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Cosmic electromagnetic fields due to perturbations in the gravitational field

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We use nonlinear gauge-invariant perturbation theory to study the interaction of an inflation produced seed magnetic field with density and gravitational wave perturbations in an almost Friedmann-Lemaître-Robertson-Walker spacetime with zero spatial curvature. We compare the effects of this coupling under the assumptions of poor conductivity, perfect conductivity and the case where the electric field is sourced via the coupling of velocity perturbations to the seed field in the ideal magnetohydrodynamic regime, thus generalizing, improving on and correcting previous results. We solve our equations for long wavelength limits and numerically integrate the resulting equations to generate power spectra for the electromagnetic field variables, showing where the modes cross the horizon. We find that the interaction can seed electric fields with nonzero curl and that the curl of the electric field dominates the power spectrum on small scales, in agreement with previous arguments.

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I. INTRODUCTION

Large scale magnetic fields of varying amplitudes are present in entire galaxy clusters, individual galaxies and high redshift condensations. Such fields are observed on characteristic scales of $\sim$1 Mpc and are of micro-Gauss strength, 10^{-7}–10^{-5} G [1,2]. Despite their ubiquity, their origin is still a mystery. There are literally tens of candidate mechanisms proposed to explain the origin and evolution of such fields, spanning different theories of physics [3]. It is now widely believed that the structure of magnetic fields in spiral galaxies is consistent with the dynamo amplification mechanism. The dynamo mechanism can produce amplification factors of up to $\sim 10^8$ but requires a seed field in order to operate and thus cannot explain the origin of magnetic fields. Additionally, adiabatic contraction of magnetic flux lines during structure formation can enhance galactic fields by a factor of $\sim 10^3$.

Among the physical mechanisms proposed to explain the origin of the seed field is one due to Harrison [4]. This mechanism rests on the fact that nonzero vorticity in the prerecombination photon-baryon plasma can generate weak magnetic fields of about $\sim 10^{-25}$ G. However, vorticity is not a generated mode at first order in perturbation theory and has to be put in as an initial condition. Second order treatments of the prerecombination plasma in terms of a kinetic theory description has also been used to generate the required seed fields [5–10]. The key idea is a preferential Thompson scattering of photons off free electrons, over the scattering off protons [the scattering off protons is suppressed by a factor $(m_e/m_p)^2$] which induces differences in the proton and electron velocity fields. Electric fields are then induced to countercharge separation between the electrons and protons. The generated electric fields will then feed in the magnetic induction equation to generate magnetic fields at second order in perturbation theory. The photon anisotropic stress also couples to the electron velocities and contributes to the magnetic field sources. In addition, other arguments relying on electroweak phase transitions [11,12], topological defects [13], velocity perturbations [14] etc. have been proposed as candidate mechanisms. The generated fields, however, are usually too weak to leave any detectable imprint on the cosmic microwave background (CMB) [5]. This is not surprising given the form of the fluid quantities of a magnetic field. In particular, the energy density $\mu_B = B^2/2$, the isotropic pressure $p_B = B^2/6$ and the anisotropic pressure $\Pi_{ab} = B_{\langle a}B_{b\rangle}$ of a field generated at second order will manifest at fourth order in perturbation theory, which is not relevant for CMB anisotropies.

In addition to meeting the right strengths, the generated fields must be of the right scale to match those observed today. One of the problems of primordial generation mechanisms in general is that although some may reach the required strengths, they are causal in nature. This means that their coherence scales cannot exceed the Hubble scale during the time of magnetic field generation. By comparison, the galactic scale today is well outside the Hubble scale at such early epochs. Moreover, the small scale fields, i.e., those that are already subhorizon before matter-radiation equality, cannot reach the recombination epoch due to microphysical mechanisms such as magnetic and photon diffusion processes [3].

Inflation and other pre-big bang models capable of causally producing superhorizon perturbations are often invoked to circumvent this scale problem. However, the residual magnetic fields surviving the exponential

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expansion accompanying many inflationary models are thought to be too weak to be of cosmological relevance [15]. New physics often has to be introduced such as exotic couplings of the electromagnetic field to other fields such as the dilaton field to avoid the accompanying exponential dilution of the magnetic fields [19]. The primordial fields are also constrained by the fact that the anisotropic stress of the produced magnetic fields contains a spin-2 component and will result in an overproduction of gravitational waves at horizon crossing which is inconsistent with standard big bang nucleosynthesis constraints [20,21].

Apart from studying the generation of magnetic fields, one can also study interactions of a preexisting magnetic field with gravitational degrees of freedom. This is often studied in the context of amplification of the seed magnetic field or gravitational wave detection. Much progress has been made in this area [22–25]. Most of these studies however have been restricted to focusing on the interaction of magnetic fields with tensor perturbations; In this work we revisit and extend the work presented in Refs. [22,24,26], to include scalar perturbations in the matter fluctuations.

When using perturbation theory about a Friedmann-Lemaître-Robertson-Walker (FLRW) background to study the interaction, one is immediately faced with the problem of how to embed the seed magnetic field into the background. The isotropy of the FLRW spacetime does not readily allow for any direction preference that may be introduced by a vector field. There are several ways to handle this and we mention briefly just three of them. One can treat the seed magnetic field as a zeroth order quantity, subject to the assumption that the energy density of the field be small compared to the energy density of matter $B^2 \ll \mu$ and that the anisotropic stress is negligible $\Pi_{ab} = B_{\langle a}B_{b\rangle} \approx 0$. With these approximations, the energy density of the magnetic field cannot alter the gravitational dynamics of the background spacetime; this approach is often referred to as the weak-field approximation. Another approach is to treat the seed field as a statistically homogeneous and isotropic random field with $\langle \mathbf{B}\rangle = 0$ but $\langle B^2\rangle \neq 0$ and so, the seed field does not introduce any directional dependence in the background spacetime. One can then easily employ statistical methods to quantify the field's behavior. Another possibility is to leave the background spacetime untouched but treat the seed field as a first order perturbation, using a two parameter approximation scheme to characterize the perturbations in the electromagnetic and gravitational field; this is the approach we adopt in this work.

One can go a long way in comparing the different perturbation schemes. For example, in the weak-field approximation, the induced magnetic field will be at first order, a well understood regime in perturbation theory. While in the two parameter case, the induced field will be at second order [27], a regime that is not so well developed. Nevertheless, for the purposes of our work, the two approaches are mathematically equivalent. The apparent differences between them is as a result of relabeling of spacetimes, i.e., *first order* in the weak-field approximation corresponds to *second order* in the two parameter case. Indeed, Maxwell's equations and thus the Einstein-Maxwell system takes the same mathematical form in both of these approaches. They both use the machinery of relativistic perturbation theory and are thus prone to gauge issues, see Refs. [28,29] for example.

The present article is structured as follows: we present details of our perturbative framework in Sec. III. After a presentation of the interaction equations in Sec. VI, we present the derivation of the equations describing the induction of electromagnetic (EM) fields in Secs. VII A and VII Bfor a general current and a note on how to evaluate the induced electrical current in Sec. VII C. We present the power spectra of the induced magnetic field variable in Sec. X and finally a summary in XI. We employ the $1 + 3$ covariant approach to perturbation theory [30] and follow [31] by adopting the more geometrically motivated metric signature $(- + + +)$ and we use geometrized units $8\pi G = c = 1$, where G is the gravitational constant and c is the speed of light in vacuum.

II. PRELIMINARIES

A. 1 + 3 spacetime splitting

One of the nice aspects of the $1 + 3$ covariant approach to general relativity is that the underlying dynamical equations have a stronger appeal from a physical point of view, as compared to the quasilinear, second-order partial differential equation form, which the Einstein field equations take in the metric based approach.

