Hawking radiation of asymptotically nonflat dyonic black holes in Einstein-Maxwell-dilaton gravity

Peter I. Slavov* and Stoytcho S. Yazadjiev†

Department of Theoretical Physics, Faculty of Physics, Sofia University, Sofia 1164, Bulgaria

(Received 11 April 2012; published 22 October 2012)

In the present paper, we investigate the Hawking radiation of asymptotically nonflat dyonic black holes in 4D Einstein-Maxwell-dilaton gravity in semiclassical approximation. We show that the problem allows an exact analytical treatment and we compute exactly the semiclassical radiation spectrum of both nonextremal and extremal black holes under consideration. In the high-frequency regime, we find that the Hawking temperature does not agree with the surface gravity when the magnetic charge is nonzero. Even more surprisingly, the Hawking temperature is independent of the black hole intrinsic characteristics, as the mass and magnetic charge, and depends only on the linear dilaton background parameter.

DOI: 10.1103/PhysRevD.86.084042 PACS numbers: 04.50.Kd, 04.70.−s

I. INTRODUCTION

The Hawking radiation of black holes is an emblematic effect in the quantum field theory in curved spacetime [1,2]. This effect, lying on the wedge of classical and quantum gravity, reveals the deep connection between the black hole physics and thermodynamics. In the semiclassical approximation, the spectrum of the Hawking radiation can be obtained by computing the Bogoliubov coefficients in two different vacua and matching them appropriately [2]. Another procedure that, in asymptotically flat spacetimes, gives the same result as the Bogoliubov coefficients method, is to compute the absorption and the transmission (or reflection) coefficients of waves defined at the asymptotic regions [3].

Ever since its discovery, the Hawking radiation has continued to be a hot area of research. The importance of the Hawking effect for the fundamental physics stimulates its investigation for various black hole solutions with different structures and asymptotics. Unfortunately, the wave equations in black hole spacetimes cannot be solved analytically and this makes the full study of the Hawking radiation hard. Only in special cases we are able to solve the wave equations exactly. The cases when the wave equation is exactly solvable are important because they enable us to study the Hawking radiation in detail and, for example, allow us to compute the radiation spectrum exactly. In the present paper, we consider one such case when the exact analytical treatment of the Hawking radiation is possible. More precisely, we study the Hawking radiation of a class of asymptotically nonflat dyonic black holes in the 4D Einstein-Maxwell-dilaton gravity.

The Einstein-Maxwell-dilaton gravity in four-dimensional spacetime is described by the following action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g} \times [\mathcal{R} - 2g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - e^{-2\alpha\varphi}F_{\mu\nu}F^{\mu\nu}], \quad (1)$$

where $\mathcal{R}$ is the scalar curvature with respect to the spacetime metric $g_{\mu\nu}$, $F_{\mu\nu}$ is the electromagnetic field, and φ is the scalar dilaton field with a coupling constant α. In the present paper, we are interested in the black hole solutions of (1). More precisely we consider the following dyonic black hole solution for $\alpha = 1$ found in [4]

$$ds^2 = -\frac{(r-r_-)(r-r_+)}{r_0 r}dt^2 + \frac{r_0 r}{(r-r_-)(r-r_+)}dr^2 + r_0 r(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

$$\Phi_e = \frac{1}{\sqrt{2}}\frac{r}{r_0}, \quad (3)$$

$$\Phi_m = \sqrt{\frac{r_-}{2r_+}}\frac{r_+}{r}, \quad (4)$$

$$e^{2\varphi} = \frac{r}{r_0}, \quad (5)$$

where r_-, r_+ and r_0 are constants. Here Φ_e and Φ_m are the electric and magnetic potentials and the Maxwell 2-form is given by

$$F = d\Phi_e \wedge dt + e^{2\varphi} \star (d\Phi_m \wedge dt), \quad (6)$$

with $\star$ being the Hodge dual.

The solution describes an asymptotically nonflat dyonic black hole with inner and outer horizons at $r = r_-$ and r_+, respectively. The electric and magnetic charges are given by

$$Q = \frac{r_0}{\sqrt{2}}, \qquad P = \sqrt{\frac{r_+ r_-}{2}}. \quad (7)$$

*pslavov@phys.uni-sofia.bg
†yazad@phys.uni-sofia.bg

In the limit $P \to 0$ ($r_- \to 0$), we recover the pure electrical linear dilaton black hole solution [5] (see also Ref. [6]). It should be noted that the parameter r_0 (or equivalently the electric charge Q) is associated not with a specific black hole, but rather with a given linear dilaton background. The linear dilaton background solution is obtained by setting $r_+ = r_- = 0$ in our solution.

