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Lorentz gauge quantization in a cosmological space-time

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It has recently been shown that it is not possible to impose the Lorentz gauge condition in a cosmological space-time using the Gupta-Bleuler method of quantization. It was also shown that it is possible to add $\nabla_\mu A^\mu$ as a new degree of freedom to the electromagnetic field and that this new degree of freedom might be the dark energy which is producing the accelerated expansion of the Universe. In this paper, I show that the Lorentz gauge condition can be imposed using Dirac's method of quantizing constrained dynamical systems. I also compute the vacuum expectation value of the energy-momentum tensor and show that it vanishes. Thus, in Dirac's approach, the electromagnetic field does not make a contribution to the dark energy.

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I. INTRODUCTION

It has recently been shown [1] that it is not possible to maintain the Lorentz gauge condition using the Gupta-Bleuler approach in a cosmological space-time. In the past, when the Universe is Minkowski, the Gupta-Bleuler condition $\partial_\mu A^{\mu(+)}_{\rm in}|\Psi\rangle = 0$ is imposed, which involves only the positive frequency part of the Lorentz gauge condition. The Universe then evolves into an out Minkowski region. The transverse in-positive frequency modes evolve into out-positive frequency modes, but the timelike and longitudinal in-positive frequency modes evolve into linear combinations of the out-positive and out-negative frequency modes. This produces a violation of the Lorentz gauge condition $\partial_\mu A^{\mu(+)}_{\rm out}|\Psi\rangle = 0$ in the out region.

The authors then considered quantizing the electromagnetic field in a cosmological space-time without imposing the Lorentz gauge condition using the action

$$S = \int\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\xi(\nabla_\mu A^\mu)^2\right]\sqrt{g}d^4x. \tag{1}$$

This gives the electromagnetic filed an additional scalar degree of freedom, namely $\nabla_\mu A^\mu$. They show that super-Hubble modes of this scalar contribute a cosmological constant to the energy-momentum tensor. This raises the interesting possibility that this additional degree of freedom may be the dark energy which is producing the accelerated expansion of the Universe.

In this paper, I use Dirac's approach [2,3] to quantize the electromagnetic field in the Lorentz gauge in a cosmological space-time. Consistency of the time evolution of the Lorentz gauge constraint introduces a second constraint $\vec{\nabla}\cdot\vec{E}\approx 0$. These constraints are first class, and in this approach, the full constraints, not just the positive frequency parts, are imposed on the state vector. Dirac's approach is shown to give a consistent way to impose the Lorentz gauge condition.

I also compute the vacuum expectation value of the energy-momentum tensor and show that it vanishes. Thus, in Dirac's approach, the electromagnetic field does not make a contribution to the dark energy.

II. DIRAC QUANTIZATION IN THE LORENTZ GAUGE

Consider a Friedman-Robertson-Walker space-time with a metric given by

$$ds^2 = a^2(t)[-dt^2 + dx^2 + dy^2 + dz^2]. \tag{2}$$

Since Maxwell's equations are conformally invariant in four space-time dimensions, the source free field equations are given by

$$\Box A^\mu - \partial^\mu(\partial_\alpha A^\alpha) = 0, \tag{3}$$

where $A^\mu = \eta^{\mu\nu}A_\nu$ and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The Lorentz gauge condition

$$\nabla_\mu(g^{\mu\nu}A_\nu) = 0 \tag{4}$$

is not conformally invariant and can be written as

$$\partial_\mu A^\mu + 2\psi A^t = 0, \tag{5}$$

where

$$\psi = \frac{1}{a}\frac{da}{dt}. \tag{6}$$

Substituting this into the field equations gives

$$\Box A^\mu + 2\partial^\mu(\psi A^t) = 0. \tag{7}$$

This equation, together with the Lorentz gauge condition, constitutes the field equations of the theory.

To quantize this theory, I will follow Dirac's method [2,3] of quantizing the electromagnetic field in Minkowski space-time in the Lorentz gauge. First, the classical theory will be developed, and then it will be quantized.

A Lagrangian density which produces (7), up to terms which vanish when the Lorentz gauge condition is imposed, is

$$L = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) - 2\psi A^\mu \partial_\mu A_t + 2(\dot{\psi} - \psi^2)A_t^2, \tag{8}$$

with the action given by $S = \int L d^3x$. The field equations which follow from this Lagrangian density are

$$\Box A^\mu + 2\partial^\mu(\psi A^t) + 2\psi \delta_t^\mu(\partial_\alpha A^\alpha + 2\psi A^t) = 0, \tag{9}$$

which are equivalent to Maxwell's equations once the Lorentz gauge condition is imposed.

