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Setting the renormalization scale in QCD: The principle of maximum conformality

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A key problem in making precise perturbative QCD predictions is the uncertainty in determining the renormalization scale μ of the running coupling $\alpha_s(\mu^2)$. The purpose of the running coupling in any gauge theory is to sum all terms involving the β function; in fact, when the renormalization scale is set properly, all nonconformal $\beta \neq 0$ terms in a perturbative expansion arising from renormalization are summed into the running coupling. The remaining terms in the perturbative series are then identical to that of a conformal theory; i.e., the corresponding theory with $\beta = 0$. The resulting scale-fixed predictions using the *principle of maximum conformality* (PMC) are independent of the choice of renormalization scheme—a key requirement of renormalization group invariance. The results avoid renormalon resummation and agree with QED scale setting in the Abelian limit. The PMC is also the theoretical principle underlying the Brodsky-Lepage-Mackenzie procedure, commensurate scale relations between observables, and the scale-setting method used in lattice gauge theory. The number of active flavors n_f in the QCD β function is also correctly determined. We discuss several methods for determining the PMC scale for QCD processes. We show that a single global PMC scale, valid at leading order, can be derived from basic properties of the perturbative QCD cross section. The elimination of the renormalization scale ambiguity and the scheme dependence using the PMC will not only increase the precision of QCD tests, but it will also increase the sensitivity of collider experiments to new physics beyond the Standard Model.

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I. INTRODUCTION

A key difficulty in making precise perturbative QCD predictions is the uncertainty in determining the renormalization scale μ of the running coupling $\alpha_s(\mu^2)$. It is common practice to simply guess a physical scale $\mu = Q$ of order of a typical momentum transfer Q in the process, and then vary the scale over a range $Q/2$ and $2Q$. This procedure is clearly problematic since the resulting fixed-order pQCD prediction will depend on the choice of renormalization scheme; it can even predict negative QCD cross sections at next-to-leading order [1].

The purpose of the running coupling in any gauge theory is to sum all terms involving the β function; in fact, when the renormalization scale μ is set properly, all nonconformal $\beta \neq 0$ terms in a perturbative expansion arising from renormalization are summed into the running coupling. The remaining terms in the perturbative series are then identical to that of a conformal theory; i.e., the theory with $\beta = 0$. The divergent *renormalon* series of order $\alpha_s^n \beta^n n!$ does not appear in the conformal series. Thus as in quantum electrodynamics, the renormalization scale μ is determined unambiguously by the principle of maximal conformality (PMC). This is also the principle underlying Brodsky-Lepage-Mackenzie (BLM) scale setting [2]

It should be recalled that there is no ambiguity in setting the renormalization scale in QED. In the standard Gell-Mann-Low scheme for QED, the renormalization scale is simply the virtuality of the virtual photon [3]. For example, in electron-muon elastic scattering, the renormalization scale is the virtuality of the exchanged photon, spacelike momentum transfer squared $\mu^2 = q^2 = t$. Thus

$$\alpha(t) = \frac{\alpha(t_0)}{1 - \Pi(t, t_0)}, \tag{1}$$

where

$$\Pi(t, t_0) = \frac{\Pi(t) - \Pi(t_0)}{1 - \Pi(t_0)}, \tag{2}$$

sums all vacuum polarization contributions to the dressed photon propagator, both proper and improper. [Here $\Pi(t) = \Pi(t, 0)$ is the sum of proper vacuum polarization insertions, subtracted at $t = 0$.] Formally, one can choose any initial renormalization scale $\mu_0^2 = t_0$, since the final result when summed to all orders will be independent of t_0. This is the invariance principle used to derive renormalization group results such as the Callan-Symanzik equations [4,5]. However, the formal invariance of physical results under changes in t_0 does not imply that there is no optimal scale. In fact, as seen in QED, the scale choice $\mu^2 = q^2$, the photon virtuality, immediately sums all vacuum polarization contributions to all orders exactly in the conventional Gell-Mann-Low scheme. With any other choice of scale, one will recover the same result, but only after summing an infinite number of vacuum polarization corrections.

Thus, although the *initial* choice of renormalization scale t_0 is arbitrary, the *final* scale t which sums the vacuum polarization corrections is unique and unambiguous. The resulting perturbative series is identical to the conformal series with zero β function. In the case of muonic atoms, the modified muon-nucleus Coulomb potential is precisely $-Z\alpha(-\vec{q}^2)/\vec{q}^2$; i.e., $\mu^2 = -\vec{q}^2$. Again, the renormalization scale is unique.

One can employ other renormalization schemes in QED, such as the $\overline{MS}$ scheme, but the physical result will be the same once one allows for the relative displacement of the scales of each scheme. For example, one can start with the result in the $\overline{MS}$ scheme for spacelike argument $q^2 = -Q^2$, for the standard one-loop charged lepton pair vacuum polarization contribution to the photon propagator using dimensional regularization:

$$\log\frac{\mu^2_{\overline{MS}}}{m_\ell^2} = 6\int_0^1 dx x(1-x)\log\frac{m_\ell^2 + Q^2 x(1-x)}{m_\ell^2}, \quad (3)$$

which becomes at large Q^2

$$\log\frac{\mu^2_{\overline{MS}}}{m_\ell^2} = \log\frac{Q^2}{m_\ell^2} - 5/3; \quad (4)$$

i.e., $\mu^2_{\overline{MS}} = Q^2 e^{-5/3}$. Thus if $Q^2 >> 4m_\ell^2$, we can identify

$$\alpha_{\overline{MS}}(e^{-5/3}q^2) = \alpha_{\mathrm{GM}-L}(q^2). \quad (5)$$

The $e^{-5/3}$ displacement of renormalization scales between the $\overline{MS}$ and Gell-Mann-Low schemes is a result of the convention [6] that was chosen to define the minimal dimensional regularization scheme. One can use another definition of the renormalization scheme, but the final physical prediction cannot depend on the convention. This invariance under choice of scheme is a consequence of the transitivity property of the renormalization group [3,7–9].

The same principle underlying renormalization scale setting in QED must also hold in QCD since the n_f terms in the QCD β function have the same role as the lepton N_ℓ vacuum polarization contributions in QED. QCD and QED share the same Yang-Mills Lagrangian. In fact, one can show [10] that QCD analytically continues as a function of N_C to Abelian theory when $N_C \to 0$ at fixed $\alpha = C_F\alpha_s$ with $C_F = \frac{N_C^2-1}{2N_C}$. For example, at lowest order $\beta_0^{\mathrm{QCD}} = \frac{1}{4\pi}(\frac{11}{3}N_C - \frac{2}{3}n_f) \to -\frac{1}{4\pi}\frac{2}{3}n_f$ at $N_C = 0$. Thus the same scale-setting procedure must be applicable to all renormalizable gauge theories.

