Evidence for nonanalytic light quark mass dependence in the baryon spectrum

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Using precise lattice QCD computations of the baryon spectrum, we present the first direct evidence for the presence of contributions to the baryon masses which are nonanalytic in the light quark masses, contributions which are often denoted *chiral logarithms*. It is demonstrated that the $SU(3)$ flavor-singlet mass combination suffers the most severe convergence issues. The flavor-octet baryon mass splittings, which are corrected by chiral logarithms at next-to-leading order in $SU(3)$ chiral perturbation theory, yield baryon-pion axial coupling constants D, F, C and H consistent with QCD values, the first evidence of chiral logarithms in the baryon spectrum. The Gell-Mann–Okubo relation, a flavor-**27** baryon mass splitting, which is dominated by chiral corrections from light quark masses, provides further evidence for the presence of nonanalytic light quark mass dependence in the baryon spectrum; we simultaneously find the Gell-Mann–Okubo relation to be inconsistent with the first few terms in a Taylor expansion in $m_s - m_l$, which must be valid for small values of this $SU(3)$ breaking parameter. Additional, more definitive tests of $SU(3)$ chiral perturbation theory will become possible with future, more precise, lattice calculations.

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I. INTRODUCTION

Quantum chromodynamics (QCD) is one of the fundamental gauge theories of the standard model of particle physics, encoding the interactions among quarks and gluons. At high energies, the theory exhibits the property of asymptotic freedom where the coupling between the quarks and gluons runs to zero as the interactions are probed with larger momentum transfer. Conversely, at low energies, at a scale of $\Lambda_{QCD} \sim 1$ GeV, the coupling between the quarks and gluons becomes $\mathcal{O}(1)$, and the theory is no longer amenable to a perturbative treatment; the quark and gluon degrees of freedom are bound into the observed hadronic degrees of freedom, the protons, neutrons, pions, etc., which leave only subtle clues about the underlying fundamental theory of QCD.

These properties of QCD, as well as many others, are now well established thanks to a variety of techniques that have been developed to understand the rich phenomena that emerge from the theory. One of the most important tools is lattice QCD, a numerical solution to the theory, performed on a discrete, Euclidean space-time lattice. With algorithmic advances and ever growing computing power, state-of-the-art lattice QCD calculations are performed at several lattice spacings, with moderate physical space-time volumes and with dynamical light quark masses at or near their physical values [1–5]. Recently, the ground state hadron spectrum, composed of up, down and strange quarks, has been reproduced from lattice calculations with a few percent uncertainty [6]. This serves as an important benchmark in demonstrating the ability for these

numerical calculations to produce precise quantitative predictions for hadronic physics observables. Indeed, lattice calculations are playing an important role in many areas of both nuclear and high-energy physics [7].

In addition to this numerical solution to QCD, a variety of analytic methods have been developed to understand the low-energy regime of the theory. The most prominent method is chiral perturbation theory (χPT) which exploits an approximate global symmetry of QCD [8]. For the *up*, *down* and *strange* quarks with masses less than Λ_{QCD}, the QCD Lagrangian is approximately invariant under global chiral transformations of the quark fields such that the theory has an approximate $SU(3)_L \times SU(3)_R$ chiral symmetry, which becomes exact in the limit the quarks are massless. This approximate chiral symmetry is spontaneously broken to the $SU(3)_V$ subgroup by the QCD vacuum giving rise to the pion-octet pseudoscalar pseudo-Nambu-Goldstone bosons, the pions, kaons and eta. The realization of this chiral symmetry, as well as its spontaneous and explicit symmetry breaking, can be described by constructing a chiral Lagrangian which contains this pion octet as well as other $SU(3)_V$ hadron multiplets as explicit degrees of freedom. In the chiral limit, the pion octet becomes exact Nambu-Goldstone bosons which have only derivative couplings to themselves and other hadrons. This theory, χPT, is nonrenormalizable and contains an infinite number of operators whose forms are constrained by the global symmetries of QCD, while the coefficients of these operators, the low-energy constants (LECs), are unconstrained and must be determined by comparing with experimental data and/or the results of numerical lattice QCD calculations. The quantitative relevance of these operators is dictated by an expansion in the soft momentum of the

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pion octet and the light quark masses suppressed by the chiral-symmetry breaking scale, Λ_χ; hadronic observables can be computed at low energies to any fixed precision by keeping operators to a given order in the chiral expansion, thus requiring determination of only a finite number of the LECs.

