Positioning systems in Minkowski space-time: Bifurcation problem and observational data

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In the framework of relativistic positioning systems in Minkowski space-time, the determination of the inertial coordinates of a user involves the *bifurcation problem* (which is the indeterminate location of a pair of different events receiving the same emission coordinates). To solve it, in addition to the user emission coordinates and the emitter positions in inertial coordinates, it may happen that the user needs to know *independently* the orientation of its emission coordinates. Assuming that the user may observe the relative positions of the four emitters on its celestial sphere, an observational rule to determine this orientation is presented. The bifurcation problem is thus solved by applying this observational rule, and consequently, *all* of the parameters in the general expression of the coordinate transformation from emission coordinates to inertial ones may be computed from the data received by the user of the relativistic positioning system.

I. INTRODUCTION

To locate the users[1] of a Global Navigation Satellite System (GNSS), several geometric methods and algebraic algorithms have been developed in the past [1–5] that are still in use [6,7]. Basically, the algebraic statement of the *location problem* is rather simple: to find the events where the emission light cones of four broadcast signals intersect. Of course, this idea is implicit in the Bancroft algorithm [2] and other similar ones [3]. In fact, Abel and Chaffee [4,5] used Minkowskian algebra to state the problem properly, making apparent that the more Lorentzian a description is, the more clear algorithm is performed.

However, in a full relativistic framework (cf., Refs. [8–14]), and even in the case of the flat space-time, an explicit form of the solution of the location problem for arbitrary emitters has not been obtained until recently [15].

In Ref. [15], an exact relativistic formula giving the inertial coordinates of an event in terms of the received emission coordinates is obtained. This formula applies in *all* the emission coordinate region and involves the orientation of the emission coordinates of the user. Nevertheless, there exists an inherent limitation on the applicability of this formula: only the users in a certain region (named the central region, see Sec. IV B) of a positioning system can obtain the orientation from the sole *standard emission data*, that is to say, from the sole set of the positions of the four emitters in inertial coordinates and of the emission coordinates of the user. Consequently, only these restricted users are able to locate themselves in inertial coordinates.

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[1]The word *user* here denotes any person or device able to receive the pertinent emitted data from the relativistic positioning system and to extract from it the corresponding information. For short, we shall refer to the user as *it*.

Here, assuming that the users out of the central region may observe the relative positions of the four emitters on their celestial sphere, we will give a simple rule allowing *any* user of the positioning system to locate itself in inertial coordinates. To show that, we will see that the orientation of the emission coordinates of a user is related to the relative positions of the emitters of the positioning system on the celestial sphere of the user.

In building current GNSS models, the usual assumption consists in picking out an approximate numerical solution. But, because gravitational effects are not taken into account at the considered leading order, one should start from the best accurate solution that nowadays we know. Such a solution is precisely the simple, exact, and covariant formula found in Ref. [15] and improved here, giving the location of a user of a relativistic positioning system in Minkowski space-time.

Let us remark that our result not only concerns GNSS around the Earth, but also general (relativistic) positioning systems anywhere in the Solar System or elsewhere. It is true that for most, but not all, of the present applications of the GNSS the users are near the Earth's surface. Therefore, they are usually in the central region of the satellites they detect, so that additional data (and in particular our observational rule) are not necessary. But for other applications of the GNSS as well as for general positioning systems, our observational rule may be a simple way to solve the bifurcation problem and hence, the location one.

A. Outline of the paper

The paper is organized as follows. In Sec. II, the inertial coordinates of a user are expressed in terms of the emitter configuration and the orientation of the positioning system. This provides a covariant formula for the transformation from emission to inertial coordinates. An analysis of the solution in terms of the configuration of the emitters is also presented. In Sec. III, some properties of the border between the two emission coordinate domains are obtained,

and an observational rule to detect it is remembered. Section IV is devoted to define the genuine regions and coordinate domains involved in the problem. We stress the geometrical meaning of the coordinate transformation formula in connection with these regions. In particular, we show that in the central region of the positioning system the orientation is computable from the sole standard emission data. In Sec. V we discuss the bifurcation problem (nonuniqueness of solutions in the determination of the location) which is related to the existence of regions whose events can not be located from the sole standard emission data. We give an *observational rule* to solve the above indetermination problem. This rule allows us to determine, at any event in the emission region, the orientation of the emission coordinates of the user from the *observational data* of the relative positions of the emitters on the celestial sphere of any user at this event. The concluding Sec. VI is devoted to summarize and discuss the results. The used notation is explained in an Appendix.

Some preliminary results of this work were presented at the Spanish Relativity meeting ERE-2010 [16].

B. Relativistic positioning terminology: brief compendium

As pointed out in Ref. [9], *relativistic positioning systems* [8–13] and the *emission coordinates* [14,17] they realize are essential elements to develop the relativistic theory of the GNSS. Starting from scratch, we present here a compendium of basic definitions about this specific subject. Anyway, we consider these definitions necessary not only to make this paper self contained, but also as an incipient piece of concepts to deal with GNSS in a full relativistic perspective.

Relativistic positioning system.—Set of four emitters A ($A = 1, 2, 3, 4$), of worldlines $\gamma_A(\tau^A)$, broadcasting their respective proper times τ^A by means of electromagnetic signals.[2]

Emission coordinates of an event.—The four times $\{\tau^A\}$ which are received at each event reached by the emitted signals.[3]

Configuration of the emitters for an event x.—Set of four events $\{\gamma_A(\tau^A)\}$ of the emitters at the emission times $\{\tau^A\}$ received at x.

Emission region.—Set $\mathcal{R}$ of events reached by the four signals broadcast by the positioning system. Every $x \in \mathcal{R}$ is labeled with the corresponding emission coordinates $\{\tau^A\}$.

Characteristic emission function.—Map Θ that to every $x \in \mathcal{R}$ associates its emission coordinates, that is $\Theta(x) = (\tau^A)$.

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[2]For simplicity the proper time is taken here, but any other time is valid. For example, the global positioning system (GPS) broadcasts the GPS time, a time which, roughly speaking, coincides up to a fixed shift with the international atomic time, a sort of mean proper time on the Earth surface.

[3]Emission coordinates have received different appellations in the past (see Ref. [18] for a brief and critical account).

The characteristic emission function describes the action of a positioning system and, hence, represents it.

Emission coordinate region.—Subset $\mathcal{C}$ of the emission region $\mathcal{R}$ where the gradients $d\tau^A$ are well defined and linearly independent.