Thus there is a close correspondence between the QCD renormalization scale and that of the analogous QED process. For example, in the case of e^+e^- annihilation to three jets, the PMC/BLM scale is set by the gluon jet virtuality, just as in the corresponding QED reaction. The specific argument of the running coupling depends on the renormalization scheme because of their intrinsic definitions; however, the actual numerical prediction is scheme independent.

The basic procedure for PMC/BLM scale setting is to shift the renormalization scale so that all terms involving the β function are absorbed into the running coupling. The remaining series is then identical with a conformal theory with $\beta = 0$. Thus, an important feature of the PMC is that its QCD predictions are independent of the choice of renormalization scheme. The PMC procedure also agrees with QED in the $N_C \to 0$ limit.

The determination of the PMC scale for exclusive processes is often straightforward. For example, consider the process $e^+e^- \to c\bar{c} \to c\bar{c}g^* \to c\bar{c}b\bar{b}$, where all the flavors and momenta of the final-state quarks are identified. The n_f terms at next-to-leading order (NLO) come from the quark loop in the gluon propagator. Thus the PMC scale for the differential cross section in the $\overline{MS}$ scheme is given simply by the $\overline{MS}$ scheme displacement of the gluon virtuality: $\mu^2_{\mathrm{PMC}} = e^{-5/3}(p_b + p_{\bar{b}})^2$.

In practice, one can identify the PMC/BLM scale for QCD by varying the initial renormalization scale μ_0^2 to identify all of the β-dependent nonconformal contributions. At lowest order $\beta_0 = \frac{1}{4\pi}(11/3N_C - 2/3n_f)$. Thus at NLO one can simply use the dependence on the number of flavors n_f that arises from the quark loops associated with ultraviolet renormalization as a marker for β_0.

In QCD, the n_f terms also arise from the renormalization of the three-gluon and four-gluon vertices as well as from gluon wave function renormalization.

It is often stated that the argument of the coupling in a renormalization scheme based on dimensional regularization has no physical meaning since the scale μ was originally introduced as a mass parameter in extended space-time dimensions. However, the QED example above shows that the $\overline{MS}$ scale is unambiguously related to invariants in physical $3 + 1$ space. The connection of $\alpha_{\overline{MS}}$ to the Gell-Mann-Low scheme can be established at all orders. This also provides the analytic extension [11] of the $\alpha_{\overline{MS}}$ scheme for finite fermion masses as well as to timelike arguments where the coupling is complex.

An example that shows how critical it is to properly fix the renormalization scale is the three-gluon vertex. The PMC/BLM scale that appears in the three-gluon vertex is a function of the virtuality of the three external gluons q_1^2, q_2^2, and q_3^2. It has been computed in detail in Ref. [12]. The results are surprising when the virtualities are very different as in the subprocess $gg \to g \to Q\bar{Q}$,

$$\hat{\mu}^2 \propto \frac{q^2_{\min} q^2_{\mathrm{med}}}{q^2_{\max}}, \quad (6)$$

where $|q^2_{\min}| < |q^2_{\mathrm{med}}| < |q^2_{\max}|$; i.e., $q^2_{\max}$ has the maximal virtuality [13]. The prediction based on simply guessing $\mu^2 \simeq q^2_{\max}$ would give misleading results.

The PMC/BLM scale that appears in the three-gluon vertex is the mass scale that controls the number of quark flavors n_f that appears in the triangle graph. This is verified by keeping the quark masses and threshold dynamics in the loop. Thus we accurately determine the number of flavors n_f that appears in the β function in the three-gluon coupling. This generalizes for all gluonic processes.

Although these results have been obtained using the pinch scheme, the final PMC/BLM result is scheme independent. The pinch scheme is used because it provides a gauge-invariant setting for the analysis. In effect one calculates a scattering amplitude with three on-shell quark currents. One then obtains 14 invariant amplitudes that describe the three-gluon vertex, only one of which is renormalized.

In fact the calculation of the PMC scale for the three-gluon vertex $g_a \to g_b g_c$ given in Eq. (6) uses the pinch scheme to obtain a gauge-invariant result. In effect, one computes the entire gauge-invariant on-shell amplitude $q_a + \bar{q}_a \to q_b \bar{q}_b + q_c \bar{q}_c$ including the triangle loop graph from quark loops with general mass. All 14 invariant amplitudes are computed analytically to one loop, only one of which is renormalized. The PMC scale for the three-gluon vertex as given in Eq. (6) also correctly sets the scale that controls the number of effective flavors that contribute to the β function for the three-gluon vertex. Details are given in Refs [12,13].

These results show that the usual method of guessing the renormalization scale for processes involving the three-gluon and four-gluon couplings typically misses this essential physics, assigns n_F incorrectly, and mischaracterizes the perturbative prediction. The error that is introduced can be in principle eliminated at infinite order, but only if one can sum the renormalon series.

The explicit result for the PMC/BLM scale is the physical scale controlling the quark threshold in the specific renormalization procedure used, but it is always possible to relate one scheme with another by the transitivity property of the renormalization group. This property is guaranteed by the PMC so there can be a constant displacement between schemes.

The PMC method is a general approach to set the renormalization scale in QCD including purely gluonic processes. It is scheme independent and void of renormalon growth due to the absence of the β-function terms in the perturbative expansion. We stress that the β function is gauge invariant in any correct renormalization scheme. The resulting conformal series is then gauge invariant. Thus the PMC is a gauge-invariant procedure.

It is sometimes argued that it is advantageous not to fix the renormalization scale at all, since its variation provides a measure of higher order contributions to the theory predictions. In fact, one obtains sensitivity only to the β-dependent nonconformal terms by this procedure. In some cases the conformal contributions may be unexpectedly large. For example, the very large electron-loop light-by-light scattering contribution [14] $\simeq 18(\alpha^3/\pi)^3$ to the muon anomalous magnetic moment is disassociated with renormalization or the β function. Of course, one can still compute the variation of the prediction around the PMC scale as an indicator of higher order nonconformal terms.

Stevenson has proposed that one should set the renormalization scale at a point where the predicted cross section has minimal variation with respect to μ—the principle of minimal sensitivity (PMS) [15]. However, unlike the PMC, the application of the PMS to jet production gives unphysical results [16] since it sums physics into the running coupling not associated with renormalization. Worse, the PMS prediction depends on the choice of renormalization scheme, and it violates the transitivity property of the renormalization group [17]. Such heuristic scale-setting methods also give incorrect results when applied to Abelian QED.

It should be emphasized that the factorization scale that enters predictions for QCD inclusive reactions is introduced to match nonperturbative and perturbative aspects of the parton distributions in hadrons; it is present even in conformal theory, and thus its determination is a completely separate issue from renormalization scale setting.