The approach is based on a $1 + 3$ decomposition of geometric quantities with respect to a fundamental four velocity u^a:

$$u_a = \frac{dx^a}{d\tau}, \qquad u_a u^a = -1, \tag{1}$$

where x^a are general coordinates and τ measures the proper time along the world line. The key equations governing the full structure of the spacetime are derived from the Ricci and the once and twice contracted Bianchi identities applied to the 4-velocity vector [30]. This splitting uniquely defines two projection tensors,

$$U^a{}_b = -u^a u_b \Rightarrow U^a{}_c U^c{}_b = U^a{}_b,\, U^a{}_a = 1,\, U_{ab}u^b = u_a, \tag{2}$$

$$h_{ab} = g_{ab} + u_a u_b \Rightarrow h^a{}_c h^c{}_b = h^a{}_b,\, h^a{}_a = 3,\, h_{ab}u^b = 0, \tag{3}$$

which project along and orthogonal to the 4-velocity u^a. We define two projected covariant derivatives, the convective time derivative along u^a and the spatially projected covariant derivative,

$$\dot{Q}^{a\cdots b}{}_{c\cdots d} \equiv u^e \nabla_e Q^{a\cdots b}{}_{c\cdots d} \quad \text{and}$$
$$\mathrm{D}_e Q^{a\cdots b}{}_{c\cdots d} \equiv h^a{}_p \cdots h^b{}_q h^r{}_c \cdots h^s{}_d h^f{}_e \nabla_f Q^{p\cdots q}{}_{r\cdots s}, \tag{4}$$

respectively. The basic equations are then characterized by the irreducible parts of the first covariant derivative of u_a,

$$\nabla_a u_b = -u_a \mathcal{A}_b + \mathrm{D}_a u_b$$
$$= -u_a \mathcal{A}_b + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}, \tag{5}$$

where $\mathcal{A}_b = u^a \nabla_a u_b$ is the relativistic acceleration vector representing the effect of inertial forces on the fluid; $\mathrm{D}_a u^a = \Theta$ is the rate of volume expansion; $\sigma_{ab} = \mathrm{D}_{\langle a} u_{b\rangle}$ is the symmetric trace-free rate of shear tensor, describing the rate of distortion of the fluid flow; $\omega_{ab} = \mathrm{D}_{[a} u_{b]}$ is the antisymmetric vorticity tensor, describing the rigid rotation of the fluid relative to a nonrotating frame.

B. FLRW background

We choose as our background the FLRW models, which are spatially homogeneous and isotropic. Thus, relative to the congruence u_a, the kinematical variables have to be locally isotropic, which implies the vanishing of the 4-acceleration $\dot{u}_a = 0$, the rate of shear $\sigma_{ab} = 0$ and the vorticity vector $\omega_a = 0$. Spatial homogeneity implies that the spatial gradients of the energy density μ, pressure p, and the expansion Θ vanish, i.e., $\mathrm{D}_a \mu = \mathrm{D}_a p = \mathrm{D}_a \Theta = 0$. Moreover, the FLRW spacetime is characterized by a perfect fluid matter tensor, i.e., $\pi = q_a = 0$. These restrictions imply that the spacetime is conformally flat, i.e., the electric and magnetic parts of the Weyl tensor vanish, $E_{ab} = H_{ab} = 0$. This leads to the key background equations, the energy conservation equation

$$\dot{\mu} = -(1+w)\Theta\mu, \tag{6}$$

the Raychaudhuri equation

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}\mu(1+3w) + \Lambda, \tag{7}$$

where $w = p/\mu$ and the Friedmann equation

$$\mu + \Lambda = \frac{1}{3}\Theta^2 + \frac{3K}{a^2}. \tag{8}$$

III. PERTURBATIVE FRAMEWORK

As already mentioned, a FLRW spacetime cannot readily host magnetic fields, as their anisotropic stresses $\Pi_{ab} = \tilde{B}_{\langle a}\tilde{B}_{b\rangle} \neq 0$ will break the isotropy. We thus treat the background magnetic field $\tilde{B}_a$ as a first order perturbation to the isotropic spacetime. This lends the energy density, the isotropic and anisotropic pressure of the field to second order in perturbation theory.

We then proceed by adopting a two parameter perturbative framework [32–35]. Fundamentally, this consists of separately parametrizing the gravitational and Maxwell field perturbations in two expansion parameters ϵ_g and $\epsilon_{\tilde{B}}$, representing the amplitudes of the gravitational and electromagnetic field perturbations, respectively [22,26,33]. Using this parametrization, any quantity $Q^{\cdots}{}_{\cdots}$ in the physical spacetime can be expanded in the form

$$Q^{\cdots}{}_{\cdots} = \epsilon_g^0 \epsilon_{\tilde{B}}^0 Q^{(0,0)\cdots}{}_{\cdots} + \epsilon_g^1 \epsilon_{\tilde{B}}^0 Q^{(1,0)\cdots}{}_{\cdots} + \epsilon_g^0 \epsilon_{\tilde{B}}^1 Q^{(0,1)\cdots}{}_{\cdots}$$
$$+ \epsilon_g^1 \epsilon_{\tilde{B}}^1 Q^{(1,1)\cdots}{}_{\cdots} + \mathcal{O}(\epsilon_g^2, \epsilon_{\tilde{B}}^2), \tag{9}$$

where the first term on the right represents the background term; the first and second terms represent the first order gravitational and electromagnetic perturbations, respectively; the fourth term represents the nonlinear coupling we are looking to investigate; the higher order terms represent self-coupling terms of order ϵ_g^m and $\epsilon_{\tilde{B}}^n$, $m, n \geq 2$. In general, terms describing the coupling will be of the form $\epsilon_g^m \epsilon_{\tilde{B}}^n$, where, in this work, we restrict the perturbative order to $\mathcal{O}(\epsilon_g^1 \epsilon_B^1)$ and therefore neglect terms of order $\mathcal{O}(\epsilon_g^2 \epsilon_B^1)$, $\mathcal{O}(\epsilon_g^1 \epsilon_B^2)$ and higher, resulting from the self-coupling of the fields; this includes gravitational couplings with the magnetic anisotropy $\Pi_{ab} = -\tilde{B}_{\langle a}\tilde{B}_{b\rangle}$, leading to $\mathcal{O}(\epsilon_g^1 \epsilon_B^2)$ terms. We will generally refer to quantities of order $\mathcal{O}(\epsilon_g^1 \epsilon_B^1)$ simply as nonlinear and reserve the designation second order for terms that are of order ϵ_g^2 and $\epsilon_{\tilde{B}}^2$. As in Refs. [22,26,33], one can visualize this framework as a hierarchy of spacetimes to label the different perturbative orders.

We make the common assumption in the literature that the perturbed spacetimes have the same manifold as the background spacetime; i.e., we consider the perturbations as fields propagating on the background spacetime [35,36]. In this treatment, therefore, we restrict the possibility that the perturbations may alter the differential structure of the background manifold and so we neglect issues of backreaction.

We are also interested in studying this coupling in a gauge-invariant manner. The gauge problem in relativistic perturbation theory has been dealt with in the literature, see for example Refs. [35,37–40]. The Stewart and Walker lemma [40] serves as a basis for the generalization of gauge invariance to arbitrary order [35,39]. It follows that a quantity Q is gauge invariant at order $\mathcal{O}(\epsilon_g^m \epsilon_{\tilde{B}}^n)$ if and only if $Q^{(0)}$ and its perturbations of order lower than $\mathcal{O}(\epsilon_g^m \epsilon_{\tilde{B}}^n)$ are either vanishing, or a constant scalar or a combination of Kronecker deltas with constant coefficients [32,35].

Since the interaction terms are of order $\mathcal{O}(\epsilon_g^1 \epsilon_B^1)$ we have that the induced magnetic field B_a will be of the same order; we also assume that the electric field E_a will be of the same order as the induced magnetic field. Clearly B_a does not satisfy the criteria for gauge invariance at $\mathcal{O}(\epsilon_g^1 \epsilon_{\tilde{B}}^1)$ since it is neither vanishing nor a constant scalar at

$\mathcal{O}(\epsilon_g^0\epsilon_{\tilde{B}}^1)$. To this end, we make use of the same auxiliary variable $\beta_a = \dot{\tilde{B}}_a + \frac{2}{3}\Theta\tilde{B}_a$ identified in Refs. [22,26,33]. We do not however integrate β_a to recover the gauge-dependent magnetic field, but treat it as the fundamental variable whose deviation from zero quantifies deviation from the adiabatic decay of the magnetic field.

IV. THE EINSTEIN-MAXWELL SYSTEM

The Einstein-Maxwell equations (A10) contain terms that couple the electromagnetic fields to the gravitational fields. These can be written at $\mathcal{O}(\epsilon_g^1\epsilon_{\tilde{B}}^1)$ by discarding higher order terms. This results in the two propagation equations,

$$\dot{B}_{\langle a\rangle} + \frac{2}{3}\Theta B_a = \sigma_{ab}\tilde{B}^b - \text{curl}\,E_a, \tag{10}$$

$$\dot{E}_{\langle a\rangle} + \frac{2}{3}\Theta E_a = \text{curl}\,B_a + \epsilon_{abc}\mathcal{A}^b\tilde{B}^c - \mathcal{J}_a, \tag{11}$$

subject to the constraints, $\mathrm{D}_aE^a = 0 = \mathrm{D}_aB^a$. Following Ref. [41] we make the following comments: (i) The magnetic field $\tilde{B}_a$ appearing in Eqs. (13) and (14) multiplied by the gravitational variables should not be the same as the B_a appearing alone. The variable B^a is a mixture of linear and nonlinear quantities (the seed magnetic field and the induced field) while the terms involving $\tilde{B}^a$ are a product of first order quantities. One has to keep this in mind when integrating the equations. (ii) The system is not gauge invariant as already mentioned in Sec. III. This can be attributed to the mixture of linear and nonlinear terms in the system. In the covariant approach to perturbation theory, the solution of perturbed differential operators is never sought, one can get around this by making sure that the differential operators involved operate on quantities of the corresponding perturbative order.

