

Notes on spinoptics in a stationary spacetimeChul-Moon Yoo^{1,*}¹*Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

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In Reference [V.P. Frolov and A.A. Shoom, Phys. Rev. D **84**, 044026 (2011)], equations of the modified geometrical optics for circularly polarized photon trajectories in a stationary spacetime are derived by using a $(1 + 3)$ -decomposed form of Maxwell's equations. We derive the same results by using a four-dimensional covariant description. In our procedure, the null nature of the modified photon trajectory naturally appears and the energy flux is apparently null. We find that, in contrast to the standard geometrical optics, the inner product of the stationary Killing vector and the tangent null vector to the modified photon trajectory is no longer a conserved quantity along light paths. This quantity is furthermore different for left- and right-handed photon. A similar analysis is performed for gravitational waves and an additional factor of 2 appears in the modification due to the spin-2 nature of gravitational waves.

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I. INTRODUCTION

Light propagation in the gravitational field of a rotating body has been a topic of study in the past several years. One phenomenon of interest is rotation of the polarization vector, known as gravitational Faraday rotation [1–7]. This effect does not manifest in the gravitational field of a nonrotating body, such as Schwarzschild spacetime [8], while it does occur for Kerr spacetime. This fact suggests that the existence of helicity-rotation coupling and the propagation of circularly polarized electromagnetic waves depends on the helicity. Its occurrence has been confirmed by analyzing Maxwell's equations in curved spacetimes created by rotating bodies [9–11]. This effect is also discussed based on the gravitational Larmor's theorem [12,13].

In Ref. [14], Frolov and Shoom reported that the spinoptics in a gravitational field created by a rotating massive compact object can be described by a modified geometrical optics approximation. They used a $(1 + 3)$ -decomposed form of Maxwell's equations and also considered a standard geometrical optics approach. In their setting of the base vector field for the circular polarization, a phase shift appears that depends on the helicity. They proposed a modification in which the ordering of the equations associated with the geometrical optics approximation is changed so that the phase shift is absorbed in the eikonal of the eikonal ansatz. This treatment also leads to a modification of the photon trajectory depending on the helicity. Using this procedure, scattering of circularly polarized light by a rotating black hole is discussed in Ref. [15].

In this paper, we derive the same equations of the modified geometrical optics as in Ref. [14] by using another description in which four-dimensional covariance is maintained. In our procedure, we can easily see the

four-dimensional picture of the photon propagation. In addition, this procedure can be easily applied to the case of gravitational waves, as will be explicitly shown (see also Ref. [13]).

This paper is organized as follows. In Sec. II A, we review the standard geometrical optics approximation for Maxwell's equations in a stationary spacetime. We introduce a circular polarization base vector field in Sec. II B using an identical method to that in Ref. [14]. We then discuss the transport equation for the polarization vector using the circular polarization base vector field in Sec. II C. The photon trajectory, transport equation and energy flux in the modified geometrical optics are given in Sec. III. In Sec. IV the procedure is applied to the case of gravitational waves. Sec. V is devoted to a summary and discussion.

II. STANDARD GEOMETRICAL OPTICS**A. Trajectory, transport equation and energy flux**

In this paper we focus on a stationary spacetime manifold (\mathcal{M}, g) , where \mathcal{M} is a four-dimensional manifold with a smooth Lorentzian metric g which has a smooth one-parameter group G of the isometry generated by the Killing vector field ξ . Following Ref. [14], we write the line element of the stationary spacetime as follows:

$$ds^2 = -h(dt - \hat{g}_i dx^i)^2 + h\hat{\gamma}_{ij} dx^i dx^j, \quad (1)$$

where i, j run from 1 to 3 and h, \hat{g}_i and $\hat{\gamma}_{ij}$ are functions of x^i . The stationary Killing vector field is given by

$$\xi = \frac{\partial}{\partial t}, \quad \xi^\mu \xi_\mu = -h < 0. \quad (2)$$

Using the action of the isometry group G on \mathcal{M} , we can define the orbit space associated with the Killing vector field ξ as $\mathcal{N} := \mathcal{M}/G$. We define the normalized Killing vector field u by

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$$u^\mu := \xi^\mu / \sqrt{h}. \quad (3)$$

For later convenience, we define the projection tensor γ by

$$\gamma_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu. \quad (4)$$

Then, $\gamma_{ij} = h\hat{\gamma}_{ij}$ and γ gives the naturally induced metric on \mathcal{N} .

Note that in this paper, we consider the region in which h is positive definite. This condition may not be satisfied for regions within the ergosphere of a Kerr black hole. Therefore, as with the formalism in Refs. [14,15], our formalism cannot be straightforwardly applied to the ergoregion with the Killing vector field which is tangent to the world line of the static observer at the infinity.

