

**Negative parity baryons in the  $1/N_c$  expansion: The three towers of states revisited**

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We discuss the compatibility between the quark-shell picture and the meson-nucleon scattering picture in large  $N_c$  QCD for mixed symmetric  $\ell = 1$  states previously analyzed by using a simple Hamiltonian including operators up to order  $\mathcal{O}(N_c^0)$  defined in the standard ground state symmetric core + excited quark procedure. Here we introduce a Hamiltonian of order  $\mathcal{O}(N_c^0)$  defined in a new approach where the separation of the system into two parts is not required. Three degenerate sets of states (towers) with the same quantum numbers as in the scattering picture and in the standard procedure are obtained. Thus the compatibility is equally achieved. The eigenvalues of the presently chosen Hamiltonian also have simple analytic expressions, depending linearly on the three dynamical coefficients entering the Hamiltonian. This reinforces the validity of the new approach which had already successfully described excited negative parity baryons in a large energy range.

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**I. INTRODUCTION**

Many of the qualitative and quantitative predictions of the  $1/N_c$  expansion method [1–4], where  $N_c$  is the number of colors, have been proved successful when compared to experimental results of ground state baryons [5–8]. The method is based on the observation that, for  $N_f$  flavors, the ground state baryons display an exact contracted  $SU(2N_f)$  symmetry when  $N_c \rightarrow \infty$ . At large, but finite  $N_c$ , this symmetry is broken by contributions of order of  $1/N_c$ , leading to mass splittings.

Subsequently, efforts have been made to extend the method to excited states. There are two complementary pictures of large  $N_c$  for baryon resonances. One is the so-called *quark-shell picture* where the symmetry is extended to  $SU(2N_f) \times O(3)$ , which allows us to classify baryons in excitation bands  $N$  [9], like in the quark model where  $N_c = 3$  [10]. The other is the *resonance or scattering picture* derived from symmetry features of chiral soliton models. The role of large  $N_c$  QCD is to relate the scattering amplitudes in various channels with  $K$ -amplitudes, where  $K$  is the grand spin  $\vec{K} = \vec{I} + \vec{J}$ . These are linear relations in the meson-nucleon scattering amplitudes from which one can infer some patterns of degeneracy among resonances.

Naturally, there has been concern about the compatibility of the two pictures. Simultaneously Cohen and Lebed [11] and Pirjol and Schat [12] studied the  $N = 1$  band which represents the lowest  $[70, 1^-]$  multiplet and found the same three sets of degenerate states as in the *resonance or scattering picture*. In both studies there

were three leading order operators in the mass formula,  $c_1 \mathbb{1}$  of order  $\mathcal{O}(N_c)$  together with  $\ell \cdot s$  and  $\frac{1}{N_c} \ell^{(2)} \cdot g \cdot G_c$  having matrix elements of order  $\mathcal{O}(N_c^0)$ . In particular, Pirjol and Schat showed that the three sets of degenerate states correspond to irreducible representations of the contracted  $SU(4)_c$  symmetry, the three sets being called three towers of states. Moreover, to the three leading order operators in the mass formula they added  $1/N_c$  corrections and reanalyzed the mass spectrum of the lowest negative parity nonstrange baryons. They found ambiguities in the identification of physical states where  $N_c = 3$  with the degenerate large  $N_c$  tower states. Actually, in the  $SU(4)$  case the degeneracy of sets of states corresponding to irreducible representations of the contracted  $SU(4)_c$  symmetry was first discussed by Pirjol and Yan in Ref. [13].

Later on, the compatibility between the two pictures was discussed on a general basis again by Cohen and Lebed [14]. The compatibility was claimed for completely symmetric ( $S$ ), mixed symmetric ( $MS$ ) and completely antisymmetric ( $A$ ) states of  $N_c$  quarks having angular momentum up to  $\ell = 3$ . In Ref. [15] we gave an explicit proof of the degeneracy of mass eigenvalues in the quark-shell picture for  $\ell = 3$ . We thus supported the idea of full compatibility of Ref. [14] for higher parts of the spectrum. This compatibility means that any complete spin-flavor multiplet within one picture fills the quantum numbers of the other picture. In addition, we could prove that the quark-shell picture is richer in information, by making a clear distinction between degenerate sets of states of different values of the angular momentum but associated to the same grand spin  $K$ . For example, one can associate a common  $K = 2$  to both  $\ell = 1$  and  $\ell = 3$ . A similar situation appears for

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every  $K$  value associated to two distinct values of  $\ell$  satisfying the  $\delta(K\ell 1)$  rule [14].