One of the principle applications of χPT has been to determine the light quark mass dependence of various hadronic observables, the simplest of which is the light hadron spectrum. The motivation comes from the significant numerical cost of performing lattice QCD calculations at the physical values of the *up* and *down* quark masses. χPT can be used to extrapolate the numerical lattice QCD results to the physical values of the light quark masses, in the process determining some of the LECs associated with the quark mass dependent operators. This program has been very successful when applied to the Nambu-Goldstone meson spectrum and decay constants (see Ref. [9] for a review), beginning with the first significant comparison of lattice QCD results with χPT [10].

The comparison with the light baryon spectrum has been wrought with more significant difficulties and the overall convergence, and usefulness, of the $SU(3)$ baryon χPT is in question. These challenges are not unexpected: first, there is a dense spectrum of low lying excited states, introducing new scales in the theory; second, while the expansion parameters of χPT in the Nambu-Goldstone meson sector are given by $\epsilon_m \sim m_{K,\pi,\eta}^2/\Lambda_\chi^2$, the small expansion parameter when the baryon fields are included becomes $\epsilon \sim m_{K,\pi,\eta}/\Lambda_\chi$ [11]. For the physical kaon, $\epsilon \simeq 1/2$, and from general expectations of asymptotic series, one does not expect this theory to have a controlled perturbative expansion. A few recent comparisons of $SU(3)$ baryon χPT to numerical lattice QCD results have led to the conclusion the three flavor chiral expansion is failing to provide a controlled, convergent expansion [12–14]. The issues of convergence are not limited to the three flavor expansion [15] and recent analysis indicates the range of expansion of the two flavor theory, considering only an expansion about the limit of vanishing up and down quark masses, extends only to $m_\pi^{\max} \simeq 300$–350 MeV [12,16].

These challenges have led to a number of efforts to reorganize the expansion for baryon χPT. The initial approach is known as heavy baryon χPT (HBχPT) which treats the baryons as nearly static fields allowing for an expansion in inverse powers of the baryon mass [17,18], modeled after the heavy quark effective theory [19]. This led to significant phenomenological successes which are partly reviewed in Refs. [20,21]. Early on, it was recognized that the convergence of the theory would be problematic because of the large contributions from kaon and eta loops in various observables. A new regularization scheme was proposed: the introduction of a (chiral-symmetry violating) long-range regulator, e.g., a dipole regulator, to soften the contribution from the kaon and eta loops [22].

When applied to chiral extrapolations of lattice QCD results, this led to some successes in simultaneously describing both the numerical results and physical observables [23,24]. An additional reorganization of the chiral expansion, equivalent to a resummation of the leading kinetic corrections to the baryon propagators, was constructed and has become known as infrared regularized baryon χPT [25], with several offshoots to deal with renormalization of higher loop corrections [26,27]. Lattice QCD calculations in the last few years have also made feasible the use of the $SU(2)$ expansion for hyperons [28,29]. In this work, we further examine a new application of an old idea: combining the large N_c expansion [30,31] with the $SU(3)$ chiral expansion [32–36]. This approach has a few formal advantages over the other methods. In the large N_c limit, there is an extra symmetry, the contracted spin-flavor symmetry [32,33], allowing for an unambiguous field-theoretic method to include the low lying decuplet baryon resonances in the theory; in the large N_c limit, the spin-1/2 and -3/2 baryons become degenerate and infinitely heavy. Further, while the large N_c and $SU(3)$ chiral expansions on their own may not provide well converged effective theories, the combined expansions may prove sufficient for a controlled perturbative expansion. This approach was first explored in Ref. [37] where it was demonstrated that the predictions from the combined large N_c and $SU(3)$ expansions on the baryon spectrum are well met for a range of light quark masses.