In an attempt to cast it in a consistent and gauge invariant manner, we introduce the following nonlinear variables: the fundamental variable β_a measuring deviation from adiabatic decay, I_a describing the interaction with shear distortions and ξ_a describing interaction with density perturbations. These are defined as

$$\beta_a = \dot{B}_{\langle a\rangle} + \frac{2}{3}\Theta B_a, \qquad I_a = \sigma_{ab}\tilde{B}^b \quad \text{and} \quad \xi_a = \epsilon_{abc}\mathcal{A}^b\tilde{B}^c, \tag{12}$$

and results in the following system:

$$\beta_a = I_a - \mathcal{E}_a, \tag{13}$$

$$\dot{E}_{\langle a\rangle} + \frac{2}{3}\Theta E_a = \mathcal{B}_a + \xi_a - \mathcal{J}_a, \tag{14}$$

where we have written curl $E_a = \mathcal{E}_a$ and curl $B_a = \mathcal{B}_a$ for brevity.

V. THE LINEAR EQUATIONS

A. The linear magnetic field: $\mathcal{O}(\epsilon_{\tilde{B}})$

We treat the seed magnetic field as a first order perturbation to the spacetime. The seed field may have its origins in inflation or other mechanisms based on string cosmology, in which electromagnetic vacuum fluctuations are amplified due to a dynamical dilaton or an inflaton field [19]. We assume that at order $\mathcal{O}(\epsilon_g^0\epsilon_{\tilde{B}}^1)$ the electric fields are small compared to the magnetic fields, i.e., $E^2 \ll B^2$. Thus, in the absence of diffusive losses or amplification, the induction equation (13) takes the frozen-in form,

$$\dot{\tilde{B}}_{\langle a\rangle} + \frac{2}{3}\Theta\tilde{B}_a = 0, \tag{15}$$

regardless of the equation of state or plasma properties of the cosmic fluid. It follows then that the magnetic field decays adiabatically as $\tilde{B}_a \propto a^{-2}$, where a is the cosmological scale factor. This adiabatic decay arises from the expansion of the Universe which conformally dilutes the field lines due to flux conservation. The frozen-in condition (15) does not discriminate between homogeneous $(\mathrm{D}_a\tilde{B}_b = 0)$ and inhomogeneous $(\mathrm{D}_a\tilde{B}_b \neq 0)$ magnetic fields. For an inhomogeneous field the spatial gradients of the seed magnetic field $\mathrm{D}_b\tilde{B}_a$ are of the same order as $\tilde{B}_a$ and evolve as $\mathrm{D}_bB_a \propto a^{-3}$.

B. Gravitational perturbations: $\mathcal{O}(\epsilon_g)$

The Weyl tensor C_{abcd} represents the free gravitational field, enabling gravitational action at a distance. In analogy with splitting the Maxwell field tensor F_{ab} into a magnetic and an electric field, C_{abcd} can be split covariantly into a *magnetic* part $H_{ab} = \frac{1}{2}\epsilon_{ade}C^{de}{}_{bc}u^c$ and an *electric* part $E_{ab} = C_{abcd}u^cu^d$. The electric part of the Weyl tensor describes tidal effects, akin to the tidal tensor associated with the Newtonian potential, while the magnetic part describes the propagation of gravitational radiation. The Weyl tensor vanishes in the conformally flat FLRW spacetime and so E_{ab} and H_{ab} are covariant first order gauge invariant quantities in the Weyl curvature. We also define the first order gauge invariant variables $\mathcal{X}_a = a\mathrm{D}_a\mu$ and $\mathcal{Z} = a\mathrm{D}_a\Theta$ to characterize density perturbations. Now, the system governing gravitational perturbations is given by the following propagation equations [42]:

$$\dot{\sigma}_{\langle ab\rangle} + \frac{2}{3}\Theta\sigma_{ab} = \mathrm{D}_{\langle a}\mathcal{A}_{b\rangle} - E_{ab}, \tag{16}$$

$$\dot{H}_{\langle ab\rangle} + \Theta H_{ab} = -\text{curl}\,E_{ab}, \tag{17}$$

$$\dot{E}_{\langle ab\rangle} + \Theta E_{ab} = \text{curl}\,H_{ab} - \frac{1}{2}\mu(1+w)\sigma_{ab}, \tag{18}$$

$$\dot{\mathcal{X}}_{\langle a\rangle} - \Theta w\mathcal{X}_a = -(1+w)\mathcal{Z}_a, \tag{19}$$

$$\dot{\mathcal{Z}}_{\langle a\rangle} + \frac{2}{3}\Theta\mathcal{Z}_a = -\frac{1}{2}\mu\mathcal{X}_a - \frac{w}{3(1+w)}\left(-\frac{1}{3}\Theta^2 + \mu + \Lambda\right)\mathcal{X}_a - \frac{w}{1+w}\mathrm{D}^2\mathcal{X}_a. \tag{20}$$

In addition to the propagation equations above, the following constraints have to be satisfied:

$$a\mathrm{D}^c\sigma_{bc} = \frac{2}{3}\mathcal{Z}_b, \qquad aD^cE_{bc} = \frac{1}{3}\mu\mathcal{X}_b \quad \text{and} \quad H_{ab} = \mathrm{curl}\,\sigma_{ab}, \tag{21}$$

where we have set the vorticity to zero ($\omega_a = 0$); see also Ref. [41]. Note that at first order in gravitational perturbations, the only source of vector modes is the vorticity ω_a; since, we neglect the effects of vorticity, $\omega_a = 0$, all the vector modes vanish. The shear tensor σ_{ab} can then be irreducibly split into scalar and tensor contributions as [43]

$$\sigma_{ab} = \sigma^S_{ab} + \sigma^T_{ab}, \quad \text{where } \mathrm{curl}\,\sigma^S_{ab} = 0, \quad \text{and} \quad \mathrm{D}^a\sigma^T_{ab} = 0. \tag{22}$$

The pure tensor modes can be used to characterize gravitational waves [44]. The scalar part of the shear couples to density perturbations and is related to the clumping of matter via the constraints (21).

By differentiating (16) and using (18) and one of the constraints (21) to substitute for E_{ab} and H_{ab}, one arrives at a forced wave equation for the shear,

$$\ddot{\sigma}_{\langle ab\rangle} - \mathrm{D}^2\sigma_{ab} + \frac{5}{3}\Theta\dot{\sigma}_{\langle ab\rangle} + \left[\frac{1}{9}\Theta^2 + \frac{1}{6}\mu - \frac{3}{2}p + \frac{5}{3}\Lambda\right]\sigma_{ab} = -\frac{w}{a^2(1+w)}\left[\dot{\mathcal{X}}_{ab} + \frac{1}{3}\Theta\mathcal{X}_{ab}\right], \tag{23}$$

where $\mathcal{X}_{ab} = -(1+w)a^2\mathrm{D}_{\langle a}\mathcal{A}_{b\rangle}/w = a\mathrm{D}_{\langle a}\mathcal{X}_{b\rangle}$. We need an evolution equation for $\mathcal{X}_{ab}$ in order to close Eq. (23). One can start from (19) and (20) to write a wave equation for $\mathcal{X}_a$ then taking the comoving spatial gradient of the resulting wave equation will yield the following:

$$\ddot{\mathcal{X}}_{ab} - w\mathrm{D}^2\mathcal{X}_{ab} - \left(w - \frac{2}{3}\right)\Theta\dot{\mathcal{X}}_{ab} + \frac{1}{2}\mu(3w+1)(w-1)\mathcal{X}_{ab} - 2w\Lambda\mathcal{X}_{ab} = 0. \tag{24}$$

In including scalar perturbations, we have explicitly coupled the shear tensor to density perturbations. This shows that density gradients source distortions in the Weyl curvature and vice versa. Hence, knowing the shear allows one to compute density gradients and knowing density gradients one can compute the scalar part of the shear [45].

