat-}\rangle}, \tag{A49}$$

and their parity conjugates

$$P_1^\bullet = -\frac{K_5^\flat \cdot k_6 + (K_5^\flat + k_6)\cdot K_1^\flat}{\langle K_1^{\flat -}|\not{K}_5^\flat + \not{k}_6|K_2^{\flat -}\rangle}, \qquad P_2^\bullet = -\frac{[K_1^\flat k_1]}{2[K_2^\flat k_1]},$$
$$P_3^\bullet = -\frac{[K_1^\flat k_6]}{2[K_2^\flat k_6]} \qquad P_4^\bullet = -\frac{[K_1^\flat k_5]}{2[K_2^\flat k_5]}, \qquad P_5^\bullet = -\frac{\langle K_5^{\flat -}|\not{K}_1^\flat + \not{k}_6|K_4^{\flat -}\rangle}{2[K_2^\flat K_4^\flat]\langle K_5^\flat K_1^\flat\rangle}$$
$$Q_1^\bullet = -\frac{K_1^\flat \cdot k_6 + K_5^\flat\cdot (K_1^\flat + k_6)}{\langle K_4^{\flat -}|\not{K}_1^\flat + \not{k}_6|K_5^{\flat -}\rangle}, \qquad Q_2^\bullet = -\frac{\langle K_5^\flat k_6\rangle}{2\langle K_4^\flat k_6\rangle}, \qquad Q_3^\bullet = -\frac{K_5^\flat\cdot k_{61} + \frac{1}{2}s_{61}}{\langle K_4^{\flat -}|\not{k}_{61}|K_5^{\flat -}\rangle}. \tag{A50}$$

This heptacut belongs to case I treated in Sec. III A, and there are thus six kinematical solutions (shown in Fig. 2). Parametrizing the loop momenta according to Eqs. (3.12) and (3.13), the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) are solved by setting the parameters equal to the values

$$\alpha_1 = 1, \qquad \alpha_2 = 0, \qquad \beta_1 = 0, \qquad \beta_2 = 1 \tag{A51}$$

and those given in Fig. 2 with

$$\beta_3(z) = \frac{Q_2^\bullet(P_3 - P_5)(z - P_1)}{(P_3 - P_1)(z - P_5)}, \tag{A52}$$

for kinematical solution $\mathcal{S}_5$. The heptacut double-box integral I_{6,σ_1} is $\sum_{i=1}^6 \oint_{\Gamma_i} dz J_i(z)$ where

$$J_i(z) = \frac{1}{32\gamma_1\gamma_2}\times \begin{cases} (\langle K_1^{\flat -}|\not{K}_5^\flat + \not{k}_6|K_2^{\flat -}\rangle z(z - P_1^\bullet))^{-1} & \text{for } i = 2, 6 \\ (\langle K_2^{\flat -}|\not{K}_5^\flat + \not{k}_6|K_1^{\flat -}\rangle z(z - P_1))^{-1} & \text{for } i = 4, 5 \\ (\langle K_4^{\flat -}|\not{K}_1^\flat + \not{k}_6|K_5^{\flat -}\rangle z(z - Q_1^\bullet))^{-1} & \text{for } i = 1 \\ (\langle K_5^{\flat -}|\not{K}_1^\flat + \not{k}_6|K_4^{\flat -}\rangle z(z - Q_1))^{-1} & \text{for } i = 3 \end{cases}. \tag{A53}$$

a. Heptacut #5 of the right-hand side of Eq. (4.1)

The result of applying heptacut #5 to the right-hand side of Eq. (4.1) is

$$\frac{1}{4}\sum_{i=1}^6 \oint_{\Gamma_i} dz J_i(z) K_i(z), \tag{A54}$$

where the kernels evaluated on the six kinematical solutions are

$$\begin{aligned}
K_1(z) = c_{6,\sigma_1} &- \frac{1}{2}\frac{c_{21,\sigma_{10}}}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)} + \frac{1}{2}\frac{c_{19,\sigma_6}}{\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(P_1^\bullet - P_3^\bullet)} + \frac{1}{2}\frac{c_{20,\sigma_4}}{\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z - Q_2^\bullet)} \\
&- \frac{1}{4}\frac{c_{23,\sigma_1}}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} \\
&- \frac{1}{2}\frac{c_{24,\sigma_1}\langle K_4^{\flat -}|\not{k}_{612}|K_5^{\flat -}\rangle(z - Q_1^\bullet) + c_{24,\sigma_4}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(P_1^\bullet - P_3^\bullet) + c_{24,\sigma_{10}}\langle K_4^{\flat -}|\not{k}_{61}|K_5^{\flat -}\rangle(z - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} \\
&- \frac{c_{13,\sigma_4}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{61}|K_5^{\flat -}\rangle(P_1^\bullet - P_3^\bullet)(z - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} + \frac{c_{9,\sigma_4}\langle K_4^{\flat -}|\not{k}_{612}|K_5^{\flat -}\rangle(z - Q_1^\bullet)}{\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z - Q_2^\bullet)} \\
&- \frac{c_{12,\sigma_7}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{612}|K_5^{\flat -}\rangle(P_1^\bullet - P_3^\bullet)(z - Q_1^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)(z - Q_2^\bullet)} - \frac{c_{11,\sigma_{10}}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(P_1^\bullet - P_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle(P_1^\bullet - P_2^\bullet)}
\end{aligned} \tag{A55}$$

$$K_2(z) = c_{6,\sigma_1} - \frac{1}{2}\frac{c_{21,\sigma_{10}}}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle(z-P_2^\bullet)} + \frac{1}{2}\frac{c_{19,\sigma_6}}{\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(z-P_3^\bullet)} + \frac{1}{2}\frac{c_{20,\sigma_4}}{\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(Q_1^\bullet - Q_2^\bullet)}$$
$$- \frac{1}{4}\frac{c_{23,\sigma_1}}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z-P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)} + \frac{c_{8,\sigma_6}\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle(z-P_1^\bullet)}{\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(z-P_3^\bullet)}$$
$$- \frac{1}{2}\frac{c_{24,\sigma_4}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(z-P_3^\bullet) + c_{24,\sigma_7}\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle(z-P_1^\bullet) + c_{24,\sigma_{10}}\langle K_4^{\flat -}|\not{k}_{61}|K_5^{\flat -}\rangle(Q_1^\bullet - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z-P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)}$$
$$- \frac{c_{13,\sigma_4}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{61}|K_5^{\flat -}\rangle(z-P_3^\bullet)(Q_1^\bullet - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z-P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)} - \frac{c_{10,\sigma_{10}}\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle(z-P_1^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle(z-P_2^\bullet)}$$
$$- \frac{c_{12,\sigma_1}\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{61}|K_5^{\flat -}\rangle(z-P_1^\bullet)(Q_1^\bullet - Q_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z-P_2^\bullet)(Q_1^\bullet - Q_2^\bullet)} - \frac{c_{11,\sigma_{10}}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(z-P_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_1|K_2^{\flat -}\rangle(z-P_2^\bullet)} \quad \text{(A56)}$$

$$K_3(z) = \text{parity conjugate of } K_1(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}] \quad \text{(A57)}$$

$$K_4(z) = \text{parity conjugate of } K_2(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}] \quad \text{(A58)}$$

$$K_5(z) = c_{6,\sigma_1} - \frac{1}{2}\frac{c_{21,\sigma_{10}}}{\langle K_2^{\flat -}|\not{k}_1|K_1^{\flat -}\rangle(z-P_2)} + \frac{1}{2}\frac{c_{19,\sigma_6}}{\langle K_2^{\flat -}|\not{k}_6|K_1^{\flat -}\rangle(z-P_3)} + \frac{1}{2}\frac{c_{20,\sigma_4}}{\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(\beta_3(z) - Q_2^\bullet)}$$
$$- \frac{1}{4}\frac{c_{23,\sigma_1}}{\langle K_2^{\flat -}|\not{k}_1|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z) - Q_2^\bullet)} + \frac{c_{8,\sigma_6}\langle K_2^{\flat -}|\not{k}_{56}|K_1^{\flat -}\rangle(z-P_1)}{\langle K_2^{\flat -}|\not{k}_6|K_1^{\flat -}\rangle(z-P_3)}$$
$$+ \frac{c_{9,\sigma_4}\langle K_4^{\flat -}|\not{k}_{612}|K_5^{\flat -}\rangle(\beta_3(z) - Q_1^\bullet)}{\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(\beta_3(z) - Q_2^\bullet)} - \frac{1}{2}\frac{c_{24,\sigma_1}\langle K_4^{\flat -}|\not{k}_{612}|K_5^{\flat -}\rangle(\beta_3(z) - Q_1^\bullet)}{\langle K_2^{\flat -}|\not{k}_1|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z) - Q_2^\bullet)}$$
$$- \frac{1}{2}\frac{c_{24,\sigma_4}\langle K_2^{\flat -}|\not{k}_6|K_1^{\flat -}\rangle(z-P_3) + c_{24,\sigma_7}\langle K_2^{\flat -}|\not{k}_{56}|K_1^{\flat -}\rangle(z-P_1) + c_{24,\sigma_{10}}\langle K_4^{\flat -}|\not{k}_{61}|K_5^{\flat -}\rangle(\beta_3(z) - Q_3^\bullet)}{\langle K_2^{\flat -}|\not{k}_1|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z) - Q_2^\bullet)}$$
$$- \frac{c_{13,\sigma_1}\langle K_2^{\flat -}|\not{k}_{56}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{612}|K_5^{\flat -}\rangle(z-P_1)(\beta_3(z) - Q_1^\bullet)}{\langle K_2^{\flat -}|\not{k}_1|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z) - Q_2^\bullet)} - \frac{c_{10,\sigma_{10}}\langle K_2^{\flat -}|\not{k}_{56}|K_1^{\flat -}\rangle(z-P_1)}{\langle K_2^{\flat -}|\not{k}_1|K_1^{\flat -}\rangle(z-P_2)}$$
$$- \frac{c_{13,\sigma_4}\langle K_2^{\flat -}|\not{k}_6|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{61}|K_5^{\flat -}\rangle(z-P_3)(\beta_3(z) - Q_3^\bullet)}{\langle K_2^{\flat -}|\not{k}_1|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z) - Q_2^\bullet)} - \frac{c_{11,\sigma_{10}}\langle K_2^{\flat -}|\not{k}_6|K_1^{\flat -}\rangle(z-P_3)}{\langle K_2^{\flat -}|\not{k}_1|K_1^{\flat -}\rangle(z-P_2)}$$
$$- \frac{c_{12,\sigma_1}\langle K_2^{\flat -}|\not{k}_{56}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{61}|K_5^{\flat -}\rangle(z-P_1)(\beta_3(z) - Q_3^\bullet)}{\langle K_2^{\flat -}|\not{k}_1|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z) - Q_2^\bullet)}$$
$$- \frac{c_{12,\sigma_7}\langle K_2^{\flat -}|\not{k}_6|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{612}|K_5^{\flat -}\rangle(z-P_3)(\beta_3(z) - Q_1^\bullet)}{\langle K_2^{\flat -}|\not{k}_1|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z) - Q_2^\bullet)} \quad \text{(A59)}$$

$$K_6(z) = \text{parity conjugate of } K_5(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}], \quad \text{(A60)}$$

where $\beta_3(z)$ is given in Eq. (A52).

b. Heptacut #5 of the left-hand side of Eq. (4.1)

The result of applying heptacut #5 to the left-hand side of Eq. (4.1) is

$$i\sum_{i=1}^{6}\oint_{\Gamma_i} dz J_i(z)\prod_{j=1}^{6} A_j^{\text{tree}}(z)\bigg|_{\mathcal{S}_i}, \quad \text{(A61)}$$

where, assuming without loss of generality the external helicities are $(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$, the cut amplitude evaluated on the six different kinematical solutions yields