Having a controlled expansion is necessary but not sufficient to claim success. The principle prediction from χPT is the contributions to hadronic observables which are nonanalytic in the light quark masses. The masses of the Nambu-Goldstone boson is given to leading order by the Gell-Mann–Oakes–Renner relation [38], $m_{i,j}^2 = B(m_i + m_j)$, with a meson composed of a quark-antiquark pair of (anti)flavors i and j, and m_i the mass of a quark with flavor i. Therefore, in χPT, the nonanalytic light quark mass dependence arises from pion-octet loops, which often contribute $\ln(m_{K,\pi,\eta}^2)$ terms to hadronic observables, and are commonly referred to as *chiral logs*. These contributions cannot arise from a finite number of local counterterms but only from the long-range contributions from the light pion-octet degrees of freedom, the *pion cloud*. Isolating this predicted light quark mass dependence in lattice QCD results has been a major challenge for many years. The definitive identification of these contributions is hailed as a signal that the *up* and *down* (and *strange*) quarks are sufficiently light that the lattice results can be described accurately by χPT. This task has proved to be very challenging, as often these nonanalytic light quark mass contributions are subleading, or masked by other systematics.

In this work, we present for the first time, direct evidence of nonanalytic light quark mass dependence in the baryon spectrum. As will be discussed in this article, this work is not the definitive work on the subject, as there are many

systematics which must be resolved, but this is an important first step in the quest for chiral logs.

II. THE HEAVY BARYON CHIRAL LAGRANGIAN AND THE LARGE N_c EXPANSION

A. Heavy baryon chiral Lagrangian in the $1/N_c$ expansion

The three flavor heavy baryon chiral Lagrangian at leading order (LO) in the momentum expansion and to first order in the chiral-symmetry breaking quark mass matrix $\mathcal{M}_q \equiv \mathrm{diag}(m_u, m_d, m_s)$ is given by [17,18]

$$\mathcal{L} = \mathrm{Tr}\bar{B}_v (iv \cdot \mathcal{D}) B_v - \bar{T}_v^\mu (iv \cdot \mathcal{D}) T_{v\mu} - \frac{1}{4}\Delta_0 \mathrm{Tr}\bar{B}_v B_v$$
$$+ \frac{5}{4}\Delta_0 \bar{T}_v^\mu T_{v\mu} + 2D\,\mathrm{Tr}(\bar{B}_v S_v^\mu \{\mathcal{A}_\mu, B_v\})$$
$$+ 2F\,\mathrm{Tr}(\bar{B}_v S_v^\mu [\mathcal{A}_\mu, B_v]) + \mathcal{C}(\bar{T}_v^\mu \mathcal{A}_\mu B_v + \bar{B}_v \mathcal{A}_\mu T_v^\mu)$$
$$+ 2\mathcal{H} \bar{T}_v^\mu S_v^\nu \mathcal{A}_\nu T_{v\mu} + 2\sigma_B \mathrm{Tr}(\bar{B}_v B_v)\mathrm{Tr}\mathcal{M}_+$$
$$- 2\sigma_T \bar{T}_v^\mu T_{v\mu}\mathrm{Tr}\mathcal{M}_+ + 2b_D \mathrm{Tr}(\bar{B}_v\{\mathcal{M}_+, B_v\})$$
$$+ 2b_F \mathrm{Tr}(\bar{B}_v[\mathcal{M}_+, B_v]) + 2b_T \bar{T}_v^\mu \mathcal{M}_+ T_{v\mu}, \quad (1)$$

where the spin-1/2 octet baryon fields B_v and spin-3/2 decuplet baryon fields T_v^μ are two-component velocity-dependent baryon fields which are related to the usual four-component relativistic Dirac spin baryon fields B and T^μ by

$$B_v(x) = \frac{1 + \not{v}}{2} e^{iM_0 v \cdot x} B(x),$$
$$T_v^\mu(x) = \frac{1 + \not{v}}{2} e^{iM_0 v \cdot x} T^\mu(x). \quad (2)$$