II. IDENTIFYING THE RENORMALIZATION SCALE USING THE PRINCIPLE OF MAXIMUM CONFORMALITY

Given the analytic form of the hard process amplitude or cross section as a series in $\alpha_s(\mu_0^2)$ calculated at an initial scale μ_0^2 and at a certain order [NLO, next-to-next-to-leading order (NNLO), and so on], one can identify the PMC scale, order by order, in a systematic way:

(1) The variation of the cross section with respect to $\log\mu_0^2$ can be used to distinguish the conformal terms versus the nonconformal terms proportional to the β function.

(2) The identified nonconformal terms have the form $\beta \times \log p_{ij}/\mu_0^2$ where $p_{ij} = p_i \cdot p_j$ are the scalar product invariants $i \neq j$ that enter the hard subprocess. In practice, these terms can be identified as coefficients of n_f, the number of flavors appearing in the β function; i.e., the flavor dependence arising from quark loops associated with coupling constant renormalization. The n_f terms in QCD arise from the renormalization in the three-gluon and four-gluon vertices as well as from gluon wave function renormalization.

(3) The scale is then shifted $\mu_0^2 \to \mu^2$ in order to absorb the nonconformal terms. Thus when the scale is correctly set, the coefficients of $\alpha_s(\mu^2)$ become independent of the β function and $\log\mu^2$.

(4) The series is then identical to that of the conformal theory where $\beta = 0$ as given by the Banks-Zaks method [18].

(5) The PMC scale is fixed for an observable (such as a differential cross section). PMC then can give a single effective global scale for the whole set of skeleton graphs entering the calculations that sums all the nonconformal β terms associated with renormalization into the running coupling.

Other examples of this procedure will be given in the next sections.

A. The global PMC scale

Ideally, as in the BLM method, one should allow for separate scales for each skeleton graph; e.g., for electron-electron scattering, one takes $\alpha(t)$ and $\alpha(u)$ for the t- and u-channel amplitudes, respectively.

Setting separate renormalization scales can be a challenging task for complicated processes in QCD where there are many final-state particles and thus many possible Lorentz scalars $p_{ij}^2 = p_i \cdot p_j$. However, one can obtain a useful first approximation to the full PMC/BLM scale-setting procedure by using a single *global* scale μ^2 that appropriately weights the individual BLM scales.

The global scale can be determined by varying the subprocess amplitude with respect to each invariant, thus determining the coefficients f_{ij} of $\log p_{ij}^2/\mu_0^2$ in the nonconformal terms in the amplitude. The global PMC scale is then

$$\mu^2 = C \times \Pi_{ij}[p_{ij}^2]^{w_{ij}}, \tag{7}$$

i.e.,

$$\log \mu^2 = \sum_{i\neq j} w_{ij} \log p_{ij}^2 + \log C, \tag{8}$$

where the weight for each invariant is

$$w_{ij} = \frac{f_{ij}}{\sum_{i\neq j} f_{ij}}, \tag{9}$$

and $\sum_{i\neq j} w_{ij} = 1$. The constant C is the scheme displacement; e.g., $C = e^{-5/3}$ for $\overline{MS}$ for $\mu^2 >> 4m_f^2$.

As a specific example of the application of a PMC global scale, consider the electron-electron scattering amplitude in QED. (For simplicity, we will just take the contribution of the convection current to the amplitude, as in scalar QED.) The Lorentz-invariant Born amplitude at the initial scale t_0 is then

$$M^0(t,u) = 4\pi\alpha(t_0)\left(\frac{s-u}{t} + \frac{s-t}{u}\right). \tag{10}$$

The running QED coupling $\alpha(q^2)$ in QED sums all proper and improper vacuum polarization graphs

$$M(t,u) = 4\pi\alpha(t)\left(\frac{s-u}{t}\right) + 4\pi\alpha(u)\left(\frac{s-t}{u}\right), \tag{11}$$

where to leading order

$$\alpha(t) = \alpha(t_0)\left(1 + n_\ell \frac{\alpha(t_0)}{3\pi}\log\frac{-t}{t_0}\right). \tag{12}$$

Aside from power-suppressed contributions involving the lepton masses, the resulting series is identical to the corresponding conformal theory with $\beta = 0$.

In this process we have contributions from both the t- and u-channel amplitudes that require separate renormalization scales for each skeleton graph. However, at leading order we can weight the amplitudes to obtain a single PMC/BLM scale that still sums the nonconformal β terms into the running coupling $\alpha(\mu^2)$ at leading order. For example, using the standard Gell-Mann—Low scheme, we can write

$$M(t,u) = f(t)\alpha(t) + g(u)\alpha(u) = (f(t)+g(u))\alpha(\hat{\mu}^2), \tag{13}$$

where $f(t) = 4\pi(s-u)/t$ and $g(u) = 4\pi(s-t)/u$ are the Born amplitudes for the t and u channels, respectively.

Then in this case we have two basic PMC scales $\alpha(t)$ and $\alpha(u)$ for each skeleton graph in the standard Gell-Mann—Low scheme used in QED. These couplings then sum all of the vacuum polarization corrections to the skeleton graphs to infinite order. The result is then gauge invariant and the logarithm of the global scale is

$$\log \hat{\mu}^2 = \frac{f(t)}{f(t)+g(u)}\log(-t) + \frac{g(u)}{f(t)+g(u)}\log(-u), \tag{14}$$

which duplicates the multiscale result at NLO.

One can also use the mean value theorem to obtain an effective single scale that analytically reproduces the exact multiscale result to next-to-leading order. Since it matches the exact result at NLO, it also retains gauge invariance at this order. Moreover, the PMC single or multiscale result is independent of the choice of scheme. The single scale result illustrates why it is wrong to guess a single scale like $\mu^2 = p_T^2$ since it fails to agree with this simple example.

