

Interpreting the conformal cousin of the Husain-Martinez-Nuñez spacetime

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A 2-parameter inhomogeneous cosmology in Brans-Dicke theory, obtained by conformally transforming the Husain-Martinez-Nuñez scalar field solution of the Einstein equations is studied and interpreted physically. According to the values of the parameters it describes a wormhole or a naked singularity. The reasons why there is not a one-to-one correspondence between conformal copies of this metric are discussed.

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I. INTRODUCTION

In their low-energy limit, most theories attempting to quantize gravity produce modifications of general relativity in the form of nonminimally coupled dilaton fields and/or higher derivative terms in the gravitational sector (this is the case, for example, of the bosonic string theory which reduces to an $\omega = -1$ Brans-Dicke theory [1]). Attempts to explain the present-day cosmological acceleration discovered using the luminosity distance-redshift relation of type Ia supernovae [2] without introducing the *ad hoc* dark energy has led, among other scenarios, to infrared modifications of gravity [3]. This “ $f(R)$ ” gravity is nothing but a Brans-Dicke theory with a special scalar field potential (see Ref. [4] for reviews).

Several alternative theories of gravity have been proposed and studied recently, as low-energy effective actions or as toy models for quantum or emergent gravity, or in the context of early or late universe cosmology (see Ref. [5] for a recent review). In addition, varying “constants” of nature hypothesized by Dirac [6] can be implemented naturally in scalar-tensor gravity, in which the gravitational coupling depends on the spacetime point [7,8].

When approaching a theory of gravity, it is important to understand its spherically symmetric solutions and, in particular, its black holes. Solutions of the field equations describing inhomogeneities in cosmological spaces have been studied with the specific purpose of modeling spatial variations of the gravitational coupling [9,10]. Spherically symmetric inhomogeneous solutions of Brans-Dicke gravity which describe a central condensation embedded in a Friedmann-Lemaître-Robertson-Walker (FLRW) background have been found, but not studied or interpreted, in Ref. [10]. Extra value is added to the study of spacetimes describing central objects in cosmological backgrounds by

the fact that such metrics are not well understood even in the context of Einstein theory [11–15]. Moreover, the old problem of the influence of the cosmic expansion on local dynamics (and *vice-versa*), which originally led to the study of such solutions [11], is not completely solved [16].

In this paper we analyze a Brans-Dicke solution found by Clifton *et al.* [10] and describing an inhomogeneity embedded in a FLRW universe. This solution is generated using a conformal transformation and the Husain-Martinez-Nuñez scalar field solution of general relativity [17] as a seed. The conformal copy is not a perfect mirror image of the original solution, however, because it is found (in Sec. II) that it describes a spacetime with properties quite different from the original one. This fact should not lead to superficial statements on the physical inequivalence between conformal frames because the usual conformal mapping between Brans-Dicke’s and Einstein’s theories (and their solutions) prescribes also a scaling of units in the Einstein frame [18] which went lost in Ref. [10], where the authors intended only to generate a new spherical and inhomogeneous solution of the Brans-Dicke field equations. Section IV contains a discussion on this subject.

II. UNDERSTANDING THE CLIFTON-MOTA-BARROW SPACETIME

Clifton *et al.* [10] conformally mapped the spherically symmetric and dynamical Husain-Martinez-Nuñez [17] scalar field solution of general relativity to obtain the 2-parameter class of Brans-Dicke spacetimes

$$ds^2 = -A^{\alpha(1-\frac{1}{\sqrt{3}\beta})}(r)dt^2 + A^{-\alpha(1+\frac{1}{\sqrt{3}\beta})}(r)t^{\frac{2(\beta-\sqrt{3})}{3\beta-\sqrt{3}}} \times [dr^2 + r^2 A(r) d\Omega_{(2)}^2], \quad (1)$$

$$\phi(t, r) = A^{\frac{1}{2\beta}}(r)t^{\frac{2}{\sqrt{3}\beta-1}}, \quad (2)$$

where $d\Omega_{(2)}^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on the unit 2-sphere,