The emitter worldlines are excluded from $\mathcal{C}$ because every $d\tau^A$ is not defined at the emission event $\gamma_A(\tau^A)$ (this event being the vertex of the emission light cone $\tau^A = $ Constant).

Orientation of a relativistic positioning system at the event x.—Orientation of its emission coordinates at x. It is given by the sign $\hat{\epsilon}$ of the Jacobian determinant $j_\Theta(x)$ of Θ at x, $\hat{\epsilon} \equiv \text{sgn} j_\Theta(x)$.

In terms of the gradients of the emission coordinates, one has

$$\hat{\epsilon} = \text{sgn}[*(d\tau^1 \wedge d\tau^2 \wedge d\tau^3 \wedge d\tau^4)], \qquad (1)$$

where $*$ stands for the Hodge dual operator, and $\wedge$ is the exterior product (see the Appendix for transcription into index notation).

II. THE LOCATION PROBLEM IN MINKOWSKI SPACE-TIME

Suppose a given specific coordinate system $\{x^\alpha\}$ covering the emission region $\mathcal{R}$, let $\gamma_A(\tau^A)$ be the worldlines of the emitters referred to this particular coordinate system, and let $\{\tau^A\}$ be the values of the emission coordinates received by a user. The data set $E \equiv \{\gamma_A(\tau^A), \{\tau^A\}\}$ is called the *standard emission data set*.

The *location problem with respect to E*, also called the *standard location problem* for short, is the problem of finding the coordinates $\{x^\alpha\}$ of the user from the sole data E.

In Ref. [15], the above standard location problem was analyzed for *arbitrary* relativistic positioning systems in Minkowski space-time, assuming that the specific coordinate system $\{x^\alpha\}$ is an *inertial* one. There, the explicit expression $x^\alpha = \kappa^\alpha(\tau^A)$ was found, giving the coordinate transformation from emission coordinates to inertial ones [Eq. (3) below].

Particular simple cases have already been studied: considering a two-dimensional [19–21] or a three-dimensional [22] space-time, or for special motions of the emitters in the Schwarzschild geometry from analytical [23] and numerical [24] approaches. For a recent approach to emission coordinates using the integration of the eikonal equation, and some numerical simulations, see also Ref. [25].

In this and the following sections, we are mainly dealing with relativistic positioning systems in Minkowski space-time.

A. Covariant expression of the solution

From now on, we shall suppose that any user in the emission coordinate region $\mathcal{C}$ receives the standard emission data set E. Let us denote by x the position vector (with respect the origin O of this inertial system) of an event P in
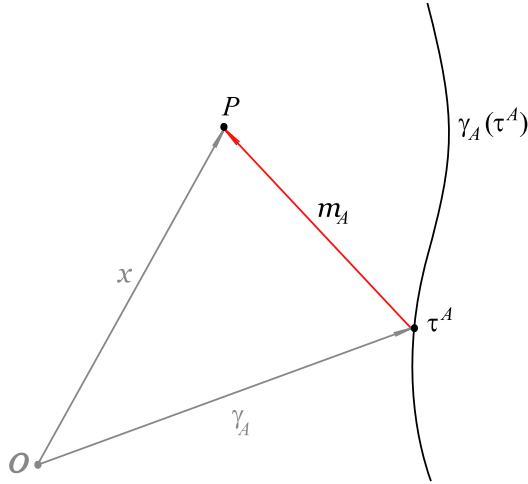
FIG. 1 (color online). The emission of an electromagnetic signal from a satellite $\gamma_A(\tau^A)$ at proper time τ^A, and its reception by a user at $x \equiv OP$. These events define the future pointing null vector m_A.

the emission region $\mathcal{R}$, $x \equiv OP$. If a user at P receives the broadcast times $\{\tau^A\}$, then γ_A denotes the position vectors of the emitters at the emission times, $\gamma_A \equiv O\gamma_A(\tau^A)$. Then

$$m_A \equiv x - \gamma_A, \qquad (A = 1, \ldots, 4) \qquad (2)$$

are future-oriented light-like vectors that represent the trajectories followed by the electromagnetic signals from the emitters $\gamma_A(\tau^A)$ to the reception event $x \in \mathcal{R}$ (Fig. 1).

In the standard emission data set E, the emission data $\{\tau^A\}$ received at x are the emission coordinates of the event $x \in \mathcal{R}$ and were broadcast when the emitters were at the events $\{\gamma_A(\tau^A)\}$, the configuration of the emitters for the event x. Generically, these four events determine the *configuration hyperplane* for x.[4]

For the events x in the emission coordinate region $\mathcal{C}$, the transformation $x = \kappa(\tau^A)$ from emission to inertial coordinates is locally well defined. In Ref. [15], we have obtained a covariant expression of this transformation, given by the following formula:

$$x = \gamma_4 + y_* - \frac{y_*^2 \chi}{(y_* \cdot \chi) + \hat{\epsilon}\sqrt{(y_* \cdot \chi)^2 - y_*^2 \chi^2}}, \qquad (3)$$

where $\gamma_4(\tau^4)$ has been chosen as the reference emitter.[5]

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[4]Here, we always consider that the emitter configuration is *regular*, i.e., the four events $\{\gamma_A(\tau^A)\}$ are noncoplanar. Nonregular or *degenerate* configurations are considered elsewhere (for some remarks concerning these situations, see Refs. [4,5]).

[5]The transformation (3) from emission to inertial coordinates may be written in a totally symmetric form without the choice of any emitter worldline as reference origin line. For this purpose, one has to consider the barycenter of the emitters as the convenient reference event rather than one of the emitters. This issue will be addressed elsewhere [26], in connection with the symmetric formulation of the location problem in flat space-time.
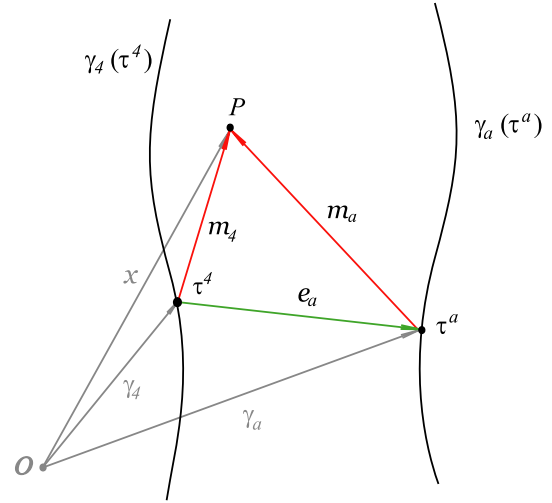

FIG. 2 (color online). If we choose the emitter four as origin (reference emitter), the relative positions of the others (referred emitters) are $e_a = \gamma_a - \gamma_4$, $a = 1, 2, 3$, and the position vector of the event P is m_4.