In order to find the physical mass of the black hole solution under consideration, we use the quasilocal formalism [7]. Since our spacetime is not asymptotically flat, a suitable substraction procedure is needed to obtain a finite mass. In our case, the linear dilaton background is the most natural and unique choice for the substraction background. The explicit calculations give the following result for the mass [4]:

$$M = \frac{1}{4}(r_+ + r_-). \tag{8}$$

The surface gravity is given by

$$\kappa = \frac{r_+ - r_-}{2r_0 r_+}. \tag{9}$$

In the case with $r_+ = r_-$, we obtain an extremal black hole solution with zero surface gravity.

Following [8], one can formally derive the first law for the black holes under consideration, namely

$$dM = \frac{\kappa}{2\pi} d\left(\frac{A_{\mathcal{H}}}{4}\right) + \Phi_m^{\mathcal{H}} dP, \tag{10}$$

where $A_{\mathcal{H}} = 4\pi r_0 r_+$ is the horizon area and $\Phi_m^{\mathcal{H}} = \sqrt{\frac{r_-}{2r_+}}$ is the magnetic potential evaluated on the horizon $r = r_+$. Here the parameter r_0, (respectively Q) is kept fixed since it is associated with the background. General comments about the black hole thermodynamics in asymptotically nonflat spacetimes will be given in the last section.

In the next section, we study the Hawking radiation of the asymptotically nonflat black hole solutions presented above in semiclassical approximation in both the nonextremal and extremal case. The Hawking radiation of pure electrical linear dilaton black holes (i.e., corresponding to $P = 0$) was studied in semiclassical approximation in Ref. [9].

II. HAWKING RADIATION IN SEMICLASSICAL APPROXIMATION

A. Hawking radiation of nonextremal black holes

In order to study the Hawking radiation of our black holes, we consider a test scalar field ψ satisfying the wave equation

$$\Box\psi = 0, \tag{11}$$

where $\Box$ is the curved spacetime D'alambert operator.

For the static spherically symmetric metric (2), the D'alambert operator takes the following explicit form:

$$\begin{aligned}\Box &= \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\\ &= -\frac{r_0 r}{(r-r_-)(r-r_+)}\partial_{tt}^2 + \frac{1}{r_0 r}\partial_r[(r-r_-)(r-r_+)\partial_r]\\ &\quad + \frac{1}{r_0 r}\Delta_{(\theta,\varphi)},\end{aligned}$$

where $\Delta_{(\theta,\varphi)}$ is the Laplace operator on the unit sphere $S^2_{r=1}$. So the scalar wave equation $\Box\psi = 0$ multiplied by $r_0 r$ becomes

$$\begin{aligned}&-\frac{r_0^2 r^2}{(r-r_-)(r-r_+)}\partial_{tt}^2\psi + \partial_r[(r-r_-)(r-r_+)\partial_r]\psi\\ &\quad + \Delta_{(\theta,\varphi)}\psi = 0.\end{aligned} \tag{12}$$

Consider now a harmonic eigenmode as a partial solution of (12) in separate variables

$$\psi_{\omega lm}(t, r, \theta, \varphi) := R_{\omega l}(r) Y_{lm}(\theta, \varphi) e^{-i\omega t}, \tag{13}$$

where $Y_{lm}(\theta, \varphi)$ are the spherical harmonics. Then the Fourier coefficients $R_{\omega l}(r)$ satisfy the equation

$$\begin{aligned}&\frac{d}{dr}[(r-r_-)(r-r_+)\dot{R}_{\omega l}(r)]\\ &\quad + \left[\frac{\tilde{\omega}^2 r^2}{(r-r_-)(r-r_+)} - l(l+1)\right]R_{\omega l}(r) = 0,\end{aligned} \tag{14}$$

where $\tilde{\omega} := r_0\omega$ is a dimensionless frequency.

After the substitutions

$$z := \frac{r_+ - r}{r_+ - r_-}, \qquad z_0 := \frac{r_+}{r_+ - r_-} > 1, \tag{15}$$

the equation for $Z_{\omega l}(z) := R_{\omega l}(r)$ becomes

$$\begin{aligned}&z(1-z)\ddot{Z}_{\omega l}(z) + (1-2z)\dot{Z}_{\omega l}(z)\\ &\quad + \left[\frac{\tilde{\omega}^2(z-z_0)^2}{z(1-z)} + l(l+1)\right]Z_{\omega l}(z) = 0.\end{aligned} \tag{16}$$