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$$A(r) = 1 - \frac{2C}{r}, \quad (3)$$

$$\alpha = \pm \frac{\sqrt{3}}{2}, \quad (4)$$

$$\beta = \sqrt{2\omega + 3}, \quad (5)$$

C is a parameter related to the mass of the central inhomogeneity, and ω is the Brans-Dicke coupling parameter which is required to be larger than $-3/2$. We adopt the notations of Ref. [19]. There are spacetime singularities at $r = 2C$ and at $t = 0$, therefore, the relevant coordinate range is $2C < r < +\infty$ and $t > 0$ [10]. The scale factor of the spatially flat FLRW background universe is

$$a(t) = \frac{t^{\beta-\sqrt{3}}}{t^{3\beta-\sqrt{3}}} \equiv t^\gamma. \quad (6)$$

The line element (1) can be rewritten as

$$ds^2 = -A^\sigma(r)dt^2 + A^\Theta(r)a^2(t)dr^2 + R^2(t, r)d\Omega_{(2)}^2, \quad (7)$$

where

$$\sigma = \alpha \left(1 - \frac{1}{\sqrt{3}\beta}\right), \quad (8)$$

$$\Theta = -\alpha \left(1 + \frac{1}{\sqrt{3}\beta}\right), \quad (9)$$

and

$$R(t, r) = A^{\frac{\Theta+1}{2}}(r)a(t)r, \quad (10)$$

is the areal radius.

Let us examine the behavior of the area $4\pi R^2$ of 2-spheres of symmetry by studying how the areal radius behaves as a function of r . We have

$$\begin{aligned} \frac{\partial R}{\partial r} &= a(t)A^{\frac{\Theta-1}{2}}(r) \left[1 - \frac{2C}{r} \frac{(1-\Theta)}{2}\right] \\ &\equiv a(t)A^{\frac{\Theta-1}{2}}(r) \left(1 - \frac{r_0}{r}\right), \end{aligned} \quad (11)$$

where

$$r_0 = (1 - \Theta)C, \quad (12)$$

or, in terms of the proper (areal) radius,

$$R_0(t) = \left(\frac{\Theta + 1}{\Theta - 1}\right)^{\frac{\Theta+1}{2}} (1 - \Theta)a(t)C. \quad (13)$$

The critical value r_0 exists in the relevant spacetime region $r_0 > 2C$ if $\Theta < -1$. In this case the areal radius can be written as

$$R(r) = \frac{ra(t)}{\left(1 - \frac{2C}{r}\right)^{\frac{|\Theta+1|}{2}}}, \quad (14)$$

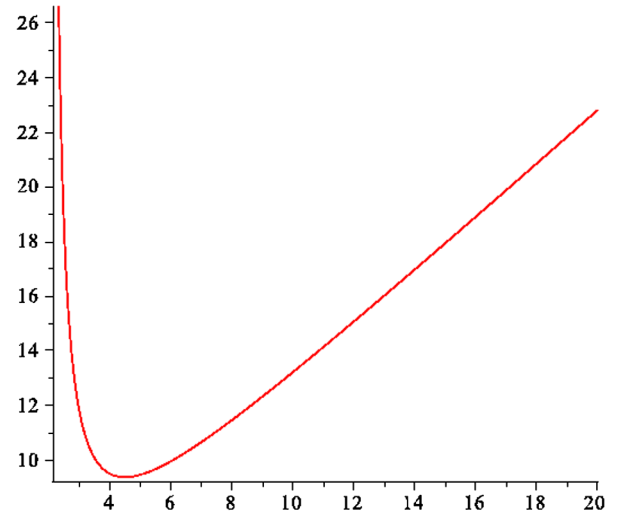


FIG. 1 (color online). The areal radius R (vertical axis) as a function of r (horizontal axis) for the parameter values $\alpha = \sqrt{3}/2$, $\Theta = -3/2$, and $C = 1$ (in units of C) at a time t_* at which $a(t_*) = 1$.