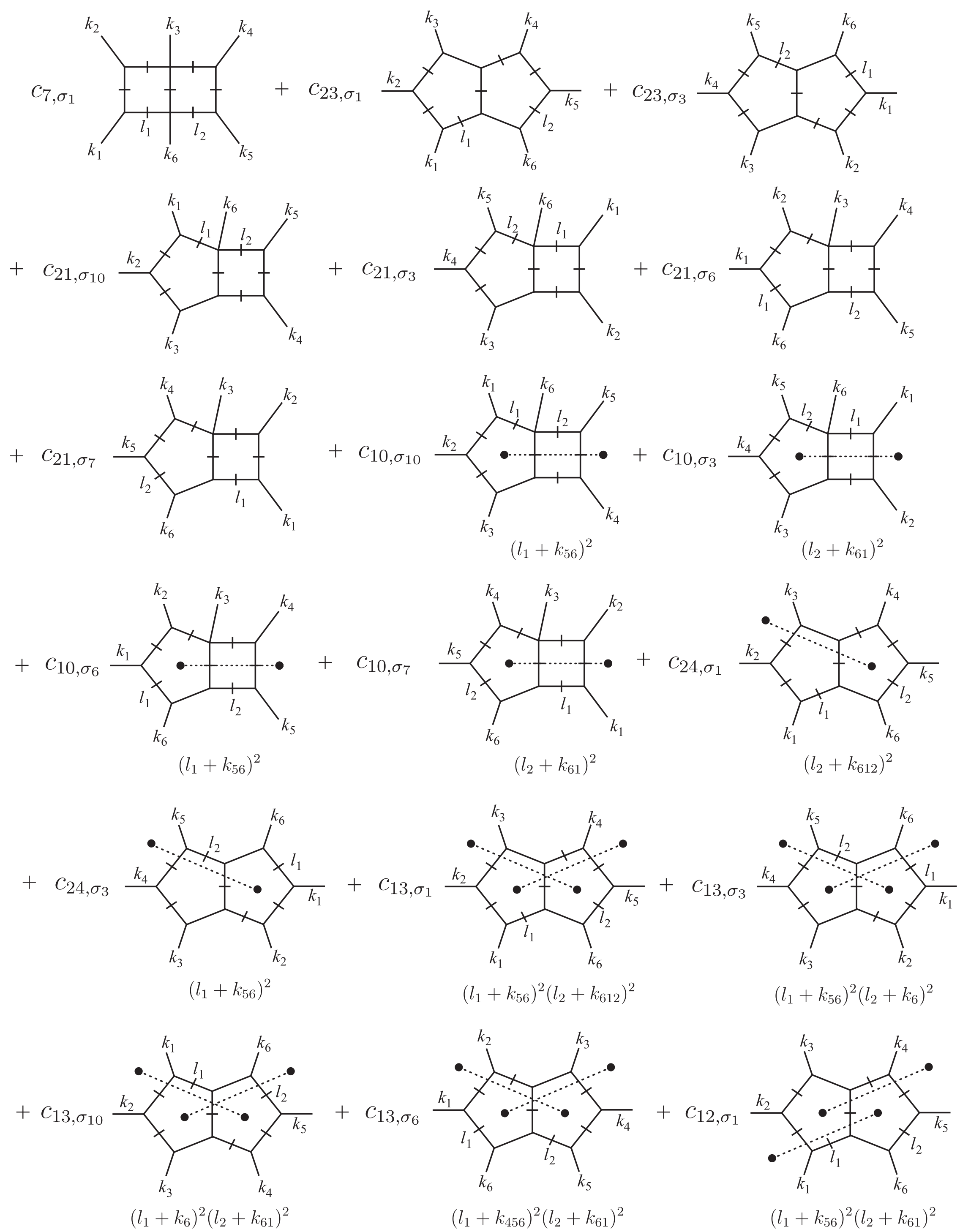
$$\prod_{j=1}^{6} A_j^{\text{tree}}(z)\bigg|_{\mathcal{S}_i} = -\frac{i}{16}A^{\text{tree}}_{--++++} \times \begin{cases} \frac{1}{J_4(z)}\left(\frac{1}{z-P_2} - \frac{1}{z-P_3}\right) & \text{for } i = 4 \\ 0 & \text{for } i = 1, 2, 3, 5, 6. \end{cases} \quad \text{(A62)}$$

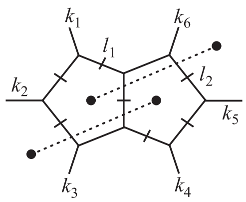
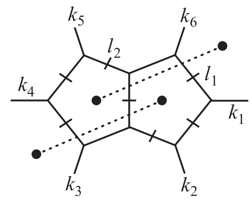
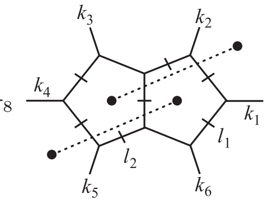
6. Heptacut #6

This heptacut is defined by the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) with the vertex momenta

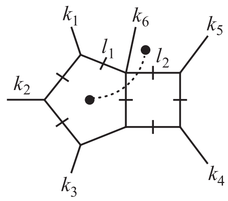
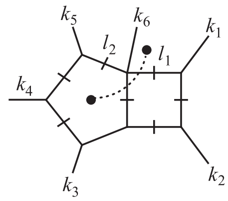
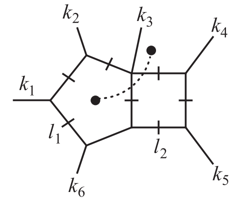
$$K_1 = k_1 \qquad K_2 = k_2 \qquad K_3 = k_3 \qquad K_4 = k_4 \qquad K_5 = k_5 \qquad K_6 = k_6. \tag{A63}$$

Applying this heptacut to the right-hand side of Eq. (4.1) leaves the following linear combination of cut integrals

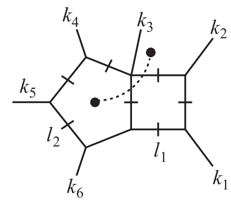


$+\ c_{12,\sigma_{10}}$  $+\ c_{12,\sigma_3}$  $+\ c_{12,\sigma_8}$ 

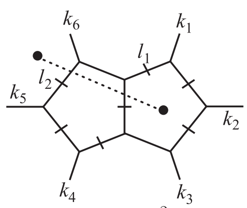
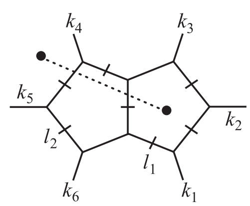
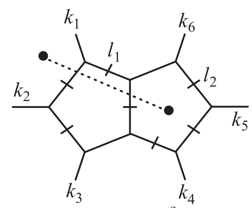
$(l_1+k_6)^2(l_2+k_{612})^2 \qquad (l_1+k_{456})^2(l_2+k_6)^2 \qquad (l_1+k_{56})^2(l_2+k_{61})^2$

$+\ c_{11,\sigma_{10}}$  $+\ c_{11,\sigma_3}$  $+\ c_{11,\sigma_6}$ 

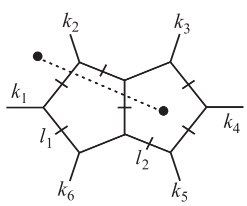
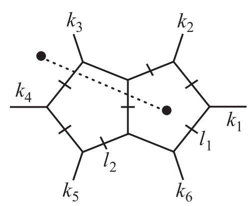
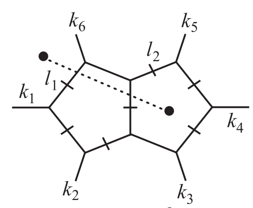
$(l_1+k_6)^2 \qquad (l_2+k_6)^2 \qquad (l_1+k_{456})^2$

$+\ c_{11,\sigma_7}$ 

$(l_2+k_{612})^2$

$+\ c_{24,\sigma_4}$  $+\ c_{24,\sigma_7}$  $+\ c_{24,\sigma_{10}}$ 

$(l_1+k_6)^2 \qquad (l_1+k_{56})^2 \qquad (l_2+k_{61})^2$

$+\ c_{24,\sigma_6}$  $+\ c_{24,\sigma_8}$  $+\ c_{24,\sigma_{11}}$ 

$(l_2+k_{61})^2 \qquad (l_1+k_{456})^2 \qquad (l_2+k_6)^2$

We define the spinor ratios

$$
\begin{aligned}
&P_1=-\frac{K_5^\flat\cdot(K_1^\flat+k_6)+K_1^\flat\cdot k_6}{\langle K_2^{\flat-}|\not{K}_5^\flat+\not{k}_6|K_1^{\flat-}\rangle}, \qquad P_2=-\frac{K_1^\flat\cdot k_{123}-\frac{1}{2}s_{123}}{\langle K_2^{\flat-}|\not{k}_{123}|K_1^{\flat-}\rangle}, \qquad P_3=-\frac{\langle K_1^\flat k_6\rangle}{2\langle K_2^\flat k_6\rangle} \qquad P_4=-\frac{\langle K_1^\flat k_5\rangle}{2\langle K_2^\flat k_5\rangle},\\
&P_5=-\frac{\langle K_5^{\flat-}|\not{K}_1^\flat+\not{k}_6|K_4^{\flat-}\rangle}{2[K_1^\flat K_4^\flat]\langle K_5^\flat K_2^\flat\rangle}, \qquad P_6=-\frac{\langle K_4^{\flat-}|\not{K}_1^\flat+\not{k}_6|K_5^{\flat-}\rangle}{2\langle K_2^\flat K_4^\flat\rangle[K_5^\flat K_1^\flat]} \qquad Q_1=-\frac{K_5^\flat\cdot(K_1^\flat+k_6)+K_1^\flat\cdot k_6}{\langle K_5^{\flat-}|\not{K}_1^\flat+\not{k}_6|K_4^{\flat-}\rangle},\\
&Q_2=-\frac{K_5^\flat\cdot k_{345}-\frac{1}{2}s_{345}}{\langle K_5^{\flat-}|\not{k}_{345}|K_4^{\flat-}\rangle}, \qquad Q_3=-\frac{[K_5^\flat k_6]}{2[K_4^\flat k_6]} \qquad Q_4=-\frac{[K_5^\flat k_1]}{2[K_4^\flat k_1]}, \qquad Q_5=-\frac{\langle K_2^{\flat-}|\not{K}_5^\flat+\not{k}_6|K_1^{\flat-}\rangle}{2[K_1^\flat K_4^\flat]\langle K_5^\flat K_2^\flat\rangle},
\end{aligned}
\tag{A64}
$$

and their parity conjugates

$$P_1^\bullet = -\frac{K_5^\flat\cdot(K_1^\flat + k_6) + K_1^\flat\cdot k_6}{\langle K_1^{\flat -}|\not{K}_5^\flat + \not{k}_6|K_2^{\flat -}\rangle}, \qquad P_2^\bullet = -\frac{K_1^\flat\cdot k_{123} - \frac{1}{2}s_{123}}{\langle K_1^{\flat -}|\not{k}_{123}|K_2^{\flat -}\rangle},$$
$$P_3^\bullet = -\frac{[K_1^\flat k_6]}{2[K_2^\flat k_6]} \qquad P_4^\bullet = -\frac{[K_1^\flat k_5]}{2[K_2^\flat k_5]},$$
$$P_5^\bullet = -\frac{\langle K_4^{\flat -}|\not{K}_1^\flat + \not{k}_6|K_5^{\flat -}\rangle}{2\langle K_1^\flat K_4^\flat\rangle[K_5^\flat K_2^\flat]}, \qquad P_6^\bullet = -\frac{\langle K_5^{\flat -}|\not{K}_1^\flat + \not{k}_6|K_4^{\flat -}\rangle}{2[K_2^\flat K_4^\flat]\langle K_5^\flat K_1^\flat\rangle}$$
$$Q_1^\bullet = -\frac{K_5^\flat\cdot(K_1^\flat + k_6) + K_1^\flat\cdot k_6}{\langle K_4^{\flat -}|\not{K}_1^\flat + \not{k}_6|K_5^{\flat -}\rangle}, \qquad Q_2^\bullet = -\frac{K_5^\flat\cdot k_{345} - \frac{1}{2}s_{345}}{\langle K_4^{\flat -}|\not{k}_{345}|K_5^{\flat -}\rangle},$$
$$Q_3^\bullet = -\frac{\langle K_5^\flat k_6\rangle}{2\langle K_4^\flat k_6\rangle} \qquad Q_4^\bullet = -\frac{\langle K_5^\flat k_1\rangle}{2\langle K_4^\flat k_1\rangle}, \qquad Q_5^\bullet = -\frac{\langle K_1^{\flat -}|\not{K}_5^\flat + \not{k}_6|K_2^{\flat -}\rangle}{2\langle K_1^\flat K_4^\flat\rangle[K_5^\flat K_2^\flat]}. \tag{A65}$$

This heptacut belongs to case II treated in Sec. III A, and there are thus four kinematical solutions (shown in Fig. 3). Parametrizing the loop momenta according to Eqs. (3.12) and (3.13), the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) are solved by setting the parameters equal to the values

$$\alpha_1 = 1, \qquad \alpha_2 = 0, \qquad \beta_1 = 0, \qquad \beta_2 = 1 \tag{A66}$$

and those given in Fig. 3. The heptacut double-box integral I_{7,σ_1} is $\sum_{i=1}^4 \oint_{\Gamma_i} dz J_i(z)$ where

$$J_i(z) = \frac{1}{32\gamma_1\gamma_2}\times\begin{cases} (\langle K_4^{\flat -}|\not{K}_1^\flat + \not{k}_6|K_5^{\flat -}\rangle z(z - Q_1^\bullet))^{-1} & \text{for } i = 1\\ (\langle K_5^{\flat -}|\not{K}_1^\flat + \not{k}_6|K_4^{\flat -}\rangle z(z - Q_1))^{-1} & \text{for } i = 2\\ (\langle K_2^{\flat -}|\not{K}_5^\flat + \not{k}_6|K_1^{\flat -}\rangle z(z - P_1))^{-1} & \text{for } i = 3\\ (\langle K_1^{\flat -}|\not{K}_5^\flat + \not{k}_6|K_2^{\flat -}\rangle z(z - P_1^\bullet))^{-1} & \text{for } i = 4. \end{cases} \tag{A67}$$

a. Heptacut #6 of the right-hand side of Eq. (4.1)