Quantity y_* is given by

$$y_* = \frac{1}{\xi \cdot \chi} i(\xi)H, \qquad (4)$$

where χ is the *configuration vector*

$$\chi = *(e_1 \wedge e_2 \wedge e_3) \qquad (5)$$

and H is the *configuration bivector*

$$H = *(\Omega_1 e_2 \wedge e_3 + \Omega_2 e_3 \wedge e_1 + \Omega_3 e_1 \wedge e_2), \qquad (6)$$

with (Fig. 2)

$$\Omega_a = \frac{1}{2}(e_a)^2, \quad e_a = \gamma_a - \gamma_4, \quad (a = 1, 2, 3), \qquad (7)$$

and where ξ is any vector transversal to the configuration, $\xi \cdot \chi \neq 0$, and $i(\xi)H$ stands for the tensor contraction of ξ and the first slot of H (see Appendix).

Quantity $\hat{\epsilon}$ is the *orientation* of the positioning system at x, that is now equivalently expressed as

$$\hat{\epsilon} = \text{sgn}[*(m_1 \wedge m_2 \wedge m_3 \wedge m_4)]. \qquad (8)$$

It is worthy to remark that χ and H are determined by the relative positions $e_a = \gamma_a - \gamma_4$ associated with a given configuration of the emitters. Therefore, y_* is directly computable from the sole standard emission data.

Nevertheless, if we want to obtain x from (3) we also need to determine the orientation $\hat{\epsilon}$, which involves, by substituting (2) in (8), the unknown x. In fact, from Eqs. (2) and (7) it is clear that $m_a = m_4 - e_a$ (see Fig. 2) and one obtains:

$$m_1 \wedge m_2 \wedge m_3 \wedge m_4 = -(e_1 \wedge e_2 \wedge e_3 \wedge m_4), \qquad (9)$$

that taking into account (5) allows us to express Eq. (8) as

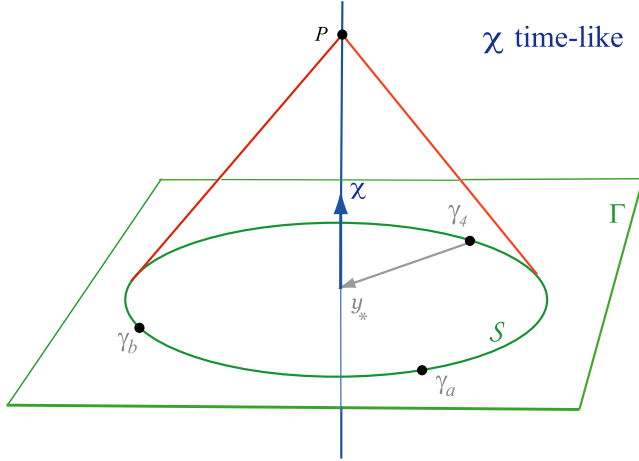$$\hat{\epsilon} = \text{sgn}(\chi \cdot m_4), \qquad (10)$$

FIG. 3 (color online). For the event P, the configuration hyperplane Γ is light-like, $\chi^2 < 0$. In this case, the emitters remain on a two-sphere $\mathcal{S}$ laying in Γ, and a sole emission solution P exists. In this three-dimensional representation for three satellites, the two-sphere reduces to a circle.

FIG. 4 (color online). For the event P, the configuration hyperplane Γ is light-like, $\chi^2 = 0$. In this case, the emitters remain on a two-paraboloid $\mathcal{P}$ laying in Γ, and a sole emission solution P exists. In this three-dimensional representation for three satellites, the two-paraboloid reduces to a parabola.

which by (2) depends on x. Therefore, in order to show that Eq. (3) does not chase its own tail, we must be able to determine the orientation $\hat{\epsilon}$ at x by using a procedure not involving the previous knowledge of x.

B. Analysis of the solution

In Ref. [15], Eq. (3) was obtained by separately analyzing three different cases, and gluing together their different solutions in a sole covariant an analytic expression. In gluing them, the role played by the external element ξ is essential.[6]

The three cases correspond to the different causal characters of the configuration vector χ. In space-time metric signature $(-, +, +, +)$, one has for each case:

(i) χ time-like, $\chi^2 < 0$: there is a sole emission solution x (the other one is a reception solution). The orientation $\hat{\epsilon}$ corresponding to the emission solution remains to be calculated;

(ii) χ light-like, $\chi^2 = 0$: there is a sole emission solution x (the other one being degenerate). The orientation $\hat{\epsilon}$ corresponding to the emission solution remains to be calculated;

(iii) χ space-like, $\chi^2 > 0$: there are two emission solutions, x and x'. They only differ by their orientation $\hat{\epsilon}$. The problem is how to determine the one corresponding to the real user.

The above cases are illustrated in Figs. 3–5, respectively.

For cases (i) and (ii), the matter to determine $\hat{\epsilon}$ was solved in Ref. [15] (see Sec. IV B below). Figure 3 shows the emission solution for the case (i). The configuration hyperplane, being space-like, cuts the past light cone of the solution in a two-sphere containing the configuration of the emitters. Figure 4 shows case (ii), where the emitter
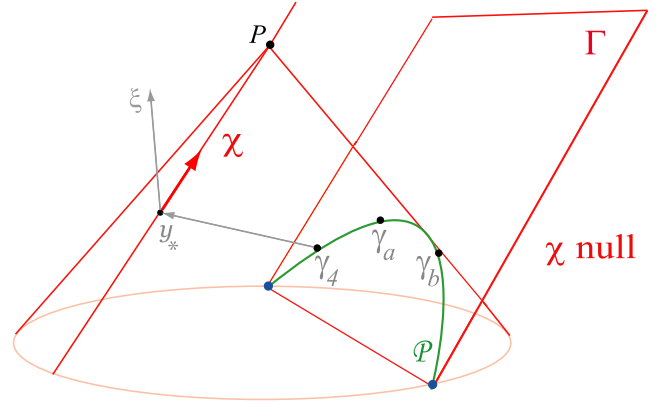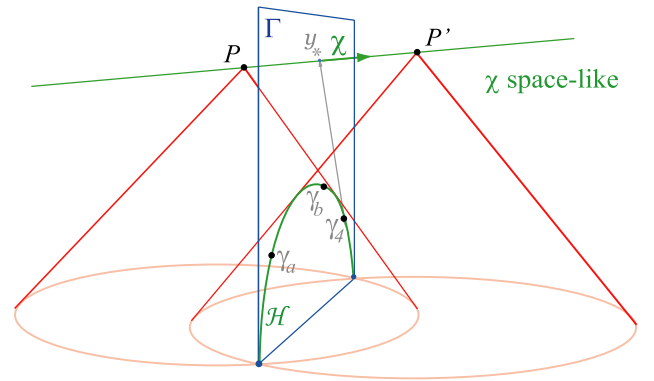
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[6]For a detailed discussion about this point, see Ref. [15].