VI. THE INTERACTION: $\mathcal{O}(\epsilon_g\epsilon_{\tilde{B}})$

The Maxwell fields couple to Weyl curvature through the shear term and density perturbations through the acceleration terms and the nonlinear identity (28). In the case of pure tensor modes in the shear tensor, the interaction variable $I_a = \sigma^T_{ab}\tilde{B}^b$ was shown to satisfy a closed wave equation, for both a homogeneous [22] and an inhomogeneous [26] seed field $\tilde{B}_a$. Here, we include contributions from scalar perturbations in the shear, which give rise to source terms due to coupling with density perturbations. In this case I_a satisfies a forced wave equation,

$$\ddot{I}_{\langle a\rangle} - \mathrm{D}^2I_a + 3\Theta\dot{I}_{\langle a\rangle} + \left[\frac{13}{9}\Theta^2 - \frac{1}{6}\mu - \frac{5}{2}\mu w + \frac{7}{3}\Lambda\right]I_a = \mathcal{C}^I_a, \tag{25}$$

where the forcing term $\mathcal{C}^I_a$ is given by

$$\mathcal{C}^I_a = -\frac{w}{a^2(1+w)}(\dot{S}_{\langle a\rangle} + \Theta S_a). \tag{26}$$

To close the above system, we give the companion wave equation for $S_a = a\tilde{B}^b\mathrm{D}_{\langle a}\mathcal{X}_{b\rangle}$ as

$$\ddot{S}_a - w\mathrm{D}^2S_a + (2-w)\Theta\dot{S}_a + \left[\frac{2}{3}(1-w)(\Lambda + \Theta^2) + \frac{1}{6}\mu(1+3w)(3w-5)\right]S_a = 0. \tag{27}$$

We note, for later convenience (Sec. VII A) that the forcing term $\mathcal{C}^I_a = 0$ in a matter dominated universe ($w = 0$); i.e., I_a decouples from S_a when $w = 0$.

VII. INDUCTION OF EM FIELDS

We introduce nonlinear gravitationally induced *effective current* terms $\mathcal{C}^E_a$, $\mathcal{C}^{\mathcal{E}}_a$ and $\mathcal{C}^{\beta}_a$ which are made up of the coupling between density and gravitational wave perturbations; these will act as driving forces of the induced Maxwell fields.

A. The electric field

We show how the coupling of gravitational perturbations with the seed magnetic field can induce electric fields. Here we give wave equations for the induced electric field E_a and its rotation $\mathcal{E}_a$. In deriving the wave equation for E_a, we differentiate (14) and equate the result to the nonlinear identity,

$$(\mathrm{cu\dot{r}l}\,B_a) = \mathrm{curl}\,\beta_a - \Theta\,\mathrm{curl}\,B_a + H_{ab}\tilde{B}^b + \frac{1}{3a(1+w)}\epsilon_{abc}\tilde{B}^b(\Theta w\mathcal{X}^c - 2\dot{\mathcal{X}}^c), \tag{28}$$

obtained from the commutation relations (the Appendix) and we have used Eq. (19) to rewrite the acceleration terms. The resulting wave equation is found to be

$$\ddot{E}_{\langle a\rangle} - \mathrm{D}^2 E_a + \frac{5}{3}\Theta \dot{E}_a + \left[\frac{2}{9}\Theta^2 + \frac{1}{3}\mu(1-3w) + \frac{4}{3}\Lambda\right] E_a = \mathcal{C}^E_a, \tag{29}$$

where $\mathcal{C}^E_a$ is a gravitationally induced source term given by

$$\mathcal{C}^E_a = \mathrm{curl}\, I_a + H_{ab}\tilde{B}^b + \frac{1}{a(1+w)}\epsilon_{abc} \times \left[\left(w - \frac{2}{3}\right)(\tilde{B}^b \mathcal{X}^c)^{\cdot} + \Theta\left(w - \frac{4}{9}\right)\tilde{B}^b \mathcal{X}^c\right] - \Theta \mathcal{J}_a - \dot{\mathcal{J}}_a, \tag{30}$$

and $\mathcal{J}_a$ is the 3-current. The terms involving ϵ_{abc} in $\mathcal{C}^E_a$ vanish when the magnetic field $\tilde{B}^a$ is parallel to the fractional density gradient $\mathcal{X}^a$. Taking the curl of (29) results in the equation governing the rotation of E_a,

$$\ddot{\mathcal{E}}_a - \mathrm{D}^2 \mathcal{E}_a + \frac{7}{3}\Theta \dot{\mathcal{E}}_a + \left[\frac{7}{9}\Theta^2 + \frac{1}{6}\mu(1-9w) + \frac{5}{3}\Lambda\right]\mathcal{E}_a = \mathcal{C}^{\mathcal{E}}_a, \tag{31}$$

where the source term $\mathcal{C}^{\mathcal{E}}_a = \mathrm{curl}\, \mathcal{C}^E_a$ is given by

$$\mathcal{C}^{\mathcal{E}}_a = -(\mathrm{curl}\, \mathcal{J}_a)^{\cdot} - \frac{4}{3}\Theta\, \mathrm{curl}\, \mathcal{J}_a + 2\mathrm{D}^b \mathrm{D}_{[a} I_{b]} + \epsilon_{acd}\tilde{B}_b \mathrm{D}^c H^{db} + \frac{2}{a^2(1+w)}\left[\left(w - \frac{2}{3}\right)(a\tilde{B}_{[a}\mathrm{D}^b \mathcal{X}_{b]})^{\cdot} + \Theta\left(w - \frac{4}{9}\right) a\tilde{B}_{[a}\mathrm{D}^b \mathcal{X}_{b]}\right]. \tag{32}$$

B. The magnetic field

As already mentioned, the induced magnetic field will be characterized via the variable $\beta_a = \dot{B}_a + \frac{2}{3}\Theta B_a$. On using (13), (25), and (31), one can write a second-order equation governing the evolution of the fundamental variable β_a. This can be written in either of two forms: in terms of I_a or $\mathcal{E}_a$, corresponding to using (13) as a constraint to either of (31) or (25), respectively. Recall that both I_a and $\mathcal{E}_a$ satisfy wave equations of the form $\mathcal{L}[I_a] = \mathcal{C}^I_a$ and $\mathcal{L}[\mathcal{E}_a] = \mathcal{C}^{\mathcal{E}}_a$, where the $\mathcal{C}^i_a$s are source terms.

Using covariant harmonics [45], one can already notice from (25) and (31) that the eigenfunctions used to harmonically decompose I_a and $\mathcal{E}_a$ are not the same for a general perturbation [46]. Consider the induction equation (13), and write it as $\beta_a = \sum_k (\mathcal{P}_a I_{(k)} - \mathcal{Q}_a \mathcal{E}_{(k)})$, where $\mathcal{P}_a$ and $\mathcal{Q}_a$ are distinct eigenfunctions of the Laplace-Beltrami operator, i.e., $\mathcal{P}_a \neq \mathcal{Q}_a$. For the separation of variables technique to work for β_a, one must eliminate either $I_a = \mathcal{P}_a I_{(k)}$, along with its source terms $\mathcal{C}^I_a$ or $\mathcal{E}_a = \mathcal{Q}_a \mathcal{E}_{(k)}$ along with its source terms $\mathcal{C}^{\mathcal{E}}_a$. In this way, β_a can then be expanded in terms of one set of complete eigenfunctions. This presents a problem: since both I_a and $\mathcal{E}_a$ are coupled to source terms $\mathcal{C}^I_a$ and $\mathcal{C}^{\mathcal{E}}_a$ respectively at second order, both $\mathcal{C}^I_a$ and $\mathcal{C}^{\mathcal{E}}_a$ will still couple to the β_a equation at this order, thereby introducing the differing set of eigenfunctions $\mathcal{P}_a$ and $\mathcal{Q}_a$. A similar problem arose in Ref. [47], due to the inclusion of a vorticity term.