It is important to mention that the above studies were based on the procedure to consider an excited state as a single quark excitation about a ground state symmetric core [16,17]. In fact the operators mentioned above follow the notation of Refs. [16,17], namely lower case indicates operators acting on the excited quark and the subscript  $c$  is attached to those acting on the symmetric core.

The symmetric core + excited quark was originally proposed [16] as an extension of the ground state treatment to excited states and was inspired by the Hartree picture. In this way, in the flavor-spin space, the problem was reduced to the knowledge of matrix elements of the  $SU(2N_f)$  generators between symmetric states, already known from the ground state studies. Accordingly, the wave function was approximately given by the coupling of an excited quark to a ground state core of  $N_c - 1$  quarks, without performing antisymmetrisation. Subsequently the symmetric core + an excited quark approach was supported by Pirjol and Schat [18] within a general large  $N_c$  constituent quark model Hamiltonian starting from an exact wave function. A practical problem is that the number of operators entering the mass formula is exceedingly large. For example, for the  $[70, 1^-]$  multiplet in  $SU(4)$  there are 12 linearly independent operators in powers of  $1/N_c$  included, while in the  $N = 1$  band there are seven experimentally known resonances + two mixing angles. It is therefore difficult, if not impossible, to find out the most dominant operators. In addition, the symmetric core always has equal spin and isospin, therefore some information is lost regarding the baryons structure.

As an alternative, in Ref. [19] we have proposed a method where all identical quarks are treated on the same footing and we have an exact wave function in the orbital-flavor-spin space. The procedure has been successfully applied to the  $N = 1$  band [19–21] and recently to the  $N = 3$  band [22], where data are very scarce. We found out that the most dominant operators of order  $1/N_c$  were both the spin and flavor, the latter being neglected in all studies based on the symmetric core + an excited quark approach. They are  $\frac{1}{N_c} S^i S^i$  and  $\frac{1}{N_c} [T^a T^a - \frac{1}{12} N_c (N_c + 6)]$ , respectively, where the latter is compatible both with  $SU(4)$  and  $SU(6)$  [20]. The generators  $S^i$  and  $T^a$  act on the whole system.

It is precisely in our approach that we wish to analyze the compatibility of the *quark-shell picture* and of the *resonance or scattering picture*. It will be shown that the present results are as simple as those of the symmetric core + an excited quark approach. It is remarkable that in the *quark-shell picture* described below the Hamiltonian eigenvalues are linear analytic functions of the dynamical coefficients  $c_i$  entering the Hamiltonian, like in Refs. [11,12]. In this way one can easily identify the three sets of degenerate states which are identical in the

quantum numbers with those of the *resonance or scattering picture* and those of Refs. [11,12]. The compatibility is therefore confirmed and this gives strong support to our procedure.

## II. A SIMPLIFIED MASS OPERATOR

In the quark-shell picture the leading-order Hamiltonian corresponding to the procedure of Refs. [19–21] has the following form

$$H = c_1 O_1 + c_2 O_2 + c_6 O_6, \quad (1)$$

where the operators presently under consideration are

$$O_1 = N_c \mathbb{1}, \quad O_2 = \ell \cdot s, \quad O_6 = \frac{15}{N_c} L^{(2)ij} G^{ia} G^{ja}, \quad (2)$$

in the notation of Ref. [21] for the operators  $O_i$ . The first two terms are identical to those of Refs. [11] or [12]. The matrix elements of the first term are  $N_c$  on all baryons and the second term is a one-body operator defined in the spirit of the Hartree picture [2] and its matrix elements are of order  $\mathcal{O}(N_c^0)$ . The third term is new and consistent with our procedure described in the introduction. It is a two-body operator but has matrix elements of order  $\mathcal{O}(N_c^0)$ . It contains the tensor  $L^{(2)ij}$  of  $SO(3)$  defined as