FIG. 5 (color online). For the events, P and P', the configuration hyperplane Γ is time-like, $\chi^2 > 0$. In this case, the emitters remain on a two-hyperboloid $\mathcal{H}$ laying in Γ, and both P and P' are emission solutions corresponding to the same emission coordinates (τ^A). In this three-dimensional representation for three satellites, the two-hyperboloid reduces to a hyperbola.

configuration stays on a two-paraboloid contained in the null configuration hyperplane.

For case (iii), $\hat{\epsilon}$ cannot be determined from the sole emission data. Figure 5 shows a pair of emission solutions receiving the same emission coordinates. This 'indetermination' is known as the *bifurcation problem*. To solve it is the main subject of this paper (see Sec. V).

III. THE BORDER BETWEEN THE EMISSION COORDINATE DOMAINS

The emission coordinate region contains two emission coordinate domains (see Sec. IV below). The border between these domains is the hypersurface $\mathcal{J}$, where the Jacobian determinant of the characteristic emission function Θ vanishes,

$$\mathcal{J} = \{x | j_\Theta(x) = 0\}. \tag{11}$$

We are going to obtain some related properties showing its interest in relativistic positioning.

First, let us note that, in an adequate and condensed form Eq. (3) reads as

$$x = \gamma_4 + y_* - \lambda \chi, \tag{12}$$

where

$$\lambda \equiv \frac{y_*^2}{(y_* \cdot \chi) + \hat{\epsilon} \sqrt{\Delta}}, \qquad \Delta \equiv (y_* \cdot \chi)^2 - y_*^2 \chi^2. \tag{13}$$

As $m_4 = y_* - \lambda \chi$ is a null vector, and vectors $\{y_*, \chi\}$ and $\{m_4, \chi\}$ generate the same two-plane, the following relation holds:

$$\text{sgn}(\Delta) = \text{sgn}[(\chi \cdot m_4)^2], \tag{14}$$

and then $\Delta \geq 0$, assuring consistence for the above definition of λ. Consequently, one has the following result, made already evident by Eqs. (14) and (10).

Proposition 1.—$j_\Theta(x) = 0$ if, and only if, $\Delta = 0$.

The fact that Δ is non-negative says that the two-plane generated by y_* and χ is everywhere time-like, except in the border $\mathcal{J}$, where this plane is light-like.

Coming back to Eq. (6), let us note that H is a simple bivector, that is

$$H = \chi \wedge a \tag{15}$$

for some vector a, because of $i(\chi) * H = 0$, which is a direct consequence of Eqs. (5) and (6). Therefore, the invariant $(H, *H)$ vanishes, $(H, *H) = 0$, and the invariant (H, H) takes the expression (see the Appendix):

$$(H, H) = \chi^2 a^2 - (\chi \cdot a)^2. \tag{16}$$

On the other hand, substituting (15) into (4), y_* is expressed as

$$y_* = \frac{1}{\xi \cdot \chi} [(\xi \cdot \chi) a - (\xi \cdot a) \chi], \tag{17}$$

and then Eq. (13) for Δ becomes

$$\Delta = (\chi \cdot a)^2 - \chi^2 a^2. \tag{18}$$

Consequently, Δ really does not depend on the choice of the transversal vector ξ and by comparing (16) and (18) the following result has been proved.

Proposition 2.—Up to sign, the quantity Δ defined in (13) is the scalar invariant (H, H) of the configuration bivector H:

$$\Delta = -(H, H). \tag{19}$$

Moreover, from Eq. (19), the user can determine Δ from the sole standard emission data E. Thus, taking into account Proposition 1, the user is able to know from the sole standard set E it receives, when it is crossing the border $\mathcal{J}$ of the two emission coordinate domains.

Furthermore, it is worth remarking that on the border $\mathcal{J}$ the location of a user may be unambiguously solved. There, its location is obtained from (12) by taking $\Delta = 0$ in Eq. (13).

On the way, taking into account that $(H, H) = 0$, H will be a null bivector only when the invariant $(H, *H)$ vanishes. Then, we have also proven the following result.

Proposition 3.—For an event $x \in \mathcal{R}$, the configuration bivector H is a null bivector if, and only if $j_\Theta(x) = 0$.

On the other hand, an *observational method* allowing the user to detect when it is on the border $\mathcal{J}$ has been previously studied by Coll and Pozo, who stated the following result [27,28].

Proposition 4.—The border $\mathcal{J}$ consists in those events for which any user at them can see the four emitters on a circle on its celestial sphere.

This result is rather counterintuitive. When the GPS satellites are all near the horizon, or are all too close together on our zenith, the error in positioning is great. It would seem then that the optimal conditions for a precise location would be obtained when all the satellites are situated on an intermediate circle of the celestial sphere (say, among 30 or 60 degrees with respect to the zenith). Nevertheless, Proposition 4 shows that the circle corresponds to the most degenerate distribution that a set of satellites may have.

Proposition 4 also makes clear that the border $\mathcal{J}$ may be plotted from the sole observational data, a result that was not, *a priori*, evident.

IV. REGIONS AND COORDINATE DOMAINS IN RELATIVISTIC POSITIONING

This section provides a geometrical background to analyze the space-time regions which are relevant in relativistic positioning. In particular, we study the subset of the emission coordinate region $\mathcal{C}$, where the orientation $\hat{\epsilon}$ is computable from the standard emission data E (the central region of the positioning system).

A. Emission configuration regions $\mathcal{C}_s$, $\mathcal{C}_\ell$, and $\mathcal{C}_t$

The emission coordinate region $\mathcal{C}$ is constituted by three disjoint regions, and one can write $\mathcal{C} = \mathcal{C}_s \cup \mathcal{C}_\ell \cup \mathcal{C}_t$. They are the *space-like* $\mathcal{C}_s$, the *null* $\mathcal{C}_\ell$, and the *time-like* $\mathcal{C}_t$ emission configuration regions defined by the conditions $\chi^2 < 0$, $\chi^2 = 0$, and $\chi^2 > 0$, respectively.