An appropriate substitution like $Z_{\omega l}(z) := z^p(z-1)^q h_{\omega l}(z)$ will help us to obtain a more familiar linear differential equation

$$\begin{aligned}&z(1-z)\ddot{h}_{\omega l}(z) + [1+2p-2(p+q+1)z]\dot{h}_{\omega l}(z)\\ &\quad - \Bigg[(p+q+1)(p+q) + \tilde{\omega}^2 - l(l+1)\\ &\quad - \frac{p^2 + z_0^2\tilde{\omega}^2}{z} - \frac{q^2 + (z_0-1)^2\tilde{\omega}^2}{1-z}\Bigg]h_{\omega l}(z) = 0,\end{aligned}$$

where p and q are determined by the conditions that eliminate the simple rational fraction $1/z$ and $1/(1-z)$ in the coefficient in front of $h_{\omega l}(z)$. There are four combinations for p and q, that can do it, but the most convenient one is to take $p := iz_0\tilde{\omega}$ and $q := -i(z_0 - 1)\tilde{\omega}$

$$\begin{aligned}&z(1-z)\ddot{h}_{\omega l}(z) + [1 + 2iz_0\tilde{\omega} - 2(1+i\tilde{\omega})z]\dot{h}_{\omega l}(z)\\ &\quad - [i\tilde{\omega} - l(l+1)]h_{\omega l}(z) = 0.\end{aligned} \tag{17}$$

This is just the hypergeometric equation and, after identifying (17) with the canonical form,

$$z(1-z)\ddot{h}_{\omega l}(z)+[c-(1+a+b)z]\dot{h}_{\omega l}(z)-abh_{\omega l}(z)=0, \tag{18}$$

one can easily obtain the canonical parameters:

$$\left| \begin{array}{l} a = 1/2 + i(\tilde{\omega} + \lambda_{\omega l}) \\ b = 1/2 + i(\tilde{\omega} - \lambda_{\omega l}), \\ c = 1 + 2iz_0\tilde{\omega} \end{array} \right. \quad \text{where } \lambda_{\omega l} := \sqrt{\tilde{\omega}^2 - (l+1/2)^2}. \tag{19}$$

In our dimensionless variable (15), the singular point $z = 1$ corresponds to the inner horizon $r = r_-$, while the singular point $z = 0$ corresponds to the outer horizon $r = r_+$. We are interested in the general solution around the outer horizon $r = r_+$ that can be continued to the spatial infinity $z \to -\infty$ $(r \to +\infty)$

$$h_{\omega l}(z) = C_1F(a, b, c; z) + C_2z^{1-c}F(1 + a - c, 1 + b - c, 2 - c; z). \tag{20}$$

Consequently for

$$Z_{\omega l}(z) := z^{iz_0\tilde{\omega}}(z-1)^{-i(z_0-1)\tilde{\omega}}h_{\omega l}(z), \tag{21}$$

we obtain the expression

$$\begin{aligned} Z_{\omega l}(z) = (1-z)^{-i(z_0-1)\tilde{\omega}}[&C_1(-z)^{iz_0\tilde{\omega}}F(1/2 + i(\tilde{\omega}+\lambda_{\omega l}), 1/2 + i(\tilde{\omega}-\lambda_{\omega l}), 1+2iz_0\tilde{\omega}; z) \\ &+ C_2(-z)^{-iz_0\tilde{\omega}}F(1/2 - i[(2z_0-1)\tilde{\omega}-\lambda_{\omega l}], 1/2 - i[(2z_0-1)\tilde{\omega}+\lambda_{\omega l}], 1-2iz_0\tilde{\omega}; z)], \end{aligned}$$

using (19) and (20).

For the asymptotic of (20) near the outer horizon (i.e., $z \to 0_-$), the zero order expansion of $F(a, b, c; z)$ is enough $(F(a, b, c, 0) = 1)$. Also $\lim_{z\to 0_-}(1-z)^{-i(z_0-1)\tilde{\omega}} = 1$. Then we have

$$Z_{\omega l}(z \to 0_-) \simeq C_1(-z)^{iz_0\tilde{\omega}} + C_2(-z)^{-iz_0\tilde{\omega}}. \tag{22}$$

For physical interpretation, it is appropriate to define a new real spatial variable x, by the relation

$$-z = \frac{r-r_+}{r_+-r_-} := \exp\frac{x}{z_0r_0}, \tag{23}$$

with $r \to r_+ \Rightarrow x \to -\infty$ and $r \to +\infty \Rightarrow x \to +\infty$. Now taking into account that $x = x(r)$ is an uniformly growing function and replacing (23) in (22), the asymptotic solution (22) multiplied by $e^{-i\omega t}$ can be considered as a superposition of an outgoing and an ingoing wave

$$R_{\omega l}(r \to r_+)e^{-i\omega t} \simeq A_{\text{out}}e^{i\omega(x-t)} + A_{\text{in}}e^{-i\omega(x+t)}, \quad \text{where} \left| \begin{array}{l} A_{\text{out}} := C_1 \\ A_{\text{in}} := C_2. \end{array} \right.$$

In the case $l = 0$, $Y_{00}(\theta, \varphi) = 1$, and the upper expression is just the solution for the eigenmode $\psi_{\omega 00}(t, r, \theta, \varphi)$ for $r = r_+$.