In Ref. [14], Maxwell's equations are reduced to the master equations on the orbit space \mathcal{N} with the metric $\hat{\gamma}$. We do not follow the same procedure and instead use the four-dimensional covariant form of the equations. We consider the vector potential A_μ which satisfies the Lorenz gauge condition given by

$$\nabla_\mu A^\mu = 0, \quad (5)$$

and the wave equation given by

$$\nabla^\nu \nabla_\nu A_\mu - R_{\mu}{}^\nu A_\nu = 0, \quad (6)$$

where $R_{\mu\nu}$ is the Ricci curvature tensor. Following the standard method (e.g., Refs. [16,17]), we write the eikonal ansatz as follows:

$$A_\mu = (a_\mu + \epsilon b_\mu + \mathcal{O}(\epsilon^2))e^{iS/\epsilon}, \quad (7)$$

where ϵ is a bookkeeping parameter that we take to be small during our manipulations; at the end of our calculations we reset it to $\epsilon \rightarrow 1$, so that S becomes the actual phase function.

Substituting the ansatz (7) into Eq. (5), we obtain the following equation from the order of ϵ^{-1} :

$$a^\mu k_\mu = 0, \quad (8)$$

where k_μ is defined by

$$k_\mu := \nabla_\mu S. \quad (9)$$

From the order of ϵ^{-2} in Eq. (6), we obtain

$$k^\mu k_\mu = 0. \quad (10)$$

Rewriting Eq. (10) as

$$\mathcal{H} := \frac{1}{2}g^{\mu\nu}\nabla_\mu S\nabla_\nu S = 0, \quad (11)$$

we can regard this equation as a Hamilton-Jacobi equation for S . Since the four-velocity of the corresponding dynamical system to Eq. (11) is given by k^μ , we can regard the Hamiltonian equation for the Hamiltonian (11) as the equation for the ray trajectory generated by k^μ . The Hamiltonian equations are given by

$$k^\nu \nabla_\nu k^\mu = 0. \quad (12)$$

Hence trajectories are given by null geodesics. This equation can be simply derived by differentiating Eq. (10) and using Eq. (9).

The order of ϵ^{-1} in Eq. (6) gives the following transport equation:

$$k^\nu \nabla_\nu a^\mu + \frac{1}{2}a^\mu \nabla_\nu k^\nu = 0. \quad (13)$$

Following convention, we divide a^μ into the real amplitude a and the complex polarization vector ℓ^μ as follows:

$$a_\mu = a\ell_\mu, \quad \ell^\mu \bar{\ell}_\mu = 1, \quad a \in \mathbb{R}, \quad (14)$$

where $\bar{\ell}_\mu$ denotes the complex conjugate of ℓ_μ . Then, contracting $\bar{\ell}_\mu$ with the transport equation (13), from the real part, we obtain

$$\nabla_\mu (a^2 k^\mu) = 0. \quad (15)$$

Substituting this equation into (13), we obtain

$$k^\nu \nabla_\nu \ell^\mu = 0. \quad (16)$$

Eq. (15) describes the conservation of the photon number and Eq. (16) indicates that the polarization vector ℓ^μ is parallel-transported along the ray trajectory.

The field strength $F_{\mu\nu}$ of the vector potential (7) is given by

$$F_{\mu\nu} = \text{Re}\{\nabla_\mu A_\nu - \nabla_\nu A_\mu\} \simeq 2a\text{Re}\{ie^{iS}k_{[\mu}\ell_{\nu]}\}, \quad (17)$$

at the leading order of the geometrical optics approximation, where square brackets denote anti-symmetrization. Then, the energy-momentum tensor $T^{\mu\nu}$ is given by

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{4\pi} \left(F^\mu{}_\lambda F^{\nu\lambda} - \frac{1}{4}g^{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} \right) \\ &= \frac{a^2}{8\pi} k^\mu k^\nu (1 - \text{Re}\{e^{2iS}\ell_\lambda \ell^\lambda\}). \end{aligned} \quad (18)$$

Averaging over several wavelengths, we obtain

$$\langle T^{\mu\nu} \rangle = \frac{a^2}{8\pi} k^\mu k^\nu. \quad (19)$$

This expression indicates that the energy flux is proportional to k^μ and null at the leading order of the standard geometrical optics approximation.