The result of applying heptacut #6 to the right-hand side of Eq. (4.1) is

$$\frac{1}{4}\sum_{i=1}^4 \oint_{\Gamma_i} dz J_i(z) K_i(z), \tag{A68}$$

where the kernels evaluated on the four kinematical solutions are

$$\begin{aligned} K_1(z) = c_{7,\sigma_1} &- \frac{1}{4}\frac{c_{23,\sigma_1}}{\langle K_1^{\flat -}|\not{k}_{123}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(\alpha_3(z) - P_2^\bullet)(z - Q_3^\bullet)} - \frac{1}{2}\frac{c_{21,\sigma_{10}}}{\langle K_1^{\flat -}|\not{k}_{123}|K_2^{\flat -}\rangle(\alpha_3(z) - P_2^\bullet)}\\ &- \frac{1}{4}\frac{c_{23,\sigma_3}}{\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{345}|K_5^{\flat -}\rangle(\alpha_3(z) - P_3^\bullet)(z - Q_2^\bullet)} - \frac{1}{2}\frac{c_{21,\sigma_3}}{\langle K_4^{\flat -}|\not{k}_{345}|K_5^{\flat -}\rangle(z - Q_2^\bullet)}\\ &+ \frac{1}{2}\frac{c_{21,\sigma_6}}{\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(\alpha_3(z) - P_3^\bullet)} + \frac{1}{2}\frac{c_{21,\sigma_7}}{\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(z - Q_3^\bullet)} - \frac{c_{10,\sigma_{10}}\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle(\alpha_3(z) - P_1^\bullet)}{\langle K_1^{\flat -}|\not{k}_{123}|K_2^{\flat -}\rangle(\alpha_3(z) - P_2^\bullet)}\\ &- \frac{1}{2}\frac{c_{24,\sigma_1}\langle K_4^{\flat -}|\not{k}_{612}|K_5^{\flat -}\rangle(z - Q_2^\bullet) + c_{24,\sigma_4}\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle(\alpha_3(z) - P_3^\bullet)}{\langle K_1^{\flat -}|\not{k}_{123}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(\alpha_3(z) - P_2^\bullet)(z - Q_3^\bullet)}\\ &- \frac{1}{2}\frac{c_{24,\sigma_7}\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle(\alpha_3(z) - P_1^\bullet) + c_{24,\sigma_{10}}\langle K_4^{\flat -}|\not{k}_{61}|K_5^{\flat -}\rangle(z - Q_1^\bullet)}{\langle K_1^{\flat -}|\not{k}_{123}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_6|K_5^{\flat -}\rangle(\alpha_3(z) - P_2^\bullet)(z - Q_3^\bullet)}\\ &- \frac{c_{10,\sigma_3}\langle K_4^{\flat -}|\not{k}_{61}|K_5^{\flat -}\rangle(z - Q_1^\bullet)}{\langle K_4^{\flat -}|\not{k}_{345}|K_5^{\flat -}\rangle(z - Q_2^\bullet)}\\ &- \frac{1}{2}\frac{c_{24,\sigma_3}\langle K_1^{\flat -}|\not{k}_{56}|K_2^{\flat -}\rangle(\alpha_3(z) - P_1^\bullet) + c_{24,\sigma_6}\langle K_4^{\flat -}|\not{k}_{61}|K_5^{\flat -}\rangle(z - Q_1^\bullet)}{\langle K_1^{\flat -}|\not{k}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{k}_{345}|K_5^{\flat -}\rangle(\alpha_3(z) - P_3^\bullet)(z - Q_2^\bullet)} \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\frac{c_{24,\sigma_8}\langle K_1^{\flat-}|\not{k}_{456}|K_2^{\flat-}\rangle(\alpha_3(z)-P_2^\bullet)+c_{24,\sigma_{11}}\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-Q_3^\bullet)}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(\alpha_3(z)-P_3^\bullet)(z-Q_2^\bullet)}\\
&+\frac{c_{10,\sigma_6}\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle(\alpha_3(z)-P_1^\bullet)}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle(\alpha_3(z)-P_3^\bullet)}\\
&-\frac{c_{13,\sigma_1}\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{612}|K_5^{\flat-}\rangle(\alpha_3(z)-P_1^\bullet)(z-Q_2^\bullet)}{\langle K_1^{\flat-}|\not{k}_{123}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\alpha_3(z)-P_2^\bullet)(z-Q_3^\bullet)}+\frac{c_{10,\sigma_7}\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(z-Q_1^\bullet)}{\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-Q_3^\bullet)}\\
&-\frac{c_{13,\sigma_3}\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\alpha_3(z)-P_1^\bullet)(z-Q_3^\bullet)}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(\alpha_3(z)-P_3^\bullet)(z-Q_2^\bullet)}\\
&-\frac{c_{13,\sigma_{10}}\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(\alpha_3(z)-P_3^\bullet)(z-Q_1^\bullet)}{\langle K_1^{\flat-}|\not{k}_{123}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\alpha_3(z)-P_2^\bullet)(z-Q_3^\bullet)}\\
&-\frac{c_{13,\sigma_6}\langle K_1^{\flat-}|\not{k}_{456}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(\alpha_3(z)-P_2^\bullet)(z-Q_1^\bullet)}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(\alpha_3(z)-P_3^\bullet)(z-Q_2^\bullet)}-\frac{c_{11,\sigma_{10}}\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle(\alpha_3(z)-P_3^\bullet)}{\langle K_1^{\flat-}|\not{k}_{123}|K_2^{\flat-}\rangle(\alpha_3(z)-P_2^\bullet)}\\
&-\frac{c_{12,\sigma_1}\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(\alpha_3(z)-P_1^\bullet)(z-Q_1^\bullet)}{\langle K_1^{\flat-}|\not{k}_{123}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\alpha_3(z)-P_2^\bullet)(z-Q_3^\bullet)}-\frac{c_{11,\sigma_3}\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-Q_3^\bullet)}{\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(z-Q_2^\bullet)}\\
&-\frac{c_{12,\sigma_{10}}\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{612}|K_5^{\flat-}\rangle(\alpha_3(z)-P_3^\bullet)(z-Q_2^\bullet)}{\langle K_1^{\flat-}|\not{k}_{123}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\alpha_3(z)-P_2^\bullet)(z-Q_3^\bullet)}+\frac{c_{11,\sigma_6}\langle K_1^{\flat-}|\not{k}_{456}|K_2^{\flat-}\rangle(\alpha_3(z)-P_2^\bullet)}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle(\alpha_3(z)-P_3^\bullet)}\\
&-\frac{c_{12,\sigma_3}\langle K_1^{\flat-}|\not{k}_{456}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\alpha_3(z)-P_2^\bullet)(z-Q_3^\bullet)}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(\alpha_3(z)-P_3^\bullet)(z-Q_2^\bullet)}+\frac{c_{11,\sigma_7}\langle K_4^{\flat-}|\not{k}_{612}|K_5^{\flat-}\rangle(z-Q_2^\bullet)}{\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-Q_3^\bullet)}\\
&-\frac{c_{12,\sigma_8}\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(\alpha_3(z)-P_1^\bullet)(z-Q_1^\bullet)}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(\alpha_3(z)-P_3^\bullet)(z-Q_2^\bullet)}
\end{aligned}
\tag{A69}
$$

$$K_2(z)=\text{parity conjugate of }K_1(z)\qquad[\text{obtained by applying Eqs.(4.12)–(4.17)}],\tag{A70}$$

where $\alpha_3(z)$ can be read off from the on-shell values quoted below solution $\mathcal{S}_1$ in Fig. 3.

$$
\begin{aligned}
K_3(z)={}&c_{7,\sigma_1}-\frac{1}{4}\frac{c_{23,\sigma_1}}{\langle K_2^{\flat-}|\not{k}_{123}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_3^\bullet)}-\frac{1}{2}\frac{c_{21,\sigma_{10}}}{\langle K_2^{\flat-}|\not{k}_{123}|K_1^{\flat-}\rangle(z-P_2)}\\
&-\frac{1}{4}\frac{c_{23,\sigma_3}}{\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(z-P_3)(\beta_3(z)-Q_2^\bullet)}-\frac{1}{2}\frac{c_{21,\sigma_3}}{\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(\beta_3(z)-Q_2^\bullet)}\\
&+\frac{1}{2}\frac{c_{21,\sigma_6}}{\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle(z-P_3)}-\frac{1}{2}\frac{c_{24,\sigma_1}\langle K_4^{\flat-}|\not{k}_{612}|K_5^{\flat-}\rangle(\beta_3(z)-Q_2^\bullet)+c_{24,\sigma_4}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle(z-P_3)}{\langle K_2^{\flat-}|\not{k}_{123}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_3^\bullet)}\\
&+\frac{1}{2}\frac{c_{21,\sigma_7}}{\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\beta_3(z)-Q_3^\bullet)}-\frac{1}{2}\frac{c_{24,\sigma_7}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle(z-P_1)+c_{24,\sigma_{10}}\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat-}|\not{k}_{123}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_3^\bullet)}\\
&-\frac{1}{2}\frac{c_{24,\sigma_3}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle(z-P_1)+c_{24,\sigma_6}\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(z-P_3)(\beta_3(z)-Q_2^\bullet)}\\
&-\frac{1}{2}\frac{c_{24,\sigma_8}\langle K_2^{\flat-}|\not{k}_{456}|K_1^{\flat-}\rangle(z-P_2)+c_{24,\sigma_{11}}\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\beta_3(z)-Q_3^\bullet)}{\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(z-P_3)(\beta_3(z)-Q_2^\bullet)}-\frac{c_{10,\sigma_{10}}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle(z-P_1)}{\langle K_2^{\flat-}|\not{k}_{123}|K_1^{\flat-}\rangle(z-P_2)}\\
&-\frac{c_{13,\sigma_1}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{612}|K_5^{\flat-}\rangle(z-P_1)(\beta_3(z)-Q_2^\bullet)}{\langle K_2^{\flat-}|\not{k}_{123}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_3^\bullet)}-\frac{c_{10,\sigma_3}\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(\beta_3(z)-Q_1^\bullet)}{\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(\beta_3(z)-Q_2^\bullet)}
\end{aligned}
$$

$$- \frac{c_{13,\sigma_3}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_1)(\beta_3(z)-Q_3^\bullet)}{\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(z-P_3)(\beta_3(z)-Q_2^\bullet)} + \frac{c_{10,\sigma_6}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle(z-P_1)}{\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle(z-P_3)}$$
$$- \frac{c_{13,\sigma_{10}}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(z-P_3)(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat-}|\not{k}_{123}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_3^\bullet)} + \frac{c_{10,\sigma_7}\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(\beta_3(z)-Q_1^\bullet)}{\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\beta_3(z)-Q_3^\bullet)}$$
$$- \frac{c_{13,\sigma_6}\langle K_2^{\flat-}|\not{k}_{456}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(z-P_3)(\beta_3(z)-Q_2^\bullet)} - \frac{c_{11,\sigma_{10}}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle(z-P_3)}{\langle K_2^{\flat-}|\not{k}_{123}|K_1^{\flat-}\rangle(z-P_2)}$$
$$- \frac{c_{12,\sigma_1}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(z-P_1)(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat-}|\not{k}_{123}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_3^\bullet)} - \frac{c_{11,\sigma_3}\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\beta_3(z)-Q_3^\bullet)}{\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(\beta_3(z)-Q_2^\bullet)}$$
$$- \frac{c_{12,\sigma_{10}}\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{612}|K_5^{\flat-}\rangle(z-P_3)(\beta_3(z)-Q_2^\bullet)}{\langle K_2^{\flat-}|\not{k}_{123}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_3^\bullet)} + \frac{c_{11,\sigma_6}\langle K_2^{\flat-}|\not{k}_{456}|K_1^{\flat-}\rangle(z-P_2)}{\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle(z-P_3)}$$
$$- \frac{c_{12,\sigma_3}\langle K_2^{\flat-}|\not{k}_{456}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(z-P_2)(\beta_3(z)-Q_3^\bullet)}{\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(z-P_3)(\beta_3(z)-Q_2^\bullet)} + \frac{c_{11,\sigma_7}\langle K_4^{\flat-}|\not{k}_{612}|K_5^{\flat-}\rangle(\beta_3(z)-Q_2^\bullet)}{\langle K_4^{\flat-}|\not{k}_6|K_5^{\flat-}\rangle(\beta_3(z)-Q_3^\bullet)}$$
$$- \frac{c_{12,\sigma_8}\langle K_2^{\flat-}|\not{k}_{56}|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{61}|K_5^{\flat-}\rangle(z-P_1)(\beta_3(z)-Q_1^\bullet)}{\langle K_2^{\flat-}|\not{k}_6|K_1^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{345}|K_5^{\flat-}\rangle(z-P_3)(\beta_3(z)-Q_2^\bullet)} \tag{A71}$$

$$K_4(z) = \text{parity conjugate of } K_3(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}], \tag{A72}$$

where $\beta_3(z)$ can be read off from the on-shell values quoted below solution $\mathcal{S}_3$ in Fig. 3.