The mass M_0 is the flavor-singlet mass of the baryon octet and decuplet baryons in the $SU(3)$ chiral limit $m_q \to 0$. Specifically,

$$M_0 = \frac{5}{4}\langle M_8 \rangle - \frac{1}{4}\langle M_{10} \rangle, \quad (3)$$

where $\langle M_8 \rangle$ and $\langle M_{10} \rangle$ are the average flavor-singlet masses of the spin-1/2 flavor-octet baryons and the spin-3/2 flavor-decuplet baryons, respectively, in the chiral limit. In the large N_c expansion, M_0 is $\mathcal{O}(N_c)$ for baryons with N_c quarks. The leading heavy baryon chiral Lagrangian also contains the flavor-singlet hyperfine mass splitting

$$\Delta_0 = \langle M_{10} \rangle - \langle M_8 \rangle, \quad (4)$$

which is proportional to the total spin-squared J_v^2 of each baryon multiplet. The mass parameter Δ_0 is $\mathcal{O}(1/N_c)$ in the $1/N_c$ expansion. The $SU(3)$ flavor representations of the QCD baryons are the flavor-octet

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}, \quad (5)$$

and the completely symmetric rank-3 flavor-decuplet T_{ijk}, normalized such that $T_{uuu} = \Delta^{++}$. The heavy baryon chiral Lagrangian also contains four independent baryon-pion couplings, the axial couplings D, F, $\mathcal{C}$ and $\mathcal{H}$. The couplings D and F describe the usual baryon-octet pion couplings; $\mathcal{C}$ describes pion couplings between octet and decuplet baryons; and $\mathcal{H}$ describes the pion coupling of the decuplet baryons. The pion-octet fields

$$\Pi \equiv \pi^a T^a$$
$$= \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}, \quad (6)$$

appear in the heavy baryon chiral Lagrangian in the nonlinear representation $\xi^2 = \Sigma = e^{2i\Pi/f}$, where $f \sim 130$ MeV is the pion decay constant in the chiral limit. The vector and axial vector pion combinations

$$\mathcal{A}_\mu = \frac{i}{2}(\xi \partial_\mu \xi^\dagger - \xi^\dagger \partial_\mu \xi),$$
$$\mathcal{V}_\mu = \frac{1}{2}(\xi \partial_\mu \xi^\dagger + \xi^\dagger \partial_\mu \xi), \quad (7)$$

appear in the baryon-pion couplings and through the baryon covariant derivative $\mathcal{D}_\mu = \partial_\mu + i\mathcal{V}_\mu$. In the heavy baryon chiral Lagrangian, S_v^μ is the spin operator which acts on the spinor portion of the baryon field.

Additional dependence on the pion field enters through the quark mass matrix spurion

$$\mathcal{M}_+ = \frac{1}{2}(\xi \mathcal{M}_q^\dagger \xi + \xi^\dagger \mathcal{M}_q \xi^\dagger). \quad (8)$$

In this work, we compare with lattice computations performed with degenerate u and d quark masses $m_u = m_d = m_l$, so the quark mass matrix reduces to

$$\mathcal{M}_q = \frac{1}{3}(2m_l + m_s)\mathbb{1} + \frac{2}{\sqrt{3}}(m_l - m_s)T^8. \quad (9)$$

There are two flavor-singlet contributions to the baryon masses with one insertion of the quark mass matrix coming from the terms proportional to σ_B and σ_T. There are also three flavor-octet contributions to the baryon masses with a single insertion of the quark mass matrix, proportional to b_D, b_F and b_T (called b_C previously [39]).