B. Base vector fields

Taking stationarity into account, we impose

$$\mathcal{L}_\xi k^\mu = \xi^\nu \nabla_\nu k^\mu - k^\nu \nabla_\nu \xi^\mu = 0, \quad (20)$$

where \mathcal{L}_ξ is the Lie derivative with respect to ξ . Using this equation and $\nabla_\mu k_\nu = \nabla_\nu k_\mu$, we obtain

$$\nabla_\mu (\xi^\nu k_\nu) = 0. \quad (21)$$

We define the frequency ω as follows:

$$\omega := -\xi^\mu k_\mu. \quad (22)$$

We introduce the spacelike unit vector along the ray direction n^μ , given by

$$n^\mu := \frac{\sqrt{h}}{\omega} k^\mu - u^\mu. \quad (23)$$

This satisfies

$$n^\mu n_\mu = 1, \quad n^\mu u_\mu = 0. \quad (24)$$

To set an orthonormal base system, we define two additional unit spacelike vector fields e_1^μ and e_2^μ , given below. First, at a point, we set e_A^μ such that the following conditions are satisfied:

$$g_{\mu\nu} e_A^\mu e_B^\nu = \delta_{AB}, \quad u_\mu e_A^\mu = n_\mu e_A^\mu = 0, \quad (25)$$

where $A = 1, 2$. Then, following Ref. [14], we extend e_A^μ along the integral curve of n^μ by imposing the following condition:

$$\begin{aligned} \mathcal{F}_n e_A^\mu &:= n^\nu D_\nu e_A^\mu + e_A^\nu (n^\lambda D_\lambda n_\nu) n^\mu - (e_A^\nu n_\nu) n^\lambda D_\lambda n^\mu \\ &= 0 \Leftrightarrow n^\nu D_\nu e_A^\mu = -e_A^\nu (n^\lambda D_\lambda n_\nu) n^\mu \\ &= n^\mu n_\nu n^\lambda D_\lambda e_A^\nu, \end{aligned} \quad (26)$$

where the action of D_μ on a vector field v^μ is defined by

$$D_\mu v_\nu = \gamma_\mu^\rho \gamma_\nu^\lambda \nabla_\rho v_\lambda. \quad (27)$$

We can check that the condition Eq. (26) is equivalent to Eq. (85) in Ref. [14], which gives Fermi transport on $(\mathcal{N}, \hat{\gamma})$. In addition, we extend the base vector fields along the integral curve of ξ by the Lie transport as follows:

$$\mathcal{L}_\xi e_A^\mu = 0. \quad (28)$$

Then, Eq. (25) is satisfied at any point of the spacetime.

Finally, we define the circular polarization base vector field as follows:

$$m^\mu = (e_1^\mu + i\sigma e_2^\mu)/\sqrt{2}, \quad (29)$$

where $\sigma = \pm 1$ specifies circular polarization. Then, we have

$$e_1^\mu = \sqrt{2}(m^\mu + \bar{m}^\mu), \quad (30)$$

$$e_2^\mu = -i\sigma\sqrt{2}(m^\mu - \bar{m}^\mu). \quad (31)$$

m^μ and \bar{m}^μ also satisfy Eq. (26) and are Lie-transported along the ξ direction.

C. Parallel transport of the polarization vector

As shown in Eq. (16), the polarization vector ℓ^μ is parallel-transported along the null geodesic generated by k^μ . Using the circular polarization base vector field m^μ , we can write

$$\ell^\mu = m^\mu e^{i\varphi}, \quad (32)$$

where φ is a real function of x^i . Then, Eq. (16) can be rewritten as

$$k^\nu \nabla_\nu (m^\mu e^{i\varphi}) = 0 \Leftrightarrow m^\mu k^\nu \nabla_\nu (e^{i\varphi}) = -e^{i\varphi} k^\nu \nabla_\nu m^\mu. \quad (33)$$

Contracting with \bar{m}^μ , we obtain

$$ik^\nu \nabla_\nu \varphi = m^\mu k^\nu \nabla_\nu \bar{m}_\mu. \quad (34)$$

Using (23), we find

$$\begin{aligned} ik^\nu \nabla_\nu \varphi &= \frac{\omega}{\sqrt{h}} m^\mu (n^\nu + u^\nu) \nabla_\nu \bar{m}_\mu \\ &= \frac{\omega}{\sqrt{h}} m^\mu n^\nu \nabla_\nu \bar{m}_\mu + \frac{\omega}{h} m^\mu \xi^\nu \nabla_\nu \bar{m}_\mu \\ &= \frac{\omega}{\sqrt{h}} m^\mu n^\nu D_\nu \bar{m}_\mu + \frac{\omega}{h} m^\mu \xi^\nu \nabla_\nu \bar{m}_\mu \\ &= \frac{\omega}{h} m^\mu \xi^\nu \nabla_\nu \bar{m}_\mu \\ &= \frac{\omega}{h} m^\mu \bar{m}^\nu \nabla_\nu \xi_\mu, \end{aligned} \quad (35)$$

where we have used $\mathcal{F}_n \bar{m}^\mu = 0$ and $\mathcal{L}_\xi \bar{m}^\mu = 0$. Since $\nabla_\mu \xi_\nu$ is antisymmetric, we obtain