$$L^{(2)ij} = \frac{1}{2} \{L^i, L^j\} - \frac{1}{3} \delta_{i,-j} \vec{L} \cdot \vec{L}, \quad (3)$$

which acts on the orbital wave function of the whole system of  $N_c$  quarks and is normalized as in Ref. [9]. Note that when the angular momentum acts on the whole system we use capital  $L_i$  to distinguish it from  $\ell_i$ , in the spin-orbit operator, which acts on a single quark. In our approach  $O_2$  and  $O_6$  are the only operators of order  $\mathcal{O}(N_c^0)$ . The neglect of  $1/N_c$  corrections in the  $1/N_c$  expansion makes sense for the comparison with the scattering picture in the large  $N_c$  limit, as already discussed in Ref. [11].

## III. THE BASIS STATES

The Hamiltonian (1) is diagonalized in the bases described below. We remind the reader that each orbital-spin-flavor state must be symmetric, to be combined with an antisymmetric color state in order to satisfy the Pauli principle. For an excited state with  $\ell = 1$ , the partition must be  $[N_c - 1, 1]$  both in the orbital and spin-flavor spaces. By using inner products of the permutation group  $S_{N_c}$  we can construct spin-flavor states with the desired symmetry from spin  $S$  and flavor  $F$  states. In this section we describe the spin and flavor spaces by their permutation symmetry, i.e., by a partition  $[f] = [f_1, f_2]$  where  $f_1$  and  $f_2$  represent the number of boxes in the first and second row of a Young tableau, respectively. The relation to  $SU(3)$  flavor space notation is  $f_1 = \lambda + \mu$  and  $f_2 = \mu$  (see Appendix A).

### A. The nucleon case

We have the following  $[N_c - 1, 1]$  spin-flavor states which form a symmetric state with the orbital  $\ell = 1$  state of the same partition. These are

$$[N_c - 1, 1]_{\text{SF}} = \left[ \frac{N_c + 1}{2}, \frac{N_c - 1}{2} \right]_S \times \left[ \frac{N_c + 1}{2}, \frac{N_c - 1}{2} \right]_F, \quad (4)$$

where  $N_c \geq 3$  and  $S = 1/2, J = 3/2$ ,

$$[N_c - 1, 1]_{\text{SF}} = \left[ \frac{N_c + 3}{2}, \frac{N_c - 3}{2} \right]_S \times \left[ \frac{N_c + 1}{2}, \frac{N_c - 1}{2} \right]_F, \quad (5)$$

where  $N_c \geq 3$  and  $S = 3/2, J = 1/2, 3/2, 5/2$ . As one can easily see, they give rise to matrices associated to a given  $J$  which are either  $2 \times 2$  or  $1 \times 1$ . For  $N_c = 3$  the states (4) and (5) correspond to  ${}^2_8$  and  ${}^4_8$  multiplets, respectively. The case  $N_c = 7$  is illustrated with Young diagrams in Appendix A.

### B. The $\Delta$ case

In a similar way we use inner product of spin and flavor states to obtain basis states in the spin-flavor space compatible with the orbital state  $[N_c - 1, 1]$  with  $\ell = 1$ . For the  $\Delta$  case we have

$$[N_c - 1, 1]_{\text{SF}} = \left[ \frac{N_c + 1}{2}, \frac{N_c - 1}{2} \right]_S \times \left[ \frac{N_c + 3}{2}, \frac{N_c - 3}{2} \right]_F, \quad (6)$$

where  $N_c \geq 3$  and  $S = 1/2, J = 3/2$ , denoted in the following as  ${}^2_{10}_J$ ,

$$[N_c - 1, 1]_{\text{SF}} = \left[ \frac{N_c + 3}{2}, \frac{N_c - 3}{2} \right]_S \times \left[ \frac{N_c + 3}{2}, \frac{N_c - 3}{2} \right]_F, \quad (7)$$

where  $N_c \geq 5$  and  $S = 3/2, J = 1/2, 3/2, 5/2$ , denoted in the following as  ${}^4_{10}_J$ ,

$$[N_c - 1, 1]_{\text{SF}} = \left[ \frac{N_c + 5}{2}, \frac{N_c - 5}{2} \right]_S \times \left[ \frac{N_c + 3}{2}, \frac{N_c - 3}{2} \right]_F, \quad (8)$$

where  $N_c \geq 7$  and  $S = 5/2$  and  $J = 1/2, 3/2, 5/2, 7/2$ , denoted in the following as  ${}^6_{10}_J$ .