The $1/N_c$ expansion [30] for baryons [31] leads to the emergence of a spin-flavor symmetry [32,33,35] for large N_c baryons. In Ref. [40], the heavy baryon Lagrangian was formulated in the $1/N_c$ expansion. Relations among the coefficients in the heavy baryon chiral Lagrangian occur at leading and subleading orders in the $1/N_c$ expansion, which reduces the number of independent chiral coefficients in the heavy baryon chiral Lagrangian at leading and subleading orders in $1/N_c$. In addition, there exists a planar flavor symmetry [40] at leading order in $1/N_c$, which

relates flavor-singlet to flavor-octet parameters at this order, further reducing the number of independent chiral coefficients in the heavy baryon chiral Lagrangian. In particular, planar QCD flavor symmetry relates the flavor-singlet quark mass parameters σ_B and σ_T to the flavor-octet quark mass parameters b_D, b_F and b_T at leading orders in $1/N_c$. The flavor-octet and flavor-singlet quark mass parameters are given in terms of the coefficients b_n of the spin-0 flavor-octet $1/N_c$ expansion,[1] where the subscript n refers to the fact that the corresponding operator $\mathcal{O}_{(n)}$ is an n-body quark operator which is accompanied by an explicit factor of N_c^{1-n}. To first subleading order in the $1/N_c$ expansion, the mass matrix parameters of the heavy baryon chiral Lagrangian for QCD with $N_c = 3$ are given by

$$b_D = \frac{1}{4}b_2, \qquad b_F = \frac{1}{2}b_1 + \frac{1}{6}b_2, \qquad b_T = -\frac{3}{2}b_1 - \frac{5}{4}b_2,$$

$$\sigma_B = \frac{1}{2}b_1 + \frac{1}{12}b_2, \qquad \sigma_T = \frac{1}{2}b_1 + \frac{5}{12}b_2. \qquad (10)$$

The axial couplings D, F, $\mathcal{C}$ and H also have an expansion in terms of spin-1 flavor-octet coefficients a_n of the $1/N_c$ expansion. To first subleading order in $1/N_c$, the pion-baryon couplings of the heavy baryon chiral Lagrangian for QCD with $N_c = 3$ are related to the $1/N_c$ coefficients by [40,41][2]

$$D = \frac{1}{2}a_1, \qquad F = \frac{1}{3}a_1 + \frac{1}{6}a_2,$$

$$\mathcal{C} = -a_1, \qquad \mathcal{H} = -\frac{3}{2}a_1 - \frac{3}{2}a_2. \qquad (11)$$

B. Mass relations R_1, R_3 and R_4

In Ref. [37], it was argued that a better approach to exploring the baryon spectrum was to utilize our knowledge of both large N_c as well as $SU(3)$ symmetry which is known to work well for the experimental spectrum [43]; instead of considering the individual baryon masses directly, one should explore the light quark mass dependence of various linear combinations of the baryon masses, chosen to have definite scaling in terms of $1/N_c$ and $SU(3)$ symmetry breaking.[3] The various linear combinations were determined in Ref. [43]. In Ref. [37], it was demonstrated that the predicted scaling with both $1/N_c$ and $(m_s - m_l)$ was clearly visible in the lattice data. The first few mass combinations had statistically meaningful values over the range of quark masses, but there were not enough

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[1]Here, we adopt a simplified notation for the operator coefficients compared to Ref. [40].
[2]The $1/N_c$ operator analysis has recently been extended to the two-body axial current operators [42], such as $\mathrm{Tr}(\bar{B}\mathcal{A} \cdot \mathcal{A}B)$.
[3]Reference [44] utilized the large N_c relations between operators in baryon χPT to study the baryon spectrum, but not the linear combinations constructed to have definite scaling in $m_s - m_l$ and $1/N_c$.

statistics to resolve all of them. In this work, we focus our attention on three of these mass relations, R_1, R_3 and R_4.[4] These mass relations are given by

$$R_i = \frac{\sum_j c_{ij}M_j}{\sum_j |c_{ij}|}, \qquad (12)$$

where

$$\begin{aligned}
\mathsf{M}_1 &= \sum_j c_{1j}M_j \\
&= 25(2M_N + M_\Lambda + 3M_\Sigma + 2M_\Xi) \\
&\quad - 4(4M_\Delta + 3M_{\Sigma^*} + 2M_{\Xi^*} + M_\Omega), \\
\mathsf{M}_3 &= \sum_j c_{3j}M_j \\
&= 5(6M_N + M_\Lambda - 3M_\Sigma - 4M_\Xi) \\
&\quad - 2(2M_\Delta - M_{\Xi^*} - M_\Omega), \\
\mathsf{M}_4 &= \sum_j c_{4j}M_j = (M_N + M_\Lambda - 3M_\Sigma + M_\Xi),
\end{aligned} \qquad (13)$$

and for example $R_4 = \mathsf{M}_4/6$.