$$\begin{aligned} k^\nu \nabla_\nu \varphi &= \sigma \frac{\omega}{h} e_1^{[\mu} e_2^{\nu]} \nabla_\nu \xi_\mu \\ &= \frac{1}{2} \sigma \frac{\omega}{h} u_\rho n_\lambda \varepsilon^{\mu\nu\rho\lambda} \nabla_\nu \xi_\mu \\ &= \frac{1}{2} \sigma u_\rho k_\lambda \varepsilon^{\mu\nu\rho\lambda} \nabla_\nu u_\mu, \end{aligned} \quad (36)$$

where $\varepsilon^{\mu\nu\rho\lambda}$ is the completely antisymmetric tensor with $\varepsilon^{0123} = 1/\sqrt{-\det g}$. Performing (1 + 3) decomposition, we can check that Eq. (36) is equivalent to Eq. (102) in Ref. [14]. This is the well-known gravitational analogue of the Faraday rotation [1–7].

III. MODIFIED GEOMETRICAL OPTICS

A. Modification of the eikonal

The guiding principle of the modification is that the phase term $e^{i\varphi}$ should be included in the eikonal (see Eqs. (7), (14), and (32)), that is, the modified eikonal \tilde{S} should be given by

$$S \rightarrow \tilde{S} \sim S + \varphi. \quad (37)$$

Before modification, the Hamiltonian of the ray trajectory is given by

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu} k_\mu k_\nu = \frac{1}{2} g^{\mu\nu} \nabla_\mu S \nabla_\nu S. \quad (38)$$

The modification of the eikonal (36) and (37) suggest the following Hamiltonian:

$$\tilde{\mathcal{H}} = \frac{1}{2} g^{\mu\nu} (\nabla_\mu \tilde{S} - \sigma \varphi_\mu) (\nabla_\nu \tilde{S} - \sigma \varphi_\nu), \quad (39)$$

where we have defined

$$\varphi_\mu := \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} u^\nu \nabla^\rho u^\lambda. \quad (40)$$

This expression was first derived by a group at Osaka City University [18] in a different way.

Our aim is to modify the ordering of the field equations so that Eq. (39) is obtained. It will be seen in Eq. (55) that our procedure eliminates the phase shift Eq. (36). We do not change the form of the eikonal ansatz (7) but formally put the tilde “~” on all quantities, as follows:

$$A_\mu = (\tilde{a}_\mu + \epsilon \tilde{b}_\mu + \mathcal{O}(\epsilon^2)) e^{i\tilde{S}/\epsilon}. \quad (41)$$

To obtain the Hamiltonian (39) we change the orders of significance in the geometrical optics approximation by rewriting the gradient operator as follows:

$$\nabla_\mu \rightarrow \nabla_\mu - i\epsilon^{-1} \sigma \varphi_\mu + i\sigma \varphi_\mu. \quad (42)$$

Then, the Lorenz gauge equation and wave equation become

$$\nabla_\mu A^\mu = 0 \rightarrow (\nabla_\mu - i\epsilon^{-1} \sigma \varphi_\mu + i\sigma \varphi_\mu) A^\mu = 0, \quad (43)$$

$$\begin{aligned} \nabla_\nu \nabla^\nu A_\mu &= \mathcal{O}(\epsilon^0) \\ &\rightarrow (\nabla_\nu - i\epsilon^{-1} \sigma \varphi_\nu + i\sigma \varphi_\nu) \\ &\quad \times (\nabla^\nu - i\epsilon^{-1} \sigma \varphi^\nu + i\sigma \varphi^\nu) A_\mu \\ &= \mathcal{O}(\epsilon^0). \end{aligned} \quad (44)$$

This modification is trivial if we take $\epsilon \rightarrow 1$, but this enhances the effect of the circular polarization to the leading order.

From the order of ϵ^{-1} in Eq. (43), we obtain

$$\tilde{a}^\mu q_\mu = 0, \quad (45)$$

where

$$q_\mu = \nabla_\mu \tilde{S} - \sigma \varphi_\mu. \quad (46)$$

This means that the polarization vector \tilde{a}^μ is perpendicular to the ray direction given by q^μ .

From the order of ϵ^{-2} in Eq. (44), we have

$$q^\mu q_\mu = 0. \quad (47)$$

This equation is identical to $\tilde{\mathcal{H}} = 0$ and $\tilde{\mathcal{H}}$ is simply the Hamiltonian for the ray trajectory.

In the same way as with Eq. (20), we extend the vector q^μ with the Lie transport along the integral curves of ξ^μ , that is,

$$\mathcal{L}_\xi q^\mu = 0. \quad (48)$$

Then, we define the frequency $\tilde{\omega}$ as follows:

$$\tilde{\omega} := -\xi^\mu q_\mu. \quad (49)$$

It should be noted that this frequency is not constant in general, in contrast to $\omega = -\xi^\mu k_\mu$. Since q^μ depends on the helicity σ , $\tilde{\omega}$ also depends on σ . We define the space-like unit vector along the modified ray direction \tilde{n}^μ as follows:

$$\tilde{n}^\mu := \frac{\sqrt{h}}{\tilde{\omega}} q^\mu - u^\mu. \quad (50)$$

Following the same procedure as in Sec. II B, we can define the modified circular polarization base vector \tilde{m}^μ associated with \tilde{n}^μ .