The canonical momenta densities are given by

$$\Pi^\mu = \dot{A}^\mu + 2\psi A_t \delta_t^\mu, \tag{10}$$

the Hamiltonian density is given by

$$h = \frac{1}{2}\Pi^\mu \Pi_\mu + \frac{1}{2}(\partial_k A_\mu)(\partial^k A^\mu) + 2\psi \Pi^t A_t + 2\psi A^k \partial_k A_t - 2\dot{\psi} A_t^2 \tag{11}$$

and the Lorentz gauge condition is given by

$$\chi_1 = \Pi^t + \partial_m A^m - 4\psi A_t = 0. \tag{12}$$

For consistency, it is necessary that

$$\dot{\chi}_1 = \{\chi_1, H\} + \frac{\partial \chi_1}{\partial t} = \partial_m(\partial^m A^t + \Pi^m) + 2\psi \chi_1 \approx 0, \tag{13}$$

where { } denotes the Poisson bracket, $H = \int h d^3x$, and $\approx$ denotes a weak equality (i.e., an equality on the constraint hypersurface). Thus, there is an additional constraint

$$\chi_2 = \partial_m(\partial^m A^t + \Pi^m) \approx 0. \tag{14}$$

It is interesting to note that this new secondary constraint can also be written as $\chi_2 = -\vec{\nabla} \cdot \vec{E} \approx 0$. The consistency condition $\dot{\chi}_2 = \nabla^2 \chi_1 \approx 0$ does not produce any new constraints. It is easy to show that $\{\chi_1, \chi_2\} = 0$, so that χ_1 and χ_2 are first class constraints.

To quantize the theory, in the Schrodinger picture, the dynamical variables A_μ and Π^μ become time independent operators satisfying

$$[A_\mu(\vec{x}), A_\nu(\vec{y})] = [\Pi^\mu(\vec{x}), \Pi^\nu(\vec{y})] = 0 \tag{15}$$

and

$$[A_\mu(\vec{x}), \Pi^\nu(\vec{y})] = i\delta_\mu^\nu \delta^3(\vec{x} - \vec{y}), \tag{16}$$

where [ ] denotes the commutator and I have set $\hbar = 1$. A state vector is introduced which satisfies the Schrodinger equation

$$i\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle, \tag{17}$$

where $H = \int h d^3x$ and h is given by Eq. (11). In the standard approach, $2\psi \Pi^t A_t$ would be replaced by $\psi(\Pi^t A_t + A_t \Pi^t)$. However, it will be shown that this replacement leads to a divergence in the Hamiltonian when acting on physical states (defined below). This replacement will therefore not be done here. The resulting Hamiltonian will not be Hermitian, but the non-Hermitian part will vanish when acting on physical states.

The constraints are imposed on the wave function as follows:

$$\chi_1|\Psi\rangle = 0 \quad \text{and} \quad \chi_2|\Psi\rangle = 0. \tag{18}$$

States which satisfy these constraints are said to be physical states. The space of physical states will be denoted by $\mathcal{H}_{\rm phys}$. For a consistent quantum theory, we require that $[\chi_1, \chi_2] = \alpha\chi_1 + \beta\chi_2$ where α and β are operators which appear to the left of the constraints. This is satisfied since $[\chi_1, \chi_2] = 0$.

To preserve the constraints under time evolution, it is necessary that

$$\frac{d}{dt}[\chi_k|\Psi\rangle] = \left\{\frac{\partial \chi_k}{\partial t} - i[\chi_k, H]\right\}|\Psi\rangle = 0. \tag{19}$$

Thus, we require that

$$\frac{\partial \chi_k}{\partial t} - i[\chi_k, H] \approx 0, \tag{20}$$

where, in the quantum theory, $A \approx 0$ implies that $A|\Psi\rangle = 0$. For χ_1, we have

$$\frac{\partial \chi_1}{\partial t} - i[\chi_1, H] = \chi_2 + 2\psi\chi_1 \approx 0, \tag{21}$$

and for χ_2 we have

$$\frac{\partial \chi_2}{\partial t} - i[\chi_2, H] = \nabla^2 \chi_1 \approx 0. \tag{22}$$

Thus, the constraints are preserved under time evolution.

The constraint χ_2 can be simplified. The term $\partial_m \Pi^m$ involves only the longitudinal part of Π^m, and this longitudinal part can be written as the gradient of a scalar U. Thus, $\partial_m \Pi^m = \nabla^2 U$. The constraint χ_2 can therefore be written as

$$\chi_2 = \nabla^2(A^t + U). \tag{23}$$

Now, $\nabla^2(A^t + U) \approx 0$ over all space has the unique solution $A^t + U \approx 0$, if the fields vanish at infinity.