b. Heptacut #6 of the left-hand side of Eq. (4.1)

The result of applying heptacut #6 to the left-hand side of Eq. (4.1) is

$$i\sum_{i=1}^{4}\oint_{\Gamma_i} dz J_i(z)\prod_{j=1}^{6}A_j^{\text{tree}}(z)\bigg|_{\mathcal{S}_i}, \tag{A73}$$

where, assuming without loss of generality the external helicities are $(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$, the cut amplitude evaluated on the four different kinematical solutions yields

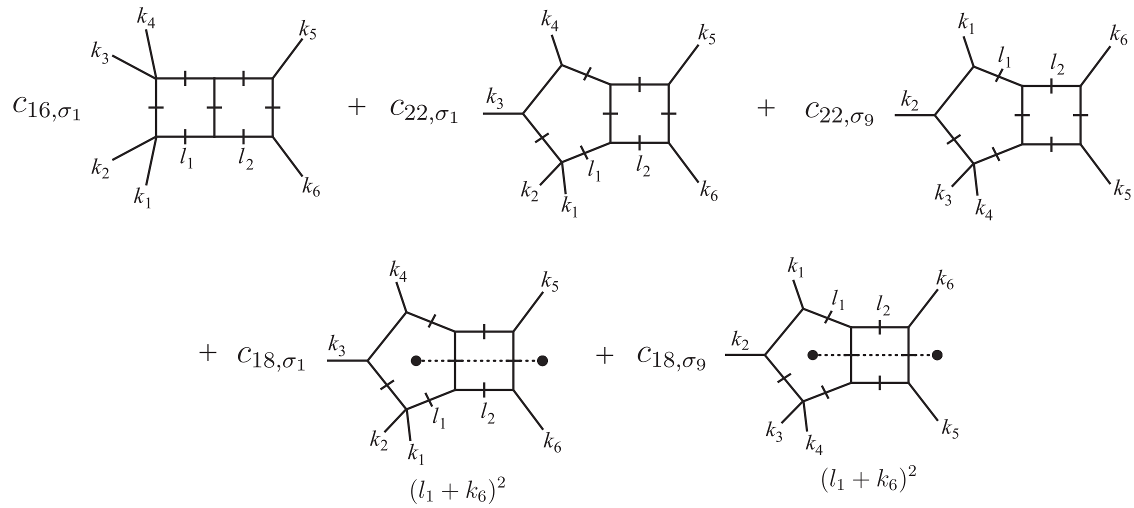
$$\prod_{j=1}^{6}A_j^{\text{tree}}(z)\bigg|_{\mathcal{S}_i} = 0 \quad \text{for } i = 1, \ldots, 4. \tag{A74}$$

7. Heptacut #7

Note that in this section we will leave the $\pm$ in $\gamma_1^\pm$, $P_i^\pm$, $Q_i^\pm$ etc. implicit and simply write γ_1, P_i, Q_i etc. for notational simplicity. The heptacut considered here is defined by the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) with the vertex momenta

$$K_1 = k_{12} \qquad K_2 = k_{34} \qquad K_3 = 0 \qquad K_4 = k_5 \qquad K_5 = k_6 \qquad K_6 = 0. \tag{A75}$$

Applying this heptacut to the right-hand side of Eq. (4.1) leaves the following linear combination of cut integrals



We define the spinor ratios

$$P_1 = -\frac{\langle K_1^\flat K_5^\flat\rangle}{2\langle K_2^\flat K_5^\flat\rangle}\frac{\gamma_1(S_2+\gamma_1)}{\gamma_1^2 - S_1S_2}, \qquad P_2 = \frac{[K_2^\flat K_5^\flat]}{2[K_1^\flat K_5^\flat]}\frac{S_1S_2(1+S_1/\gamma_1)}{\gamma_1^2 - S_1S_2},$$

$$P_3 = \frac{[K_2^\flat K_4^\flat]}{2[K_1^\flat K_4^\flat]}\frac{S_1S_2(1+S_1/\gamma_1)}{\gamma_1^2 - S_1S_2} \qquad P_4 = -\frac{\langle K_1^\flat k_1\rangle}{2\langle K_2^\flat k_1\rangle}\frac{\gamma_1(S_2+\gamma_1)}{\gamma_1^2 - S_1S_2},$$

$$P_6 = -\frac{\frac{\gamma_1(S_2+\gamma_1)}{\gamma_1^2-S_1S_2}K_1^\flat\cdot k_3 - \frac{S_1S_2(S_1+\gamma_1)}{\gamma_1(\gamma_1^2-S_1S_2)}K_2^\flat\cdot k_3 - \frac{1}{2}(s_{13}+s_{23}) - \sqrt{\Delta}}{2\langle K_2^{\flat -}|\not{k}_3|K_1^{\flat -}\rangle}, \qquad P_5 = \frac{[K_2^\flat k_1]}{2[K_1^\flat k_1]}\frac{S_1S_2(1+S_1/\gamma_1)}{\gamma_1^2 - S_1S_2},$$

$$P_7 = -\frac{\frac{\gamma_1(S_2+\gamma_1)}{\gamma_1^2-S_1S_2}K_1^\flat\cdot k_3 - \frac{S_1S_2(S_1+\gamma_1)}{\gamma_1(\gamma_1^2-S_1S_2)}K_2^\flat\cdot k_3 - \frac{1}{2}(s_{13}+s_{23}) + \sqrt{\Delta}}{2\langle K_2^{\flat -}|\not{k}_3|K_1^{\flat -}\rangle},$$

$$Q_1 = -\frac{\gamma_1(S_2+\gamma_1)\langle K_5^\flat K_1^\flat\rangle[K_1^\flat K_5^\flat] + S_1S_2(1+S_1/\gamma_1)\langle K_5^\flat K_2^\flat\rangle[K_2^\flat K_5^\flat]}{2(\gamma_1(S_2+\gamma_1)\langle K_5^\flat K_1^\flat\rangle[K_1^\flat K_4^\flat] + S_1S_2(1+S_1/\gamma_1)\langle K_5^\flat K_2^\flat\rangle[K_2^\flat K_4^\flat])}, \qquad Q_2 = -\frac{[K_1^\flat K_5^\flat]}{2[K_1^\flat K_4^\flat]} \tag{A76}$$

and their parity conjugates

$$P_1^\bullet = -\frac{[K_1^\flat K_5^\flat]}{2[K_2^\flat K_5^\flat]}\frac{\gamma_1(S_2+\gamma_1)}{\gamma_1^2 - S_1S_2}, \qquad P_2^\bullet = \frac{\langle K_2^\flat K_5^\flat\rangle}{2\langle K_1^\flat K_5^\flat\rangle}\frac{S_1S_2(1+S_1/\gamma_1)}{\gamma_1^2 - S_1S_2},$$

$$P_3^\bullet = \frac{\langle K_2^\flat K_4^\flat\rangle}{2\langle K_1^\flat K_4^\flat\rangle}\frac{S_1S_2(1+S_1/\gamma_1)}{\gamma_1^2 - S_1S_2} \qquad P_4^\bullet = -\frac{[K_1^\flat k_1]}{2[K_2^\flat k_1]}\frac{\gamma_1(S_2+\gamma_1)}{\gamma_1^2 - S_1S_2},$$

$$P_6^\bullet = -\frac{\frac{\gamma_1(S_2+\gamma_1)}{\gamma_1^2-S_1S_2}K_1^\flat\cdot k_3 - \frac{S_1S_2(S_1+\gamma_1)}{\gamma_1(\gamma_1^2-S_1S_2)}K_2^\flat\cdot k_3 - \frac{1}{2}(s_{13}+s_{23}) - \sqrt{\Delta}}{2\langle K_1^{\flat -}|\not{k}_3|K_2^{\flat -}\rangle},$$

$$P_5^\bullet = \frac{\langle K_2^\flat k_1\rangle}{2\langle K_1^\flat k_1\rangle}\frac{S_1S_2(1+S_1/\gamma_1)}{\gamma_1^2 - S_1S_2}, \qquad P_7^\bullet = -\frac{\frac{\gamma_1(S_2+\gamma_1)}{\gamma_1^2-S_1S_2}K_1^\flat\cdot k_3 - \frac{S_1S_2(S_1+\gamma_1)}{\gamma_1(\gamma_1^2-S_1S_2)}K_2^\flat\cdot k_3 - \frac{1}{2}(s_{13}+s_{23}) + \sqrt{\Delta}}{2\langle K_1^{\flat -}|\not{k}_3|K_2^{\flat -}\rangle},$$

$$Q_1^\bullet = -\frac{\gamma_1(S_2+\gamma_1)\langle K_5^\flat K_1^\flat\rangle[K_1^\flat K_5^\flat] + S_1S_2(1+S_1/\gamma_1)\langle K_5^\flat K_2^\flat\rangle[K_2^\flat K_5^\flat]}{2(\gamma_1(S_2+\gamma_1)\langle K_4^\flat K_1^\flat\rangle[K_1^\flat K_5^\flat] + S_1S_2(1+S_1/\gamma_1)\langle K_4^\flat K_2^\flat\rangle[K_2^\flat K_5^\flat])}, \qquad Q_2^\bullet = -\frac{\langle K_1^\flat K_5^\flat\rangle}{2\langle K_1^\flat K_4^\flat\rangle}, \tag{A77}$$

where the discriminant appearing in P_6, P_7, $P_6^\bullet$, $P_7^\bullet$ is given by

$$\Delta = \left(\frac{\gamma_1(S_2+\gamma_1)}{\gamma_1^2 - S_1S_2}K_1^\flat\cdot k_3 - \frac{S_1S_2(1+S_1/\gamma_1)}{\gamma_1^2 - S_1S_2}K_2^\flat\cdot k_3 - \frac{1}{2}(s_{13}+s_{23})\right)^2 + \frac{4S_1S_2(S_1+\gamma_1)(S_2+\gamma_1)}{(\gamma_1^2 - S_1S_2)^2}(K_1^\flat\cdot k_3)(K_2^\flat\cdot k_3). \tag{A78}$$

This heptacut belongs to case III treated in Sec. III A, and there are thus four kinematical solutions (shown in Fig. 4).

Parametrizing the loop momenta according to Eqs. (3.12) and (3.13), the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) are solved by setting the parameters equal to the values

$$\alpha_1 = \frac{\gamma_1(S_2+\gamma_1)}{\gamma_1^2 - S_1 S_2}, \qquad \alpha_2 = \frac{S_1 S_2 (S_1+\gamma_1)}{\gamma_1(S_1S_2-\gamma_1^2)}, \qquad \beta_1 = 0, \qquad \beta_2 = 1 \tag{A79}$$

and those given in Fig. 4. The heptacut double-box integral I_{16,σ_1} is $\frac{1}{2}\sum_\pm \sum_{i=1}^4 \oint_{\Gamma_i} dz J_i^\pm(z)$ where

$$J_i^\pm(z) = \begin{cases} \Big(32\gamma_2\Big((S_2+\gamma_1)\langle K_4^{\flat-}|\not{K}_1^\flat|K_5^{\flat-}\rangle + \frac{S_1S_2}{\gamma_1}\Big(1+\frac{S_1}{\gamma_1}\Big)\langle K_4^{\flat-}|\not{K}_2^\flat|K_5^{\flat-}\rangle\Big)z(z-Q_1^\bullet)\Big)^{-1} & \text{for } i=1,2 \\ \Big(32\gamma_2\Big((S_2+\gamma_1)\langle K_5^{\flat-}|\not{K}_1^\flat|K_4^{\flat-}\rangle + \frac{S_1S_2}{\gamma_1}\Big(1+\frac{S_1}{\gamma_1}\Big)\langle K_5^{\flat-}|\not{K}_2^\flat|K_4^{\flat-}\rangle\Big)z(z-Q_1)\Big)^{-1} & \text{for } i=3,4 \end{cases} \tag{A80}$$

and where we recall that the $\pm$ in $\gamma_1^\pm$, $K_{1\pm}^\flat$, $Q_1^\pm$ etc. have here been left implicit.