From the order of ϵ^{-1} in Eq. (44), we obtain

$$q^\nu \nabla_\nu \tilde{a}^\mu + \frac{1}{2} \tilde{a}^\mu \nabla_\nu q^\nu + i\sigma q^\nu \varphi_\nu \tilde{a}^\mu = 0. \quad (51)$$

We divide \tilde{a}^μ into the real scalar amplitude \tilde{a} and the circular polarization vector $\tilde{\ell}^\mu := \tilde{a}^\mu / \tilde{a} = \tilde{m}^\mu e^{i\tilde{\varphi}}$. Contracting with \tilde{m}^μ , we obtain

$$\begin{aligned} q^\nu \nabla_\nu \tilde{a} + i\tilde{a} q^\nu \nabla_\nu \tilde{\varphi} + \tilde{a} \tilde{m}_\mu q^\nu \nabla_\nu \tilde{m}^\mu + \frac{1}{2} \tilde{a} \nabla_\nu q^\nu \\ + i\sigma \tilde{a} q^\nu \varphi_\nu = 0. \end{aligned} \quad (52)$$

Similar to Eqs. (35) and (36), the third term of this equation can be rewritten as

$$\begin{aligned} \tilde{m}_\mu q^\nu \nabla_\nu \tilde{m}^\mu &= -\frac{\tilde{\omega}}{\sqrt{h}} \tilde{m}_\mu (\tilde{n}^\nu + u^\nu) \nabla_\nu \tilde{m}^\mu \\ &= -\frac{\tilde{\omega}}{\sqrt{h}} \tilde{m}_\mu u^\nu \nabla_\nu \tilde{m}^\mu \\ &= -\frac{\tilde{\omega}}{h} \tilde{m}_\mu \xi^\nu \nabla_\nu \tilde{m}^\mu \\ &= -\frac{\tilde{\omega}}{h} \tilde{m}^\mu \tilde{m}^\nu \nabla_\nu \xi_\mu \\ &= -i\sigma q^\mu \varphi_\mu. \end{aligned} \quad (53)$$

Then, from the real and imaginary parts of Eq. (52), we obtain the following two equations:

$$\nabla_\mu (\tilde{a}^2 q^\mu) = 0, \quad (54)$$

$$q^\mu \nabla_\mu \tilde{\varphi} = 0. \quad (55)$$

Eq. (54) describes the photon number conservation and Eq. (55) indicates that the phase $\tilde{\varphi}$ is constant along the ray trajectory. Eq. (55) is the desired result for the modification.

From the Hamiltonian (39), we obtain the following equation of motion for the ray trajectory:

$$q^\nu \nabla_\nu q^\mu = \sigma f^\mu{}_\nu q^\nu, \quad (56)$$

where

$$f_{\mu\nu} = \nabla_\mu \varphi_\nu - \nabla_\nu \varphi_\mu. \quad (57)$$

Performing (1 + 3)-decomposition, we can derive Eq. (112) in Ref. [14].

We also perform the replacement (42) in the expression (17). We obtain

$$\begin{aligned} F_{\mu\nu} &= \text{Re}\{(\nabla_\mu - i\epsilon^{-1}\sigma\varphi_\mu + i\sigma\varphi_\mu)A_\nu \\ &\quad - (\nabla_\nu - i\epsilon^{-1}\sigma\varphi_\nu + i\sigma\varphi_\nu)A_\mu\} \\ &\simeq 2a\text{Re}\{ie^{i\tilde{s}}q_{[\mu}\tilde{\ell}_{\nu]}\} \end{aligned} \quad (58)$$

at the leading order of the modified geometrical optics approximation. Then, the energy-momentum tensor $T^{\mu\nu}$ is given by

$$T^{\mu\nu} \simeq \frac{\tilde{a}^2}{8\pi} q^\mu q^\nu (1 - \text{Re}\{e^{2i\tilde{s}}\tilde{\ell}_\lambda\tilde{\ell}^\lambda\}). \quad (59)$$

Averaging over several wavelengths, we obtain

$$\langle T^{\mu\nu} \rangle \simeq \frac{\tilde{a}^2}{8\pi} q^\mu q^\nu. \quad (60)$$

This expression indicates that the energy flux is proportional to q^μ and null at the leading order of the modified geometrical optics approximation.