The Hamiltonian can be decomposed into transverse and longitudinal/time-like parts (see Dirac [2] for the details of the calculation in Minkowski space-time):

$$H_T = \frac{1}{2}\int [\Pi^m_{(T)} \Pi^{(T)}_m + (\partial_s A^{(T)}_m)(\partial^s A^m_{(T)})] d^3x, \tag{24}$$

$$H_L^{(1)} = \frac{1}{2}\int [\partial_r(U - A^t)\partial^r(U + A^t) + (\partial_m A^m - \Pi^t)(\partial_m A^m + \Pi^t - 4\psi A_t)] d^3x \tag{25}$$

and

$$H_L^{(2)} = -2\dot{\psi} \int A_t^2 d^3x. \tag{26}$$

Note that $H_L^{(1)} \approx 0$. In flat space-time, with $\psi = 0$, $H_L^{(2)} = 0$, so that the longitudinal and timelike degrees of freedom do not contribute to the field energy. This is not the case in a cosmological space-time with $\dot{\psi} \neq 0$. The transverse part of the Hamiltonian is the same as it is in Minkowski space-time.

It is interesting to note that if $2\psi \Pi^t A_t$ were replaced by $\psi(\Pi^t A_t + A_t \Pi^t)$, there would be an additional term $\psi[A_t(x), \Pi^t(x)] \sim i\psi \delta^3(0)$ in $H_L^{(2)}$, so that when the Universe is expanding (or contracting), the evolution of $|\Psi\rangle$ would not be well defined. It is also important to note that although H is not Hermitian, it satisfies

$$H \approx \frac{1}{2} \int [\Pi^m_{(T)} \Pi^{(T)}_m + (\partial_s A^{(T)}_m)(\partial^s A^m_{(T)})] - 4\dot{\psi} A_t^2] d^3x. \tag{27}$$

Thus, when acting on physical states, H is Hermitian.

To simplify the constraints and dynamics, consider the ket

$$|\Psi_M\rangle = \exp\left[-2i\psi \int A_t^2 d^3x\right] |\Psi\rangle. \tag{28}$$

The constraints satisfied by $|\Psi_M\rangle$ are

$$(\Pi^t + \partial_m A^m)|\Psi_M\rangle = 0 \quad \text{and} \quad (\partial_m \Pi^m + \nabla^2 A^t)|\Psi_M\rangle = 0, \tag{29}$$

and $|\Psi_M\rangle$ satisfies the equations of motion

$$i\frac{d}{dt}|\Psi_M\rangle = H_T |\Psi_M\rangle. \tag{30}$$

This ket therefore satisfies the Minkowski space constraints and equation of motion. Thus, we see that $|\Psi\rangle$ is equal to a Minkowski state vector multiplied by an operator valued phase factor. It is interesting to note that $\langle\Psi|\Psi\rangle = \langle\Psi_M|\Psi_M\rangle$ and that

$$\langle\Psi|\Omega(\Pi^m, A_\mu)|\Psi\rangle = \langle\Psi_M|\Omega(\Pi^m, A_\mu)|\Psi_M\rangle, \tag{31}$$

where Ω is any operator which depends on Π^m and A_μ.

An operator Ω is said to be physical if $\Omega|\Psi\rangle \in \mathcal{H}_{\text{phys}}$ for all $|\Psi\rangle \in \mathcal{H}_{\text{phys}}$. Physical operators must then satisfy $[\Omega, \chi_k] = 0$. Since the constraints involve only timelike and longitudinal components, any operator which depends only on the transverse components will be physical. The electromagnetic field $F_{\mu\nu}$ is also physical, but the potentials A_μ are not.

As is well known, there exists a residual gauge freedom of the form $\bar{A}^\mu = A^\mu + \nabla^\mu \chi$ with $\Box\chi = 0$ even once the Lorentz gauge is imposed. It is easy to show that H_T and $H_L^{(1)}$ are invariant under this transformation while $H_L^{(2)}$ is not. From Eq. (27), one can see that a gauge transformation of this type only changes the phase of the state vector.