These relations are designed to isolate various operators in the combined $1/N_c$ and $SU(3)$ breaking expansions. At $\mathcal{O}(m_q)$, only relations R_1–R_4 are nonvanishing. For this reason, the relations R_5–R_8 are particularly interesting to use with light quark mass extrapolations, as the leading contribution begins with the chiral loops at $\mathcal{O}(m_q^{3/2})$. However, even more precise results of the baryon spectrum than exist are needed for these relations. Using the large N_c expansions through second nontrivial order, and working through next-to-leading order (NLO) in the chiral expansion, the relation R_1 is given by

$$\begin{aligned}
\frac{3}{2}R_1(m_l, m_s) = M_0 &- \frac{3}{4}\left(b_1 + \frac{5}{18}b_2\right)(2m_l + m_s) \\
&- \frac{35a_1^2 - 5a_2^2}{96}\frac{3\mathcal{F}_\pi^0 + 4\mathcal{F}_K^0 + \mathcal{F}_\eta^0}{(4\pi f)^2} \\
&- \frac{a_1^2}{96}\left[50\frac{3\mathcal{F}_\pi^\Delta + 4\mathcal{F}_K^\Delta + \mathcal{F}_\eta^\Delta}{(4\pi f)^2}\right. \\
&\quad \left. - 4\frac{3\mathcal{F}_\pi^{-\Delta} + 4\mathcal{F}_K^{-\Delta} + \mathcal{F}_\eta^{-\Delta}}{(4\pi f)^2}\right]. \qquad (14)
\end{aligned}$$

The nonanalytic function $\mathcal{F}_\phi^\Delta = \mathcal{F}(m_\phi, \Delta, \mu)$ is defined as

$$\begin{aligned}
\mathcal{F}(m, \Delta, \mu) = (\Delta^2 - m^2 + i\epsilon)^{3/2}&\ln\left(\frac{\Delta + \sqrt{\Delta^2 - m^2 + i\epsilon}}{\Delta - \sqrt{\Delta^2 - m^2 + i\epsilon}}\right) \\
&- \frac{3}{2}\Delta m^2\ln\left(\frac{m^2}{\mu^2}\right) - \Delta^3\ln\left(\frac{4\Delta^2}{m^2}\right), \qquad (15)
\end{aligned}$$

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[4]The relation R_2 gives at leading order the hyperfine splitting Δ_0. For the current lattice data set, this quantity provides no further information over the use of R_1.

which has the limits and properties

$$\mathcal{F}(0, \Delta, \mu) = 0$$

$$\mathcal{F}(m, 0, \mu) = \pi m^3 \tag{16}$$

$$\mathcal{F}(m, -\Delta, \mu) = \begin{cases} -\mathcal{F}(m, \Delta, \mu) + 2i\pi(\Delta^2 - m^2)^{3/2}, & m < |\Delta| \\ -\mathcal{F}(m, \Delta, \mu) + 2\pi(m^2 - \Delta^2)^{3/2}, & m > |\Delta|. \end{cases}$$

For the baryon spectrum, the leading nonanalytic light quark mass dependence is encoded in this function. As such, it is of particular interest to find evidence of this behavior in the spectrum.