IV. GRAVITATIONAL SPINOPTICS

A. Standard geometrical optics

The eikonal ansatz for the metric perturbation is given by follows:

$$h_{\mu\nu} = (a_{\mu\nu} + \epsilon b_{\mu\nu} + \mathcal{O}(\epsilon^2))e^{iS/\epsilon}. \quad (61)$$

Hereafter we work in the transverse-traceless gauge. The transverse gauge equation and the wave equation are given by

$$\nabla_\mu h^{\mu\nu} = 0, \quad (62)$$

$$\nabla_\rho \nabla^\rho h_{\mu\nu} + 2R_{\rho\mu\lambda\nu} h^{\rho\lambda} = 0. \quad (63)$$

From the order of ϵ^{-1} in Eq. (62), we obtain

$$a_{\mu\nu} k^\nu = 0, \quad (64)$$

where k_μ is given by (9). From the order of ϵ^{-2} in Eq. (63), we obtain the same equation as Eq. (10) and (12). From the order of ϵ^{-1} in Eq. (63), we obtain

$$k^\rho \nabla_\rho a_{\mu\nu} + \frac{1}{2} a_{\mu\nu} \nabla_\rho k^\rho = 0. \quad (65)$$

As in Eq. (14), we divide $a_{\mu\nu}$ as follows:

$$a_{\mu\nu} = \alpha \ell_{\mu\nu}, \quad \ell^{\mu\nu} \tilde{\ell}_{\mu\nu} = 1, \quad \alpha \in \mathbb{R}. \quad (66)$$

We can then obtain the following two equations:

$$\nabla_\rho (\alpha^2 k^\rho) = 0, \quad (67)$$

$$k^\rho \nabla_\rho \ell_{\mu\nu} = 0. \quad (68)$$

Eq. (67) describes the graviton number conservation and Eq. (68) indicates parallel transport of the polarization tensor $\ell_{\mu\nu}$ along the null geodesic generated by k^μ .

From Isaacson's formula [19,20], the effective energy-momentum tensor for gravitational waves can be written as

$$\langle T_{\mu\nu}^{(\text{GW})} \rangle = \frac{1}{32\pi} \langle \text{Re}\{\nabla_\mu h_{\rho\lambda}\} \text{Re}\{\nabla_\nu h^{\rho\lambda}\} \rangle. \quad (69)$$

At the leading order of the geometrical optics approximation, we obtain

$$\langle T_{\mu\nu}^{(\text{GW})} \rangle \simeq \frac{1}{64\pi} \alpha^2 k_\mu k_\nu. \quad (70)$$

This expression indicates that the energy flux of the gravitational waves is proportional to k^μ and null at the leading order of the standard geometrical optics approximation.

B. Base setting and parallel transport of the polarization tensor

Let us consider the base tensor fields for linear polarization tensors given by

$$e_+^{\mu\nu} = \frac{1}{\sqrt{2}} \delta^{AB} e_A^\mu e_B^\nu, \quad e_\times^{\mu\nu} = \sqrt{2} e_1^{(\mu} e_2^{\nu)}, \quad (71)$$

where round brackets around indices denote symmetrization. These satisfy

$$\begin{aligned} g_{\mu\nu} g_{\rho\lambda} e_\times^{\mu\rho} e_\times^{\nu\lambda} &= 1, & g_{\mu\nu} g_{\rho\lambda} e_+^{\mu\rho} e_+^{\nu\lambda} &= 1, \\ g_{\mu\nu} g_{\rho\lambda} e_\times^{\mu\rho} e_+^{\nu\lambda} &= 0, & g_{\rho\lambda} e_+^{\mu\rho} e_\times^{\lambda\nu} &= e_1^{[\mu} e_2^{\rho]}. \end{aligned} \quad (72)$$

For the circular polarization specified by σ , we can define the polarization base tensor $m_{\mu\nu}$ by

$$m^{\mu\nu} = \frac{1}{\sqrt{2}} (e_+^{\mu\nu} + i\sigma e_\times^{\mu\nu}). \quad (73)$$

This satisfies

$$\begin{aligned} m_{\mu\nu} m^{\mu\nu} &= 1, & \bar{m}_{\mu\nu} \bar{m}^{\mu\nu} &= 1, \\ \bar{m}_{\mu\nu} m^{\mu\nu} &= 0, & m^{\mu\rho} \bar{m}_\rho{}^\nu &= i\sigma e_1^{[\mu} e_2^{\nu]}. \end{aligned} \quad (74)$$

Using this circular polarization base tensor, we can write

$$\ell_{\mu\nu} = m_{\mu\nu} e^{i\psi}. \quad (75)$$

Parallel transport of the polarization vector means that

$$k^\rho \nabla_\rho (m_{\mu\nu} e^{i\psi}) = 0 \Leftrightarrow m_{\mu\nu} k^\rho \nabla_\rho (e^{i\psi}) = -e^{i\psi} k^\rho \nabla_\rho m_{\mu\nu}. \quad (76)$$