At the spatial infinity $(r \to +\infty, 1/z \to 0)$, the asymptotic solution can be written by using the known relation between hypergeometric functions $F(a, b, c; z)$ and $F(a', b', c'; 1/z)$. Taking only the leading-order expansion of $F(a', b', c'; 1/z)$ with respect to $1/z$, we have the following expressions:

$$\begin{aligned} F(b, c; z \to -\infty) &\simeq \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}, \\ (-z)^{1-c}F(1+a-c, 1+b-c, 2-c; z \to -\infty) &\simeq= \frac{\Gamma(2-c)\Gamma(b-a)}{\Gamma(1+b-c)\Gamma(1-a)}(-z)^{-a} + \frac{\Gamma(2-c)\Gamma(a-b)}{\Gamma(1+a-c)\Gamma(1-b)}(-z)^{-b}. \end{aligned} \tag{24}$$

Applying the transformations (24) to (20), replacing $h_{\omega l}(z)$ in (21) and finally taking into account that $\lim_{z\to-\infty}[-z/(1-z)]^{i(z_0-1)\tilde{\omega}} = 1$, the general asymptotic solution becomes

$$Z_{\omega l}(z \to -\infty) \simeq \frac{1}{\sqrt{-z}}[B_{\rm out}(-z)^{i\lambda_{\omega l}} + B_{\rm in}(-z)^{-i\lambda_{\omega l}}], \tag{25}$$

where

$$B_{\rm out} := C_1 \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} + C_2 \frac{\Gamma(2-c)\Gamma(a-b)}{\Gamma(1+a-c)\Gamma(1-b)},$$
$$B_{\rm in} := C_1 \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} + C_2 \frac{\Gamma(2-c)\Gamma(b-a)}{\Gamma(1+b-c)\Gamma(1-a)}.$$

Following (23) and applying the substitution

$$k_{\omega l} := \frac{\lambda_{\omega l}}{z_0 r_0}, \tag{26}$$

when $\lambda_{\omega l} \in \mathbb{R}$, the radial function in x-variable can be considered again like a superposition of 1D modes with wave vectors $\pm k_{\omega l}$ ($(-z)^{\pm i\lambda_{\omega l}} = e^{\pm i k_{\omega l} x}$):

$$R_{\omega l}(r \to +\infty) \simeq \sqrt{\frac{r_+ - r_-}{r}}[B_{\rm out} e^{ik_{\omega l}x} + B_{\rm in} e^{-ik_{\omega l}x}], \tag{27}$$

$$B_{\rm out} = \Gamma(2i\lambda_{\omega l})\left[\frac{\Gamma(1+2iz_0\tilde{\omega})A_{\rm out}}{\Gamma(1/2+i(\tilde{\omega}+\lambda_{\omega l}))\Gamma(1/2+i[(2z_0-1)\tilde{\omega}+\lambda_{\omega l}])} + \frac{\Gamma(1-2iz_0\tilde{\omega})A_{\rm in}}{\Gamma(1/2-i(\tilde{\omega}-\lambda_{\omega l}))\Gamma(1/2-i[(2z_0-1)\tilde{\omega}-\lambda_{\omega l}])}\right],$$
$$B_{\rm in} = \Gamma(-2i\lambda_{\omega l})\left[\frac{\Gamma(1+2iz_0\tilde{\omega})A_{\rm out}}{\Gamma(1/2+i(\tilde{\omega}-\lambda_{\omega l}))\Gamma(1/2+i[(2z_0-1)\tilde{\omega}-\lambda_{\omega l}])} + \frac{\Gamma(1-2iz_0\tilde{\omega})A_{\rm in}}{\Gamma(1/2-i(\tilde{\omega}+\lambda_{\omega l}))\Gamma(1/2-i[(2z_0-1)\tilde{\omega}+\lambda_{\omega l}])}\right]. \tag{28}$$

Black hole radiation is considered as a specific boundary condition when only an outgoing mode at the spatial infinity exists, $B_{\rm in} = 0$. This condition determines the ratio of the coefficients $A_{\rm in}/A_{\rm out}$ or the reflection coefficient R

$$R = \frac{|A_{\rm in}|^2}{|A_{\rm out}|^2}\bigg|_{B_{\rm in}=0} = \frac{|\Gamma(1+2iz_0\tilde{\omega})|^2|\Gamma(1/2-i(\tilde{\omega}+\lambda_{\omega l}))|^2|\Gamma(1/2-i[(2z_0-1)\tilde{\omega}+\lambda_{\omega l}])|^2}{|\Gamma(1-2iz_0\tilde{\omega})|^2|\Gamma(1/2+i(\tilde{\omega}-\lambda_{\omega l}))|^2|\Gamma(1/2+i[(2z_0-1)\tilde{\omega}-\lambda_{\omega l}])|^2}.$$