For  $N_c = 3$  the first state belongs to the  ${}^2_{10}$  multiplet. The other two types of states do not appear in the real

world with  $N_c = 3$ . As above, it is easy to find out the size of the matrix of a fixed  $J$ . For example, for  $J = 3/2$  we have a  $3 \times 3$  matrix defined in the space of all  $\Delta$  states.

## IV. RESULTS AND DISCUSSION

The analytic expressions of the resulting matrix elements of the operators contained in the Hamiltonian (1) are given in Tables I and II, as a function of  $N_c$ . Details of the calculation of the matrix elements are given in Appendix A.

To obtain the matrices to be diagonalized we have to take the limit  $N_c \rightarrow \infty$  in the matrix elements of  $O_2$  and  $O_6$  given in Tables I and II. As an example we present the largest possible matrix, corresponding to  $\Delta_{3/2}$  states, which is

TABLE I. Diagonal matrix elements of  $O_1, O_2$  and  $O_6$  for all states belonging to the  $[70, 1^-]$  multiplet.

	$O_1$	$O_2$	$O_6$
${}^2_8 \frac{1}{2}$	$N_c$	$-\frac{2N_c-3}{3N_c}$	0
${}^4_8 \frac{1}{2}$	$N_c$	$-\frac{5}{6}$	$-\frac{25(N_c-1)}{8N_c}$
${}^2_8 \frac{3}{2}$	$N_c$	$\frac{2N_c-3}{6N_c}$	0
${}^4_8 \frac{3}{2}$	$N_c$	$-\frac{1}{3}$	$\frac{5(N_c-1)}{2N_c}$
${}^4_8 \frac{5}{2}$	$N_c$	$\frac{1}{2}$	$-\frac{5(N_c-1)}{8N_c}$
${}^2_{10} \frac{1}{2}$	$N_c$	$\frac{1}{3}$	0
${}^2_{10} \frac{3}{2}$	$N_c$	$-\frac{1}{6}$	0
${}^4_{10} \frac{1}{2}$	$N_c$	$-\frac{1}{6}$	$\frac{5(N_c+5)}{2N_c}$
${}^4_{10} \frac{3}{2}$	$N_c$	$-\frac{2}{15}$	$-\frac{2(N_c+5)}{N_c}$
${}^4_{10} \frac{5}{2}$	$N_c$	$\frac{1}{5}$	$\frac{N_c+5}{2N_c}$
${}^6_{10} \frac{3}{2}$	$N_c$	$-\frac{7}{10}$	$-\frac{7(3N_c-25)}{12N_c}$
${}^6_{10} \frac{5}{2}$	$N_c$	$-\frac{1}{5}$	$\frac{2(3N_c-25)}{3N_c}$
${}^6_{10} \frac{7}{2}$	$N_c$	$\frac{1}{2}$	$-\frac{5(3N_c-25)}{24N_c}$

TABLE II. Off-diagonal matrix elements of  $O_1, O_2$  and  $O_6$  for all states belonging to the  $[70, 1^-]$  multiplet.