It is possible to do away with $\mathcal{C}^I_a$ in Eq. (25) by requiring that $w = 0$ and this alleviates the problem [48]. We shall then henceforth restrict to the pressureless dust ($w = 0$) case and write the β_a wave equation in terms of $\mathcal{E}_a$:

$$\ddot{\beta}_{\langle a\rangle} - \mathrm{D}^2 \beta_a + 3\Theta \dot{\beta}_{\langle a\rangle} + \left[\frac{13}{9}\Theta^2 - \frac{1}{6}\mu + \frac{7}{3}\Lambda\right]\beta_a = \mathcal{C}^{\beta}_a, \tag{33}$$

where

$$\mathcal{C}^{\beta}_a = -\frac{2}{3}\Theta \dot{\mathcal{E}}_a + \left[-\frac{2}{3}\Theta^2 + \frac{1}{3}\mu - \frac{2}{3}\Lambda\right]\mathcal{E}_a + (\mathrm{curl}\, \mathcal{J}_a)^{\cdot} + \frac{4}{3}\Theta\, \mathrm{curl}\, \mathcal{J}_a - 2\mathrm{D}^b \mathrm{D}_{[a} I_{b]} - \epsilon_{acd}\tilde{B}_b \mathrm{D}^c H^{db} - \frac{2}{a^2}\left[-\frac{2}{3}(a\tilde{B}_{[a}\mathrm{D}^b \mathcal{X}_{b]})^{\cdot} - \frac{4}{9}\Theta(a\tilde{B}_{[a}\mathrm{D}^b \mathcal{X}_{b]})\right]. \tag{34}$$

Note that while we keep $S_a = a\tilde{B}^b \mathrm{D}_{\langle a}\mathcal{X}_{b\rangle}$ distinct from $a\tilde{B}_{[a}\mathrm{D}^b \mathcal{X}_{b]}$ in real space, their evolution equations can be made equivalent in harmonic space by a suitable choice of eigenfunctions [49]. We shall thus write $S_{(\ell)}$ in place of $\tilde{B}_{(n)}\mathcal{X}_{(k)}$ to avoid introducing another letter to denote the latter. This should not lead to any ambiguities.

C. The electric current

1. Limiting cases: Poor and perfect conductivity

To close the above system, one needs to take care of the current term $\mathcal{J}_a$ appearing in (30), (32), and (34). This term depends on the electrical properties of the medium. It is given in terms of the electric field E_a via Ohm's law,

$$\mathcal{J}_a = \varsigma E_a, \tag{35}$$

where ς is the electrical conductivity of the medium. In this section, we consider only the limiting cases of very high ($\varsigma \to \infty$) and very poor conductivity ($\varsigma \to 0$). Under the assumption of poor conductivity, the currents vanish $\mathcal{J}_a = 0$, despite the presence of a nonzero electric field. In this case, one solves Eqs. (29), (31), and (33), with the current terms set to zero. At the opposite end, the case of perfect conductivity, the electric fields vanish and the currents keep the magnetic field frozen in with the fluid. In this case, the current term satisfies

$$(\mathrm{curl}\, \mathcal{J}_a)^{\cdot} + \frac{4}{3}\Theta\, \mathrm{curl}\, \mathcal{J}_a = 2\mathrm{D}^b \mathrm{D}_{[a} I_{b]} + \epsilon_{acd}\tilde{B}_b \mathrm{D}^c H^{db} + \frac{2}{a^2}\left[-\frac{2}{3}(a\tilde{B}_{[a}\mathrm{D}^b \mathcal{X}_{b]})^{\cdot} - \frac{4}{9}\Theta(a\tilde{B}_{[a}\mathrm{D}^b \mathcal{X}_{b]})\right], \tag{36}$$

and (29) and (31) are no longer relevant. One can verify that using this relation reduces Eq. (33) to $\beta_a = I_a$, as can be confirmed also from the induction equation (13).

One can also invoke the magnetohydrodynamic (MHD) approximation, which is valid for cold plasmas (pressureless dust can be well approximated by a cold plasma treatment) [50]. Cold plasmas have components with nonrelativistic velocities and are thus mathematically easier to deal with [26,51,52]. We consider a two component electron-ion plasma and assume that the motion properties of the plasma on macroscopic scales are captured by the center of mass 3-velocity v^a of the system; i.e., the difference in mean velocities of the individual species is small compared with the fluid velocity. We also assume charge neutrality of the cosmic plasma; i.e., the number densities of the electrons and ions n_e and n_i are roughly equal, $n_e \approx n_i$; this guarantees the vanishing of the total charge $\rho_c = -e(n_e - n_i) \approx 0$ and the background 3-current $\mathcal{J}_a \approx 0$. In this case, the generalized Ohm's law is given by

$$\mathcal{J}_{\langle a\rangle} = \varsigma(E_a + \epsilon_{abc}v^b\tilde{B}^c), \qquad v^a = \frac{\mu_e v_e^a + \mu_i v_i^a}{\mu_e + \mu_i}, \tag{37}$$

where the subscripts e and i denote quantities for electrons and ions, respectively. The center of mass 3-velocity v_a of the electron-ion plasma can be shown to satisfy the linearized Euler equation,

$$\dot{v}_{\langle a\rangle} + \frac{1}{3}\Theta v_a = 0. \tag{38}$$

In the ideal-MHD environment, the conductivity of the medium is very high ($\varsigma \to \infty$), then $E_a + \epsilon_{abc}v^b\tilde{B}^c \to 0$ in order to keep the current $\mathcal{J}_a$ finite. This readily gives the electric field E_a and its rotation $\mathcal{E}_a$ as $E_a = -\epsilon_{abc}v^b\tilde{B}^c$ and $\mathcal{E}_a = 2\tilde{B}_{[a}\mathrm{D}^b v_{b]}$. Using (15) and (38), one can show that

$$\dot{E}_{\langle a\rangle} + \Theta E_a = 0, \quad \text{and} \quad \dot{\mathcal{E}}_a + \frac{4}{3}\Theta\mathcal{E}_a = 0. \tag{39}$$

With these, the 3-current $\mathcal{J}_a$ satisfies

$$\begin{aligned}(\mathrm{curl}\,\mathcal{J}_a)^{\cdot} + \frac{4}{3}\Theta\,\mathrm{curl}\,\mathcal{J}_a &= 2\mathrm{D}^b\mathrm{D}_{[a}I_{b]} + \epsilon_{acd}\tilde{B}_b\mathrm{D}^c H^{db} \\ &\quad + \frac{2}{a^2}\left[-\frac{2}{3}(a\tilde{B}_{[a}\mathrm{D}^b\mathcal{X}_{b]})^{\cdot} - \frac{4}{9}\Theta(a\tilde{B}_{[a}\mathrm{D}^b\mathcal{X}_{b]})\right] \\ &\quad + \mathrm{D}^2\mathcal{E}_a - \left(-\frac{1}{9}\Theta^2 + \frac{5}{6}\mu + \frac{1}{3}\Lambda\right)\mathcal{E}_a.\end{aligned} \tag{40}$$

Substituting (40) into (33) results in

$$\begin{aligned}\ddot{\beta}_{\langle a\rangle} - \mathrm{D}^2\beta_a + 3\Theta\dot{\beta}_{\langle a\rangle} + \left[\frac{13}{9}\Theta^2 - \frac{1}{6}\mu + \frac{7}{3}\Lambda\right]\beta_a \\ = \mathrm{D}^2\mathcal{E}_a + \left[\frac{1}{3}\Theta^2 - \frac{1}{2}\mu - \Lambda\right]\mathcal{E}_a.\end{aligned} \tag{41}$$

The application of the ideal MHD approximation in cosmology has often been criticized as being of a practical appeal rather than of a physical one [53]. Ideally, the curl of E_a should be the outcome of a rigorous treatment of the physics of the particle interactions in terms of a kinetic theory description, see for example Refs. [9,10].

2. Intermediate case: finite conductivity

The case of poor conductivity may not be very relevant in the post-recombination epoch as the universe then acquires very high conductivity. The perfect conductivity case, while relevant, may be thought of as an idealized notion. We thus turn to the finite conductivity case. The conductivity of the post decoupling era can be modeled by

$$\varsigma = \frac{n_e^2 e^2}{m_e n_\gamma \sigma_T} \approx 10^{11}\ \mathrm{s}^{-1}, \tag{42}$$

where n_e is the density of free electrons, e is the electric charge of an electron, m_e is the mass of an electron, n_γ is the density of photons and σ_T is the collision cross section. For a perfect fluid, the ratio n_γ/n_e is constant; see Ref. [54] for example.