Using kinematical constraints such as the total momentum conservation $s + t + u = 0$, the weighted scale dependence can be confined into the $\log(t/u)$ term inside the running coupling. The global scale $\hat{\mu}^2$ is maximal at $\theta_{CM} = \pi/2$ ($\hat{\mu}^2 = \sqrt{tu} = -t = -u$) and vanishes at the boundaries $(0, \pi)$ where $\tan^2(\theta_{CM}/2) = t/u$. The effective renormalization scale for electron-electron scattering in Eq. (14)

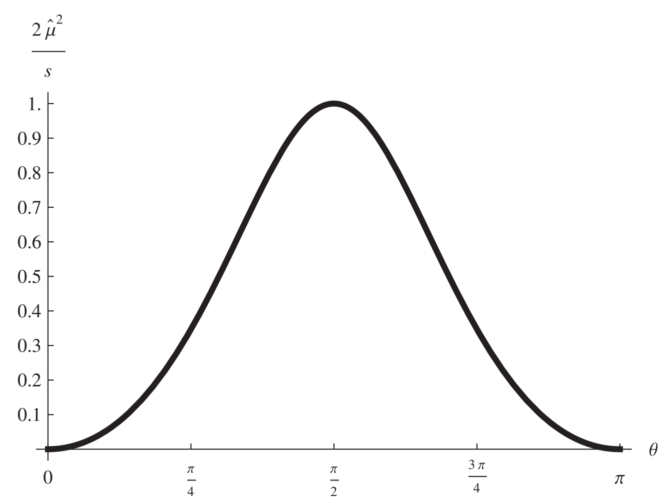

FIG. 1. The PMC/BLM scale as a function of the CM angle θ_{CM}: $ee \to ee$ scalar QED.

is weighted by the respective scattering amplitudes. The t-channel amplitude strongly dominates at $\Theta_{CM} = 0$, and the renormalization scale is thus t. Similarly, the u-channel amplitude strongly dominates at $\Theta_{CM} = \pi$, and the effective renormalization scale in that domain is u. Thus in both limits the effective renormalization scale $\hat{\mu}$ vanishes.

The results are shown in Fig. 1.

A PMC EXAMPLE FOR QCD: APPLICATION TO JET CROSS SECTIONS IN ELECTRON-POSITRON ANNIHILATION

As an example of the application of the PMC to QCD, we will show how the renormalization scale can be determined for the cross sections for e^+e^- annihilation into two and three jets in the $\overline{MS}$ scheme.

The two-jet cross section has only infrared divergences:

$$\sigma^{(2)} = \sigma_0\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2}(1-\lambda/2)\frac{\Gamma(1-\lambda/2)}{\Gamma(2-\lambda)}, \qquad (15)$$

where $\sigma_0 = 4\pi\frac{\alpha^2}{3q^2}N_C\sum_{i=1}^{N_f}e_i^2$.

Here $\lambda \equiv 4 - n$ is the number of extra space-time dimensions used to regulate infrared and ultraviolet divergent integrals. Eventually all of the infrared divergences and the factors involving λ will cancel out. In dimensional regularization the scale μ is introduced as a mass scale to restore the correct dimension of the coupling. The gauge coupling g_R is related to the renormalized coupling constant α_R by

$$\frac{g_R^2}{(4\pi)^{(4-\lambda)/2}} = \frac{\alpha_s(\mu^2)}{4\pi}(\mu^2)^{\lambda/2}e^{\gamma_E\lambda/2}, \qquad (16)$$

and here γ_E is the Euler constant.

As discussed in the Introduction, the mass scale of schemes defined by dimensional regularization attains its physical meaning when it is applied to QED. The renormalized gauge coupling is also related to the bare coupling by

$$g_R = \sqrt{Z_3}Z_2/Z_1 g_0, \qquad (17)$$

where Z_1 is the renormalization constant for the quark-antiquark-gluon vertex, Z_2 for the quark field, and Z_3 for the gluon field. The renormalization constants are

$$Z_1 = 1 - \frac{g_0^2}{16\pi^2}(N_c + C_F)\left(\frac{2}{\lambda_{\rm UV}} - \frac{2}{\lambda_{\rm IR}}\right), \qquad (18)$$

$$Z_2 = 1 - \frac{g_0^2}{16\pi^2}C_F\left(\frac{2}{\lambda_{\rm UV}} - \frac{2}{\lambda_{\rm IR}}\right), \qquad (19)$$

$$Z_3 = 1 + \frac{g_0^2}{16\pi^2}\left(\frac{5}{3}N_c - \frac{2}{3}N_f\right)\left(\frac{2}{\lambda_{\rm UV}} - \frac{2}{\lambda_{\rm IR}}\right), \qquad (20)$$

where $\lambda_{\rm UV}$, $\lambda_{\rm IR}$ are related, respectively, to the UV and IR poles. In the MS only the pole associated with UV renormalization is subtracted out, and this leads us to a redefinition of the gauge coupling:

$$\frac{1}{g_R}\delta g_0 = \frac{g_R^2}{16\pi^2}\left(\frac{2}{3}N_f - \frac{11}{3}N_c\right)\frac{1}{\lambda_{\rm UV}}. \qquad (21)$$

A suitable renormalization scheme is the $\overline{MS}$ that differs from MS by a constant term and the respective counterterm can be inserted in the Born cross section by shifting the coupling constant:

$$\begin{aligned}\alpha_s^0 &= \alpha_s^{\overline{MS}}\left\{1 - \left(\frac{11}{6}N_c - \frac{2}{3}T_R\right)\frac{\alpha_s^{\overline{MS}}}{2\pi}\left(\frac{1}{\epsilon} + (\ln 4\pi - \gamma_E)\right)\right\}\\ &= \alpha_s^{\overline{MS}}\left\{1 - \beta_0\alpha_s^{\overline{MS}}\left(\frac{1}{\bar{\epsilon}}\right)\right\},\end{aligned} \qquad (22)$$

where:

$$\frac{1}{\bar{\epsilon}} = \frac{1}{\epsilon} + (\ln 4\pi - \gamma_E), \qquad (23)$$

$$\beta_0 = \frac{1}{2\pi}\left(\frac{11}{6}N_c - \frac{2}{3}T_R\right), \qquad (24)$$

with $T_R = N_f/2$, $\epsilon = \lambda_{\rm UV}/2$.

The Born cross section for $e^+e^- \to q(p_1)\bar{q}(p_2)g(p_3)$ for massless quarks and gluons is

$$\left.\frac{d\sigma^{(3)}(\mu^2)}{dx_1dx_2}\right|_{\rm Born} = \sigma^{(2)}\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2}\frac{1}{\Gamma(1-\lambda/2)}F_\lambda(x_1,x_2) \times \frac{\alpha_s^{MS}(\mu^2)}{2\pi}C_F B^{V-\lambda/2S}(x_1,x_2). \qquad (25)$$

Here

$$F_\lambda(x_1,x_2) = [(x_1+x_2-1)(1-x_1)(1-x_2)]^{-\lambda/2}, \qquad (26)$$

and

$$B^{V-\lambda/2S}(x_1,x_2) = B^V(x_1,x_2) - \frac{\lambda}{2}B^S(x_1,x_2), \qquad (27)$$

$$B^V(x_1,x_2) = \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}, \qquad (28)$$

$$B^S(x_1,x_2) = \frac{x_3^2}{(1-x_1)(1-x_2)}, \qquad (29)$$

where $x_i = \frac{2E_i}{\sqrt{q^2}}$ in the e^+e^- CM. In terms of invariants, $y_{ij} = s_{ij}/q^2 = (p_i+p_j)^2/q^2$. Then $x_1 = 1 - y_{23}$, $x_2 = 1 - y_{13}$, $x_3 = 1 - y_{12}$, $x_1 + x_2 + x_3 = 2$.

