Cooling study of Dirac sheets in $SU(3)$ lattice gauge theory below T_c

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Using a standard cooling method for $SU(3)$ lattice gauge fields, constant Abelian magnetic field configurations are extracted after dyon-antidyon constituents forming metastable $Q = 0$ configurations have annihilated. These so-called Dirac sheets, standard and nonstandard ones, corresponding to the two $U(1)$ subgroups of the $SU(3)$ group, have been found to be stable if emerging from the confined phase, close to the deconfinement phase transition, with sufficiently nontrivial Polyakov loop values. On a finite lattice we find a nice agreement of the numerical observations with the analytic predictions concerning the stability of Dirac sheets depending on the value of the Polyakov loop.

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I. INTRODUCTION

In lattice gauge theories the cooling method is used to remove short distance fluctuations in order to search for (approximate) classical solutions of the Euclidean field equations [1–4]. We consider this technique as a device [5] (like smearing or filtering based on low-lying modes of the Dirac operator) that may help to identify topological excitations generically present in the sample configurations representing the zero-temperature (or thermal) ensemble of gauge fields [6–8].

Cooling studies of nonzero-temperature $SU(2)$ lattice fields [5] have identified as topological excitations both calorons with nontrivial holonomy [9–11] or dyon-antidyon pairs which finally annihilate. Sometimes this annihilation process provides a constant Abelian magnetic field called *Dirac sheet* (DS), which turns out to be either stable or unstable under further cooling [12]. The stability is strongly correlated with the spatial average value of the Polyakov loop (the holonomy) in the given stage of cooling. In Ref. [13] an explanation for this observation was presented.

Some time ago we started cooling studies of $SU(3)$ gluodynamics, applying the Cabibbo-Marinari procedure in the cooling mode for the standard Wilson action [14]. On plateaus characterized by values of the action within the range 0.5–1.5 times the one-instanton action $S_{\rm inst}$ the emerging topological objects turned out to be either calorons or anticalorons (dissociated or not dissociated into their three respective dyon or antidyon constituents) or one or two dyon-antidyon pairs. Sometimes [similar to the $SU(2)$ case] the annihilation process of a dyon-antidyon pair leaves behind a constant Abelian magnetic field. In the $SU(3)$ case the structure of such Dirac sheets is somewhat richer than in the $SU(2)$ case. Below we will describe their analytic construction following a seminal paper by Gerard 't Hooft [15]. We will expand the concept of marginal stability [16–18] to the $SU(3)$ case. We shall find agreement between the analytically worked-out preconditions—in terms of the holonomy—for stability of the Dirac sheets in a finite volume on one hand and the numerical observations for Monte Carlo generated—and subsequently cooled—lattice gauge fields.

II. DIRAC SHEET SOLUTIONS

In lattice gauge theories usually periodic boundary conditions are applied for the gauge fields (by default, if no special needs suggest something else). Thus, the DS configurations that can be obtained by the cooling procedure are periodic as well. The simplest way, however, to present analytic solutions with a constant color-magnetic field on a hypertorus uses twisted boundary conditions [15]. In this case most of the structure of the solutions is absorbed into twists (the gauge transformations that the gauge fields acquire over the periods on a hypertorus). They look rather complicated and are even non-Abelian while the gauge fields themselves are rather simple. To have periodic solutions we should make clear that the twists can be removed by appropriate gauge transformations. The necessary condition for this is the commutativity of twists in different directions. Below we will apply this condition to find those solutions that allow to be made periodic.

Discussing the special self-dual solutions, 't Hooft was considering the general $SU(N)$ case. The gauge field $A_\mu(x)$ and the field strength $F_{\mu\nu}(x)$ are strictly Abelian while the twists are non-Abelian. The gauge field $A_\mu(x)$ is proportional to the diagonal traceless matrix $\omega = 2\pi\,\mathrm{diag}(l, \ldots, l, -k, \ldots, -k)$ with positive integers l and k such that $l + k = N$,

$$A_\mu(x) = \omega \sum_\nu \alpha_{\mu\nu} x_\nu / L_\mu L_\nu,$$
$$F_{\mu\nu}(x) = -\omega(\alpha_{\mu\nu} - \alpha_{\nu\mu})/L_\mu L_\nu, \tag{1}$$

where L_μ, $\mu = 1, \ldots, 4$ are the linear extensions of the hypertorus,

$$\alpha_{\mu\nu} - \alpha_{\nu\mu} = n^{(2)}_{\mu\nu}/Nl - n^{(1)}_{\mu\nu}/Nk. \tag{2}$$

The integers $n^{(2)}_{\mu\nu}$ and $n^{(1)}_{\mu\nu}$ summed to $n_{\mu\nu} = n^{(1)}_{\mu\nu} + n^{(2)}_{\mu\nu}$ define the so-called twist tensor $n_{\mu\nu}$. For $n_{\mu\nu} = 0$ (mod. N) the twists are commuting and can be removed by appropriate gauge transformations such that gauge fields become periodic.