and $R(r) \rightarrow +\infty$ as $r \rightarrow 2C^+$: the area of 2-spheres of symmetry diverges as $r \rightarrow 2C^+$. Because of Eq. (11), for $\Theta < -1$ we have $\frac{1-\Theta}{2} > 1$ and $\partial R/\partial r > 0$ for r larger than

$$r_0 = 2C \left(\frac{1 - \Theta}{2}\right) > 2C, \quad (15)$$

$\partial R/\partial r = 0$ at $r = r_0$, and $\partial R/\partial r < 0$ for $r < r_0$. The function $R(r)$ has a minimum at r_0 (Fig. 1). The area of 2-spheres of symmetry decreases between $2C$ and r_0 , where it is minimum, then it increases again. There is a wormhole throat joining two spacetime regions (cf. Ref. [20] for a detailed wormhole theory). Note that, since

$$\Theta = \mp \frac{\sqrt{3}}{2} \left(1 + \frac{1}{\sqrt{3}\sqrt{2\omega + 3}}\right), \quad (16)$$

for $\alpha = \pm\sqrt{3}/2$, the condition $\Theta < -1$ requires $\alpha = +\sqrt{3}/2$. This value of α , however, is a necessary but not a sufficient condition for the apparent horizon to exist. The sufficient condition $\Theta < -1$ imposes the constraint on the Brans-Dicke parameter

$$\omega < \frac{1}{2} \left[\frac{1}{(2 - \sqrt{3})^2} - 3 \right] \equiv \omega_0, \quad (17)$$

which can be satisfied in the allowed parameter region $\omega > -3/2$. Therefore, for $-3/2 < \omega < \omega_0$ there is an apparent horizon at $r_0 > 2C$ and the solution can be taken to represent a Brans-Dicke wormhole. Here by *wormhole* we simply refer to a spacetime containing a smooth wormhole throat connecting two spacetime regions. Other definitions of wormhole (for example, a generalization to the dynamical case of the definition of Ref. [21]) are more

stringent and would not allow this spacetime to be called a wormhole.

The region $2C < r < r_0$ is not a FLRW region and the scalar field (2) is finite and nonzero at r_0 :

$$\phi(t, r_0) = t^{\frac{2}{\sqrt{3}\beta-1}} \left(\frac{\Theta + 1}{\Theta - 1} \right)^{\frac{1}{2\beta}}. \quad (18)$$

The proper radius (13) of the wormhole throat is exactly comoving with the cosmic substratum and disappears if the central inhomogeneity is removed, which is formally described by the limit $C \rightarrow 0$.

Let us now investigate the presence of apparent horizons in the metric (1). Using Eq. (10) and substituting the relation between differentials

$$dr = \frac{dR - A^{\frac{\Theta+1}{2}}(r)\dot{a}(t)rdt}{A^{\frac{\Theta-1}{2}}a(t)\frac{C(\Theta+1)}{r} + A^{\frac{\Theta+1}{2}}(r)a(t)}, \quad (19)$$

into the line element (7), one obtains

$$ds^2 = -A^\sigma dt^2 + \left[\frac{dR^2 - 2A^{\frac{\Theta+1}{2}}r\dot{a}dt dR + A^{\frac{\Theta+1}{2}}r^2\dot{a}^2 dt^2}{D_1(r)} \right] + R^2 d\Omega_{(2)}^2, \quad (20)$$

where

$$D_1(r) = A(r) \left[1 + \frac{C(\Theta + 1)}{rA(r)} \right]^2. \quad (21)$$

Collecting similar terms yields

$$ds^2 = -\frac{(D_1 A^\sigma - H^2 R^2)}{D_1} dt^2 - \frac{2HR}{D_1} dt dR + \frac{dR^2}{D_1} + R^2 d\Omega_{(2)}^2, \quad (22)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter of the background universe. The inverse of the metric $g_{\mu\nu}$ of Eq. (22) is

$$(g^{\mu\nu}) = \begin{pmatrix} -\frac{1}{A^\sigma} & -\frac{HR}{A^\sigma} & 0 & 0 \\ -\frac{HR}{A^\sigma} & \frac{(D_1 A^\sigma - H^2 R^2)}{A^\sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2 \sin^2 \theta} \end{pmatrix}. \quad (23)$$

In the presence of spherical symmetry the apparent horizons are located by the roots of the equation $\nabla^c R \nabla_c R = 0$ (e.g., Ref. [22]) or $g^{RR} = 0$, which here yields

$$D_1(r)A(r) = H^2(t)R^2(t, r). \quad (24)$$

The left-hand side of this equation depends only on r while the right-hand side depends on both r and t . This equation can only be satisfied when the right-hand side is time-independent and the only possibility for this to occur is when $H = \gamma/t = 0$, corresponding to $\gamma = 0$, $\beta = \sqrt{3}$, and $\omega = 0$. This value of the Brans-Dicke parameter gives a static solution describing a spherical inhomogeneity in a

Minkowski background, which is discussed in the next section. With this exception, Eq. (24) has no solutions and there are no apparent horizons in the spacetime (1). In particular, for $\Theta < -1$ the wormhole throat is not an apparent horizon.