The renormalized one-loop corrected cross section for $e^+e^- \to q(p_1)\bar{q}(p_2)g(p_3)$ is given by Eq. (2.11) of Fabricius *et al.* [19] For our purposes it is sufficient to quote only the term proportional to β_0 in the $\overline{MS}$ scheme:

$$\frac{d\sigma^{(3)}}{dx_1 dx_2}\bigg|_{\text{oneloop}} = \frac{d\sigma^{(3)}(\mu^2)}{dx_1 dx_2}\bigg|_{\text{Born}} \times \left[1 + \alpha_s(\mu^2)\frac{\Gamma(1-\lambda/2)}{\Gamma(1-\lambda)}\left(\frac{4\pi\mu^2}{q^2}\right)^{\lambda/2} \times \beta_0\left(\log\frac{\mu^2}{q^2}\right) + \cdots\right], \quad (30)$$

where the coupling is defined as in Eq. (22): $\alpha_{MS}(e^{\log 4\pi - \gamma_E}\mu^2) \equiv \alpha_{\overline{MS}}(\mu^2)$. The remaining contributions are independent of n_f and β_0

We can eliminate the nonconformal log term proportional to β_0 by shifting the renormalization scale $\alpha_{MS}(\mu^2)$ in the Born cross section Eq. (25)

$$\alpha_s(\mu^2) \simeq \alpha_s(q^2)\left(1 - \alpha_s(q^2)\beta_0 \log\left[\frac{\mu^2}{q^2}\right]\right);$$

however, it is first convenient to shift the scale to $\mu^2 \to (\mu_0^2)$.

Then

$$\frac{d\sigma^{(3)}}{dx_1 dx_2}\bigg|_{\text{oneloop}} = \frac{d\sigma^{(3)}(\mu_0^2)}{dx_1 dx_2}\bigg|_{\text{Born}} \times \left[1 + \alpha_s(\mu_0^2)\frac{\Gamma(1-\lambda/2)}{\Gamma(1-\lambda)}\left(\frac{4\pi\mu_0^2}{q^2}\right)^{\lambda/2} \times \beta_0\left(\log\frac{\mu_0^2}{q^2}\right) + \cdots\right]. \quad (31)$$

Naively one could simply fix the scale to $\sqrt{q^2}$, but the three-jet cross section will still be affected by IR divergences; in order to apply the PMC/BLM prescription we will first need to include the four-jet contributions.

IV. NUMERICAL SCALE FIXING

The complete differential three-jet cross section has been calculated by Fabricius *et al.* [19], and we quote here the results for the β_0-dependent terms:

$$\frac{d^2\sigma^{(3)}(\epsilon,\delta)}{dx_1 dx_2} = \sigma_0 \frac{\alpha_s(q^2)}{2\pi} C_F \times \left\{B^V(x_1,x_2)\left[1 - \alpha_s(q^2) \times \beta_0\left(\log\left(\frac{1-\cos\delta}{2}\right) + \log\hat{x}_3^2 - \frac{13}{3}\right)\right] - B^S(x_1,x_2)\alpha_s(q^2)\frac{\beta_0}{2}\right\} + \mathcal{O}(\delta^2)) + \cdots, \quad (32)$$

where $\hat{x}_3 = (2 - x_1 - x_2)$ and

$$d\sigma^{(3)}(\epsilon,\delta) = d\sigma^{(3)} + d\sigma^{(4)}(\epsilon,\delta) \quad (33)$$

is the sum of the three- and the four-jets contributions. The cancellation of the IR poles is guaranteed by the KLN theorem [20,21].

The variables (ϵ, δ) are small quantities introduced in the virtual amplitude in order to define the soft and collinear four-jet contributions to the three-jet cross section. In particular, these quantities refer, respectively, to the fraction of the total energy and to the cone opening angle that defines the phase volume for a thee-jet event (for more details, see Ref. [19]).

In order to extract the PMC/BLM scale we first work in the $\overline{MS}$ scheme, fixing an arbitrary renormalization scale: $\mu^2 = \mu_0^2$. It turns out that the β_0 term of the three-jet differential IR safe cross section has the form

$$\frac{d^2\sigma^{(3)}(\epsilon,\delta)}{dx_1 dx_2} = \sigma_0 \frac{\alpha_s(\mu_0^2)}{2\pi} C_F \times \left\{B^V(x_1,x_2)\left[1 - \alpha_s(\mu_0^2) \times \beta_0\left(\log\left(\frac{1-\cos\delta}{2}\right) + 2\log(2-x_1-x_2) - \frac{13}{3} + \log\frac{q^2}{\mu_0^2}\right)\right] - B^S(x_1,x_2)\alpha_s(\mu_0^2)\frac{\beta_0}{2}\right\} + \mathcal{O}(\delta^2)) + \cdots. \quad (34)$$

In principle, we can extract information on the terms in this formula performing a detailed analysis of the dependence of the β_0 coefficient on the invariants. Performing a blindfold study, we can single out the β_0 coefficient by means of the β_0 derivative of the whole cross section or either by the n_f derivative since

$$\frac{df}{d\beta_0} = \frac{df}{dn_f} \times \frac{d\beta_0^{-1}}{dn_f}. \quad (35)$$

Then we can factorize out the Born amplitude Eq. (25):

$$\frac{d\sigma^{(3)}(\mu_0^2)}{dx_1 dx_2}\bigg|^{-1}_{\text{Born}} \cdot \frac{d}{d\beta_0}\frac{d^2\sigma^{(3)}(\epsilon,\delta;\mu_0^2)}{dx_1 dx_2} = \left[-\alpha_s(\mu_0^2)\left(\log\left(\frac{1-\cos\delta}{2}\right) + 2\log(2-x_1-x_2) - \frac{13}{3} + \log\frac{q^2}{\mu_0^2} + \frac{B^S(x_1,x_2)}{2B^V(x_1,x_2)}\right)\right] + \mathcal{O}(\delta^2)) + \cdots,$$

and at the first order approximation the PMC/BLM scale can be fixed numerically imposing

$$\left[\frac{d\sigma^{(3)}(\mu^2)}{dx_1 dx_2}\bigg|^{-1}_{\text{Born}} \cdot \left(\frac{d}{dn_f}\frac{d^2\sigma^{(3)}(\epsilon,\delta;\mu^2)}{dx_1 dx_2}\right)\bigg|_{n_f=0}\right]\bigg|_{\mu^2=\mu^2_{\text{PMC}}} = 0. \quad (36)$$

In the numerical procedure at NLO, the analytic form of the cross section is not needed; one must only keep track of the appearance of number of flavors n_f arising from loop diagrams involving renormalization. This procedure, which has been shown at NLO here, can also be iterated to higher orders in α_s, by keeping track of the n_f terms entering the β function, leading us to an improvement of the accuracy of the PMC/BLM scale $\mu^2_{\rm PMC}$.