	$O_1$	$O_2$	$O_6$
${}^2_8 \frac{1}{2} - {}^4_8 \frac{1}{2}$	0	$-\frac{1}{3} \sqrt{\frac{N_c+3}{2N_c}}$	$-\frac{25}{4} \sqrt{\frac{N_c+3}{2N_c}}$
${}^2_8 \frac{3}{2} - {}^4_8 \frac{3}{2}$	0	$-\frac{1}{6} \sqrt{\frac{5(N_c+3)}{N_c}}$	$\frac{5}{8} \sqrt{\frac{5(N_c+3)}{N_c}}$
${}^2_{10} \frac{1}{2} - {}^4_{10} \frac{1}{2}$	0	$\frac{1}{6} \sqrt{\frac{5(N_c-3)}{N_c}}$	$-\frac{5}{8} \sqrt{\frac{5(N_c-3)}{N_c}}$
${}^2_{10} \frac{3}{2} - {}^4_{10} \frac{3}{2}$	0	$\frac{5}{6} \sqrt{\frac{N_c-3}{2N_c}}$	$\frac{5}{8} \sqrt{\frac{N_c-3}{2N_c}}$
${}^2_{10} \frac{5}{2} - {}^6_{10} \frac{3}{2}$	0	0	$\frac{5}{8N_c} \sqrt{3(N_c-3)(N_c+5)}$
${}^4_{10} \frac{3}{2} - {}^6_{10} \frac{3}{2}$	0	$-\frac{3\sqrt{6}}{20}$	$-\frac{21}{16} \sqrt{\frac{6(N_c+5)}{N_c}}$
${}^4_{10} \frac{5}{2} - {}^6_{10} \frac{5}{2}$	0	$-\frac{\sqrt{21}}{10}$	$\frac{3}{8} \sqrt{\frac{21(N_c+5)}{N_c}}$

$$M_{\Delta_{3/2}}^{\ell=1} = \begin{pmatrix} c_1 N_c - \frac{1}{6} c_2 & \frac{5}{\sqrt{2}} \left( \frac{c_2}{6} + \frac{c_6}{8} \right) & \frac{15\sqrt{3}}{8} c_6 \\ \frac{5}{\sqrt{2}} \left( \frac{c_2}{6} + \frac{c_6}{8} \right) & c_1 N_c - \frac{2}{15} c_2 - 2c_6 & -\frac{3\sqrt{6}}{20} c_2 - \frac{21\sqrt{6}}{16} c_6 \\ \frac{15\sqrt{3}}{8} c_6 & -\frac{3\sqrt{6}}{20} c_2 - \frac{21\sqrt{6}}{16} c_6 & c_1 N_c - \frac{7}{10} c_2 - \frac{7}{4} c_6 \end{pmatrix}, \quad (9)$$

The eigenvalues of this matrix are

$$m'_0 = c_1 N_c - c_2 - \frac{25}{4} c_6, \quad (10)$$

$$m'_1 = c_1 N_c - \frac{1}{2} c_2 + \frac{25}{8} c_6, \quad (11)$$

$$m'_2 = c_1 N_c + \frac{1}{2} c_2 - \frac{5}{8} c_6. \quad (12)$$

Another useful example is the matrix corresponding to  $N_{3/2}$  states. This is

$$M_{N_{3/2}}^{\ell=1} = \begin{pmatrix} c_1 N_c + \frac{1}{3} c_2 & -\frac{\sqrt{5}}{6} c_2 + \frac{5\sqrt{5}}{8} c_6 \\ -\frac{\sqrt{5}}{6} c_2 + \frac{5\sqrt{5}}{8} c_6 & c_1 N_c - \frac{1}{3} c_2 + \frac{5}{2} c_6 \end{pmatrix}, \quad (13)$$

having as eigenvalues  $m'_1$  and  $m'_2$  from above. The matrix corresponding to  $N_{1/2}$  states is shown in the Appendix B.

The other matrices follow straightforwardly from Tables I and II. By diagonalizing all matrices we found that the eigenvalues (10)–(12) are the only possible ones for all  $N_J$  and  $\Delta_J$  presented above. Accordingly the following sets of degenerate negative parity multiplets were found for  $\ell = 1$  orbital excitations

$$N_{1/2}, \Delta_{3/2}, \quad (m'_0), \quad (14)$$

$$N_{1/2}, \Delta_{1/2}, N_{3/2}, \Delta_{3/2}, \Delta_{5/2}, \quad (m'_1), \quad (15)$$

$$\Delta_{1/2}, N_{3/2}, \Delta_{3/2}, N_{5/2}, \Delta_{5/2}, \Delta_{7/2}, \quad (m'_2), \quad (16)$$

where, on the right side we indicate the mass of each degenerate set. These degenerate multiplets are identical to those found in Refs. [11,12]. The masses  $m'_i$  of Eqs. (10)–(12) are naturally different from  $m_i$  of the above references because the Hamiltonian is different in structure, it contains different dynamical coefficients but has similar large  $N_c$  properties. Simple forms as those of Eqs. (10)–(12) hold only for a Hamiltonian of order  $\mathcal{O}(N_c^0)$ . Other choices within the procedure of Refs. [19–21] would lead to the inclusion of  $1/N_c$  corrections which will necessarily imply numerical calculations.