Assuming that Ohm's law holds [Eq. (35), we may write the current terms in (34) as

$$(\mathrm{curl}\,\mathcal{J}_a)^{\cdot} + \frac{4}{3}\Theta\,\mathrm{curl}\,\mathcal{J}_a = \varsigma\dot{\mathcal{E}}_a + \frac{4}{3}\Theta\varsigma\mathcal{E}_a, \tag{43}$$

where we have assumed that spatial gradients of the conductivity may be neglected ($\mathrm{D}_a\varsigma \approx 0$) and that the conductivity is constant in time ($\dot{\varsigma} \approx 0$). Substituting (43) in the wave equation (33) for β_a results in

$$\ddot{\beta}_{\langle a\rangle} - \mathrm{D}^2\beta_a + 3\Theta\dot{\beta}_{\langle a\rangle} + \left[\frac{13}{9}\Theta^2 - \frac{1}{6}\mu + \frac{7}{3}\Lambda\right]\beta_a = \mathcal{C}_a^{\beta}, \tag{44}$$

where the source term $\mathcal{C}_a^\beta$ is now given by

$$\begin{aligned}\mathcal{C}_a^\beta &= \left(\frac{\varsigma}{\Theta} - \frac{2}{3}\right)\Theta\dot{E}_a + \left[\left(2\frac{\varsigma}{\Theta} - 1\right)\frac{2}{3}\Theta^2 + \frac{1}{3}\mu - \frac{2}{3}\Lambda\right]\mathcal{E}_a \\ &\quad - 2\mathrm{D}^b\mathrm{D}_{[a}I_{b]} - \epsilon_{acd}\tilde{B}_b\mathrm{D}^c H^{db} \\ &\quad - \frac{2}{a^2}\left[-\frac{2}{3}(a\tilde{B}_{[a}\mathrm{D}^b\mathcal{X}_{b]})^{\cdot} - \frac{4}{9}\Theta(a\tilde{B}_{[a}\mathrm{D}^b\mathcal{X}_{b]})\right].\end{aligned} \tag{45}$$

Note that the electric currents $\mathcal{J}_a$, electric fields E_a and the conductivity ς are all simultaneously finite. The simplifications that arise due to the characterization of the limiting cases ($\mathcal{J}_a = 0$ for poor conducting mediums and $E_a = 0$ for perfect conducting mediums) are no longer applicable in the case of finite conductivity. One then needs a proper model for the electric currents to ensure that the initial conditions for $\mathcal{J}_a$ and E_a are not chosen independently. There are several ways in which one can model electric currents, all resulting in terms of perturbative order ϵ_g^2; see Ref. [8] for example. While these terms can be seamlessly accommodated in our framework, they have the undesirable effect of seeding magnetic fields. This will lead us away from the isolated effects of the amplification of an already existing field. Inclusion of such terms will therefore lead us to overestimate the effect of the amplification. With this in mind, we restrict to the limiting cases of VII C 1.

VIII. THE INDUCED FIELDS

We now treat separately the induction of electromagnetic fields due to interaction with *scalar* and *tensor* perturbations. To this end, we expand the perturbation variables in terms of a helicity basis (the Appendix). In addition, we use a unified time variable whose defining equation is $\dot{\tau} = \frac{3}{2}H_i$ instead of proper time, to rewrite the relevant equations [55]. We have to substitute for μ, Θ and a, appearing in the perturbed equations, from the zeroth order equations. We restrict our treatment to zero cosmological constant $\Lambda = 0$ and flat spatial sections $K = 0$. The Friedmann equation then reduces to $\mu = \Theta^2/3$, where Θ is given by $\Theta = 3H_i/\tau$; the scale factor a evolves as $a = a_i\tau^{2/3}$.

A. EM induction due to scalar perturbations

In this case, the coupling of a seed field with gravitational perturbations is described by the variables I_a and S_a; these variables become sources of electromagnetic fields.

(i) *Interaction terms*.—Equations (25) and (27) for the interaction variables I_a and S_a, respectively become

$$\frac{9}{4}I''_{(\ell)} + \frac{27}{2\tau}I'_{(\ell)} + \frac{25}{2\tau^2}I_{(\ell)} = 0, \tag{46a}$$

$$\frac{9}{4}S''_{(\ell)} + \frac{9}{\tau}S'_{(\ell)} + \frac{7}{2\tau^2}S_{(\ell)} = 0. \tag{46b}$$

Note that since $w = 0$, the entire system has decoupled from $a\tilde{B}^b \mathrm{D}_{\langle a}\mathcal{X}_{b\rangle}$, however we still need an equation for $S_{(\ell)}$ because of the coupling with $a\tilde{B}_{[a}\mathrm{D}^b\mathcal{X}_{b]}$ in Eqs. (32) and (34). These interaction variables have the general solutions,

$$I_{(\ell)}(\tau) = C_1\tau^{-10/3} + C_2\tau^{-5/3} \quad \text{and} \quad S_{(\ell)} = \frac{1}{5}C_3\tau^{-7/3} + \frac{1}{5}C_4\tau^{-2/3}, \tag{47}$$

where the C_i's are integration constants.

(ii) *EM fields*.—Equation (29) for the electric field E_a becomes

$$\frac{9}{4}E''_{(\ell)} + \frac{15}{2\tau}E'_{(\ell)} + \left[\left(\frac{\ell}{a_iH_i}\right)^2\tau^{-4/3} + \frac{3}{\tau^2}\right]E_{(\ell)} = \pm\frac{(k+n)}{3a_iH_i^2}I_{(\ell)}\tau^{-2/3} \mp \frac{1}{H_i}S'_{(\ell)} \mp \frac{4}{3H_i\tau}S_{(\ell)}. \tag{48}$$

It is much easier to solve for $\beta_{(\ell)}$ from the induction equation

$$\beta_{(\ell)} = I_{(\ell)} \mp \frac{\ell}{a_i\tau^{2/3}}E_{(\ell)} \tag{49}$$

once $I_{(\ell)}$ and $E_{(\ell)}$ are known, rather than from the wave equation (33).

B. EM induction due to tensor perturbations

In this case, the transverse and trace-free parts of the shear tensor σ_{ab} characterize gravitational waves. The interaction with a seed field is then purely described by the variable I_a without any contribution from either density or velocity perturbations. The generalized Ohm's law (37) in the MHD approximation also reduces to (35). We thus only need the equations for β_a, I_a and E_a.

(i) *Interaction variable*.—Equation (25) for the interaction variable I_a becomes

$$\frac{9}{4}I''_{(\ell)} + \frac{27}{2\tau}I'_{(\ell)} + \left[\left(\frac{\ell}{a_iH_i}\right)^2\tau^{-4/3} + \frac{25}{2\tau^2}\right]I_{(\ell)} = 0, \tag{50}$$

with the general solution,

$$I_{(\ell)}(\tau) = \tau^{-5/2}\left[C_1J_1\left(\frac{5}{2}, \frac{\ell}{a_iH_i}\frac{2}{\tau^{1/3}}\right) + C_2J_2\left(\frac{5}{2}, \frac{\ell}{a_iH_i}\frac{2}{\tau^{1/3}}\right)\right], \tag{51}$$

where C_1 and C_2 are integration constants, J_1 and J_2 are Bessel functions of the second kind.

(ii) *EM fields*.—Equation (29) for the electric field variable E_a becomes

$$\frac{9}{4}E''_{(\ell)} + \frac{15}{2\tau}E'_{(\ell)} + \left[\left(\frac{\ell}{a_iH_i}\right)^2\tau^{-4/3} + \frac{3}{\tau^2}\right]E_{(\ell)} = \pm\frac{(2k+n)}{H_i^2a_i}I_{(\ell)}\tau^{-2/3}, \tag{52}$$

and we once again determine $\beta_{(\ell)}$ from

$$\beta_{(\ell)} = I_{(\ell)} \mp \frac{\ell}{a_i\tau^{2/3}}E_{(\ell)}, \tag{53}$$

instead of using the wave equation (33).

IX. INITIAL CONDITIONS

We need initial conditions in order to solve the equations in the previous section. The conditions are adapted as follows: for β_a we invoke Maxwell's equation (13)

$$\beta_a = I_a - \mathcal{E}_a, \qquad \dot{\beta}_a = \dot{I}_a - \dot{\mathcal{E}}_a. \tag{54}$$

For the interaction variable I_a, we use the definition (12) and Eq. (15)

$$I_a = \sigma_{ab}\tilde{B}^b \quad \dot{I}_a = \dot{\sigma}_{ab}\tilde{B}^b + \sigma_{ab}\dot{\tilde{B}}^b \quad \dot{\tilde{B}}_a = -\frac{2}{3}\Theta\tilde{B}_a. \tag{55}$$

For the rotation of the electric field $\mathcal{E}_a$ [56], we use Maxwell's equation (14) and the commutation relation (A13) to get

$$\dot{\mathcal{E}}_a = -\Theta\mathcal{E}_a + \mathcal{R}_{ab}\tilde{B}^b - \mathrm{D}^2 B_a, \tag{56}$$

where in this case B_a (without the tilde) is the induced magnetic field, and we have written the first order perturbed 3-Ricci tensor $\mathcal{R}_{ab}$ as [30]

$$\mathcal{R}_{ab} = -\dot{\sigma}_{\langle ab\rangle} - \Theta\sigma_{ab}. \tag{57}$$