Following this procedure we can include all the nonconformal β terms into the running coupling constant for every physical process, setting the renormalization scale at the PMC/BLM scale without necessarily knowing the PMC/BLM analytic form. Thus we end up with a cross section that is formally equal to the corresponding conformal expansion with $\beta = 0$. In this particular case the PMC/BLM scale has the form

$$\mu^2_{\rm PMC} \simeq q^2(2 - x_1 - x_2)^2 \frac{\delta^2}{4} e^{-\frac{13}{3}+\frac{B_S(x_1,x_2)}{2B_V(x_1,x_2)}}. \qquad (37)$$

In this case the coefficient depends on the parton energies x_1, x_2, on the angle parameter δ, and on the scale ratio q^2/μ_0^2 (all these quantities can be written in the form of Lorentz invariants). The different contributions to the coefficient can be also identified, term by term, by considering the most differential cross section (i.e., for the three-jet case the triple differential cross section), by performing the derivative (or logarithmic derivative) with respect to the corresponding invariant, and then isolating the constant term. This procedure will be discussed in detail in the next section.

V. THE PMC/BLM SCALE AS A FUNCTION OF THE JET MASS RESOLUTION PARAMETER

As shown by Kramer and Lampe [16], one can define a QCD jet by defining a resolution parameter $y \cdot s$ as its maximal virtuality. The jet then consists of particles with total invariant mass squared smaller than $y \cdot s$. Using this definition, we will perform the integration of the entire three-jet differential cross section, including real, $d\sigma^{(3)}$, and virtual, $d\sigma^{(s)}$, contributions in order to have an IR safe quantity. This gives a y-dependent integrated formula with β_0-dependent terms that can be absorbed into the argument of the running coupling, according to the PMC/BLM prescription.

The entire differential three-jet cross section [22] is

$$\begin{aligned}\frac{1}{\bar\sigma_0}\frac{d\sigma^{(s)}+d\sigma^{(3)}}{dy} &= \int_y^{1-2y} dz \int_y^{1-y-z} dx T[1-x-z,x,z]\alpha_s(Q^2) \\ &\quad\times\Big(1-\beta_0\alpha_s(Q^2)\Big(\log[x]+\log[z]-\frac{5}{3}\ldots\ldots\Big)\Big) \\ &= \alpha_s(Q^2)(T(y)-\beta_0\alpha_s(Q^2)(C(y)+\ldots.)), \qquad (38)\end{aligned}$$

$$\begin{aligned}&\equiv T(y)\alpha_s(Q^2)\Big(1-\beta_0\alpha_s(Q^2)2\log\Big[\frac{\mu_{\rm BLM}}{\sqrt{s}}\Big]\Big) \\ &= T(y)\alpha_s(\mu^2_{\rm BLM}); \qquad (39)\end{aligned}$$

where $\bar\sigma_0 = \sigma_0 C_F Q^2/2\pi$, $s = Q^2$, $x = y_{13}$, $z = y_{23}$,

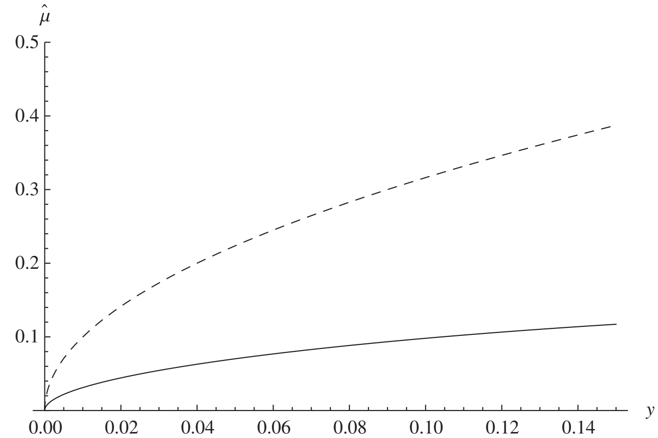

FIG. 2. The PMC/BLM scale, $\mu_{\rm PMC}$ (plane line), as a function of the jet resolution parameter y, for $e^+e^- \to q\bar q g$. For comparison, the behavior $\hat\mu \simeq \sqrt{y}$ is also shown (dashed line).

$$T[x_1,x_2,x_3] = \frac{2x_1^2+x_2^2+x_3^2+2x_1(x_2+x_3)}{x_2x_3}, \qquad (40)$$

and $T(y)$, $C(y)$ result from the partial integration of the LO and NLO terms of the three-jet cross section (for more details see Refs. [16,22]).

Then in the three-jet case, the PMC/BLM scale as function of the jet-virtuality y, has the analytic form

$$\hat\mu^2 = \mu^2_{\rm PMC/BLM} = s \times e^{-\frac{5}{3}+\frac{C(y)}{T(y)}}. \qquad (41)$$

A plot of the PMC/BLM scale against y, the virtuality resolution of the jet, in $e^+e^- \to q\bar q g$ is shown in Fig. 2. The result agrees with the BLM scale calculated by Kramer and Lampe in the $\overline{MS}$ scheme. The PMC/BLM prediction is scheme independent; the specific value of the renormalization scale is rescaled according to the choice of scheme so that all results are commensurate. The PMC/BLM scale also accurately determines n_f, the effective number of flavors in the β function. As is clear from the QED analog, the renormalization scale reflects the virtuality of the gluon jet; it thus must vanish when the resolution ys vanishes. As noted by Kramer and Lampe [16], the renormalization scales determined by the *ad hoc* PMS and fastest apparent convergence [23] procedures have the wrong physical behavior at $ys \to 0$, since they become infinite $\mu^2 \to \infty$ as the jet resolution and gluon virtuality vanish.