Complex conjugation and the Euler's reflection formula for the Gamma function give us the final result for the reflection coefficient on the outer horizon

$$R = \frac{\cosh(\pi(\tilde{\omega}-\lambda_{\omega l}))\cosh(\pi((2z_0-1)\tilde{\omega}-\lambda_{\omega l}))}{\cosh(\pi(\tilde{\omega}+\lambda_{\omega l}))\cosh(\pi((2z_0-1)\tilde{\omega}+\lambda_{\omega l}))}. \tag{29}$$

In the special case $r_- = 0 \Rightarrow z_0 = 1$, we recover the result of [9]

$$R = \frac{\cosh^2(\pi(\tilde{\omega}-\lambda_{\omega l}))}{\cosh^2(\pi(\tilde{\omega}+\lambda_{\omega l}))}. \tag{30}$$

For high frequencies $\tilde{\omega} \gg l + 1/2$ ($\Rightarrow \lambda_{\omega l} \approx \tilde{\omega}$) and $\tilde{\omega} \gg \frac{1}{z_0-1}$ we obtain

$$\begin{aligned} N := \frac{R}{1-R} &= \left[\frac{\cosh(2\pi\tilde{\omega})\cosh(2\pi z_0\tilde{\omega})}{\cosh(2\pi(z_0-1)\tilde{\omega})} - 1\right]^{-1} \\ &= 2\left[\frac{\cosh(2\pi(z_0+1)\tilde{\omega})}{\cosh(2\pi(z_0-1)\tilde{\omega})} - 1\right]^{-1} \\ &= 2\left[e^{4\pi\tilde{\omega}}\frac{1+\exp(-4\pi(z_0+1)\tilde{\omega})}{1+\exp(-4\pi(z_0-1)\tilde{\omega})} - 1\right]^{-1} \\ &\approx e^{-4\pi\tilde{\omega}}. \end{aligned} \tag{31}$$

We identify the Hawking temperature from $N \approx e^{-4\pi\tilde{\omega}} = e^{-\frac{\omega}{T_H}}$ which gives

$$T_H = \frac{1}{4\pi r_0}. \tag{32}$$

As one can see, the black hole temperature derived in semiclassical approximation does not agree with the surface gravity, i.e., $T_H \neq \frac{\kappa}{2\pi}$ where the surface gravity κ is given by (9). Only for $P = 0$ ($r_- = 0$) we have $T_H = \frac{\kappa}{2\pi}$.

B. Hawking radiation of extremal black holes

The extremal case can formally be considered as a limit of the nonextremal one, namely in the limit $r_+ \longrightarrow r_-$, i.e., $z_0 \longrightarrow \infty$. In this limit we have

$$\begin{aligned} R &= \lim_{z_0\to+\infty}\frac{\cosh(\pi(\tilde{\omega}-\lambda_{\omega l}))\cosh(\pi((2z_0-1)\tilde{\omega}-\lambda_{\omega l}))}{\cosh(\pi(\tilde{\omega}+\lambda_{\omega l}))\cosh(\pi((2z_0-1)\tilde{\omega}+\lambda_{\omega l}))} \\ &= \frac{\cosh(\pi(\tilde{\omega}-\lambda_{\omega l}))e^{-2\pi\lambda_{\omega l}}}{\cosh(\pi(\tilde{\omega}+\lambda_{\omega l}))}. \end{aligned} \tag{33}$$

Since one could doubt the legality of this limit because of the fact that $r_- = r_+$ is a singularity in our initial substitution (15) for z, and also for completeness of our investigation, we will consider this case separately.

In the extremal case, Eq. (14) becomes

$$\frac{d}{dr}[(r-r_+)^2\dot{R}_{\omega l}(r)] + \left[\frac{\tilde{\omega}^2 r^2}{(r-r_+)^2} - l(l+1)\right]R_{\omega l}(r) = 0. \tag{34}$$

The smaller number of singular points is a significant reason to require a separate investigation, so we cannot expect to obtain again hypergeometric equation. In the extremal case, our new variable will be

$$q := \frac{r_+}{r - r_+} \Rightarrow \frac{d}{dr} = -\frac{q^2}{r_+}\frac{d}{dq}. \tag{35}$$

So the equation for $Q_{\omega l}(q) := R_{\omega l}(r)$ takes the form

$$q^2\ddot{Q}_{\omega l}(q) + [\tilde{\omega}^2 - l(l+1) + 2\tilde{\omega}^2 q + \tilde{\omega}^2 q^2]Q_{\omega l}(q) = 0.$$