This means that at every event $x \in \mathcal{C}_s$ ($x \in \mathcal{C}_\ell$ or $x \in \mathcal{C}_t$, respectively) a user receives the signals from four emission events that generate a space-like (null or time-like, respectively) hyperplane.

From Eqs. (5) and (7), which only involve the emitter configuration of the standard emission data E, the user is able to determine the sign of χ^2. Consequently, from the data set E and the above definitions, the user knows in what configuration region, $\mathcal{C}_s$, $\mathcal{C}_\ell$, or $\mathcal{C}_t$, of the positioning system it is traveling.

B. The central region $\mathcal{C}^C = \mathcal{C}_s \cup \mathcal{C}_\ell$

We name $\mathcal{C}^C \equiv \mathcal{C}_s \cup \mathcal{C}_\ell$ the *central region* of the positioning system.

At every event $x \in \mathcal{C}^C$, one has $u \cdot \chi \neq 0$ for any future pointing time-like vector u, because χ is not space-like in this region. Taking into account that m_4 is a future pointing null vector, the sign of the scalar products $\chi \cdot m_4$ and $u \cdot \chi$ is the same for any future pointing time-like vector u, and from Eq. (10) this sign is precisely the orientation of the positioning system on the central region. More precisely, we can prove the following result:

Proposition 5.—In the central region $\mathcal{C}^C$, the orientation of a relativistic positioning system is constant, and may be evaluated from the sole standard emission data E:

$$\forall \; x \in \mathcal{C}^C, \qquad \hat{\epsilon} = \text{sgn}(u \cdot \chi), \qquad (20)$$

where u is any future pointing time-like vector.

Thus, from (20) any user in the central region is able to determine the orientation of the positioning system, and then from Eqs. (3)–(7) it can obtain its own position x in the inertial system from the sole standard emission data by substituting $\hat{\epsilon} = \text{sgn}(u \cdot \chi)$ in (3). The resulting sign of $\hat{\epsilon}$ will be positive or negative depending on the time orientation of the computed vector χ.

C. Front ($\mathcal{C}^F$) and back ($\mathcal{C}^B$) coordinate domains

As a consequence of Proposition 5, the Jacobian determinant does not vanish, $j_\Theta(x) \neq 0$, in the immediate vicinity of $\mathcal{C}^C$. Therefore, the border $\mathcal{J}$ divides the time-like configuration region $\mathcal{C}_t$ of $\mathcal{C}$. In other words, the whole region $\mathcal{C}_t$ cannot be recovered by a sole coordinate domain.[7] In Ref. [15] we proved the following result.

Proposition 6.—The emission coordinate region $\mathcal{C}$ is not a coordinate domain, but the union of two disjoint coordinate domains, called the front $\mathcal{C}^F$, and the back $\mathcal{C}^B$ emission coordinate domains, $\mathcal{C} = \mathcal{C}^F \cup \mathcal{C}^B$.

The front coordinate domain $\mathcal{C}^F$ contains the central region $\mathcal{C}^C$ and a proper subset $\mathcal{C}_t^F$ of the time-like configuration region, $\mathcal{C}_t^F \subset \mathcal{C}_t$. This proper subset $\mathcal{C}_t^F$ is the part of $\mathcal{C}_t$ adjacent to the central region $\mathcal{C}^C$, so that the whole front domain $\mathcal{C}^F$, $\mathcal{C}^F \equiv \mathcal{C}^C \cup \mathcal{C}_t^F$, has, by continuity, constant orientation $\hat{\epsilon}$ (the same as the central region). However, the orientation at $x \in \mathcal{C}_t^F$ cannot be determined from Eq. (20), because Proposition 5 only applies on $\mathcal{C}^C$.

The back coordinate domain $\mathcal{C}^B$ is not a simply-connected domain. In fact, the region $\mathcal{C}_t$ is not simple-connected, and its leaves are constituted by pairs of events $\{x, x'\}$ having the same emission coordinates but different inertial ones, defining two well differentiated regions: if $x \in \mathcal{C}^B$, then $x' \in \mathcal{C}_t^F \equiv \mathcal{C}_t - \mathcal{C}^B$.

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[7]Remember that, for historical reasons, the *coordinate domain* of a coordinate system is an *open* set, but not necessarily a *topological domain* (i.e., an open and connected set).
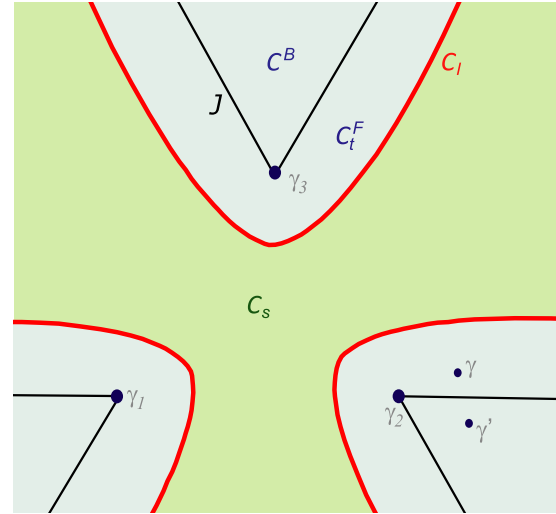

FIG. 6 (color online). Quotient space S of the stationary observers of a stationary and symmetric relativistic positioning system in flat three-dimensional space-time. Dots γ, $\gamma' \in S$ represent stationary user worldlines, and solid dots γ_a stand for stationary satellites (emitters). The emission configuration regions $\mathcal{C}_s$, $\mathcal{C}_\ell$ and $\mathcal{C}_t$ are differently colored. The border between $\mathcal{C}^B$ and $\mathcal{C}_t^F$ is $\mathcal{J}$: the surface of vanishing Jacobian. Conjugate events, (x, x'), having the same emission coordinates but different inertial ones, necessarily occur on separate parts of $\mathcal{C}_t$: if $x \in \gamma \subset \mathcal{C}_t^F$, then $x' \in \gamma' \subset \mathcal{C}^B$. In this three-dimensional situation, $\mathcal{J}$ is just the union of the shadows that each satellite produces on the signals coming from the other ones in the region $\mathcal{C}$.