VI. PMC/BLM SCALE FIXING IN THE THREE-JET CASE: THE COMPLETE DIFFERENTIAL CROSS SECTION

In the case of the complete differential cross section, i.e., the most differential cross section for a given process without any constrained variables, the PMC/BLM scales depend on the number of flavors n_f and on the independent invariants entering the process. In the case of the three jets, we notice that the cross section depends on the color and flavor parameters n_f, N_C, C_F and on the kinematical invariants s_{12}, s_{13}, s_{23} where the label 3 refers to the gluon momentum, and the indices 1, 2 refer to the quark and antiquark momenta. On the other hand, the nonconformal terms entering the running coupling depend only on the number of flavors n_f and on a reduced number of kinematical invariants. These terms can be identified by first varying the number of flavors n_f and then the invariant s_{ij}, whereas the constant term can be extracted by simply subtraction at the final step. Starting with the triple differential cross section for three jets, which is given by the sum of the singular part of four-jet differential cross section $d\sigma^{(s)}$ and the real three-jet cross section $d\sigma^{(3)}$ (for more details see Ref. [22]):

$$\frac{d\sigma^{(s)}+d\sigma^{(3)}}{dzdydx} = \tilde{\sigma}_0 \frac{\alpha_s(Q^2)}{2\pi}\delta(1-x-y-z)\Big\{T[z,x,y] \times\Big[1+\frac{\alpha_s(Q^2)}{2\pi}C_F(\ldots.)+\frac{\alpha_s(Q^2)}{2\pi}N_C(\ldots.) - \alpha_s(Q^2)\beta_0\Big(\log[x*y]-\frac{5}{3}\Big)\Big]+\frac{\alpha_s(Q^2)}{2\pi}F[z,y,x]\Big\}, \tag{42}$$

with $\tilde{\sigma}_0 = \sigma_0 C_F s$. For the sake of simplicity, we are using the notation (z, x, y) for the final-state gluon energy, quark energy, and antiquark energy, respectively. In order to extract the first order terms related to the β function, we can start performing an *ab initio* analysis of the cross section. We can first single out the β_0 coefficient by means of the β_0 derivative, or either by the number of flavors n_f derivative, using Eq. (35) and then we can factorize out the Born amplitude:

$$\frac{d\sigma^{(3)}(Q^2)}{dzdydx}\Big|^{-1}_{\mathrm{Born}}\frac{1}{\alpha_s(Q^2)}\frac{d}{d\beta_0}\Big(\frac{d\sigma^{(s)}+d\sigma^{(3)}}{dzdydx}\Big) = \Big[\log[xy]-\frac{5}{3}\Big]+O(\alpha_s),$$
$$\frac{d\sigma^{(3)}(Q^2)}{dzdydx}\Big|_{\mathrm{Born}} = \tilde{\sigma}_0\frac{\alpha_s(Q^2)}{2\pi}T[z,x,y]\delta(1-x-y-z). \tag{43}$$

Finally, we can extract the weight for each invariant by taking the logarithmic derivative:

$$\omega_i = \frac{d}{d\log(x_i)}\Big(\frac{d\sigma^{(3)}(Q^2)}{dzdydx}\Big|^{-1}_{\mathrm{Born}}\frac{1}{\alpha_s(Q^2)}\frac{d}{d\beta_0} \times\Big(\frac{d\sigma^{(s)}+d\sigma^{(3)}}{dzdydx}\Big)\Big), \tag{44}$$

where $x_i = (x, y, z)$. The constant term can be identified by subtracting out all the logarithm terms from the β_0 coefficient. Then at first order approximation in the coupling constant, the μ_{PMC} scale for the three-jet differential cross section has the analytic form

$$\mu^2_{\mathrm{PMC}} \simeq Q^2\times C\times\prod_i x_i^{\omega_i} = Q^2 xye^{-\frac{5}{3}}. \tag{45}$$

A. Commensurate scale relations

Relations between observables must be independent of the choice of scale and renormalization scheme. Such relations, called commensurate scale relations [24–26] are thus fundamental tests of theory, devoid of theoretical conventions. One can compute each observable in any convenient renormalization scheme, such as the $\overline{MS}$ scheme using dimensional regularization. However, the relation between the observables cannot depend on this choice—this is the transitivity property of the renormalization group [3,7–9]. For example, the PMC relates the effective charge $\alpha_{g_1}(Q^2)$, determined by measurements of the Bjorken sum rule, to the effective charge $\alpha_R(s)$, measured in the total e^+e^- annihilation cross section: $[1-\alpha_{g_1}(Q^2)/\pi]\times[1+\alpha_R(s^*)/\pi]=1$. The ratio of PMC scales $\sqrt{s^*}/Q \simeq 0.52$ is set by physics; it guarantees that each observable goes through each quark flavor threshold simultaneously as Q^2 and s are raised. Because all $\beta\neq 0$ nonconformal terms are absorbed into the running couplings using PMC, one recovers the conformal prediction [25]; in this case, it is the Crewther relation [27–31]. Thus by applying the PMC, the conformal commensurate scale relations between observables, such as the Crewther relation, become valid for nonconformal QCD at leading twist.

VII. CONCLUSIONS

As we have shown, the PMC provides a consistent method for setting the optimal renormalization scale in pQCD. The PMC scale is determined by identifying the β terms in the next-to-leading contributions and making the appropriate shift in order to include the β terms into the running coupling. This can be done most simply by identifying the n_f terms that come from quark loops of skeleton graphs. This includes the n_f terms that renormalize the three- and four-gluon couplings. This procedure has been used to identify the correct PMC scale for the three-gluon vertex [12,13]. The resulting series is identical to that of the corresponding conformal theory with $\beta = 0$ as given, for example, by the Banks-Zaks method [18].

The global PMC renormalization scale is particularly useful for very complex processes; one only requires the dependence of the calculated subprocess amplitudes on the initial renormalization scale μ_0^2 and n_f, the number of quark flavors appearing from quark loops associated with renormalization. The single global PMC scale, valid at leading order, can thus be derived from basic properties of the perturbative QCD cross section.

We have discussed specific methods for efficiently determining the PMC renormalization scale analytically or numerically for QCD hard subprocesses. The analytic form of the PMC renormalization scale can be determined by varying the subprocess amplitude with respect to each invariant, thus determining the coefficients f_{ij} of $\log p_{ij}^2/\mu_0^2$ in the nonconformal terms in the amplitude. This result can be used to fix the renormalization scales for each contributing skeleton graph. However, we have shown that a single PMC global scale can then be determined at NLO by appropriate weighting. Alternatively the numerical value of the PMC scale can be determined without specific information on the analytic form from the n_f derivative of the cross section. The two methods give rise to the same results at NLO.

The factorization scale, in contrast, is the scale entering the structure and fragmentation functions. Unlike the renormalization scale, a factorization scale ambiguity occurs even in a conformal theory. The factorization scale should be chosen to match the nonperturbative bound state dynamics with perturbative DGLAP evolution. This could be done explicitly using nonperturbative models such as AdS/QCD and light-front holography where the light-front wave functions of the hadrons are known.

Note that one applies the PMC method to renormalizable hard subprocesses (including the associated radiation diagrams required for IR finiteness) that enter the pQCD leading-twist factorization procedure. The initial and final quark and gluon lines are taken to be on shell so that the calculation of the hard subprocess amplitude is gauge invariant. Thus the application of the PMC to hard subprocesses does not involve the factorization scale, and thus no double or single logarithms that involve the factorization scale enter.