The mass relations R_3 and R_4 vanish in both the $SU(3)$ chiral and vector limits, making them more sensitive to the NLO nonanalytic light quark mass dependence. At NLO in the chiral expansion, and to the first two nontrivial orders in the large N_c expansion, these relations are given by

$$R_3(m_l, m_s) = \frac{20}{39} b_1(m_s - m_l) - \frac{20a_1^2 - 5a_2^2}{117} \frac{3\mathcal{F}_\pi^0 - 2\mathcal{F}_K^0 - \mathcal{F}_\eta^0}{(4\pi f)^2}$$
$$- \frac{a_1^2}{117}\left[35\frac{3\mathcal{F}_\pi^\Delta - 2\mathcal{F}_K^\Delta - \mathcal{F}_\eta^\Delta}{(4\pi f)^2} - \frac{3\mathcal{F}_\pi^{-\Delta} - 2\mathcal{F}_K^{-\Delta} - \mathcal{F}_\eta^{-\Delta}}{(4\pi f)^2}\right], \tag{17}$$

$$R_4(m_l, m_s) = -\frac{5}{18} b_2(m_s - m_l) + \frac{a_1^2 + 4a_1 a_2 + a_2^2}{36} \frac{3\mathcal{F}_\pi^0 - 2\mathcal{F}_K^0 - \mathcal{F}_\eta^0}{(4\pi f)^2}$$
$$- \frac{2a_1^2}{9} \frac{3\mathcal{F}_\pi^\Delta - 2\mathcal{F}_K^\Delta - \mathcal{F}_\eta^\Delta}{(4\pi f)^2}. \tag{18}$$

In addition to these three mass relations, the Gell-Mann–Okubo (GMO) relation is also important to examine:

$$\Delta_{\text{GMO}} = \frac{3}{4} M_\Lambda + \frac{1}{4} M_\Sigma - \frac{1}{2} M_N - \frac{1}{2} M_\Xi. \tag{19}$$

Since the quark mass operator contains pieces which transform as both an **8** as well as a **1** under $SU(3)$ transformations [Eq. (9)], there are nonvanishing contributions to the GMO relation at subleading orders in the chiral expansion. However, mass operators which transform as an **8** make vanishing contributions to Eq. (19). The leading mass operator which makes a nonzero contribution to the GMO relation transforms as a flavor-**27**. These corrections can arise either from chiral loops or from a mass operator containing two or more quark mass insertions. This makes the GMO relation particularly interesting to explore with lattice QCD calculations; the leading contribution to this mass relation comes from chiral loop effects which are nonanalytic in the light quark masses. Experimentally, the GMO relation is found to be

$$\Delta_{\text{GMO}}^{\text{phy}} = 6.45 \text{ MeV}. \tag{20}$$

Each baryon mass in the relation receives nonanalytic mass corrections which scale as $\delta M_B \propto N_c m_s^{3/2}$. These large corrections may lead to the expectation that the GMO relation receives large contributions from the loop corrections. However, one can show these $N_c m_s^{3/2}$ terms are proportional to **1** under $SU(3)$ transformations. Additionally, the $m_s^{3/2}$ contributions transform as an **8** while the $m_s^{3/2}/N_c$ corrections transform as a flavor-**27**. This provides an extra $1/N_c^2$ on top of the chiral suppression, explaining the relatively small value of the GMO relation [40].

At next-to-leading order in the chiral and large N_c expansions, the Gell-Mann–Okubo relation is

$$\Delta_{\text{GMO}}^{\text{NLO}} = \frac{a_1^2}{36(4\pi f)^2}[\mathcal{F}_\pi^0 - 4\mathcal{F}_K^0 + 3\mathcal{F}_\eta^0 + 2\mathcal{F}_\pi^\Delta - 8\mathcal{F}_K^\Delta + 6\mathcal{F}_\eta^\Delta] + \frac{4a_1 a_2 + a_2^2}{36(4\pi f)^2}[\mathcal{F}_\pi^0 - 4\mathcal{F}_K^0 + 3\mathcal{F}_\eta^0]. \tag{21}$$