For $n^{(2)}_{12} = -n^{(1)}_{12} = 1$ (with other components equal to zero) $n_{\mu\nu} = 0$, $\alpha_{12} - \alpha_{21} = 1/kl$ we get a constant magnetic field in the third direction $B_3 = F_{12}$. The action of this field on the hypertorus with $L_1 = L_2 = L_3 = L_s$ and $L_4 = L_t$ is equal to

$$S_{\rm DS} = 1/2g^2(B^a_3)^2 V_4 = 1/g^2 \,{\rm Tr}(B_3)^2 V_4$$
$$= 8\pi^2/g^2 \times N/2kl \times L_t/L_s. \tag{3}$$

Thus, for $SU(2)$ $S_{\rm DS} = S_{\rm inst} L_t/L_s$, for $SU(3)$ $S_{\rm DS} = 3/4 S_{\rm inst} L_t/L_s$, where the instanton action is $S_{\rm inst} = 8\pi^2/g^2$. In the $SU(2)$ case the magnetic field B_3 is equal to $B_3 = 2\pi\,{\rm diag}(1, -1)/L_s^2$, and its flux Φ over the 12-plane of the hypertorus is a multiple of 2π: $\Phi = 2\pi\,{\rm diag}(1, -1)$. This means that in the periodic gauge such a field could remain Abelian because of $\exp(i\Phi) = \mathbf{1}$. In the $SU(3)$ case the magnetic field $B_3 = \pi\,{\rm diag}(1, 1, -2)/L_s^2$ has a flux over the 12-plane of the hypertorus equal to $\Phi = \pi\,{\rm diag}(1, 1, -2)$. Now $\exp(i\Phi) = {\rm diag}(-1, -1, 1)$ is not equal to the unity matrix, and this means that in the periodic gauge such a field could not remain Abelian.

III. *SU*(2) EMBEDDED DIRAC SHEET SOLUTIONS

The Dirac sheet seen on the lattice in the $SU(2)$ case [12,13] is observed also in $SU(3)$ lattice simulations. We will call it *standard DS*. New, specific for the $SU(3)$ case, is the Dirac sheet with an action value equal to 3/4 of the action of the standard DS. In the following we will call it *nonstandard DS*. It is also seen in lattice simulations.

In $SU(2)$ a constant Abelian magnetic field is not stable under fluctuations of the gauge field. Charged (off diagonal) components of the gauge field have a Savvidy eigenmode [19] with negative eigenvalue

$$\lambda = -4\pi/L_s^2. \tag{4}$$

The situation can be stabilized by introducing a constant Abelian scalar potential A^3_4. Normally a constant Abelian scalar potential can be gauged away. In our case due to periodicity in time direction it can be gauged away only modulo $2\pi/L_t$. The interaction of charged (off-diagonal) components of the gauge field with this potential adds a positive term to the eigenvalue λ, turning it into

$$\lambda = -4\pi/L_s^2 + (A^3_4)^2. \tag{5}$$

The presence of the scalar potential leads to a nontrivial holonomy H that is defined as

$$H = \lim_{|\vec{x}|\to\infty} P \exp\left(i \int_0^{L_t} A_4(\vec{x}, t) dt\right). \tag{6}$$

The holonomy is parametrized as $H = {\rm diag}(e^{2\pi i\mu_1}, e^{2\pi i\mu_2})$ with $\mu_1 \le \mu_2 \le \mu_3 = 1 + \mu_1$ and $\mu_1 + \mu_2 = 0$. Thus, positive numbers $m_1 = \mu_2 - \mu_1$, $m_2 = \mu_3 - \mu_2$ sum up to unity $m_1 + m_2 = 1$. The eigenvalue λ then becomes equal to

$$\lambda = -4\pi/L_s^2 + (2\pi m_1/L_t)^2, \tag{7}$$

and its positiveness requires $L_t/L_s\sqrt{\pi} < m_{1,2} < 1 - L_t/L_s\sqrt{\pi}$. Therefore, nontrivial holonomy stabilizes DS and just this situation was observed in $SU(2)$ lattice cooling [12] and elucidated in Ref. [13].