Contracting with $\bar{m}^{\mu\nu}$, we obtain

$$ik^\mu \nabla_\mu \psi = m^{\mu\nu} k^\rho \nabla_\rho \bar{m}_{\mu\nu}. \quad (77)$$

Using Eq. (23), we have

$$\begin{aligned}
ik^\mu \nabla_\mu \psi &= \frac{\omega}{\sqrt{h}} m^{\mu\nu} (n^\rho + u^\rho) \nabla_\rho \bar{m}_{\mu\nu} \\
&= \frac{\omega}{\sqrt{h}} m^{\mu\nu} n^\rho \nabla_\rho \bar{m}_{\mu\nu} + \frac{\omega}{h} m^{\mu\nu} \xi^\rho \nabla_\rho \bar{m}_{\mu\nu} \\
&= \frac{\omega}{h} m^{\mu\nu} \xi^\rho \nabla_\rho \bar{m}_{\mu\nu} \\
&= 2 \frac{\omega}{h} m^{\mu\nu} \bar{m}_\nu{}^\rho \nabla_\rho \xi_\mu \\
&= 2i\sigma \frac{\omega}{h} e_1^{[\mu} e_2^{\rho]} \nabla_\rho \xi_\mu \\
&= i\sigma u_\rho k_\lambda \varepsilon^{\mu\nu\rho\lambda} \nabla_\nu u_\mu \\
&= 2ik^\mu \nabla_\mu \varphi, \tag{78}
\end{aligned}$$

where we have used $e_A^\mu n^\nu \nabla_\nu \bar{m}_{\mu\rho} = 0$, $\mathcal{L}_\xi \bar{m}^{\mu\nu} = 0$ and Eq. (74).

C. Modified geometrical optics

Eq. (78) suggests the Hamiltonian

$$\tilde{\mathcal{H}}_{\text{gw}} = \frac{1}{2} g^{\mu\nu} (\nabla_\mu \tilde{S}_{\text{gw}} - 2\sigma \varphi_\mu) (\nabla_\nu \tilde{S}_{\text{gw}} - 2\sigma \varphi_\nu), \tag{79}$$

for the modified geometrical optics. The extra factor 2 compared with Eq. (39) is expected from the spin-2 nature of gravitational waves. We do not change the eikonal ansatz for the metric perturbation

$$h_{\mu\nu} = (\tilde{a}_{\mu\nu} + \varepsilon \tilde{b}_{\mu\nu} + \mathcal{O}(\varepsilon^2)) e^{i\tilde{S}_{\text{gw}}/\varepsilon}. \tag{80}$$

As for Eqs. (43) and (44), we rewrite the transverse gauge equation and the wave equation as follows:

$$\nabla_\mu h^{\mu\nu} = 0 \rightarrow (\nabla_\mu - 2i\varepsilon^{-1} \sigma \varphi_\mu + 2i\sigma \varphi_\mu) h^{\mu\nu} = 0, \tag{81}$$

$$\begin{aligned}
\nabla_\rho \nabla^\rho h_{\mu\nu} &= \mathcal{O}(\varepsilon^0) \rightarrow (\nabla_\rho - 2i\varepsilon^{-1} \sigma \varphi_\rho + 2i\sigma \varphi_\rho) \\
&\quad \times (\nabla^\rho - 2i\varepsilon^{-1} \sigma \varphi^\rho + 2i\sigma \varphi^\rho) h_{\mu\nu} \\
&= \mathcal{O}(\varepsilon^0). \tag{82}
\end{aligned}$$

From the order of ε^{-1} in Eq. (81), we obtain

$$\tilde{a}_{\mu\nu} p^\nu = 0, \tag{83}$$

where

$$p_\mu = \nabla_\mu \tilde{S}_{\text{gw}} - 2\sigma \varphi_\mu. \tag{84}$$

From the order of ε^{-2} in Eq. (82), we obtain

$$p^\mu p_\mu = 0. \tag{85}$$

This equation is identical to $\mathcal{H}_{\text{gw}} = 0$. In the same way as for III A, we impose

$$\mathcal{L}_\xi p^\mu = 0, \tag{86}$$

and define the frequency $\tilde{\omega}_{\text{gw}}$ as follows:

$$\tilde{\omega}_{\text{gw}} := -\xi^\mu p_\mu. \tag{87}$$

Then, we define the spacelike unit vector along the modified ray direction $\tilde{n}_{\text{gw}}^\mu$ by

$$\tilde{n}_{\text{gw}}^\mu = \frac{\sqrt{h}}{\tilde{\omega}_{\text{gw}}} p^\mu - u^\mu. \tag{88}$$

We can then obtain the modified circular polarization base tensor $\tilde{m}_{\mu\nu}$ associated with $\tilde{n}_{\text{gw}}^\mu$.