Another remarkable aspect of the present study concerns the mixing angles. For those sectors for which large  $N_c$  and  $N_c = 3$  have the same number of quark model states, which, in particular is the nucleon case with  $J = 1/2$  and  $J = 3/2$ , see Eqs. (4) and (5), the mixing angles are identical to those obtained by Cohen and Lebed [11] or Pirjol and Schat [12].

This means  $\tan\theta_{N_{1/2}} = \sqrt{2}$  and  $\tan\theta_{N_{3/2}} = -\frac{1}{\sqrt{5}}$  and amounts to  $\theta_{N_{1/2}} = 0.96$  rad and  $\theta_{N_{3/2}} = 2.72$  rad, respectively (for details see Appendix B). The mixing angles determined from fits to  $N^*$  strong decays and photoproduction data are  $\theta_{N_{1/2}} = 0.39 \pm 0.11$  rad and  $\theta_{N_{3/2}} = 2.82 \pm 0.11$  rad, respectively [23,24]. This suggests that for  $N_{3/2}$  states the agreement with the phenomenological value is nearly achieved at order  $\mathcal{O}(N_c^0)$  but for  $N_{1/2}$  states corrections of order  $\mathcal{O}(N_c^{-1})$  are necessary. In Ref. [11] the comparison has been made with the decay data of Ref. [25], from where it has been extracted  $\theta_{N_{1/2}} = 0.56$  rad.

For the other mixing angles resulting from  $2 \times 2$  matrices we find  $\tan\theta_{\Delta_{1/2}} = \sqrt{1/5}$  and  $\tan\theta_{\Delta_{5/2}} = -\sqrt{3/7}$ . These give the same absolute values for the mixing angle as those of Cohen and Lebed [11] but of opposite signs. It may be a matter of phase convention.

We also have to compare these results with those of the meson-nucleon scattering picture, where linear relations between matrix elements  $S_{LL'RR'IJ}^\pi$  and  $S_{LRJ}^\eta$  of  $\pi$  and  $\eta$  scattering off a ground state baryon in terms of  $K$ -amplitudes were derived. They are given by the following equations

$$S_{LL'RR'IJ}^\pi = \sum_K (-1)^{R'-R} \sqrt{(2R+1)(2R'+1)(2K+1)} \\ \times \begin{Bmatrix} K & I & J \\ R' & L' & 1 \end{Bmatrix} \begin{Bmatrix} K & I & J \\ R & L & 1 \end{Bmatrix} S_{KLL'}^\pi \quad (17)$$

and

$$S_{LRJ}^\eta = \sum_K \delta_{KL} \delta(LRJ) s_K^\eta, \quad (18)$$

in terms of the reduced amplitudes  $s_{KLL'}^\pi$  and  $s_K^\eta$  respectively. These equations were first derived in the context of the chiral soliton model [26–29] where the mean-field breaks the rotational and isospin symmetries, so that  $J$  and  $I$  are not conserved but the *grand spin*  $K$  is conserved and excitations can be labeled by  $K$ . These relations are exact in large  $N_c$  QCD and are independent of any model assumption.

The explicit form of these equations can be found in Table I of Ref. [11]. That table infers a pattern of degeneracy identical to that presented in Eqs. (14)–(16). The contributing amplitudes are  $s_0^\eta$  for the resonances listed in Eq. (14),  $(s_{100}^\eta, s_{122}^\eta)$  for those of Eq. (15) and  $(s_{222}^\eta, s_2^\eta)$  for those of Eq. (16) (for details see Ref. [11]). In the resonance picture the degenerate towers of states (14)–(16)

correspond to the grand spin  $K = 0, 1$  and  $2$ , respectively. Therefore the triangular rule proposed in Ref. [14] is satisfied.