Now let us consider the embedding of this standard DS event into the $SU(3)$ group. Let vector potentials $A_{1,2}$ be proportional to ${\rm diag}(1, -1, 0)$ and the scalar potential to give the holonomy

$$H = {\rm diag}(e^{2\pi i\mu_1}, e^{2\pi i\mu_2}, e^{2\pi i\mu_3}), \tag{8}$$

with $\mu_1 \le \mu_2 \le \mu_3 \le \mu_4 = 1 + \mu_1$ and $\mu_1 + \mu_2 + \mu_3 = 0$. Now three positive numbers $m_1 = \mu_2 - \mu_1$, $m_2 = \mu_3 - \mu_2$, $m_3 = \mu_4 - \mu_3$ sum to unity $m_1 + m_2 + m_3 = 1$. Stability of the DS under fluctuations of charged (off-diagonal) $(1, 2) - (2, 1)$ components of the gauge fields requires $L_t/L_s\sqrt{\pi} < m_1 < 1 - L_t/L_s\sqrt{\pi}$. The other off-diagonal $(2, 3) - (3, 2)$ and $(3, 1) - (1, 3)$ components of the gauge fields have charges with respect to the ${\rm diag}(1, -1, 0)$ generator of the $SU(3)$ group being two times smaller than the $(1, 2) - (2, 1)$ components. Hence the stability of DS under their fluctuations requires $L_t/L_s\sqrt{2\pi} < m_2 < 1 - L_t/L_s\sqrt{2\pi}$ and $L_t/L_s\sqrt{2\pi} < m_3 < 1 - L_t/L_s\sqrt{2\pi}$, correspondingly. Taking into account that the magnetic Abelian field could lie also in other $SU(2)$ subgroups of the $SU(3)$ group, i.e., it would then be proportional to ${\rm diag}(1, 0, -1)$ or to ${\rm diag}(0, 1, -1)$ generators, we see that the standard DS in the $SU(3)$ group will be stable for values of the holonomy restricted by the following constraints on the holonomy parameters m_1, m_2, m_3:

$$L_t/L_s\sqrt{2\pi} < m_{1,2,3} < 1 - L_t/L_s\sqrt{2\pi}. \tag{9}$$

We shall visualize the stability criteria in a (X, Y) plot in the complex plane, $X = \Re(1/3\,{\rm Tr}H)$ and $Y = \Im(1/3\,{\rm Tr}H)$. The corresponding region for the standard DS configurations is shown on Fig. 1. The external curved triangle encloses all possible values of one third of the trace of a unitary matrix (the holonomy) that can be obtained by the variation of the phase parameters m_1, m_2, m_3 in the region $0 < m_{1,2,3} < 1$, while the sum is constrained by

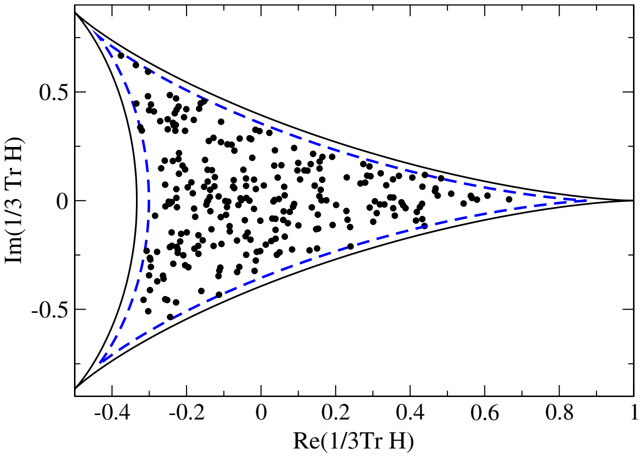

FIG. 1 (color online). The $SU(3)$ triangle and the inscribed region of stability expected for standard DS configurations (enclosed by the dashed line) compared with standard DS events found in actual lattice cooling (filled circles).

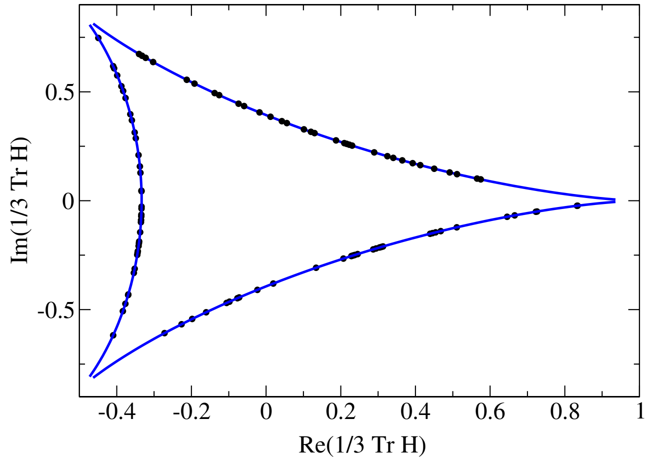

FIG. 2 (color online). The region of stability of nonstandard DS configurations [the three sides of the unclosed $SU(3)$ triangle] compared with nonstandard DS events found in actual lattice cooling (filled circles).