The usual heuristic method of guessing the renormalization scale and varying it over a range of a factor of 2 gives scheme-dependent results, leaves the nonconvergent perturbative series, and gives the wrong result when applied to QED processes. In fact, varying the renormalization scale around such a guess only exposes nonconformal contributions involving the β function; it gives no information on the conformal contributions. The PMS method [15] has similar faults—it violates the transitivity property of the renormalization group, depends on the choice of scheme, is wrong for QED, and as shown by Kramer and Lampe [16], leads to unphysical results. In contrast, the PMC method, which has no such disadvantages and satisfies all principles of renormalization theory, gives the optimal prediction for pQCD at each finite order.

The PMC is the theoretical principle underlying the BLM procedure and commensurate scale relations between observables—the rigorous scale-fixed scheme-independent relations in QCD between observables, such as the generalized Crewther relation; it is also the scale-setting method used for precision determinations of α_s in lattice gauge theory [32]. In addition, it has been recently shown that for certain observables in two-jet production, the results of using the momentum-subtraction scheme and BLM method are very similar to those of $N = 4$ super Yang-Mills theory [33,34].

In the case of the BLM method, one deals with separate renormalization scales for each skeleton diagram, as is done in QED. The PMC method provides a single effective renormalization scale that reproduces the BLM scales at NLO, even for rather complex processes that are in our list of important projects, such as W + Jets, e^+e^- annihilation, $t\bar{t}$ production, and for general observables, e.g., differential cross sections and asymmetries.

If one considers a process with high multiplicity, then one confronts a separate BLM scale for each of the multiple skeleton diagrams; thus the number of BLM scales will appear as the jet multiplicity increases. The PMC method replaces these multiple scales with an effective single scale at NLO.

We have discussed in this paper an illustration of the PMC procedure for three-jet production in e^+e^- annihilation where the n_f terms arise from the inclusive four-jet cross section after IR cancellation; these terms are included in the PMC scale with the effect of lowering its value.

The PMC method provides the correct renormalization scale from first principles without ambiguity or renormalization scheme dependence. The residual errors from the resulting conformal series provide an accurate assessment of higher order errors. The PMC/BLM uncertainty is zero at the order computed. The PMC is equivalent to the standard method used to eliminate the renormalization scale ambiguity in precision tests of QED.

The PMC method gives results that are renormalization scheme independent at each finite order. The PMC also determines the correct number of flavors n_f; this is particularly important when one uses a renormalization scheme that is analytic in the quark masses such as the analytic extension of the $\overline{MS}$ scheme [11]; one can then include the correct flavor threshold dependences and transitions as one evolves the QCD coupling. The correct displacement between the argument of the schemes is also automatically determined.

We stress that PMC does not capture all higher order effects. One still has higher order corrections in the conformal series. These can never be discovered by varying the renormalization scale, since this variation only exposes terms proportional to the β function. It is incorrect to require the scale choice to remove all higher order terms. For example, in QED, the muon anomalous moment receives a large contribution at order α^3 from the electron-loop light-by-light insertion. This is due to the physics of the higher order processes—not the running QED coupling. It is thus incorrect to vary the renormalization scale to minimize the effect of higher order corrections, since the variation of μ_R cannot expose large terms in the conformal series. Thus the PMC correctly and unambiguously exposes higher order terms that are intrinsic to physical effects, unrelated to the QCD running coupling.

We emphasize that the PMC method for setting the renormalization scale gives predictions for observables that are independent of the choice of renormalization scheme—a key requirement for a valid prediction for a physical quantity. The argument of the running coupling in a given scheme that appears in the resulting conformal series has the correct displacement so that the result is scheme independent. The number of active flavors n_f in the QCD β function is also correctly determined, and the renormalization agrees with QED scale setting in the $N_C \to 0$ Abelian limit. Furthermore, the resulting conformal series avoids the need for renormalon resummation.

A consistent application of the BLM/PMC procedure to B decays, including $B \to X_s + \gamma$, has been developed including resummation to all orders in the strong coupling constant. A review and extension of this procedure is given by Melnikov and Mitov [35].

The PMC procedure has recently been extended to the four-loop level [36], demonstrating that it provides a consistent, systematic, and scheme-independent procedure for setting the renormalization scales up to NNLO. The explicit application for determining the renormalization scale of $R_{e^+e^-}(Q)$ up to four loops has also been presented [36].

The PMC is the principle underlying the BLM scale-setting procedure, a method that has been applied to many pQCD predictions. For example, the PMC/BLM procedure for setting the renormalization scale is the standard method for determining the intercept of the BFKL pomeron [34,37].

A systematic and scheme-independent procedure for setting the PMC/BLM scales up to NNLO has also been demonstrated, including an explicit application for determining the scale for $R_{e^+e^-}(Q)$ up to four loops [36]. The PMC procedure has recently been applied to the $t\bar{t}$ hadroproduction cross section [38,39] and the $\bar{t}t$ asymmetry [40] major tests of the Standard Model at colliders [38,39]. The PMC prediction for the total cross section $\sigma_{t\bar{t}}$ agrees well with the present Tevatron and LHC data. The initial scale independence of the PMC prediction is found to be satisfied to high accuracy at the NNLO level: the total cross section remains almost unchanged even when taking very disparate initial scales. After PMC scale setting, the PQCD predictions are within 1σ of the CDF [41] and D0 measurements [42] since the relevant renormalization scale is less than the conventional estimate; the large discrepancy of the top quark forward-backward asymmetry between the Standard Model prediction and the data is thus greatly reduced.

It should also be noted that the principle of maximum conformality satisfies all of the consequences of renormalization group invariance: reflectivity, symmetry, and transitivity [43]. Using the PMC, all nonconformal terms in the perturbative expansion series are summed into the running coupling, and one obtains a unique, scale-fixed, scheme-independent prediction at any finite order. The PMC scales and the resulting finite-order PMC predictions are both to high accuracy independent of the choice of initial renormalization scale, consistent with renormalization group invariance. Moreover, after PMC scale setting, the residual initial scale dependence at fixed order due to unknown higher order $\{\beta_i\}$ terms can be substantially suppressed. The PMC thus eliminates a serious systematic scale error in pQCD predictions, greatly improving the precision of tests of the Standard Model and the sensitivity to new physics at collider and other experiments. Further discussion is given in Ref. [43].

Clearly, the elimination of the renormalization scheme ambiguity using the PMC will greatly increase the precision of QCD tests and increase the sensitivity of measurements at the LHC and Tevatron to new physics beyond the Standard Model.

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