To set up a Fock space representation, the operators

$$a^\mu_{\vec{k}} = \int e^{-i\vec{k}\cdot\vec{x}} [kA^\mu(\vec{x}) + i\Pi^\mu(\vec{x})] d^3x \tag{32}$$

and

$$a^{\dagger\mu}_{\vec{k}} = \int e^{i\vec{k}\cdot\vec{x}} [kA^\mu(\vec{x}) - i\Pi^\mu(\vec{x})] d^3x \tag{33}$$

can be introduced, where $k = |\vec{k}|$. These operators satisfy the standard commutation relations

$$[a^\mu_{\vec{k}}, a^\nu_{\vec{k}'}] = [a^{\dagger\mu}_{\vec{k}}, a^{\dagger\nu}_{\vec{k}'}] = 0 \tag{34}$$

and

$$[a^\mu_{\vec{k}}, a^{\dagger\nu}_{\vec{k}'}] = (2\pi)^3 (2k) \eta^{\mu\nu} \delta^3(\vec{k} - \vec{k}'). \tag{35}$$

The spatial parts of these operators can also be decomposed into transverse and longitudinal parts.

A vacuum state $|0\rangle$ can also be introduced which satisfies

$$(\Pi^t + \partial_m A^m - 4\psi A_t)|0\rangle = 0, \quad (\partial_m \Pi^m + \nabla^2 A^t)|0\rangle = 0 \tag{36}$$

and

$$a^{(T)}_{m\vec{k}}|0\rangle = 0. \tag{37}$$

These conditions are consistent because $a^{(T)}_{m\vec{k}}$ commutes with the two constraints. The operators $a^{(T)}_{m\vec{k}}$ act as annihilation operators, and the operators $a^{\dagger(T)}_{m\vec{k}}$ act as creation operators.

If H_T is normal ordered so that $H_T|0\rangle = H_T|0_M\rangle = 0$, then

$$|0(t)\rangle = \exp\left[2i\psi(t) \int A_t^2 d^3x\right] |0_M\rangle, \tag{38}$$

where $|0_M\rangle$ is the standard Minkowski vacuum. Now, consider a cosmological space-time with asymptotic in and out Minkowski regions. If the initial state of the field, in the past asymptotic region, is $|0_M\rangle$, the final state of the field, in the future asymptotic region, is also $|0_M\rangle$. There is, therefore, no particle production for the electromagnetic field in a cosmological space-time. This is in agreement with results obtained earlier by Parker [4,5].

III. DARK ENERGY

Since the field equations of the theory are Maxwell's equations in the Lorentz gauge, the energy-momentum tensor will be given by the standard expression

$$T_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}. \tag{39}$$

Now, consider $E = -\int T^t{}_t d^3x$. A short calculation shows that

$$E \approx \frac{1}{a^4}\left[H + 2\dot{\psi}\int A_t^2 d^3x + E_s\right], \tag{40}$$

where E_s is a surface term. This surface term will not contribute to the average energy density if we consider an infinite volume. Since the energy density must be spatially uniform in an Friedman-Robertson-Walker space-time, the surface term may be dropped. Note that E is invariant under the action of the residual gauge transformations.

Thus, in the ground state, where $\langle 0|H|0\rangle = -2\dot{\psi}\int A_t^2 d^3x$, we find that $\langle 0|E|0\rangle \approx 0$. This implies that

$$\langle 0|\rho|0\rangle \approx 0. \tag{41}$$

The vacuum expectation value of the energy density, therefore, vanishes. The pressure follows from the continuity equation (in conformal coordinates)

$$\frac{d}{dt}(a^2\rho) + a^2\psi(\rho + 3P) = 0. \tag{42}$$

Thus, when $\psi \neq 0$ the vacuum pressure must vanish.

We therefore conclude that the vacuum expectation value of the energy-momentum tensor vanishes when the field is in the ground state. Thus, there is no dark energy when the electromagnetic field is quantized in the Lorentz gauge using Dirac's procedure.

IV. CONCLUSION

It has recently been shown [1] that it is not possible to maintain the Lorentz gauge condition in a cosmological space-time using the Gupta-Bleuler approach. The authors also considered quantizing the electromagnetic field using the action (1) without imposing the Lorentz gauge condition. This introduces a new scalar degree of freedom, $\nabla_\mu A^\mu$, and they show that super-Hubble modes of this scalar contribute a cosmological constant to the energy-momentum tensor. This raises the interesting possibility that this new degree of freedom may be the dark energy which is producing an accelerated expansion of the Universe.

In this paper I quantized the electromagnetic field in the Lorentz gauge in a cosmological space-time using Dirac's method for constrained dynamical systems. Consistency of the Lorentz gauge condition under time evolution introduces a second constraint $\nabla \cdot \vec{E} \approx 0$. These two constraints are maintained under time evolution, showing that it is possible to impose the Lorentz gauge condition in a cosmological space-time. I also showed that there is no particle production in cosmological space-times, which is consistent with earlier results obtained by Parker [4,5].

I also computed the vacuum expectation value of the energy-momentum tensor and showed that it vanishes. Thus, in Dirac's approach, the electromagnetic field does not make a contribution to the dark energy.

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