We require that the gravitationally induced field variables E_a (and hence $\mathcal{E}_a$) and B_a be zero initially. This leads to the following initial conditions for the perturbation variables:

$$\begin{aligned} I^i_{(\ell)} &= \sigma^i_{(k)}\tilde{B}^i_{(n)} & I'_{i(\ell)} &= \sigma'_{i(\ell)}\tilde{B}^i_{(n)} - \frac{4}{3}\sigma^i_{(k)}\tilde{B}^i_{(n)} \\ \mathcal{E}^i_{(\ell)} &= 0 & \mathcal{E}'_{i(\ell)} &= -2\mathcal{E}^i_{(\ell)} - (\sigma'_{i(\ell)} + 2\sigma^i_{(k)})\tilde{B}^i_{(n)} \\ \beta^i_{(\ell)} &= \sigma^i_{(k)}\tilde{B}^i_{(n)} & \beta'_{i(\ell)} &= 2\sigma'_{i(k)}\tilde{B}^i_{(n)} + \frac{2}{3}\sigma^i_{(k)}\tilde{B}^i_{(n)} + 2\mathcal{E}^i_{(\ell)}. \end{aligned} \tag{58}$$

Following [25,51,57], we adopt the initial condition for the shear from $(\sigma/H)_i \sim 10^{-6}$. We choose the seed field to be $\tilde{B}^i = 10^{-20}$ G, as typical of those produced around the recombination era [10].

X. RESULTS

Given the system of initial conditions (58), one can notice that the interaction variable I_a plays the fundamental role in the interaction process. If we set $I_a = 0$ initially, then no amplification takes place. We show the time evolution $I_a(\tau)$ in Fig. 1 on a log-log scale. A noteworthy feature is the rapid decay of I_a for both scalars and tensors. Although the interaction with scalar perturbations decays slightly slower, it essentially follows the same trend as the interaction with gravitational waves. We are thus led to conclude that even including scalar perturbations in the interaction, we reach the same conclusion as [23,51] that there is no significant amplification of electromagnetic fields coming from the interaction.

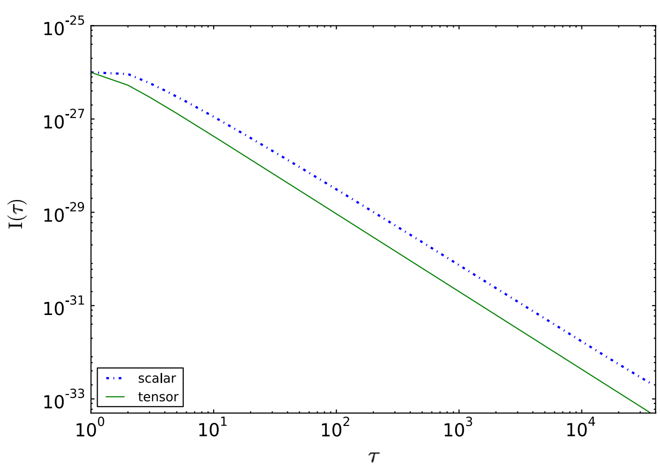

FIG. 1 (color online). Time evolution of the interaction variable in log-log axes. Note that for the interaction with scalars, the decay is slightly slower than for tensors.

The effect of the gravitational perturbations on the interaction is thought to be largest at the point where the modes enter the horizon. This is clearly evident in Figs. 2(a) and 2(b). A couple of features are worth noting from Fig. 2(a). One is that the spectrum for the interaction variable mimics that of gravitational waves. It is also consistent with the fact that gravitational waves start oscillating at horizon crossing. This is to be expected since although for a spatially inhomogeneous magnetic field $\tilde{B}_a$, the product $I_{(\ell)} = \tilde{B}_{(n)}\sigma_{(k)}$ becomes a convolution in Fourier space, $I(k) = \sum_n B(n)\sigma(k-n)$, we have only considered the mode-mode coupling case, $I(k) = B(k)\sigma(k)$.

The power spectra for the case of interaction with scalars are not as interesting. There is no scale dependence on the interaction variable I_a, cf. Eq. (46a). This is because the Laplacian term for scalar perturbations comes from the acceleration vector which is identically zero in the dust case $\mathcal{A} = 0$.

It would be interesting to generalize our treatment to include the case of nonzero pressure. This will lend us to the radiation dominated era where one can incorporate photons in the plasma and can consider collisional effects as was done in Refs. [9,10] for example. One could treat the interesting case of simultaneous generation and amplification of magnetic fields by coalescing these phenomenon.

XI. CONCLUSIONS

We have carried out an analysis of the coupling between gravitational perturbations with electromagnetic fields as a possible means for magnetic field amplification. This carries to completion the work began in Refs. [22,26]. In agreement with the work of Refs. [23,51] we argue that there is no significant amplification resulting from the interaction of magnetic field with gravitational waves. Even with the inclusion of density perturbations, the induced fields may still be orders of magnitude smaller. This justifies the perturbative treatment and our neglect of backreaction.

The induction of electromagnetic fields due to the interaction of a test magnetic field with gravitational waves was studied in Ref. [24] using the weak-field approximation. We included this study here treating the background magnetic field as a first order perturbation and recovered similar results. This shows that there is no fundamental difference between the two approaches, apart from a labeling of spacetimes, which should not affect physical results. We also extended this study by using a proper nonlinear perturbative framework. This framework was applied in Ref. [22], but an erroneous argument there led to the

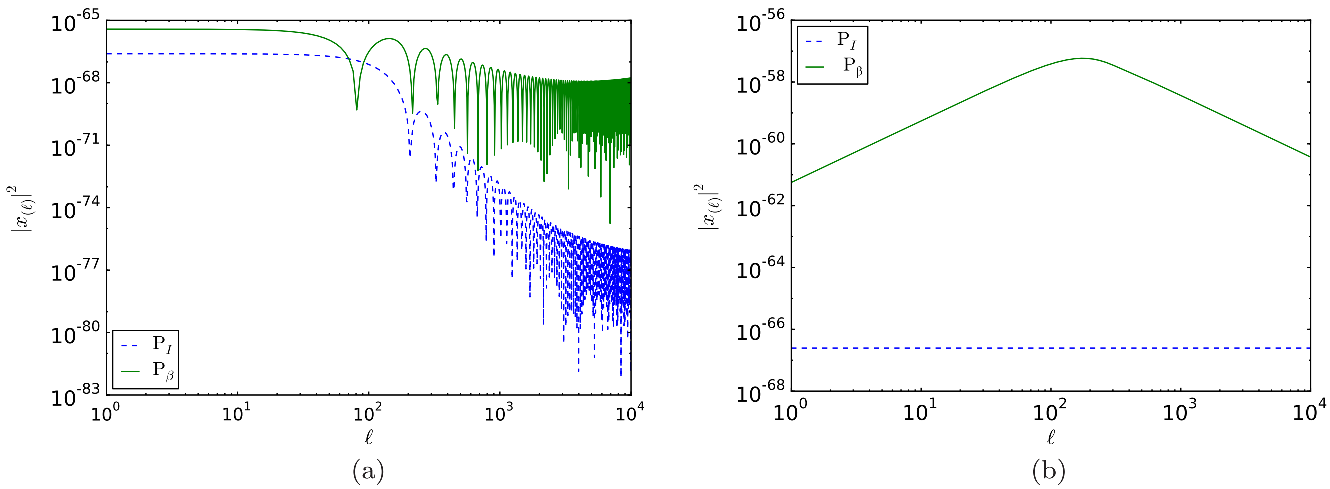

FIG. 2 (color online). Plots of power vs scale (ℓ); we define the power as $P_x = |x(\ell)^2|$. (a) Power spectra of the magnetic field variable $\beta_{(\ell)}$ (green, solid), and the interaction variable $I_{(\ell)}$ (blue, dashed) at redshift $z = 0$ for the tensor case. (b) Power spectra of the magnetic field variable $\beta_{(\ell)}$ (green, solid), and the interaction variable $I_{(\ell)}$ (blue, dashed) at redshift $z = 0$ for the scalar case.