To illustrate these coordinate domains, let us consider the simple case of a symmetric stationary positioning system in flat space-time. In this case, the four emitters define a regular tetrahedron, and $\mathcal{C}^B$ is the union of four connected components. The common boundary $\mathcal{J}$ of the domains $\mathcal{C}^F$ and $\mathcal{C}^B$ is a four-leaf hypersurface that contains the shadows that each satellite produces on the signals coming from the other ones in the region $\mathcal{C}$. The orientation $\hat{\epsilon}$ of the positioning system only changes across $\mathcal{J}$ taking a different constant value on each coordinate domain. The analogous, but simpler to draw, stationary and symmetric three-dimensional case is illustrated in Fig. 6, that shows the involved configuration regions and coordinate domains.

V. THE BIFURCATION PROBLEM. ITS OBSERVATIONAL SOLUTION

The above results show that the standard emission data E are generically insufficient to locate a user of a positioning system in an inertial system.

In the past, and in connection with GNSS, this problem was pointed out by Schmidt [1] and studied by Abel and Chaffee [4,5] by introducing a "bifurcation parameter" [equivalent to the square χ^2 of the configuration vector χ of Eq. (5)]. Afterwards, it was referred as the *bifurcation problem* [29].

In current practical situations in present day GNSS,[8] the bifurcation problem may be solved by hand: simply checking which of the two solutions satisfies an observable pertinent constraint. Thus, for example, if a user stays near the Earth's surface the right solution is the nearest to the Earth's radius. However, in extended GNSS or more general positioning systems in the Solar System, the bifurcation problem cannot be so easily avoided; it will always be present for users traveling in the time-like configuration region $\mathcal{C}_t$.

One could think that the bifurcation problem could be avoided by continuity for users traveling from the central region $\mathcal{C}^C$ (where they are able to calculate the positioning system orientation from the standard emission data) to the time-like configuration region $\mathcal{C}_t$. But the discrete character of true successive location operations, and the fact that it suffices of a sole instant to cross the border $\mathcal{J}$, also make this possibility illusory. Not only for theoretical reasons, but also for future practical applications where the role played by Earth based coordinate systems could become secondary (cf., Refs. [9,14,19]), it is essential to learn to solve this important part of the location problem, the bifurcation problem.

We have seen that, from the sole standard emission data E, the users can know the configuration region that they are traveling. The bifurcation problem appears when this configuration region is the time-like one, $\mathcal{C}_t$, because this region is constituted by pairs of conjugate events, x and x', separated by the border $\mathcal{J}$, receiving the same standard emission data (see Figs. 5 and 6). Conjugate events belong to different (back and front) coordinate domains, of different orientation. As Eq. (3) shows, the knowledge of the orientation in addition to the data set E solves completely the bifurcation problem. Thus, how to extend the standard emission data E so as to be able to determine the orientation of the positioning system for the user?

We shall suppose here that, in addition to the standard emission data E, the users are able to *observe* the relative positions of the emitters on their celestial sphere. We shall denote this *extended data set* by E^*.

Consider an arbitrary user of unit velocity u, at the event x of $\mathcal{C}$. With respect to this user, the null vectors m_A may be decomposed as

$$m_A = (m_A)_u u + \vec{m}_A, \qquad (21)$$

where $(m_A)_u = -u \cdot m_A > 0$ and $\vec{m}_A$ denote vectors of the proper three-dimensional space S_u of the user, $\vec{m}_A \in S_u$ [cf. Eq. (A1)].

Let us consider the unit vectors $\vec{n}_A = \vec{m}_A/(m_A)_u$ giving the relative directions of propagation of the signals. Because the vectors $-\vec{n}_A$ point to the positions of the emitters A, i.e., are the unit vectors along the apparent line of sight of the emitters A, we say that $\{\vec{n}_A\}$ is a set of *observational data*. It is this set of data (or any equivalent one) which, added to the standard emission data, is included in E^*.

By direct substitution of (21) in the expression (8) of $\hat{\epsilon}$, one has

$$\hat{\epsilon} = \text{sgn}\{*[u \wedge (-\vec{n}_1 \wedge \vec{n}_2 \wedge \vec{n}_3 + \vec{n}_1 \wedge \vec{n}_2 \wedge \vec{n}_4$$
$$- \vec{n}_1 \wedge \vec{n}_3 \wedge \vec{n}_4 + \vec{n}_2 \wedge \vec{n}_3 \wedge \vec{n}_4)]\}, \qquad (22)$$

where we have taken into account that $\prod_{A=1}^{4} m_A^0 > 0$ for emission vectors m_A. And because any three-form $\mathcal{F}$ in S_u satisfies $i(u) * \mathcal{F} = *(u \wedge \mathcal{F})$, the above expression gives (see the Appendix),

$$\hat{\epsilon} = \text{sgn}[(\vec{n}_1, \vec{n}_2, \vec{n}_3) - (\vec{n}_2, \vec{n}_3, \vec{n}_4)$$
$$+ (\vec{n}_3, \vec{n}_4, \vec{n}_1) - (\vec{n}_4, \vec{n}_1, \vec{n}_2)], \qquad (23)$$

where the triple product is defined according to Eqs. (A3) and (A4). Then, the following result holds.

Proposition 7.—The orientation $\hat{\epsilon}$ of a relativistic positioning system is given by

$$\hat{\epsilon} = \text{sgn}[(\vec{\mu}_1, \vec{\mu}_2, \vec{\mu}_3)] \qquad (24)$$

with $\vec{\mu}_a \equiv \vec{n}_a - \vec{n}_4$.

Thus, from the relative positions of the emitters on the celestial sphere of the user, we can obtain the orientation $\hat{\epsilon}$. For instance, if the referred emitters 1, 2, 3 are counterclockwise aligned on a circle of the celestial sphere of the user and the fourth emitter is inside this circle, then $(\vec{\mu}_1, \vec{\mu}_2, \vec{\mu}_3) > 0$. Then, analyzing separately all the possible situations we arrive to the following rule to obtain the orientation.

Observational rule to determine $\hat{\epsilon}$.—For any user in the coordinate region $\mathcal{C}$ receiving the extended data set E^*, the orientation $\hat{\epsilon}$ of the positioning system may be obtained as follows:

(i) consider the circle of the celestial sphere of the user containing the three referred emitters, $a = 1, 2, 3$,

(ii) turn this circle around its center in the increasing sense $1 \rightarrow 2 \rightarrow 3$ to orient the visual axis of the user by the rule of the right-hand screw,

(iii) if the fourth emitter $A = 4$ is in the spherical cap pointing out by this oriented axis, then the orientation is $\hat{\epsilon} = -1$, otherwise $\hat{\epsilon} = +1$.