Here the same substitution $\lambda_{\omega l} := \sqrt{\tilde{\omega}^2 - (l+1/2)^2}$ is appropriate for recognizing the above equation as Whittaker equation for $W_{\omega l}(z) := Q_{\omega l}(q)$, where $z := 2i\tilde{\omega}q$

$$\ddot{W}_{\omega l}(z) + \left[\frac{1/4 - (i\lambda_{\omega l})^2}{z^2} + \frac{-i\tilde{\omega}}{z} - \frac{1}{4}\right]W_{\omega l}(z) = 0. \tag{36}$$

The general solution can be represented in terms of the confluent hypergeometric functions (Kummer functions [10])

$$W_{\omega l}(z) = e^{-z/2}z^{\frac{1}{2}+i\lambda_{\omega l}}(C_1M(a,b,z) + C_2U(a,b,z)), \quad \text{where} \quad \left| \begin{array}{l} a = 1/2 + i(\tilde{\omega} + \lambda_{\omega l}), \\ b = 1 + 2i\lambda_{\omega l}. \end{array} \right. \tag{37}$$

The asymptotic solution on the horizon ($r \to r_+ + 0 \Rightarrow q \to +\infty$) follows the asymptotic expansion of the Kummer functions [10]

$$\frac{M(a,b,z)|_{|z|\to+\infty}}{\Gamma(b)} \simeq \frac{e^{\pm i\pi a}z^{-a}}{\Gamma(b-a)}\left[\sum_{n=0}^{R-1}\frac{(a)_n(1+a-b)_n}{n!}(-z)^{-n} + O(|z|^{-R})\right] + \frac{e^z z^{a-b}}{\Gamma(a)}\left[\sum_{n=0}^{S-1}\frac{(b-a)_n(1-a)_n}{n!}(-z)^{-n} + O(|z|^{-S})\right],$$

where $+i\pi a$ is for $-\pi/2 < \arg z < 3\pi/2$ and $-i\pi a$ is for $-3\pi/2 < \arg z < -\pi/2$. We also have

$$U(a,b,z)|_{|z|\to+\infty} \simeq z^{-a}\left[\sum_{n=0}^{R-1}\frac{(a)_n(1+a-b)_n}{n!}(-z)^{-n} + O(|z|^{-R})\right], \qquad \left(-\frac{3\pi}{2} < \arg z < \frac{3\pi}{2}\right).$$

In our case the zero order terms in the sums ($R = 1$, $S = 1$) are sufficient again. Taking into account that $\arg z = \pi/2$ and also the expressions (37), one can obtain the solution on the horizon

$$Q(q \to +\infty) \simeq C_1\Gamma(1 + 2i\lambda_{\omega l})\left[\frac{ie^{-\pi(\lambda_{\omega l}+\tilde{\omega})}e^{-i\tilde{\omega}q}(2i\tilde{\omega}q)^{-i\tilde{\omega}}}{\Gamma(1/2 - i(\tilde{\omega} - \lambda_{\omega l}))} + \frac{e^{i\tilde{\omega}q}(2i\tilde{\omega}q)^{i\tilde{\omega}}}{\Gamma(1/2 + i(\tilde{\omega} + \lambda_{\omega l}))}\right] + C_2e^{-i\tilde{\omega}q}(2i\tilde{\omega}q)^{-i\tilde{\omega}}.$$

Here the suitable substitution that transforms the general solution in terms of 1D wave modes in $\mathbb{R}$ is

$$2qe^q := e^{-x/r_0} \Rightarrow x = \frac{-r_0r_+}{r - r_+} + r_0\ln\frac{r - r_+}{2r_+}, \tag{38}$$

which gives

$$\begin{aligned} R_{\omega l}(r \to r_+) \simeq A_{\text{out}}e^{i\omega x} + A_{\text{in}}e^{-i\omega x}, \qquad A_{\text{out}} &:= C_1\frac{ie^{-\pi\lambda_{\omega l}}e^{-\frac{\pi\tilde{\omega}}{2}}\tilde{\omega}^{-i\tilde{\omega}}\Gamma(1 + 2i\lambda_{\omega l})}{\Gamma(1/2 - i(\tilde{\omega} - \lambda_{\omega l}))} + C_2e^{\pi\tilde{\omega}/2}\tilde{\omega}^{-i\tilde{\omega}}, \\ A_{\text{in}} &:= C_1\frac{\Gamma(1 + 2i\lambda_{\omega l})e^{-\frac{\pi\tilde{\omega}}{2}}\tilde{\omega}^{i\tilde{\omega}}}{\Gamma(1/2 + i(\tilde{\omega} + \lambda_{\omega l}))}. \end{aligned} \tag{39}$$

For the asymptotic solution at the radial infinity ($r \to +\infty \Rightarrow q \to 0+ \Rightarrow z \to 0$), we will use the relation between $M(a,b,z)$ and $U(a,b,z)$ [10]

$$U(a,b,z) = \frac{\pi}{\sin\pi b}\left[\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M((1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right].$$