Heptacut #7 of the right-hand side of Eq. (4.1)

The result of applying heptacut #7 to the right-hand side of Eq. (4.1) is

$$\frac{1}{8}\sum_\pm \sum_{i=1}^4 \oint_{\Gamma_i} dz J_i^\pm(z) K_i^\pm(z), \tag{A81}$$

where the kernels evaluated on the four kinematical solutions are

$$K_1^\pm(z) = c_{16,\sigma_1} - \frac{1}{2}\frac{c_{22,\sigma_1}P_1^\bullet}{\langle K_1^{\flat-}|\not{K}_3|K_2^{\flat-}\rangle(P_1^\bullet - P_6^\bullet)(P_1^\bullet - P_7^\bullet)} - \frac{1}{2}\frac{c_{22,\sigma_9}P_1^\bullet}{\langle K_1^{\flat-}|\not{K}_1|K_2^{\flat-}\rangle(P_1^\bullet - P_4^\bullet)(P_1^\bullet - P_5^\bullet)} \tag{A82}$$

$$\begin{aligned} K_2^\pm(z) = c_{16,\sigma_1} &- \frac{1}{2}\frac{c_{22,\sigma_1}\alpha_3(z)}{\langle K_1^{\flat-}|\not{K}_3|K_2^{\flat-}\rangle(\alpha_3(z) - P_6^\bullet)(\alpha_3(z) - P_7^\bullet)} - \frac{1}{2}\frac{c_{22,\sigma_9}\alpha_3(z)}{\langle K_1^{\flat-}|\not{K}_1|K_2^{\flat-}\rangle(\alpha_3(z) - P_4^\bullet)(\alpha_3(z) - P_5^\bullet)} \\ &- \frac{c_{18,\sigma_1}\langle K_1^{\flat-}|\not{K}_6|K_2^{\flat-}\rangle(\alpha_3(z)-P_1^\bullet)(\alpha_3(z)-P_2^\bullet)}{\langle K_1^{\flat-}|\not{K}_3|K_2^{\flat-}\rangle(\alpha_3(z) - P_6^\bullet)(\alpha_3(z) - P_7^\bullet)} - \frac{c_{18,\sigma_9}\langle K_1^{\flat-}|\not{K}_6|K_2^{\flat-}\rangle(\alpha_3(z)-P_1^\bullet)(\alpha_3(z)-P_2^\bullet)}{\langle K_1^{\flat-}|\not{K}_1|K_2^{\flat-}\rangle(\alpha_3(z) - P_4^\bullet)(\alpha_3(z) - P_5^\bullet)} \end{aligned} \tag{A83}$$

$$K_3^\pm(z) = \text{parity conjugate of } K_1^\pm(z) \qquad [\text{obtained by applying Eqs.}(4.12)\text{–}(4.17)] \tag{A84}$$

$$K_4^\pm(z) = \text{parity conjugate of } K_2^\pm(z) \qquad [\text{obtained by applying Eqs.}(4.12)\text{–}(4.17)], \tag{A85}$$

where $\alpha_3(z)$ is given in Fig. 4.

Heptacut #7 of the left-hand side of Eq. (4.1)

The result of applying heptacut #7 to the left-hand side of Eq. (4.1) is

$$\frac{i}{2}\sum_\pm \sum_{i=1}^4 \oint_{\Gamma_i} dz J_i^\pm(z) \prod_{j=1}^6 A_j^{\text{tree}}(z)\Big|_{\mathcal{S}_i}, \tag{A86}$$

where, assuming without loss of generality the external helicities are $(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$, the cut amplitude evaluated on the four different kinematical solutions yields

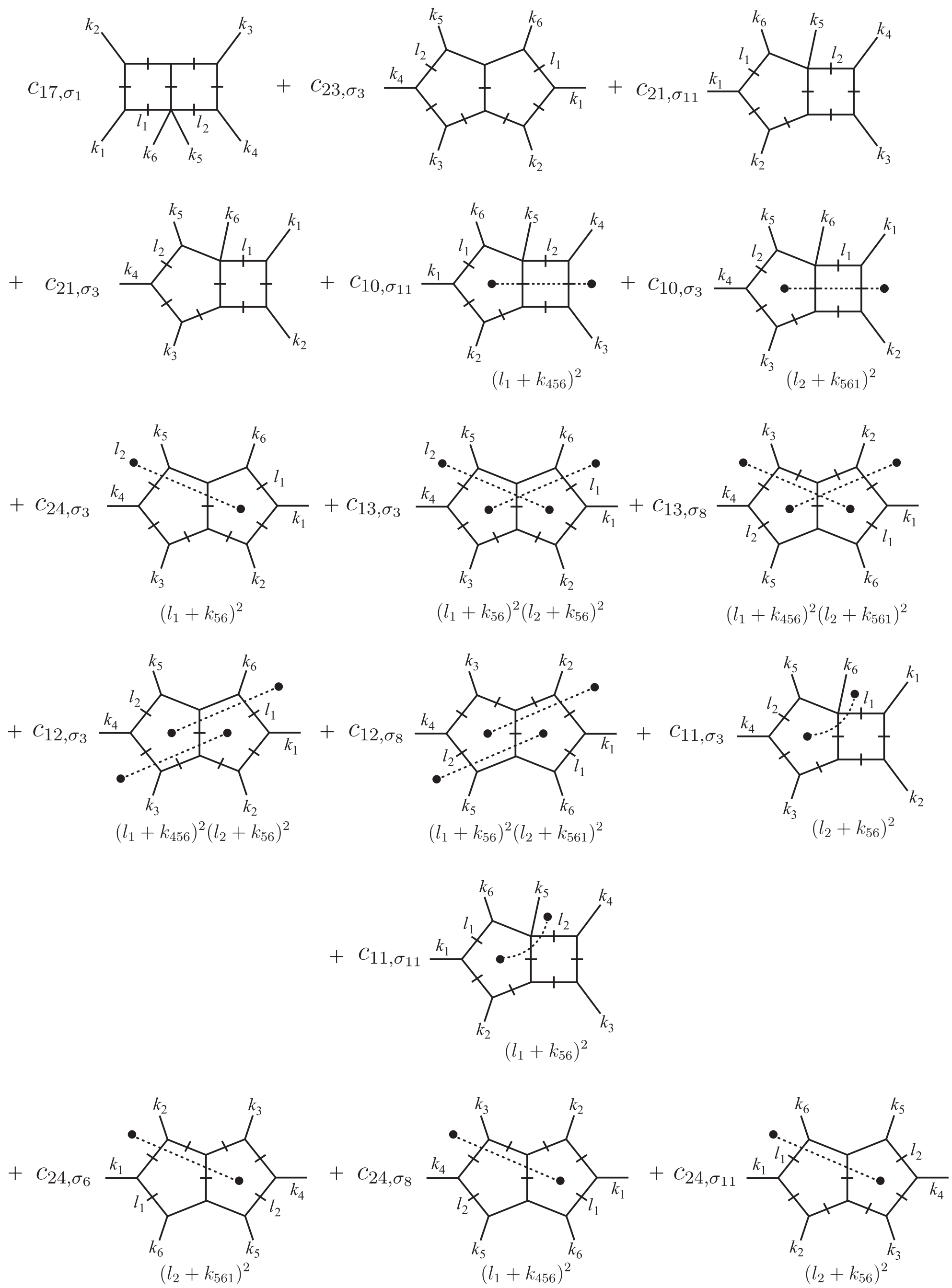
$$\prod_{j=1}^6 A_j^{\text{tree}}(z)\Big|_{\mathcal{S}_i} = 0 \quad \text{for } i = 1, \ldots, 4. \tag{A87}$$

Heptacut #8

This heptacut is defined by the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) with the vertex momenta

$$K_1 = k_1 \qquad K_2 = k_2 \qquad K_3 = 0 \qquad K_4 = k_3 \qquad K_5 = k_4 \qquad K_6 = k_{56}. \tag{A88}$$

Applying this heptacut to the right-hand side of Eq. (4.1) leaves the following linear combination of cut integrals



We define the spinor ratios

$$P_1 = -\frac{K_5^\flat \cdot K_6 + \frac{1}{2}S_6 + (K_5^\flat + K_6)\cdot K_1^\flat}{\langle K_2^{\flat -}|\not{K}_5^\flat + \not{K}_6|K_1^{\flat -}\rangle}, \qquad P_2 = -\frac{K_1^\flat \cdot k_6}{\langle K_2^{\flat -}|\not{k}_6|K_1^{\flat -}\rangle},$$
$$P_3 = -\frac{K_1^\flat \cdot k_{56} + \frac{1}{2}s_{56}}{\langle K_2^{\flat -}|\not{k}_{56}|K_1^{\flat -}\rangle} \qquad P_4 = -\frac{\langle K_4^{\flat -}|\not{K}_1^\flat + \not{K}_6|K_5^{\flat -}\rangle}{2\langle K_2^\flat K_4^\flat\rangle[K_5^\flat K_1^\flat]}$$
$$Q_1 = -\frac{K_1^\flat \cdot K_6 + \frac{1}{2}S_6 + (K_1^\flat + K_6)\cdot K_5^\flat}{\langle K_5^{\flat -}|\not{K}_1^\flat + \not{K}_6|K_4^{\flat -}\rangle}, \qquad Q_2 = -\frac{K_5^\flat \cdot k_5}{\langle K_5^{\flat -}|\not{k}_5|K_4^{\flat -}\rangle}, \qquad Q_3 = -\frac{K_5^\flat \cdot k_{56} + \frac{1}{2}s_{56}}{\langle K_5^{\flat -}|\not{k}_{56}|K_4^{\flat -}\rangle}, \qquad \text{(A89)}$$

and their parity conjugates

$$P_1^\bullet = -\frac{K_5^\flat\cdot K_6 + \frac{1}{2}S_6 + (K_5^\flat + K_6)\cdot K_1^\flat}{\langle K_1^{\flat-}|\not{K}_5^\flat + \not{K}_6|K_2^{\flat-}\rangle}, \qquad P_2^\bullet = -\frac{K_1^\flat\cdot k_6}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle},$$
$$P_3^\bullet = -\frac{K_1^\flat\cdot k_{56} + \frac{1}{2}s_{56}}{\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle} \qquad P_4^\bullet = -\frac{\langle K_5^{\flat-}|\not{K}_1^\flat + \not{K}_6|K_4^{\flat-}\rangle}{2[K_2^\flat K_4^\flat]\langle K_5^\flat K_1^\flat\rangle}$$
$$Q_1^\bullet = -\frac{K_1^\flat\cdot K_6 + \frac{1}{2}S_6 + (K_1^\flat + K_6)\cdot K_5^\flat}{\langle K_4^{\flat-}|\not{K}_1^\flat + \not{K}_6|K_5^{\flat-}\rangle}, \qquad Q_2^\bullet = -\frac{K_5^\flat\cdot k_5}{\langle K_4^{\flat-}|\not{k}_5|K_5^{\flat-}\rangle}, \qquad Q_3^\bullet = -\frac{K_5^\flat\cdot k_{56} + \frac{1}{2}s_{56}}{\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle}. \tag{A90}$$

This heptacut belongs to case I treated in Sec. III A, and there are thus six kinematical solutions (shown in Fig. 2). Parametrizing the loop momenta according to Eqs. (3.12) and (3.13), the on-shell constraints in Eqs. (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) are solved by setting the parameters equal to the values

$$\alpha_1 = 1, \qquad \alpha_2 = 0, \qquad \beta_1 = 0, \qquad \beta_2 = 1 \tag{A91}$$

and those given in Fig. 2 with

$$\beta_3(z) = -\frac{\langle K_2^{\flat-}|\not{K}_5^\flat + \not{K}_6|K_1^{\flat-}\rangle(z-P_1)}{2\langle K_2^\flat K_4^\flat\rangle[K_5^\flat K_1^\flat](z-P_4)}, \tag{A92}$$

for kinematical solution $\mathcal{S}_5$. The heptacut double-box integral I_{17,σ_1} is $\sum_{i=1}^6 \oint_{\Gamma_i} dz J_i(z)$ where

$$J_i(z) = \frac{1}{32\gamma_1\gamma_2}\times\begin{cases} (\langle K_1^{\flat-}|\not{K}_5^\flat + \not{K}_6|K_2^{\flat-}\rangle z(z-P_1^\bullet))^{-1} & \text{for } i = 2,6 \\ (\langle K_2^{\flat-}|\not{K}_5^\flat + \not{K}_6|K_1^{\flat-}\rangle z(z-P_1))^{-1} & \text{for } i = 4,5 \\ (\langle K_4^{\flat-}|\not{K}_1^\flat + \not{K}_6|K_5^{\flat-}\rangle z(z-Q_1^\bullet))^{-1} & \text{for } i = 1 \\ (\langle K_5^{\flat-}|\not{K}_1^\flat + \not{K}_6|K_4^{\flat-}\rangle z(z-Q_1))^{-1} & \text{for } i = 3 \end{cases}. \tag{A93}$$

Heptacut #8 of the right-hand side of Eq. (4.1)