By applying this observational rule, the users receiving the extended emission data set E^* can determine the orientation $\hat{\epsilon}$ and from Eq. (3) their position in inertial coordinates.

For a better geometric comprehension of the above observational rule, we can consider an alternative approach to its proof. Indeed, let us focus on the generic situation in which $\{\vec{n}_a\}$ is a basis of S_u. Then the solution of the linear system

$$\vec{n}_a \cdot \vec{s} = \omega_a, \qquad (a = 1, 2, 3) \qquad (25)$$

is given by $\vec{s} = \omega_a \vec{L}^a$, with the vectors $\vec{L}^a$ expressed in terms of the know data $\vec{n}_A$ as
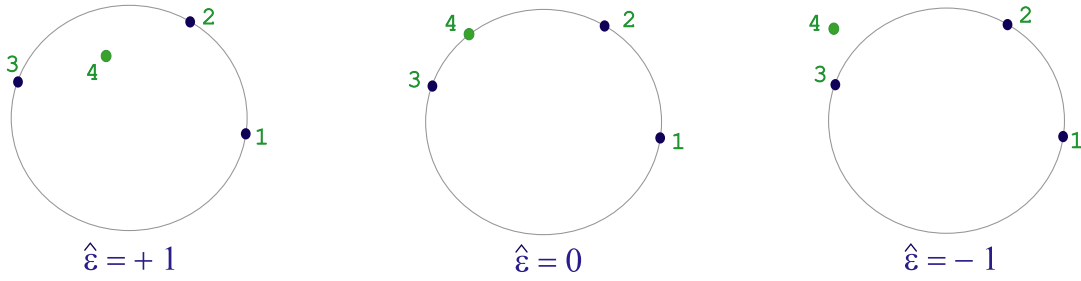
FIG. 7 (color online). Observational rule to determine the orientation $\hat{\epsilon}$ at the location of a user from the relative positions of the emitters on its celestial sphere. In the left diagram, the visual axis is oriented towards the spherical cap that does not contain the fourth emitter, and thus $\hat{\epsilon} = +1$. In the middle diagram, the four emitters are on a circle of the celestial sphere of the user, and thus $\hat{\epsilon} = 0$ (Proposition 4). In the right diagram, the visual axis is oriented towards the spherical cap that contains the fourth emitter, and thus $\hat{\epsilon} = -1$.

$$\vec{L}^a = \frac{\epsilon^{abc} \vec{n}_b \times \vec{n}_c}{2(\vec{n}_1, \vec{n}_2, \vec{n}_3)}. \tag{26}$$

Now, by substituting (26) into (23) we arrive to the following expression for the orientation,

$$\hat{\epsilon} = \text{sgn}[(1 - \vec{n}_4 \cdot \vec{L})(\vec{n}_1, \vec{n}_2, \vec{n}_3)], \tag{27}$$

where $\vec{L} \equiv \vec{L}^1 + \vec{L}^2 + \vec{L}^3$.

That $\vec{n}_4 \cdot \vec{L} = 1$ when and only when the Jacobian $j_\Theta(x)$ vanishes has been known since [27,28]. From Eq. (27), and according to the result stated in Proposition 4, the events of the emission coordinate region $\mathcal{C}$ are all those for which the four emitters are not aligned on a circle of the celestial sphere of the users at these events.

Then, it is possible to state that the factor $(\vec{n}_4 \cdot \vec{L} - 1)$ in (27) is positive or negative if $\vec{n}_4$ is interior or exterior, respectively, to the oriented half cone containing the three emitters $\{\vec{n}_a\}$ ($a = 1, 2, 3$). The unit vector axis $\vec{s}$ of this cone is given by

$$\vec{n}_a \cdot \vec{s} = \cos\varphi > 0. \tag{28}$$

Moreover, in terms of the basis $\{\vec{L}^a\}$ given by Eq. (26), the unit axis $\vec{s}$ has the expression

$$\vec{s} = \vec{L} \cos\varphi, \tag{29}$$

as can be directly verified.

Therefore, a unit vector $\vec{v}$ is in the interior of the half cone or at its exterior if the quantity $\vec{v} \cdot \vec{s}$ is greater or less than $\cos\varphi$, respectively, or by (29) if $\vec{v} \cdot \vec{L} > 1$, or $\vec{v} \cdot \vec{L} < 1$. Thus, by taking $\vec{v} = \vec{n}_4$, from (27) one has the following result.

Proposition 8.—Consider the oriented half cone containing $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$. If $\vec{n}_4$ is in its interior, then $\hat{\epsilon} = -\text{sgn}[(\vec{n}_1, \vec{n}_2, \vec{n}_3)]$. Otherwise, $\hat{\epsilon} = \text{sgn}[(\vec{n}_1, \vec{n}_2, \vec{n}_3)]$.

From this proposition we can recover the observational rule by considering all the possible relative positions of the unit vectors $\vec{n}_1, \vec{n}_2, \vec{n}_3$. Fig. 7 illustrates the application of the rule when $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ is a negative-oriented basis of S_u, that is for $(\vec{n}_1, \vec{n}_2, \vec{n}_3) < 0$.

Let us remark that the relative positions of the emitters in the celestial sphere of a user are *Lorentz invariant*: by Lorentz transformations between users at an event, the diameter of the circle as well as the positions of the emitters on it may change, but their increasing sense as well as the interior or exterior position of the fourth emitter will remain unchanged.

VI. DISCUSSION AND ENDING COMMENTS

The main result of this paper is the observational rule giving the orientation $\hat{\epsilon}$ of the emission coordinates for the user. Together with the standard emission data, it gives a full operational character to formula (3), allowing any user to obtain the coordinate transformation from emission coordinates to inertial ones and, in particular, to locate itself in inertial coordinates.

In the central region $\mathcal{C}^C$, where the orientation may be *deduced* from the sole standard emission data (Proposition 6), both the observed and the computed orientations may be contrasted.

It is worth it to remark here that the sole standard emission data allows the users to detect when they are on the border $\mathcal{J}$ separating the two coordinate domains (Proposition 1), a situation that may be also contrasted with the limit of the observational rule (when the four emitters are on a circle of the celestial sphere of each user). In spite of the fact that the border does not belong to any coordinate domain, the user can also locate itself in it (taking $\Delta = 0$ in Eqs. (12) and (13)).