After applying the upper relation for $z = 0$ where $M(a,b,0) = 1$ we reach the asymptotic solution at the spatial infinity.

$$Q(q \to 0) \simeq (2i\tilde{\omega}q)^{1/2}\left\{\left[C_1 + \frac{C_2 i\pi}{\sinh(2\pi\lambda_{\omega l})\Gamma(1/2 + i(\tilde{\omega} - \lambda_{\omega l}))\Gamma(1 + 2i\lambda_{\omega l})}\right](2i\tilde{\omega}q)^{i\lambda_{\omega l}} - \frac{C_2 i\pi}{\sinh(2\pi\lambda_{\omega l})\Gamma(1/2 + i(\tilde{\omega} + \lambda_{\omega l}))\Gamma(1 - 2i\lambda_{\omega l})}(2i\tilde{\omega}q)^{-i\lambda_{\omega l}}\right\}.$$

The substitutions (38) and $k_{\omega l} := \lambda_{\omega l}/r_0$ again give the asymptotic solution in the form of 1D wave in $\mathbb{R}$

$$R_{\omega l}(r \to \infty) \simeq \sqrt{\frac{r_+}{r}}(B_{\mathrm{out}}e^{ik_{\omega l}x} + B_{\mathrm{in}}e^{-ik_{\omega l}x}),$$

where

$$\begin{aligned} B_{\mathrm{out}} &= \frac{-C_2 i\pi(2i\tilde{\omega})^{1/2}(i\tilde{\omega})^{-i\lambda_{\omega l}}}{\sinh(2\pi\lambda_{\omega l})\Gamma(1/2 + i(\tilde{\omega} + \lambda_{\omega l}))\Gamma(1 - 2i\lambda_{\omega l})}, \\ B_{\mathrm{in}} &= \left[C_1 + \frac{C_2 i\pi}{\sinh(2\pi\lambda_{\omega l})\Gamma(1/2 + i(\tilde{\omega} - \lambda_{\omega l}))\Gamma(1 + 2i\lambda_{\omega l})}\right](2i\tilde{\omega})^{1/2}(i\tilde{\omega})^{i\lambda_{\omega l}}. \end{aligned} \tag{40}$$

We should note that due to the time-reversal symmetry of wave equation it is permissible to work with the time-reversed formulation of the radiation boundary condition. In the normal picture the radiation boundary condition means missing of the ingoing mode at the spatial infinity $B_{\mathrm{in}} := 0$. The reflection coefficient in this case is $R := |A_{\mathrm{in}}|^2/|A_{\mathrm{out}}|^2$.

In the time-reversed picture the radiation mode becomes an ingoing mode so in this treatment $B_{\mathrm{out}} := 0$. Respectively on the horizon the falling mode becomes an ingoing mode and the reflected mode becomes an outgoing mode. So, in this case we should take $R := |A_{\mathrm{out}}|^2/|A_{\mathrm{in}}|^2$.

It is possible to turn out that

$$\left.\frac{A_{\mathrm{in}}}{A_{\mathrm{out}}}\right|_{B_{\mathrm{in}}:=0} \neq \left.\frac{A_{\mathrm{out}}}{A_{\mathrm{in}}}\right|_{B_{\mathrm{out}}:=0}, \quad \text{but always} \quad \left|\frac{A_{\mathrm{in}}}{A_{\mathrm{out}}}\right|^2_{B_{\mathrm{in}}:=0} = \left|\frac{A_{\mathrm{out}}}{A_{\mathrm{in}}}\right|^2_{B_{\mathrm{out}}:=0}.$$

In the work [9] the time-reversed picture is chosen despite the fact that there is no difference in the complexity of further calculations between both approaches. It is the same for our nonextremal black holes. But one could see from the expressions (39) and (40) that the time-reversed formulation gives a shorter way to R. All the results for R were made by us using both approaches for checking the correctness of all previous calculations. Following the time-reversed picture $B_{\mathrm{out}} = 0 \Rightarrow C_2 = 0$ we find

$$\begin{aligned} R = \frac{|A_{\mathrm{out}}|^2}{|A_{\mathrm{in}}|^2} &= \left|\frac{ie^{-\pi\lambda_{\omega l}}e^{-\frac{\pi\tilde{\omega}}{2}}\Gamma(1 + 2i\lambda_{\omega l})\tilde{\omega}^{-i\tilde{\omega}}}{\Gamma(1/2 - i(\tilde{\omega} - \lambda_{\omega l}))}\right|^2 \left|\frac{\Gamma(1/2 + i(\tilde{\omega} + \lambda_{\omega l}))}{\Gamma(1 + 2i\lambda_{\omega l})e^{-\frac{\pi\tilde{\omega}}{2}}\tilde{\omega}^{i\tilde{\omega}}}\right|^2 = \frac{|\Gamma(1/2 + i(\tilde{\omega} + \lambda_{\omega l}))|^2}{|\Gamma(1/2 - i(\tilde{\omega} - \lambda_{\omega l}))|^2}e^{-2\pi\lambda_{\omega l}} \\ &= \frac{\cosh(\pi(\lambda_{\omega l} - \tilde{\omega}))}{\cosh(\pi(\lambda_{\omega l} + \tilde{\omega}))}e^{-2\pi\lambda_{\omega l}}. \end{aligned} \tag{41}$$