## V. CONCLUSIONS

The compatibility between the quark-shell picture of the  $1/N_c$  expansion method and the meson-nucleon resonance picture has been previously analyzed [11,12,14] by starting from a Hamiltonian containing operators of order  $\mathcal{O}(N_c^0)$  defined in the symmetric core + an excited quark method [16,17] and full compatibility has been found. Here we have used an alternative description of the mixed symmetric states where the separation of  $SU(6)$  generators into two terms, one acting on the core and the other on the excited quark, is avoided and the orbital-flavor-spin wave function is exactly symmetric under the permutation group [19,21], as it should be for identical quarks. Interestingly we found an identical pattern of degeneracy in the quantum numbers with that obtained from the symmetric core + an excited quark method, thus have proven that the full compatibility holds in this procedure as well. This supports once more the method we have proposed in Refs. [19,21]

where a good fit to the experiment has been found for the  $N = 1$  band, and more recently for mixed symmetric multiplets in the  $N = 3$  band [22].

The importance of the compatibility of the two pictures has been clearly pointed out in Ref. [11] where it was also stressed that it does not justify all aspects of the *quark-shell picture*, in particular the dynamical details, but it justifies those aspects of the model that essentially follow from the contracted  $SU(2N_f)$  symmetry.

## APPENDIX A

For the calculation of the matrix elements of the spin-orbit operator  $O_2 = \ell \cdot s$  one should refer to the Appendix of Ref. [15] where the notations of Ref. [30] were used for the isoscalar factors of the permutation group. In Ref. [15] the matrix elements of  $O_2$  for orbital excitation with  $\ell = 3$  have been calculated.

For the reader's benefit here we give some useful details for the calculation of the matrix elements of the operator  $O_6 = 15/N_c L^2 \cdot G \cdot G$ . The expression of the matrix elements of the operator  $L^2 \cdot G \cdot G$  as obtained in Ref. [21] is

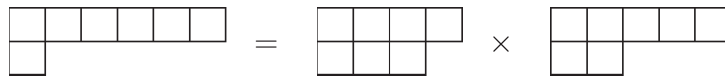
$$\begin{aligned}
 & \langle (\lambda' \mu') Y' I' I'_3; \ell' S' J J_3 | (-1)^{i+j+a} L^{(2)ij} G^{-ia} G^{-j,-a} | (\lambda \mu) Y I I_3; \ell S J J_3 \rangle \\
 &= \delta_{\ell' \ell} \delta_{\lambda \lambda'} \delta_{\mu \mu'} \delta_{Y' Y} \delta_{I' I} \delta_{I'_3 I_3} (-1)^{J+\ell-S} \frac{1}{2} C_{[f]}^{SU(6)} \sqrt{\frac{5\ell(\ell+1)(2\ell-1)(2\ell+1)(2\ell+3)}{6}} \\
 & \times \sqrt{(2S+1)(2S'+1)} \left\{ \begin{matrix} \ell & \ell & 2 \\ S & S' & J \end{matrix} \right\} \sum_{S''} (-1)^{(S-S'')} \left\{ \begin{matrix} 1 & 1 & 2 \\ S & S' & S'' \end{matrix} \right\} \\
 & \times \sum_{\rho, \lambda'', \mu''} \left( \begin{matrix} [f] & [21^4] & [f] \\ (\lambda'' \mu'') S'' & (11)1 & (\lambda \mu) S \end{matrix} \right)_{\rho} \left( \begin{matrix} [f] & [21^4] & [f] \\ (\lambda'' \mu'') S'' & (11)1 & (\lambda \mu) S' \end{matrix} \right)_{\rho}, \tag{A1}
 \end{aligned}$$

where the partition of mixed symmetric states under consideration is  $[f] = [N_c - 1, 1]$  and

$$C_{[f]}^{SU(6)} = \frac{N_c(5N_c + 18)}{12}, \tag{A2}$$

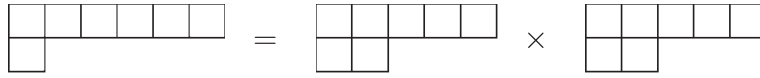
is the Casimir operator of  $SU(6)$  for the partition  $[f]$ . The two factors appearing in the sum over  $\rho$ ,  $\lambda''$  and  $\mu''$  are isoscalar factors of  $SU(6)$  to be found in Ref. [21] as well. As

an example we consider the case of  $\Delta_{3/2}$  states defined by Eqs. (6)–(8). For states of type (6)–(8) the isoscalar factors should be taken from Table VII, V and VI, respectively of Ref. [21]. These tables are very general and the meaning is as follows. Table VII corresponds to spin-flavor (FS) states where the Young diagram associated to spin has  $\lambda - 2$  boxes in the first row if the Young diagram associated to flavor has  $\lambda$  boxes, as shown below for  $N_c = 7$ ,  $(\lambda, \mu) = (3, 2)$