$m_1 + m_2 + m_3 = 1$. The smaller, inscribed curved triangle (bounded by the dashed line) is the region of stability of standard DS events.

IV. NONSTANDARD DIRAC SHEETS

Coming now to the discussion of the stability of nonstandard DS solutions, one should first stress that by construction constant Abelian magnetic fields can be supplemented only by a constant Abelian scalar potential proportional to the same diagonal $SU(3)$ generator to which the magnetic field is proportional. If the magnetic field is equal to $B_3 = \pi\,\mathrm{diag}(1, 1, -2)/L_s^2$, then in a constant Abelian scalar potential

$$A_4 = \mathrm{diag}(2\pi\mu_1/L_t, 2\pi\mu_2/L_t, 2\pi\mu_3/L_t), \qquad (10)$$

the holonomy parameters μ_1 and μ_2 should be equal to each other: $\mu_1 = \mu_2$ ($m_1 = 0$). The fluctuations of the $(1, 2) - (2, 1)$ components of gauge fields in this case do not interact with both the magnetic field and the static scalar potential. For fluctuations of charged $(2, 3) - (3, 2)$ and $(3, 1) - (1, 3)$ components the lowest modes have eigenvalues

$$\lambda_{23} = -3\pi/L_s^2 + (2\pi m_2/L_t)^2 \qquad (11)$$

and

$$\lambda_{13} = -3\pi/L_s^2 + (2\pi m_3/L_t)^2, \qquad (12)$$

correspondingly. So, the stability of such nonstandard DS solutions is possible for

$$m_1 = 0, \quad \sqrt{3/4\pi}L_t/L_s < m_{2,3} < 1 - \sqrt{3/4\pi}L_t/L_s. \qquad (13)$$

For other nonstandard DS solutions the region of stability can be obtained by the permutations of holonomy parameters m_1,m_2,m_3, The stability region is shown in the (X, Y) plot of Fig. 2 and happens to coincide with the boundary of the unclosed $SU(3)$ triangle of Fig. 1.

V. NUMERICAL RESULTS

For a numerical study of standard and nonstandard DS solutions we have employed the standard Wilson plaquette action S_W, creating an ensemble with $\beta = 6/g^2$ where g denotes the bare coupling constant. On a lattice for $L_t = 4$, $L_s = 16$ the coupling constant related to the first order deconfinement transition is equal to $\beta_c \simeq 5.69$. The initial Monte Carlo ensemble was generated in the confined phase at $\beta = 5.63$. As expected, this has guaranteed that in the process of cooling the holonomy has remained sufficiently nontrivial, such that the emerging DS configurations were stable. We have found configurations stable against further cooling with the action $S = 1/4S_{\mathrm{inst}}$ and $S = 3/16S_{\mathrm{inst}}$ in perfect agreement with analytical knowledge. We have stopped cooling at the moment, when the relative variation of action density inside the configuration became smaller than 10^{-4} (homogeneous configurations) and have measured the value of holonomy (the average Polyakov loop). The Polyakov loop also has happened homogeneously. The distance of local values of it from the average value was not larger than 10^{-5}. The scatter plots of DS events in the (X, Y) plane of the real and imaginary part of the Polyakov loop are shown in Figs. 1 and 2. The dots lie perfectly inside the regions of stability for the respective type of DS configurations. The configurations obtained turned out to be purely magnetic and—applying maximally Abelian gauge—show constant Abelian magnetic fluxes.

We did not particularly attempt to find Dirac sheets at higher temperature, $\beta > \beta_c$. We know from other simulations that the holonomy of such equilibrium configurations under cooling rapidly evolves towards central elements where Dirac sheets are unstable and therefore would have escaped observation.

VI. CONCLUSION

In conclusion, purely Abelian constant magnetic field configurations have been observed emerging from the process of cooling equilibrium (Monte Carlo) lattice fields representing the confined phase of $SU(3)$ gluodynamics. They were found to be absolutely stable provided their Polyakov loop was sufficiently nontrivial. We have shown here that this fact is related to the notion of marginal stability of the appropriate constant magnetic field configurations.

Finally we have to admit that the Dirac sheet configurations discussed in this paper will not play any role in the thermodynamic limit of the theory since their action tends to zero in this limit.

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