neglect of the rotation of the electric field, thus restricting the study to perfectly conducting environments. This was refuted in Ref. [58]. In fact, upon inspection of (58) one can conclude that even if one initially sets the rotation of the electric field to zero, $\mathcal{E}_i = 0$ there are nonzero terms on the right-hand side of the initial conditions for $\dot{\mathcal{E}}$ that will seed a nonzero $\mathcal{E}$. We also carry to completion the work in Ref. [26] by doing a proper extraction of the scalar and tensor modes and numerical integrations. In terms of the conductivity of the cosmic medium, [24] restricted their study to poor conducting mediums, [22] to perfectly conducting mediums and [26] treated the MHD approximation. We carried our analysis for all three cases. We find that for tensor perturbations, the ideal MHD approximation is just the same as the perfect conductivity assumption of the fluid treatment. For scalar perturbations, we find an additional source term in the induced field (compared with perfectly conducting environments) due to the coupling of the seed field with scalar velocity perturbations. The current term $\mathcal{J}_a$ was neglected at all orders in Ref. [22], in an attempt presumably to uphold the background magnetic field's homogeneity condition $\mathrm{D}_a\tilde{B}_b = 0$. However, this is not necessary since introducing the current term at the nonlinear order does not break the condition $\mathrm{D}_a\tilde{B}_b = 0$. Also, one cannot consistently invoke Ohm's law for poor and perfect conducting environments without a current term. In Ref. [26], an inhomogeneous seed field was assumed thereby requiring a first order current $\mathcal{J}^a = \rho_e v_e^a + \rho_i v_i^a = -e(n_e v_e^a - n_i v_i^a)$ to uphold the condition $\mathrm{D}_a\tilde{B}_b \neq 0$. However, after decoupling (which is the era considered there), Thompson scattering is no longer efficient. Thus electrons and ions are tightly coupled by Coulomb scattering at first order. Their velocity fields are therefore equal as they form a perfectly coupled baryon fluid [59,60]. There can be therefore no currents generated at this order and the condition curl $\tilde{B}_a = \mathcal{J}_a$ will render the seed field homogeneous.

Both Refs. [22,26] integrate β_a to recover the amplified magnetic field, after specifying a frame u^a. While this takes into account the frame dependence of the magnetic field B_a, it invalidates gauge invariance as the recovered B_a remains gauge dependent and takes the same value and form as it would have without the introduction of β_a. This is already pointed out in Ref. [58], See also Refs. [61,62]. We do not integrate β_a but simply note that one can assign a physical meaning to the magnetic field variable β_a by noting that $\beta_a = 0$ describes the background adiabatic decay of the fields. Any deviation from $\beta_a = 0$ would then imply amplification of the background field. Moreover, β_a is a linear combination of terms that source magnetic fields through the induction equation (13). Thus we can estimate the relative importance of each source term through β_a without having to integrate it to recover the gauge-dependent B^a. For example, we see from Fig. 2(a) that the rotation of the electric field dominates at small scales compared to the interaction term. Observations of cosmological magnetic fields are difficult enough as it is, a new cosmological observable would lead to better understanding of studies in magnetic fields. While β_a may not be that quantity, it does arise naturally from Maxwell's equations.

Also, one can readily write our key equations in terms of metric variables by adoption of a suitable tetrad as was done in Ref. [5].

Mechanisms that seek to generate magnetic fields, relying on nonlinear perturbation theory, are attractive for several reasons [63]. Among these is that they can easily blend in with known physics as they become relevant around the recombination era. This makes it possible to quantitatively evaluate the generated fields using CMB constraints. Progress in nonlinear perturbation theory will allow us to investigate these nonlinear effects in a manner that is free of spurious gauge modes [41,64].

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APPENDIX

1. Harmonic splitting

It is standard to decompose the perturbed variables harmonically in Fourier space, separating out the time and space variations [45,65,66]. The idea is to expand the quantities in terms of eigenfunctions of the Laplace-Beltrami operator. To this end, we introduce the helicity basis vectors $\mathbf{e}^{(-)}$, $\mathbf{e}^{(0)}$ and $\mathbf{e}^{(+)}$ defined by

$$e_a^{(\pm)} = -\frac{i}{\sqrt{2}}(e_a^1 \pm i e_a^2), \tag{A1}$$

where $(\mathbf{e}^1, \mathbf{e}^2, \hat{\mathbf{k}})$ form a right-handed orthonormal system with $\mathbf{e}_2 = \hat{\mathbf{k}} \times \mathbf{e}_1$ and we align $\mathbf{e}^0$ with $\hat{\mathbf{k}}$.

Using this basis, the scalar harmonic functions are given by

$$Q^{(0)} = e^{ik_j x^j}. \tag{A2}$$

Scalar-type components of vectors and tensors are expanded in terms of harmonic functions defined from $Q^{(0)}$ as follows:

$$Q_a^{(0)} = -\frac{a}{k}\mathrm{D}_a Q^{(0)} = a i \hat{k}_a e^{ik_j x^j}, \tag{A3}$$

$$Q_{ab}^{(0)} = \frac{a^2}{k^2}\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle} Q^{(0)} = -a^2\left(\hat{k}_a\hat{k}_b - \frac{1}{3}\delta_{ab}\right)e^{ik_j x^j}. \tag{A4}$$

Vector harmonics are given by

$$Q_a^{(\pm)} = e_a^{(\pm)} Q^{(0)}, \tag{A5}$$

$$Q_{ab}^{(\pm)} = -\frac{a}{k}\mathrm{D}_{(a} e_{b)}^{(\pm)} Q^{(0)} = a i \hat{k}_{(a} e_{b)}^{(\pm)} e^{ik_j x^j}. \tag{A6}$$

While tensor harmonics are defined as

$$Q_{ab}^{\pm 2} = \sqrt{\frac{3}{2}} e_a^{(\pm)} e_b^{(\pm)} Q^{(0)}. \tag{A7}$$

2. Maxwell's equations

The Maxwell field tensor F_{ab} decomposes relative to the fundamental observer as

$$F_{ab} = 2u_{[a}E_{b]} + \epsilon_{abc}B^c, \tag{A8}$$

where $E_a = F_{ab}u^b$ and $B_a = \frac{1}{2}\epsilon_{abc}F^{bc}$ are respectively the electric and magnetic field as measured by the fundamental observer moving with 4-velocity u^a. These are 3-vectors on the spacelike hypersurface, $E_a u^a = 0 = B_a u^a$. The Maxwell's equations are given by

$$\nabla_{[a}F_{bc]} = 0 \quad \text{and} \quad \nabla^b F_{ab} = J_a, \tag{A9}$$

where J is the 4-current. These equations can be decomposed covariantly into the following [30,67,68]:

$$\dot{E}_{\langle a\rangle} - \mathrm{curl}\, B_a = -\frac{2}{3}\Theta E_a + \sigma_{ab}E^b + \epsilon_{abc}(\mathcal{A}^b B^c + \omega^b E^c) - \mu_0 \mathcal{J}_{\langle a\rangle}, \tag{A10a}$$

$$\dot{B}_{\langle a\rangle} + \mathrm{curl}\, E_a = -\frac{2}{3}\Theta B_a + \sigma_{ab}B^b + \epsilon_{abc}(\mathcal{A}^b E^c + \omega^b B^c), \tag{A10b}$$

$$0 = D_a E^a - 2\omega_a B^a - \frac{\rho_c}{\epsilon_0}, \tag{A10c}$$

$$0 = D_a B^a + 2\omega_a E^a. \tag{A10d}$$

The EM fields are solenoidal in the absence of gravitational vector perturbations.

3. Commutation relations

$$(\mathrm{D}_a f)^{\cdot}_{\perp} = \mathrm{D}_a \dot{f} - \frac{1}{3}\Theta \mathrm{D}_a f + \dot{f}\mathcal{A}_a, \tag{A11}$$

$$(\mathrm{D}_a V_b)^{\cdot}_{\perp} = \mathrm{D}_a \dot{V}_b - \frac{1}{3}\Theta \mathrm{D}_a V_b - \sigma_a^c \mathrm{D}_c V_b + \epsilon_{bcd} H_a^c V^d + \mathcal{A}_a \dot{V}_b, \tag{A12}$$

$$(\mathrm{curl}\, V_a)^{\cdot}_{\perp} = \mathrm{curl}\, \dot{V}_a - \frac{1}{3}\Theta\, \mathrm{curl}\, V_a - \epsilon_{abc}\sigma^{bd}\mathrm{D}_d V^c + H_{ab}V^b - \frac{1}{3}\Theta \epsilon_{abc} V^b \mathcal{A}^c, \tag{A13}$$

$$\mathrm{curl}\,\mathrm{curl}\, S_{ab} = -\mathrm{D}^2 S_{ab} + \left(\mu + \Lambda - \frac{1}{3}\Theta^2\right)S_{ab} + \frac{3}{2}\mathrm{D}_{\langle a}\mathrm{D}^c S_{b\rangle c}. \tag{A14}$$

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