Relativistic positioning concepts have been recently implemented in an algorithm giving the Schwarzschild coordinates of the users in terms of their emission coordinates (see Ref. [24]). If the conditions of applicability of our rule are given (observation of the emitters), the rule extends the region of validity of this algorithm.

It is important to note that in dealing with approximate methods or iterative algorithms to solve the location problem in weak gravitational fields, Eq. (3) is the best zero order solution to start with.

A numerical analysis of the quantities appearing in (3) has been recently implemented [30,31]. This analysis provides a numerical test of the results obtained in Ref. [15], and a promising via to deal with numerical simulations in modeling GNSS by starting from a fully relativistic conception.

ACKNOWLEDGMENTS

APPENDIX: NOTATION

The notation of this paper has not been chosen for academic reasons, but for practical ones. This notation allows, generically, more compact and shorter expressions (in occupied space and in expended time) than index notation, improving a best understanding of the formula, but overall it suggests more compact and shorter calculations. In this subject, where expressions and calculations are determined to become more and more complicated, the choice of appropriate symbols from the beginning is more than a matter of habit or of preference. Almost all our expressions have been calculated many times in different notations, including the index one, and the symbols in the manuscript have been chosen as the best ones from the above criteria. For readers for which this notation is not usual, we indicate here the relation between tensor index notation and ours.

(i) *Interior or contracted product.*—The contraction of a vector x and the first slot of a tensor T is denoted by $i(x)T$. For instance, if T is a covariant two-tensor, $[i(x)T]_\nu = x^\mu T_{\mu\nu}$.

(ii) *Exterior or wedge product.*—If A and B are both covariant (or contravariant) antisymmetric tensors, the wedge product $A \wedge B$ is the antisymmetrized tensorial product $A \otimes B$. For instance, for vectors x and y, one has

$$(x \wedge y)^{\mu\nu} = x^\mu y^\nu - y^\mu x^\nu,$$

and for a covector θ and a two-form F,

$$(\theta \wedge F)_{\alpha\beta\gamma} = \theta_\alpha F_{\beta\gamma} + \theta_\beta F_{\gamma\alpha} + \theta_\gamma F_{\alpha\beta}.$$

(iii) *Space-time metric.*—The tensor g, defined as a four-dimensional Lorentzian metric, has components $g_{\mu\nu}$ ($\det g_{\mu\nu} < 0$). The signature of g is taken here as $(-, +, +, +)$.

The scalar, or inner, product of two vectors x and y is denoted as $x \cdot y \equiv g(x, y) = g_{\mu\nu} x^\mu y^\nu = x^\mu y_\nu$ (in particular, $x^2 \equiv x \cdot x$), and it is naturally extended to bivectors, X and Y, according to $(X, Y) \equiv \frac{1}{2} X^{\mu\nu} Y_{\mu\nu}$. Indices are raised or lowered by using the metric g.

(iv) *Metric volume element.*—The components of the metric volume element η are given by

$$\eta_{\alpha\beta\gamma\delta} = -\sqrt{-\det g}\,\epsilon_{\alpha\beta\gamma\delta},$$

where $\epsilon_{\alpha\beta\gamma\delta}$ stands for the Levi-Città permutation symbol, $\epsilon_{0123} = 1$.

(v) *Hodge dual operator.*—Let x be a vector, H a two-form, and $\mathcal{F}$ an three-form. Their associated Hodge duals, the three-form $*x$, the two-form $*H$, and the one-form $*\mathcal{F}$, are given as $(*x)_{\alpha\beta\gamma} = \eta_{\alpha\beta\gamma\mu} x^\mu$, $(*H)_{\alpha\beta} = \frac{1}{2} \eta_{\alpha\beta\mu\nu} H^{\mu\nu}$, and $(*\mathcal{F})_\alpha = \frac{1}{3!} \eta_{\alpha\beta\gamma\delta} \mathcal{F}^{\beta\gamma\delta}$, respectively.

If x, y, z, w are space-time vectors, one has

$$[*(x \wedge y)]_{\alpha\beta} = \frac{1}{2} \eta_{\alpha\beta\mu\nu}(x \wedge y)^{\mu\nu} = \eta_{\alpha\beta\mu\nu} x^\mu y^\nu,$$
$$[*(x \wedge y \wedge z)]_\alpha = \eta_{\alpha\beta\gamma\delta} x^\beta y^\gamma z^\delta,$$

and

$$*(x \wedge y \wedge z \wedge w) = \eta_{\alpha\beta\gamma\delta} x^\alpha y^\beta z^\gamma w^\delta.$$

(vi) *Invariants associated with a two-form.*—A space-time two-form H has associated two independent invariants, (H, H) and $(H, *H)$, which are given as:

$$(H, H) \equiv \frac{1}{2} H^{\mu\nu} H_{\mu\nu},$$

$$(H, *H) \equiv \frac{1}{2} H^{\mu\nu} (*H)_{\mu\nu}.$$

(vii) *Relative splitting.*—For an arbitrary user of a relativistic positioning system (space-time observer of unit 4-velocity u, $u^2 = -1$), any space-time vector m may be written as:

$$m = m_u u + \vec{m} \qquad (A1)$$

where $m_u = -m \cdot u$ and $\vec{m} \in S_u$ are the time-like and space-like components, respectively, of m relative to u, $\vec{m} \cdot u = 0$.

(viii) *Induced volume on S_u.*—The three-dimensional Euclidean space orthogonal to u, S_u, has induced volume element, η_u, given by $\eta_u \equiv -i(u)\eta$, that is, $(\eta_u)_{\beta\gamma\delta} = -u^\alpha \eta_{\alpha\beta\gamma\delta}$. The Hodge dual operator with respect to η_u is denoted as $*_u$.

(ix) *Cross and triple products in S_u.*—Vectors in S_u are denoted with an arrow above them. Thus, for vectors $\vec{a}$, $\vec{b} \in S_u$, the vector or cross product is expressed as

$$\vec{a} \times \vec{b} = *(u \wedge \vec{a} \wedge \vec{b}) = *_u(\vec{a} \wedge \vec{b}). \qquad (A2)$$

The scalar triple product is then given by

$$(\vec{a} \times \vec{b}) \cdot \vec{c} \equiv (\vec{a}, \vec{b}, \vec{c}) = *_u(\vec{a} \wedge \vec{b} \wedge \vec{c}), \qquad (A3)$$

or equivalently,

$$(\vec{a}, \vec{b}, \vec{c})u = *(\vec{a} \wedge \vec{b} \wedge \vec{c}). \qquad (A4)$$

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