The result of applying heptacut #8 to the right-hand side of Eq. (4.1) is

$$\frac{1}{4}\sum_{i=1}^{6}\oint_{\Gamma_i} dz J_i(z)K_i(z), \tag{A94}$$

where the kernels evaluated on the six kinematical solutions are

$$\begin{aligned} K_1(z) = {} & c_{17,\sigma_1} + \frac{1}{2}\frac{c_{21,\sigma_{11}}}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle(P_1^\bullet - P_2^\bullet)} + \frac{1}{2}\frac{c_{21,\sigma_3}}{\langle K_4^{\flat-}|\not{k}_5|K_5^{\flat-}\rangle(z-Q_2^\bullet)} \\ & + \frac{1}{4}\frac{c_{23,\sigma_3}}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_5|K_5^{\flat-}\rangle(P_1^\bullet - P_2^\bullet)(z-Q_2^\bullet)} + \frac{c_{10,\sigma_3}\langle K_4^{\flat-}|\not{k}_{561}|K_5^{\flat-}\rangle(z-Q_1^\bullet)}{\langle K_4^{\flat-}|\not{k}_5|K_5^{\flat-}\rangle(z-Q_2^\bullet)} \\ & + \frac{1}{2}\frac{c_{24,\sigma_3}\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle(P_1^\bullet - P_3^\bullet) + c_{24,\sigma_6}\langle K_4^{\flat-}|\not{k}_{561}|K_5^{\flat-}\rangle(z-Q_1^\bullet) + c_{24,\sigma_{11}}\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle(z-Q_3^\bullet)}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_5|K_5^{\flat-}\rangle(P_1^\bullet - P_2^\bullet)(z-Q_2^\bullet)} \\ & + \frac{c_{13,\sigma_3}\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle(P_1^\bullet - P_3^\bullet)(z-Q_3^\bullet)}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_5|K_5^{\flat-}\rangle(P_1^\bullet - P_2^\bullet)(z-Q_2^\bullet)} + \frac{c_{11,\sigma_3}\langle K_4^{\flat-}|\not{k}_{56}|K_5^{\flat-}\rangle(z-Q_3^\bullet)}{\langle K_4^{\flat-}|\not{k}_5|K_5^{\flat-}\rangle(z-Q_2^\bullet)} \\ & + \frac{c_{12,\sigma_8}\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_{561}|K_5^{\flat-}\rangle(P_1^\bullet - P_3^\bullet)(z-Q_1^\bullet)}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle\langle K_4^{\flat-}|\not{k}_5|K_5^{\flat-}\rangle(P_1^\bullet - P_2^\bullet)(z-Q_2^\bullet)} + \frac{c_{11,\sigma_{11}}\langle K_1^{\flat-}|\not{k}_{56}|K_2^{\flat-}\rangle(P_1^\bullet - P_3^\bullet)}{\langle K_1^{\flat-}|\not{k}_6|K_2^{\flat-}\rangle(P_1^\bullet - P_2^\bullet)} \end{aligned} \tag{A95}$$

$$\begin{aligned} K_2(z) = {} & c_{17,\sigma_1} + \frac{1}{2}\frac{c_{21,\sigma_{11}}}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle(z-P_2^{\bullet})} + \frac{1}{2}\frac{c_{21,\sigma_3}}{\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(Q_1^{\bullet}-Q_2^{\bullet})} \\ & + \frac{1}{4}\frac{c_{23,\sigma_3}}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(z-P_2^{\bullet})(Q_1^{\bullet}-Q_2^{\bullet})} + \frac{c_{10,\sigma_{11}}\langle K_1^{\flat -}|\not{K}_{456}|K_2^{\flat -}\rangle(z-P_1^{\bullet})}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle(z-P_2^{\bullet})} \\ & + \frac{1}{2}\frac{c_{24,\sigma_3}\langle K_1^{\flat -}|\not{K}_{56}|K_2^{\flat -}\rangle(z-P_3^{\bullet}) + c_{24,\sigma_8}\langle K_1^{\flat -}|\not{K}_{456}|K_2^{\flat -}\rangle(z-P_1^{\bullet}) + c_{24,\sigma_{11}}\langle K_4^{\flat -}|\not{K}_{56}|K_5^{\flat -}\rangle(Q_1^{\bullet}-Q_3^{\bullet})}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(z-P_2^{\bullet})(Q_1^{\bullet}-Q_2^{\bullet})} \\ & + \frac{c_{13,\sigma_3}\langle K_1^{\flat -}|\not{K}_{56}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{56}|K_5^{\flat -}\rangle(z-P_3^{\bullet})(Q_1^{\bullet}-Q_3^{\bullet})}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(z-P_2^{\bullet})(Q_1^{\bullet}-Q_2^{\bullet})} + \frac{c_{11,\sigma_3}\langle K_4^{\flat -}|\not{K}_{56}|K_5^{\flat -}\rangle(Q_1^{\bullet}-Q_3^{\bullet})}{\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(Q_1^{\bullet}-Q_2^{\bullet})} \\ & + \frac{c_{12,\sigma_3}\langle K_1^{\flat -}|\not{K}_{456}|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{56}|K_5^{\flat -}\rangle(z-P_1^{\bullet})(Q_1^{\bullet}-Q_3^{\bullet})}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(z-P_2^{\bullet})(Q_1^{\bullet}-Q_2^{\bullet})} + \frac{c_{11,\sigma_{11}}\langle K_1^{\flat -}|\not{K}_{56}|K_2^{\flat -}\rangle(z-P_3^{\bullet})}{\langle K_1^{\flat -}|\not{K}_6|K_2^{\flat -}\rangle(z-P_2^{\bullet})} \end{aligned} \tag{A96}$$

$$K_3(z) = \text{parity conjugate of } K_1(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}] \tag{A97}$$

$$K_4(z) = \text{parity conjugate of } K_2(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}] \tag{A98}$$

$$\begin{aligned} K_5(z) = {} & c_{17,\sigma_1} + \frac{1}{4}\frac{c_{23,\sigma_3}}{\langle K_2^{\flat -}|\not{K}_6|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^{\bullet})} \\ & + \frac{1}{2}\frac{c_{24,\sigma_3}\langle K_2^{\flat -}|\not{K}_{56}|K_1^{\flat -}\rangle(z-P_3) + c_{24,\sigma_6}\langle K_4^{\flat -}|\not{K}_{561}|K_5^{\flat -}\rangle(\beta_3(z)-Q_1^{\bullet})}{\langle K_2^{\flat -}|\not{K}_6|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^{\bullet})} + \frac{1}{2}\frac{c_{21,\sigma_{11}}}{\langle K_2^{\flat -}|\not{K}_6|K_1^{\flat -}\rangle(z-P_2)} \\ & + \frac{1}{2}\frac{c_{24,\sigma_8}\langle K_2^{\flat -}|\not{K}_{456}|K_1^{\flat -}\rangle(z-P_1) + c_{24,\sigma_{11}}\langle K_4^{\flat -}|\not{K}_{56}|K_5^{\flat -}\rangle(\beta_3(z)-Q_3^{\bullet})}{\langle K_2^{\flat -}|\not{K}_6|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^{\bullet})} + \frac{1}{2}\frac{c_{21,\sigma_3}}{\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(\beta_3(z)-Q_2^{\bullet})} \\ & + \frac{c_{13,\sigma_3}\langle K_2^{\flat -}|\not{K}_{56}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{56}|K_5^{\flat -}\rangle(z-P_3)(\beta_3(z)-Q_3^{\bullet})}{\langle K_2^{\flat -}|\not{K}_6|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^{\bullet})} + \frac{c_{10,\sigma_{11}}\langle K_2^{\flat -}|\not{K}_{456}|K_1^{\flat -}\rangle(z-P_1)}{\langle K_2^{\flat -}|\not{K}_6|K_1^{\flat -}\rangle(z-P_2)} \\ & + \frac{c_{13,\sigma_8}\langle K_2^{\flat -}|\not{K}_{456}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{561}|K_5^{\flat -}\rangle(z-P_1)(\beta_3(z)-Q_1^{\bullet})}{\langle K_2^{\flat -}|\not{K}_6|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^{\bullet})} + \frac{c_{10,\sigma_3}\langle K_4^{\flat -}|\not{K}_{561}|K_5^{\flat -}\rangle(\beta_3(z)-Q_1^{\bullet})}{\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(\beta_3(z)-Q_2^{\bullet})} \\ & + \frac{c_{12,\sigma_3}\langle K_2^{\flat -}|\not{K}_{456}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{56}|K_5^{\flat -}\rangle(z-P_1)(\beta_3(z)-Q_3^{\bullet})}{\langle K_2^{\flat -}|\not{K}_6|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^{\bullet})} + \frac{c_{11,\sigma_3}\langle K_4^{\flat -}|\not{K}_{56}|K_5^{\flat -}\rangle(\beta_3(z)-Q_3^{\bullet})}{\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(\beta_3(z)-Q_2^{\bullet})} \\ & + \frac{c_{12,\sigma_8}\langle K_2^{\flat -}|\not{K}_{56}|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_{561}|K_5^{\flat -}\rangle(z-P_3)(\beta_3(z)-Q_1^{\bullet})}{\langle K_2^{\flat -}|\not{K}_6|K_1^{\flat -}\rangle\langle K_4^{\flat -}|\not{K}_5|K_5^{\flat -}\rangle(z-P_2)(\beta_3(z)-Q_2^{\bullet})} + \frac{c_{11,\sigma_{11}}\langle K_2^{\flat -}|\not{K}_{56}|K_1^{\flat -}\rangle(z-P_3)}{\langle K_2^{\flat -}|\not{K}_6|K_1^{\flat -}\rangle(z-P_2)} \end{aligned} \tag{A99}$$

$$K_6(z) = \text{parity conjugate of } K_5(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}] \tag{A100}$$

where $\beta_3(z)$ is given in Eq. (A92).

Heptacut #8 of the left-hand side of Eq. (4.1)

The result of applying heptacut #8 to the left-hand side of Eq. (4.1) is

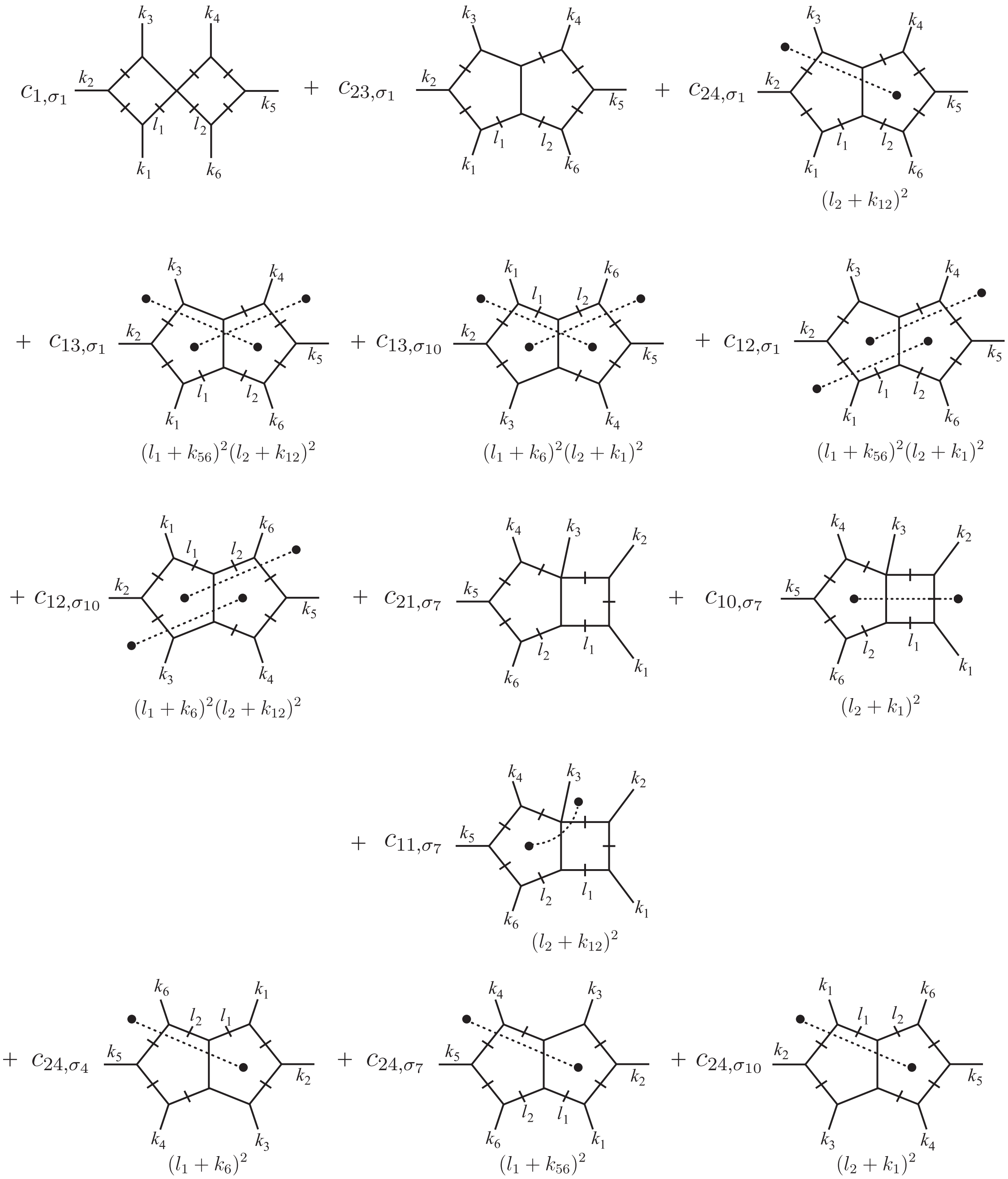
$$i\sum_{i=1}^{6}\oint_{\Gamma_i} dz J_i(z)\prod_{j=1}^{6} A_j^{\text{tree}}(z)\bigg|_{\mathcal{S}_i}, \tag{A101}$$

where, assuming without loss of generality the external helicities are $(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$, the cut amplitude evaluated on the six different kinematical solutions yields

$$\prod_{j=1}^{6} A_j^{\text{tree}}(z)\bigg|_{\mathcal{S}_i} = -\frac{i}{16} A^{\text{tree}}_{--++++} \times \begin{cases} \frac{1}{J_1(z)}\left(\frac{1}{z} - \frac{1}{z - Q_2^\bullet}\right) & \text{for} \quad i = 1 \\ \frac{1}{J_4(z)}\left(\frac{1}{z} - \frac{1}{z - P_2}\right) & \text{for} \quad i = 4 \\ \frac{1}{J_6(z)}\left(\frac{1}{z} - \frac{1}{z - P_1^\bullet}\right) & \text{for} \quad i = 6 \\ 0 & \text{for} \quad i = 2, 3, 5 \end{cases} . \tag{A102}$$