From the order of ε^{-1} in Eq. (82), we obtain

$$p^\rho \nabla_\rho \tilde{a}_{\mu\nu} + \frac{1}{2} \tilde{a}_{\mu\nu} \nabla_\rho p^\rho + 2i\sigma p^\rho \varphi_\rho \tilde{a}_{\mu\nu} = 0. \tag{89}$$

We replace $\tilde{a}_{\mu\nu}$ by $\alpha \tilde{m}_{\mu\nu} e^{i\tilde{\psi}}$. Contracting Eq. (89) with $\tilde{m}^{\mu\nu}$, we obtain

$$\begin{aligned}
p^\nu \nabla_\nu \alpha + i\alpha p^\nu \nabla_\nu \tilde{\psi} + \alpha \tilde{m}_{\mu\nu} p^\rho \nabla_\rho \tilde{m}^{\mu\nu} + \frac{1}{2} \alpha \nabla_\nu p^\nu \\
+ 2i\sigma \alpha p^\nu \varphi_\nu = 0. \tag{90}
\end{aligned}$$

Similar to Eq. (78), the third term of this equation can be rewritten as

$$\begin{aligned}
\tilde{m}^{\mu\nu} p^\rho \nabla_\rho \tilde{m}_{\mu\nu} &= -\frac{\tilde{\omega}_{\text{gw}}}{\sqrt{h}} \tilde{m}_{\mu\nu} (\tilde{n}_{\text{gw}}^\rho + u^\rho) \nabla_\rho \tilde{m}^{\mu\nu} \\
&= -\frac{\tilde{\omega}_{\text{gw}}}{\sqrt{h}} \tilde{m}_{\mu\nu} u^\rho \nabla_\rho \tilde{m}^{\mu\nu} \\
&= -\frac{\tilde{\omega}_{\text{gw}}}{h} \tilde{m}_{\mu\nu} \xi^\rho \nabla_\rho \tilde{m}^{\mu\nu} \\
&= -2 \frac{\tilde{\omega}_{\text{gw}}}{h} \tilde{m}^{\mu\nu} \tilde{m}_\nu{}^\rho \nabla_\rho \xi_\mu = -2i\sigma p^\mu \varphi_\mu. \tag{91}
\end{aligned}$$

Then, from the real and imaginary parts of Eq. (90), we obtain the following two equations:

$$\nabla_\mu (\alpha^2 p^\mu) = 0, \tag{92}$$

$$p^\mu \nabla_\mu \tilde{\psi} = 0. \tag{93}$$

Eq. (92) describes the graviton number conservation and Eq. (93) indicates that the phase $\tilde{\psi}$ is constant along the ray trajectory. The equation of motion for the ray trajectory can be derived from the Hamiltonian (79) as follows:

$$p^\nu \nabla_\nu p^\mu = 2\sigma f^\mu{}_\nu p^\nu, \tag{94}$$

where $f^\mu{}_\nu$ is defined in Eq. (57).

We also perform the replacement (42) in the expression (69), and find that

$$\langle T_{\mu\nu}^{(\text{GW})} \rangle \simeq \frac{1}{64\pi} \alpha^2 p^\mu p^\nu. \tag{95}$$

This expression indicates that the energy flux is proportional to p^μ and null at the leading order of the modified geometrical optics approximation.

V. SUMMARY AND DISCUSSION

Using a four-dimensional covariant description, we have derived the equations of the modified geometrical optics in a stationary spacetime, previously derived in Ref. [14]. In the modified geometrical optics, the three-dimensional photon trajectory is modified depending on the photon helicity. In Ref. [14], the authors used a reduced form of Maxwell's equations on the three-dimensional orbit space associated with the stationary Killing vector field. In this description, the four-dimensional picture is not clear. In contrast, in our procedure, the null nature of the photon trajectory in the modified geometrical optics naturally appears and the energy flux is apparently null. We can also see that, in contrast to the standard geometrical optics, the inner product of the stationary Killing vector and the tangent null vector to the modified photon trajectory is no longer a conserved quantity along light paths. This quantity is furthermore different for left- and right-handed photon. The same procedure can be easily applied to the case of gravitational waves and we found that an additional factor of 2 appears in the modification between the circularly polarized photon and the graviton because of the spin-1 and -2 nature

of electromagnetic waves and gravitational waves, respectively.

It is clear that the origin of the modification is in the choice of the circular polarization base vector field. Following Ref. [14], we have taken a circular polarization base vector field based on the Fermi transport along the photon trajectory projected on the three-dimensional orbit space. However, it is still not clear whether this choice of base vector field is valid for describing the circularly polarized photon trajectory. Other choices of the base vector field may give rise to different modifications. Clearly we need to justify the choice of base vector field based on observations. This could be done by comparing the electromagnetic fields given by the modified geometrical optics with those given by directly solving the wave equations. We leave this issue to a future work.

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