For high frequencies $\tilde{\omega} \gg l + 1/2$ $(\Rightarrow \lambda_{\omega l} \approx \tilde{\omega})$ we have

$$R \approx \frac{e^{-2\pi\tilde{\omega}}}{\cosh(2\pi\tilde{\omega})} \approx e^{-4\pi\tilde{\omega}} \Rightarrow N \approx e^{-4\pi\tilde{\omega}}. \tag{42}$$

Hence we find the Hawking temperature in the extremal case in semiclassical approximation

$$T_H = \frac{1}{4\pi r_0}. \tag{43}$$

Contra-intuitively the temperature of the extremal case is nonzero and, as in the nonextremal case, is independent of the intrinsic characteristics of the black hole and depends only on the background parameter r_0.

III. DISCUSSION

In the present paper we studied the Hawking radiation of asymptotically nonflat dyonic black holes in 4D Einstein-Maxwell-dilaton gravity in semiclassical approximation. It was shown that the problem can be solved exactly and we computed exactly the semiclassical radiation spectrum of both nonextremal and extremal black holes.

Our results show that the Hawking temperature, calculated in the semiclassical approximation, is not compatible with the first law (10). The reason for this discrepancy is that the spacetime is asymptotically nonflat. Other examples for discrepancy between the Hawking temperature and the surface gravity in asymptotically nonflat spacetimes can be found in Ref. [11] and references therein. In principle, the relation between the Hawking radiation and the first law in asymptotically nonflat spacetimes is controversial and depends on the particular case. In general, the first law in asymptotically nonflat spacetime is not directly connected to the temperature of the particle flux at infinity. For example, the black holes considered in the present work cannot emit massive particles because the mass term in the wave equation (11) leads to the appearance of confining potential (growing unboundedly to infinity), which prevents the particles from escaping to infinity. In other words, the Hawking temperature for massive particles, measured by an asymptotic observer, is zero.

The asymptotical nonflatness leads to ambiguous thermodynamical characteristics and thermodynamics as a whole. For example, the surface gravity is given by the formula

$$\xi^{\mu}\nabla_{\mu}\xi^{\nu} = \kappa\xi^{\nu}, \tag{44}$$

on the horizon. The above definition, however, gives the surface gravity up to a constant, because there is a freedom to rescale ξ by a constant. In the asymptotically flat case this rescaling freedom can be fixed by choosing a unit norm for the Killing field at infinity. In linear dilaton spacetimes there is no natural way to fix the rescaling freedom. The choice $\xi = \partial/\partial t$ made in the papers devoted to the linear dilaton black holes is thus *ad hoc*. The ambiguity in choosing the time vector field is present also in the Hamiltonian formalism for linear dilaton spacetimes [8], where the time vector field is again $\xi = \partial/\partial t$.

One possible way to overcome the problem of the rescaling freedom in our case is to replace the Killing field ξ with the Kodama vector field $K = \sqrt{\frac{r_0}{r}}\frac{\partial}{\partial t}$, which has a unit norm at infinity [12]. This investigation is in progress and the results will be presented elsewhere.

ACKNOWLEDGMENTS

The partial financial supports from the Bulgarian National Science Fund under Grant No. DMU-03/6, and by Sofia University Research Fund under Grant No. 148/2012 are gratefully acknowledged.

[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[2] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[3] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).
[4] S. Yazadjiev, Classical Quantum Gravity **22**, 3875 (2005).
[5] K. C. K. Chan, J. H. Horne, and R. B. Mann, Nucl. Phys. **B447**, 441 (1995).
[6] G. Clement, D. Galt'sov, and C. Leygnac, Phys. Rev. D **67**, 024012 (2003).
[7] J. Brown and J. York, Phys. Rev. D **47**, 1407 (1993).
[8] G. Clement, D. Gal'tsov, C. Leygnac, and D. Orlov, Phys. Rev. D **73**, 045018 (2006).
[9] G. Clement, J. C. Fabris, and G. T. Marques, Phys. Lett. B **651**, 54 (2007).
[10] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1970).
[11] M. Park, Phys. Lett. B **663**, 259 (2008).
[12] H. Kodama, Prog. Theor. Phys. **63**, 1217 (1980).