Heptacut #9

This heptacut is defined by the on-shell constraints in Eqs. (3.47), (3.48), (3.49), (3.50), (3.51), (3.52), and (3.53). Applying it to the right-hand side of Eq. (4.1) leaves the following linear combination of cut integrals



We define the spinor ratios

$$P_1 = \frac{[k_3k_2]}{2[k_3k_1]}, \qquad P_2 = -\frac{\langle k_1k_6\rangle}{2\langle k_2k_6\rangle}, \qquad P_3 = -\frac{k_1\cdot k_{56} + \frac{1}{2}s_{56}}{\langle k_2^-|\not{k}_{56}|k_1^-\rangle}, \qquad P_4 = -\frac{\langle k_1k_6\rangle + \frac{[k_5k_4]}{[k_6k_4]}\langle k_1k_5\rangle}{2(\langle k_2k_6\rangle + \frac{[k_5k_4]}{[k_6k_4]}\langle k_2k_5\rangle)}$$
$$Q_1 = \frac{\langle k_4k_5\rangle}{2\langle k_4k_6\rangle}, \qquad Q_2 = -\frac{k_6\cdot k_{12} + \frac{1}{2}s_{12}}{\langle k_6^-|\not{k}_{12}|k_5^-\rangle}, \qquad Q_3 = -\frac{[k_1k_6]}{2[k_1k_5]}, \tag{A103}$$

and their parity conjugates

$$P_1^\bullet = \frac{\langle k_3k_2\rangle}{2\langle k_3k_1\rangle}, \qquad P_2^\bullet = -\frac{[k_1k_6]}{2[k_2k_6]}, \qquad P_3^\bullet = -\frac{k_1\cdot k_{56} + \frac{1}{2}s_{56}}{\langle k_1^-|\not{k}_{56}|k_2^-\rangle}, \qquad P_4^\bullet = -\frac{[k_1k_6] + \frac{\langle k_5k_4\rangle}{\langle k_6k_4\rangle}[k_1k_5]}{2([k_2k_6] + \frac{\langle k_5k_4\rangle}{\langle k_6k_4\rangle}[k_2k_5])}$$
$$Q_1^\bullet = \frac{[k_4k_5]}{2[k_4k_6]}, \qquad Q_2^\bullet = -\frac{k_6\cdot k_{12} + \frac{1}{2}s_{12}}{\langle k_5^-|\not{k}_{12}|k_6^-\rangle}, \qquad Q_3^\bullet = -\frac{\langle k_1k_6\rangle}{2\langle k_1k_5\rangle}. \tag{A104}$$

This heptacut was treated in Sec. III B, and there are four kinematical solutions (shown in Fig. 6). Parametrizing the loop momenta according to Eqs. (3.54) and (3.55), the on-shell constraints in Eqs. (3.47), (3.48), (3.49), (3.50), (3.51), (3.52), and (3.53) are solved by setting the parameters equal to the values

$$\alpha_1 = 1, \qquad \alpha_2 = 0, \qquad \beta_1 = 0, \qquad \beta_2 = 1 \tag{A105}$$

and those given in Fig. 6. The Jacobian associated with the heptacut (3.47), (3.48), (3.49), (3.50), (3.51), (3.52), and (3.53) is

$$J_i(z) = -\frac{1}{16s_{12}s_{45}s_{56}}\frac{1}{z} \quad \text{for } i = 1, \ldots, 4. \tag{A106}$$

Heptacut #9 of the right-hand side of Eq. (4.1)

The result of applying heptacut #9 to the right-hand side of Eq. (4.1) is

$$\frac{1}{4}\sum_{i=1}^{4}\oint_{\Gamma_i} dz J_i(z) K_i(z), \tag{A107}$$

where the kernels evaluated on the four kinematical solutions are

$$\begin{aligned}
K_1(z) = &-\frac{1}{2\langle k_1^-|\not{k}_3|k_2^-\rangle(z-P_1^\bullet)}\Bigg[c_{1,\sigma_1} + \frac{1}{2}\bigg(\langle k_1^-|\not{k}_6|k_2^-\rangle + \langle k_1k_5\rangle[k_6k_2]\frac{[k_5k_4]}{[k_6k_4]}\bigg)^{-1}\frac{1}{z-P_2^\bullet}(c_{23,\sigma_1} + 4c_{13,\sigma_{10}}\langle k_1^-|\not{k}_6|k_2^-\rangle \\
&\times \langle k_5^-|\not{k}_1|k_6^-\rangle(z-P_2^\bullet)(Q_1^\bullet - Q_3^\bullet) + 4c_{12,\sigma_1}\langle k_1^-|\not{k}_{56}|k_2^-\rangle\langle k_5^-|\not{k}_1|k_6^-\rangle(z-P_3^\bullet)(Q_1^\bullet - Q_3^\bullet) + 4c_{12,\sigma_{10}}\langle k_1^-|\not{k}_6|k_2^-\rangle \\
&\times \langle k_5^-|\not{k}_{12}|k_6^-\rangle(z-P_2^\bullet)(Q_1^\bullet - Q_2^\bullet) - 2\langle k_1^-|\not{k}_3|k_2^-\rangle(z-P_1^\bullet)(c_{21,\sigma_7} + 2c_{10,\sigma_7}\langle k_5^-|\not{k}_1|k_6^-\rangle(Q_1^\bullet - Q_3^\bullet) \\
&+ 2c_{11,\sigma_7}\langle k_5^-|\not{k}_{12}|k_6^-\rangle(Q_1^\bullet - Q_2^\bullet)) + 2c_{24,\sigma_1}\langle k_5^-|\not{k}_{12}|k_6^-\rangle(Q_1^\bullet - Q_2^\bullet) + 2c_{24,\sigma_4}\langle k_1^-|\not{k}_6|k_2^-\rangle(z-P_2^\bullet) \\
&+ 2c_{24,\sigma_7}\langle k_1^-|\not{k}_{56}|k_2^-\rangle(z-P_3^\bullet) + 2c_{24,\sigma_{10}}\langle k_5^-|\not{k}_1|k_6^-\rangle(Q_1^\bullet - Q_3^\bullet) \\
&+ 4c_{13,\sigma_1}\langle k_1^-|\not{k}_{56}|k_2^-\rangle\langle k_5^-|\not{k}_{12}|k_6^-\rangle(z-P_3^\bullet)(Q_1^\bullet - Q_2^\bullet))\Bigg]
\end{aligned} \tag{A108}$$

$$\begin{aligned}
K_2(z) = &-\frac{1}{2\langle k_1^-|\not{k}_3|k_2^-\rangle(z-P_1^\bullet)}\Bigg[c_{1,\sigma_1} + \frac{1}{2}\bigg(\langle k_1^-|\not{k}_6|k_2^-\rangle + \langle k_1k_6\rangle[k_5k_2]\frac{\langle k_5k_4\rangle}{\langle k_6k_4\rangle}\bigg)^{-1}\frac{1}{z-P_4^\bullet}(c_{23,\sigma_1} + 4c_{13,\sigma_{10}}\langle k_1^-|\not{k}_6|k_2^-\rangle \\
&\times \langle k_6^-|\not{k}_1|k_5^-\rangle(z-P_2^\bullet)(Q_1 - Q_3) + 4c_{12,\sigma_1}\langle k_1^-|\not{k}_{56}|k_2^-\rangle\langle k_6^-|\not{k}_1|k_5^-\rangle(z-P_3^\bullet)(Q_1 - Q_3) + 4c_{12,\sigma_{10}}\langle k_1^-|\not{k}_6|k_2^-\rangle \\
&\times \langle k_6^-|\not{k}_{12}|k_5^-\rangle(z-P_2^\bullet)(Q_1 - Q_2) - 2\langle k_1^-|\not{k}_3|k_2^-\rangle(z-P_1^\bullet)(c_{21,\sigma_7} + 2c_{10,\sigma_7}\langle k_6^-|\not{k}_1|k_5^-\rangle(Q_1 - Q_3) \\
&+ 2c_{11,\sigma_7}\langle k_6^-|\not{k}_{12}|k_5^-\rangle(Q_1 - Q_2)) + 2c_{24,\sigma_1}\langle k_6^-|\not{k}_{12}|k_5^-\rangle(Q_1 - Q_2) + 2c_{24,\sigma_4}\langle k_1^-|\not{k}_6|k_2^-\rangle(z-P_2^\bullet) \\
&+ 2c_{24,\sigma_7}\langle k_1^-|\not{k}_{56}|k_2^-\rangle(z-P_3^\bullet) + 2c_{24,\sigma_{10}}\langle k_6^-|\not{k}_1|k_5^-\rangle(Q_1 - Q_3) \\
&+ 4c_{13,\sigma_1}\langle k_1^-|\not{k}_{56}|k_2^-\rangle\langle k_6^-|\not{k}_{12}|k_5^-\rangle(z-P_3^\bullet)(Q_1 - Q_2))\Bigg]
\end{aligned} \tag{A109}$$

$$K_3(z) = \text{parity conjugate of } K_2(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}] \tag{A110}$$

$$K_4(z) = \text{parity conjugate of } K_1(z) \qquad [\text{obtained by applying Eqs. (4.12)–(4.17)}]. \tag{A111}$$

Heptacut #9 of the left-hand side of Eq. (4.1)

The result of applying heptacut #9 to the left-hand side of Eq. (4.1) is

$$i \sum_{i=1}^{4} \oint_{\Gamma_i} dz J_i(z) \prod_{j=1}^{6} A_j^{\text{tree}}(z) \bigg|_{\mathcal{S}_i}, \tag{A112}$$

where, assuming without loss of generality the external helicities are $(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$, the cut amplitude evaluated on the four different kinematical solutions yields

$$\prod_{j=1}^{6} A_j^{\text{tree}}(z) \bigg|_{\mathcal{S}_i} = \frac{i}{16} A^{\text{tree}}_{--++++} \times \begin{cases} \frac{1}{J_2(z)}\left(\frac{1}{z} - \frac{1}{z-P_1^\bullet}\right) & \text{for } i = 2 \\ \frac{1}{J_4(z)}\left(\frac{1}{z} - \frac{1}{z-P_2}\right) & \text{for } i = 4 \\ 0 & \text{for } i = 1, 3. \end{cases} \tag{A113}$$