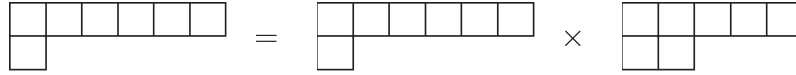
The multiplet  ${}^2 10$  from the real world ( $N_c = 3$ ) is a particular case. The number of extra boxes in the first row as compared to the second row of the Young diagram corresponding to spin is equal to one, which gives  $S = 1/2$ , and three boxes in the Young diagram corresponding to flavor, which gives  $(\lambda, \mu) = (3, 0)$ , describing a  $\Delta$  state.

Table VI corresponds to FS states where the number of extra boxes in the first row, both in the spin and flavor diagrams is the same, namely three, as shown below for  $N_c = 7$



That is why we denoted these spurious states ( $N_c \geq 5$ ) by  ${}^4 10_J$ . In the real world,  $N_c = 3$ , this case is realized for  ${}^2 8$  states with one extra box in both spin and flavor spaces, which means  $S = 1/2$  and  $(\lambda, \mu) = (1, 1)$ .

Table VI corresponds to FS states where the number of extra boxes is two units larger in the spin space than in the flavor space, see below for  $N_c = 7$



Then for  $\Delta$  states one must have  $S = 5/2$  which implies the notation  ${}^6 10_J$  for these states. The real case,  $N_c = 3$ , is the  ${}^4 8$  multiplet.

All real cases are indicated in the headings of Tables VII, V and VI, respectively of Ref. [21].

### APPENDIX B

The mixing angles extracted from electromagnetic and strong decays are defined as

$$\begin{aligned} |N_J(\text{upper})\rangle &= \cos\theta_J |{}^4 N_J\rangle + \sin\theta_J |{}^2 N_J\rangle, \\ |N_J(\text{lower})\rangle &= -\sin\theta_J |{}^4 N_J\rangle + \cos\theta_J |{}^2 N_J\rangle, \end{aligned} \quad (\text{B1})$$

where  $|{}^4 N_J\rangle$  and  $|{}^2 N_J\rangle$  are the initial states in the quark model basis and *upper* and *lower* are the physical states with upper and lower energies. Here the mixing is due to the spin-orbit and to the  $O_6$  operator because they both have off-diagonal matrix elements. To obtain the appropriate matrix, we remind the reader that in Tables I and II we have to take  $N_c \rightarrow \infty$  in the matrix elements of

$O_2$  and  $O_6$ . As an example we show here the matrix of the  $N_{1/2}$  states

$$M_{N_{1/2}}^{\ell=1} = \begin{pmatrix} c_1 N_c - \frac{2}{3} c_2 & -\frac{1}{3\sqrt{2}} c_2 - \frac{25}{4\sqrt{2}} c_6 \\ -\frac{1}{3\sqrt{2}} c_2 - \frac{25}{4\sqrt{2}} c_6 & c_1 N_c - \frac{5}{6} c_2 - \frac{25}{8} c_6 \end{pmatrix}. \quad (\text{B2})$$

This suggests that the general form of a  $2 \times 2$  matrix to be diagonalized is

$$M_{N_J}^{\ell} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}, \quad (\text{B3})$$

so the mixing angle turns out to be

$$\tan 2\theta = -\frac{2B}{C-A}. \quad (\text{B4})$$

Replacing  $A$ ,  $B$  and  $C$  by their values from Eq. (B2), we obtain  $\tan 2\theta_{N_{1/2}} = -2\sqrt{2}$ . It then follows that  $\tan\theta_{N_{1/2}} = \sqrt{2}$ . In all cases it happens that the factor in  $B$  and  $C - A$  containing the combination of  $c_2$  and  $c_6$  simplifies so that one gets simple values for  $\tan\theta$  as indicated in Sec. IV.

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