Let us discuss now the case $\omega \geq \omega_0$ and the case $\alpha = -\sqrt{3}/2$. In these situations there is no wormhole throat and no apparent horizon in the $r > 2C$ region and the Clifton-Mota-Barrow spacetime contains a naked singularity.

For $\alpha = -\sqrt{3}/2$ it is $\Theta = \frac{\sqrt{3}}{2} \left(1 + \frac{1}{\sqrt{3}\beta} \right) > 0$ and

$$R(r) = \left(1 - \frac{2C}{r} \right)^{\frac{|\Theta+1|}{2}} a(t)r, \quad (25)$$

goes to zero as $r \rightarrow 2C^+$. Since $r_0 < 2C$, the areal radius $R(r)$ is always an increasing function of r in the relevant range $2C < r < +\infty$. This spacetime contains a naked singularity at $R = 0$.

III. THE SPECIAL CASE $\omega = 0$

The value $\omega = 0$ of the Brans-Dicke coupling, corresponding to $\beta = \sqrt{3}$ and $\gamma = 0$, produces the static metric

$$ds^2 = -A^{\frac{2\alpha}{3}}(r)dt^2 + \frac{dr^2}{A^{\frac{4\alpha}{3}}(r)} + \frac{r^2}{A^{\frac{4\alpha}{3}-1}(r)} d\Omega_{(2)}^2, \quad (26)$$

and the scalar field

$$\phi(t, r) = A^{\frac{1}{2\sqrt{3}}}(r)t, \quad (27)$$

which is time-dependent even though the metric is static.¹

The metric (26) is easily identified as a member of the Campanelli-Lousto class [24]. The general Campanelli-Lousto solution has the form

$$ds^2 = -A^{b+1}(r)dt^2 + \frac{dr^2}{A^{a+1}(r)} + \frac{r^2 d\Omega_{(2)}^2}{A^a(r)}, \quad (28)$$

$$\phi(r) = \phi_0 A^{\frac{a-b}{2}}(r), \quad (29)$$

and ϕ_0 , a , and b are constants with $\phi_0 > 0$. The Brans-Dicke parameter is given by [24]

$$\omega(a, b) = -2 \frac{(a^2 + b^2 - ab + a + b)}{(a - b)^2}. \quad (30)$$

In the case of the metric (26) setting

$$(a, b) = \left(\frac{4\alpha}{3} - 1, \frac{2\alpha}{3} - 1 \right), \quad (31)$$

reproduces the Campanelli-Lousto metric (28). Then, the expression (30) gives $\omega(\frac{4\alpha}{3} - 1, \frac{2\alpha}{3} - 1) = 0$ for

¹This is not the only occurrence of this circumstance: a similar situation is known for the static limit of another separable solution of the Brans-Dicke field equations found by Clifton *et al.* [23].

$\alpha = \pm\sqrt{3}/2$. However, the scalar field (27) differs from the Campanelli-Lousto scalar (29) by the linear dependence on the time t . Thus, the static limit of the Clifton-Mota-Barrow solution provides a (rather trivial) generalization of a Campanelli-Lousto solution.

The nature of the Campanelli-Lousto spacetime depends on the sign of the parameter a [25] which, in our case, corresponds to the choice $\alpha = \sqrt{3}/2$ or $-\sqrt{3}/2$. For $a \geq 0$ (corresponding to $\alpha = +\sqrt{3}/2$, $a \simeq 0.1547$, and $\Theta = -\frac{4\alpha}{3} \simeq -1.1547 < -1$) the Campanelli-Lousto spacetime contains a wormhole throat coinciding with an apparent horizon and located at $r_0 = 2C(\frac{1-\Theta}{2}) > 2C$ [25]. This is consistent with Eq. (24) with $H = 0$ since, in this case, the equation $g^{RR} = 0$ locating the apparent horizons reduces to $D_1(r) = 0$ which yields again the root $r_0 = C(1 - \Theta)$ lying in the physical region $r > 2C$.² This is the only case in which the Clifton-Mota-Barrow solution under study contains an apparent horizon.