---

[1] R. Britto, F. Cachazo, and B. Feng, Nucl. Phys. **B715**, 499 (2005).
[2] R. Britto, F. Cachazo, B. Feng, and E. Witten, Phys. Rev. Lett. **94**, 181602 (2005).
[3] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B425**, 217 (1994).
[4] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B435**, 59 (1995).
[5] Z. Bern, L. J. Dixon, and D. A. Kosower, Annu. Rev. Nucl. Part. Sci. **46**, 109 (1996).
[6] Z. Bern and A. G. Morgan, Nucl. Phys. **B467**, 479 (1996).
[7] Z. Bern, L. J. Dixon, and D. A. Kosower, Nucl. Phys. **B513**, 3 (1998).
[8] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Phys. Lett. B **394**, 105 (1997).
[9] R. Britto, F. Cachazo, and B. Feng, Nucl. Phys. **B725**, 275 (2005).
[10] R. Britto, F. Cachazo, and B. Feng, Phys. Rev. D **71**, 025012 (2005).
[11] S. J. Bidder, N. E. J. Bjerrum-Bohr, L. J. Dixon, and D. C. Dunbar, Phys. Lett. B **606**, 189 (2005).
[12] S. J. Bidder, N. E. J. Bjerrum-Bohr, D. C. Dunbar, and W. B. Perkins, Phys. Lett. B **612**, 75 (2005).
[13] S. J. Bidder, D. C. Dunbar, and W. B. Perkins, J. High Energy Phys. 08 (2005) 055.
[14] Z. Bern, N. E. J. Bjerrum-Bohr, D. C. Dunbar, and H. Ita, J. High Energy Phys. 11 (2005) 027.
[15] N. E. J. Bjerrum-Bohr, D. C. Dunbar, and W. B. Perkins, J. High Energy Phys. 04 (2008) 038.
[16] Z. Bern, L. J. Dixon, and D. A. Kosower, Phys. Rev. D **73**, 065013 (2006).
[17] R. Britto, E. Buchbinder, F. Cachazo, and B. Feng, Phys. Rev. D **72**, 065012 (2005).
[18] R. Britto, B. Feng, and P. Mastrolia, Phys. Rev. D **73**, 105004 (2006).
[19] P. Mastrolia, Phys. Lett. B **644**, 272 (2007).
[20] A. Brandhuber, S. McNamara, B. J. Spence, and G. Travaglini, J. High Energy Phys. 10 (2005) 011.
[21] G. Ossola, C. G. Papadopoulos, and R. Pittau, Nucl. Phys. **B763**, 147 (2007).
[22] Z. Bern, L. J. Dixon, and D. A. Kosower, Ann. Phys. (N.Y.) **322**, 1587 (2007).
[23] D. Forde, Phys. Rev. D **75**, 125019 (2007).
[24] S. D. Badger, J. High Energy Phys. 01 (2009) 049.
[25] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt, and P. Mastrolia, Phys. Lett. B **645**, 213 (2007).
[26] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt, and P. Mastrolia, J. High Energy Phys. 03 (2007) 111.
[27] W. T. Giele, Z. Kunszt, and K. Melnikov, J. High Energy Phys. 04 (2008) 049.
[28] R. Britto and B. Feng, Phys. Rev. D **75**, 105006 (2007).
[29] R. Britto and B. Feng, J. High Energy Phys. 02 (2008) 095.
[30] R. Britto, B. Feng, and P. Mastrolia, Phys. Rev. D **78**, 025031 (2008).
[31] R. Britto, B. Feng, and G. Yang, J. High Energy Phys. 09 (2008) 089.
[32] C. F. Berger and D. Forde, Annu. Rev. Nucl. Part. Sci. **60**, 181 (2010).
[33] Z. Bern, J. J. Carrasco, T. Dennen, Y.-t. Huang, and H. Ita, Phys. Rev. D **83**, 085022 (2011).
[34] D. A. Kosower and K. J. Larsen, Phys. Rev. D **85**, 045017 (2012).
[35] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. **B828**, 317 (2010).
[36] N. Berkovits and J. Maldacena, J. High Energy Phys. 09 (2008) 062.
[37] N. Beisert, R. Ricci, A. A. Tseytlin, and M. Wolf, Phys. Rev. D **78**, 126004 (2008).
[38] J. M. Drummond, J. M. Henn, and J. Plefka, J. High Energy Phys. 05 (2009) 046.
[39] T. Bargheer, N. Beisert, W. Galleas, F. Loebbert, and T. McLoughlin, J. High Energy Phys. 11 (2009) 056.
[40] A. Brandhuber, P. Heslop, and G. Travaglini, J. High Energy Phys. 08 (2009) 095.
[41] G. P. Korchemsky and E. Sokatchev, Nucl. Phys. **B832**, 1 (2010).
[42] A. Sever and P. Vieira, arXiv:0908.2437.

[43] L. F. Alday, D. Gaiotto, and J. Maldacena, J. High Energy Phys. 09 (2011) 032.
[44] N. Beisert, J. Henn, T. McLoughlin, and J. Plefka, J. High Energy Phys. 04 (2010) 085.
[45] L. F. Alday, J. Maldacena, A. Sever, and P. Vieira, J. Phys. A **43**, 485401 (2010).
[46] J. M. Drummond and L. Ferro, J. High Energy Phys. 12 (2010) 010.
[47] G. P. Korchemsky and E. Sokatchev, Nucl. Phys. **B839**, 377 (2010).
[48] L. F. Alday, D. Gaiotto, J. Maldacena, A. Sever, and P. Vieira, J. High Energy Phys. 04 (2011) 088.
[49] J. M. Henn, J. Phys. A **44**, 454011 (2011).
[50] M. Bullimore and D. Skinner, arXiv:1112.1056.
[51] S. Caron-Huot and S. He, J. High Energy Phys. 07 (2012) 174.
[52] N. Arkani-Hamed, F. Cachazo, C. Cheung, and J. Kaplan, J. High Energy Phys. 03 (2010) 020.
[53] L. J. Mason and D. Skinner, J. High Energy Phys. 11 (2009) 045.
[54] N. Arkani-Hamed, F. Cachazo, and C. Cheung, J. High Energy Phys. 03 (2010) 036.
[55] M. Bullimore, L. J. Mason, and D. Skinner, J. High Energy Phys. 03 (2010) 070.
[56] J. Kaplan, J. High Energy Phys. 03 (2010) 025.
[57] N. Arkani-Hamed, J. Bourjaily, F. Cachazo, and J. Trnka, J. High Energy Phys. 01 (2011) 049.
[58] J. L. Bourjaily, J. Trnka, A. Volovich, and C. Wen, J. High Energy Phys. 01 (2011) 038.
[59] M. Bullimore, J. High Energy Phys. 01 (2011) 055.
[60] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, S. Caron-Huot, and J. Trnka, J. High Energy Phys. 01 (2011) 041.
[61] R. H. Boels, J. High Energy Phys. 11 (2010) 113.
[62] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. Hodges, and J. Trnka, J. High Energy Phys. 04 (2012) 081.
[63] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, and J. Trnka, J. High Energy Phys. 06 (2012) 125.
[64] E. I. Buchbinder and F. Cachazo, J. High Energy Phys. 11 (2005) 036.
[65] F. Cachazo and D. Skinner, arXiv:0801.4574.
[66] F. Cachazo, arXiv:0803.1988.
[67] F. Cachazo, M. Spradlin, and A. Volovich, Phys. Rev. D **78**, 105022 (2008).
[68] M. Spradlin, A. Volovich, and C. Wen, Phys. Rev. D **78**, 085025 (2008).
[69] J. Gluza, K. Kajda, and D. A. Kosower, Phys. Rev. D **83**, 045012 (2011).
[70] F. Tkachov, Phys. Lett. **100B**, 65 (1981).
[71] K. Chetyrkin and F. Tkachov, Nucl. Phys. **B192**, 159 (1981).
[72] S. Laporta, Phys. Lett. B **504**, 188 (2001).
[73] S. Laporta, Int. J. Mod. Phys. A **15**, 5087 (2000).
[74] T. Gehrmann and E. Remiddi, Nucl. Phys. **B580**, 485 (2000).
[75] R. N. Lee, J. High Energy Phys. 07 (2008) 031.
[76] C. Anastasiou and A. Lazopoulos, J. High Energy Phys. 07 (2004) 046.
[77] A. V. Smirnov, J. High Energy Phys. 10 (2008) 107.
[78] C. Studerus, Comput. Phys. Commun. **181**, 1293 (2010).
[79] A. V. Smirnov and A. V. Petukhov, Lett. Math. Phys. **97**, 37 (2011).
[80] L. M. Brown and R. P. Feynman, Phys. Rev. **85**, 231 (1952).
[81] L. M. Brown, Nuovo Cimento **22**, 178 (1961).
[82] B. Petersson, J. Math. Phys. (N.Y.) **6**, 1955 (1965).
[83] G. Källén and J. Toll, J. Math. Phys. (N.Y.) **6**, 299 (1965).
[84] D. B. Melrose, Nuovo Cimento A **40**, 181 (1965).
[85] G. Passarino and M. J. G. Veltman, Nucl. Phys. **B160**, 151 (1979).
[86] W. van Neerven and J. Vermaseren, Phys. Lett. **137B**, 241 (1984).
[87] G. J. van Oldenborgh and J. Vermaseren, Z. Phys. C **46**, 425 (1990).
[88] D. Kosower (private communication).
[89] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu, and A. Volovich, Phys. Rev. D **78**, 045007 (2008).
[90] S. Caron-Huot (private communication).
[91] P. Mastrolia and G. Ossola, J. High Energy Phys. 11 (2011) 014.
[92] S. Badger, H. Frellesvig, and Y. Zhang, J. High Energy Phys. 04 (2012) 055.
[93] P. Mastrolia, E. Mirabella, G. Ossola, and T. Peraro, arXiv:1205.7087.
[94] R. Kleiss, I. Malamos, C. Papadopoulos, and R. Verheyen, arXiv:1206.4180.
[95] S. Badger, H. Frellesvig, and Y. Zhang, arXiv:1207.2976.
[96] Y. Zhang, arXiv:1205.5707.
[97] F. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, and T. T. Wu, Phys. Lett. **103B**, 124 (1981).
[98] P. De Causmaecker, R. Gastmans, W. Troost, and T. T. Wu, Nucl. Phys. **B206**, 53 (1982).
[99] Z. Xu, D.-H. Zhang, and L. Chang, Report No. TUTP-84/3-TSINGHUA, 1984.
[100] R. Kleiss and W. J. Stirling, Nucl. Phys. **B262**, 235 (1985).
[101] J. F. Gunion and Z. Kunszt, Phys. Lett. **161B**, 333 (1985).
[102] Z. Xu, D.-H. Zhang, and L. Chang, Nucl. Phys. **B291**, 392 (1987).
[103] M. L. Mangano and S. J. Parke, Phys. Rep. **200**, 301 (1991).
[104] L. J. Dixon, arXiv:hep-ph/9601359.
[105] D. A. Kosower, R. Roiban, and C. Vergu, Phys. Rev. D **83**, 065018 (2011).
[106] R. Roiban, M. Spradlin, and A. Volovich, Phys. Rev. D **70**, 026009 (2004).
[107] C. Vergu, Phys. Rev. D **75**, 025028 (2007).
[108] To be more accurate, we will use the terminology "global poles" to refer to points where, in addition to the seven on-shell constraints (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and (3.9) being solved, there is an additional singularity, for example coming from the Jacobians arising from linearizing these cut constraints. A more precise definition will be given in Sec. IVA 3.
[109] A function $f = (f_1, \ldots, f_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ is said to have an isolated zero at $a \in \mathbb{C}^n$ iff by choosing a small enough neighborhood U of a one can achieve $f^{-1}(0) \cap U = \{a\}$.
[110] These identities can be shown by writing the determinants as square roots of Gram determinants (since both of these equal the volume of the spanned parallelotope, up to a complex phase) and using the special properties of the vectors $\{K_1^{\flat\mu}, K_2^{\flat\mu}, \langle K_1^{\flat-}|\gamma^\mu|K_2^{\flat-}\rangle, \langle K_2^{\flat-}|\gamma^\mu|K_1^{\flat-}\rangle\}$ and $\{K_4^{\flat\mu}, K_5^{\flat\mu}, \langle K_4^{\flat-}|\gamma^\mu|K_5^{\flat-}\rangle, \langle K_5^{\flat-}|\gamma^\mu|K_4^{\flat-}\rangle\}$. This will fix

the determinants up to an overall factor of i^k which can be found numerically.

[111] By "tensor integral" is meant an integral whose integrand's numerator contains powers of the loop momenta contracted into external vectors; the corresponding standard Feynman integral with a 1 in the numerator is referred to as a "scalar integral" in this terminology.

[112] Z. Bern, L. J. Dixon, and D. A. Kosower, J. High Energy Phys. 01 (2000) 027.

[113] M. T. Grisaru and H. N. Pendleton, Nucl. Phys. **B124**, 81 (1977).

[114] V. Nair, Phys. Lett. B **214**, 215 (1988).

[115] This property holds for MHV amplitudes (at any loop order) only; however, for N^{k}MHV amplitudes, the residues can be expressed economically in terms of superconformal R-invariants [35].

[116] H. Elvang, D. Z. Freedman, and M. Kiermaier, J. High Energy Phys. 10 (2010) 103.

[117] More accurately, the object we are comparing is the strictly four-dimensional part of the integrand. Indeed, as remarked above, the μ-integrals I_{14,σ_j}, I_{15,σ_j} give rise to contributions to the integrand which are $\mathcal{O}(\epsilon)$ in the dimensional regulator and hence not obtainable from evaluating leading singularities in strictly four dimensions. Thus, our findings of agreement between the integrands should be interpreted as a statement concerning the $\mathcal{O}(\epsilon^0)$ part exclusively.

[118] See the ancillary file for the arXiv version of this manuscript.

[119] To generate n rational momenta which are lightlike in $(+,-,-,-)$ signature, the first $n-2$ can be chosen as arbitrary Pythagorean quadruples (for example, generated by using the parametrization $(m_3^2+m_1^2+m_2^2, m_3^2-m_1^2-m_2^2, 2m_1m_3, 2m_2m_3)$ with $m_i \in \mathbb{Z}$) normalized by their $\|\cdot\|_1$-norm. To ensure that the n-th momentum will be lightlike and satisfy momentum conservation, the $(n-1)$-th momentum is obtained by generating an additional Pythagorean quadruple ξ of unit $\|\cdot\|_1$-norm and then rescaling it by the constant $\alpha = -\frac{(\sum_{i=1}^{n-2} k_i)^2}{2\xi^\mu \sum_{i=1}^{n-2} k_{\mu i}} \in \mathbb{Q}$ whereby $k_n = -(\sum_{i=1}^{n-2} k_i + \alpha\xi)$ is lightlike and rational.

[120] P. Mastrolia, Phys. Lett. B **678**, 246 (2009).

[121] N. Arkani-Hamed, F. Cachazo, and J. Kaplan, J. High Energy Phys. 09 (2010) 016.