For $a < 0$ (which is reproduced by the choice $\alpha = -\sqrt{3}/2$ and gives $a \simeq -2.1547$ and $\Theta \simeq 1.1547 > 0$) there are no apparent horizons and the spacetime contains a naked singularity [25]. This is consistent with the fact that Eq. (24) with $H = 0$ can only be satisfied if $D_1(r) = 0$ and in this case there are no acceptable solutions because $r_0 < 2C$.

IV. DISCUSSION AND CONCLUSIONS

According to the parameter values, the Clifton-Mota-Barrow spacetime (1) contains a wormhole or a naked singularity (black holes, wormholes, and naked singularities could in principle be distinguished observationally through gravitational lensing [26]). In the last situation, this solution of the Brans-Dicke field equations cannot be obtained as the development of regular Cauchy data.

One question which arises is the following: the Husain-Martinez-Nuñez and the Clifton-Mota-Barrow spacetimes are conformally related. As explained long ago by Dicke [18], the Jordan and the Einstein conformal frames should be different representations of the same physics (provided that the conformal transformation does not break down)—this issue has been the subject of a lively debate but has been shown to be largely a pseudo-problem (see Refs. [27–29] and the references therein). Then, why does the same solution look so different in the two different conformal frames for the parameter values for which a Jordan frame wormhole or naked singularity (1) corresponds to the Einstein frame black hole of Ref. [17]? The answer is that, by following the more ordinary route and conformally transforming the Clifton-Mota-Barrow metric and scalar field (1) to the Einstein frame would produce the Husain-Martinez-Nuñez metric *with scaling units* of

length, time and mass. What is physically relevant is the ratio of a physical quantity to its unit, and the units change with the spacetime position. Specifically, the units of length and time scale as the conformal factor Ω , while the unit of mass scales as Ω^{-1} and derived units scale accordingly [18]. In the Einstein frame, matter is coupled nonminimally to the metric while in the Jordan frame matter is minimally coupled. The scaling of units in the Einstein frame goes hand in hand with the nonminimal coupling of matter to the metric. *In vacuo* (which is the situation contemplated here), the nonminimal coupling of matter is forgotten, but the scaling of units should be remembered.

Another issue is that, contrary to event horizons (which are null surfaces and are conformally invariant), apparent horizons (which can be spacelike or even timelike) are not conformally invariant and change location under a conformal transformation [30]. In order to characterize the properties of a dynamical black hole when conformal transformations are involved, one should not consider the apparent horizons of a metric but a new surface characterized by an entropy 2-form, as explained in detail in Refs. [30,31]. The new prescription of Ref. [30] takes into account the scaling of units in the Einstein frame.

Therefore, a metric obtained from the conformal transformation to the Einstein frame of a seed Jordan frame metric with the extra information that units are scaling is quite different from the same formal metric with fixed units, which explains why conformally related spacetimes can look very different. Clifton-Mota-Barrow took the Husain-Martinez-Nuñez solution of general relativity *with fixed units* and used it as a seed to generate a new class of solutions of Brans-Dicke gravity—they did not worry about generating a *physically equivalent* solution, which would have required to take into account scaling units. This procedure is certainly legitimate and achieves the goal, but it generates physically inequivalent spacetimes when the requirement of scaling units is dropped. Indeed, there are comments in the literature about the fact that conformally related solutions of Brans-Dicke theory and of the Einstein equations do not share the same properties [32]. A similar situation occurs with the Campanelli-Lousto solutions of Brans-Dicke theory [24], which relate to Fisher-Janis-Newman-Winicour solutions of the Einstein equations in the Einstein frame [25], and with the veiled black holes of Refs. [29–31] (see Ref. [25] for a detailed discussion). To conclude, the physical nature of the Clifton-Mota-Barrow class of solutions is now clear and they do not cause problems for the interpretation of conformal frames.

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²The discussion of Ref. [25], however, does not depend on setting our $D_1 = 0$.

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