In this article, we will also be interested in the next-to-next-to-leading order (NNLO) formula. This can be determined from Refs. [45,46]. Retaining the subleading N_c relations for the quark mass operators [Eq. (10)] but only the leading relations for the axial couplings ($a_2 \to 0$), the NNLO contributions to the GMO formula are

$$\Delta_{\text{GMO}}^{\text{NNLO}} = \frac{1}{48}\frac{(b_3^M + b_4^M)(m_s - m_l)^2}{4\pi f} + \frac{23a_1^2}{384M_0}\left(\frac{m_\pi^4 - 4m_K^4 + 3m_\eta^4}{(4\pi f)^2}\right)$$

$$+ \left(\frac{b_3^A}{96\pi f} + \frac{13a_1^2}{192M_0}\right)\left(\frac{m_\pi^4 \ln(m_\pi^2/\mu^2) - 4m_K^4 \ln(m_K^2/\mu^2) + 3m_\eta^4 \ln(m_\eta^2/\mu^2)}{(4\pi f)^2}\right)$$

$$+ \frac{a_1^2}{(4\pi f)^2}\left\{b_1\left[\frac{2m_l + m_s}{12}(\mathcal{J}_\pi^\Delta - 4\mathcal{J}_K^\Delta + 3\mathcal{J}_\eta^\Delta + m_\pi^2 - 4m_K^2 + 3m_\eta^2)\right]\right.$$

$$+ b_2\left[\frac{-9m_s + 16m_l}{24}(\mathcal{J}_\pi^\Delta - 4\mathcal{J}_K^\Delta + 3\mathcal{J}_\eta^\Delta) + \frac{3}{2}(m_s - m_l)(\mathcal{J}_\eta^\Delta - \mathcal{J}_K^\Delta) - \frac{m_s - m_l}{6}(m_\eta^2 - m_k^2)\right.$$

$$\left.\left.- \frac{2}{3}(m_s - m_l)\left(m_K^2\ln\left(\frac{m_K^2}{\mu^2}\right) - m_\pi^2\ln\left(\frac{m_\pi^2}{\mu^2}\right)\right) + \frac{16m_l + 5m_s}{72}(m_\pi^2 - 4m_K^2 + 3m_\eta^2)\right]\right\}, \qquad (22)$$

where the function $\mathcal{J}_\phi^\Delta = \mathcal{J}(m_\phi, \Delta, \mu)$ encodes additional nonanalytic dependence on the light quark masses

$$\mathcal{J}(m, \Delta, \mu) = 2\Delta\sqrt{\Delta^2 - m^2 + i\epsilon}\ln\left(\frac{\Delta + \sqrt{\Delta^2 - m^2 + i\epsilon}}{\Delta - \sqrt{\Delta^2 - m^2 + i\epsilon}}\right)$$

$$+ m^2\ln\left(\frac{m^2}{\mu^2}\right) - 2\Delta^2\ln\left(\frac{4\Delta^2}{m^2}\right), \qquad (23)$$

and has the limits and properties

$$\mathcal{J}(0, \Delta, \mu) = 0,$$

$$\mathcal{J}(m, 0, \mu) = m^2\ln\left(\frac{m^2}{\mu^2}\right),$$

$$\mathcal{J}(m, -\Delta, \mu) = \begin{cases} \mathcal{J}(m, \Delta, \mu) + 4i\pi\Delta\sqrt{\Delta^2 - m^2}, & m < |\Delta| \\ \mathcal{J}(m, \Delta, \mu) - 4\pi\Delta\sqrt{m^2 - \Delta^2}, & m > |\Delta|. \end{cases}$$

$$(24)$$

III. EVIDENCE FOR NONANALYTIC LIGHT QUARK MASS DEPENDENCE IN BARYON SPECTRUM

A. Details of the lattice results

For this work, the numerical results of Ref. [12] are utilized, which are not the most recent but are still the most statistically precise data set available. The lattice calculation was performed with a mixed action composed of domain-wall fermion [47–51] propagators generated on the $n_f = 2 + 1$ asqtad-improved [52,53], rooted, staggered sea quark configurations generated by the MILC Collaboration [54]. This particular mixed-action setup has been used quite extensively by the LHP [12,55–59] and NPLQCD [14,60–74] Collaborations as well as some independent works [75–79]. The mixed-action effective field theory, which encodes the discretization effects specific to this particular mixed action, has also been thoroughly developed [80–93]. However, the baryon spectrum results used in this work exist at only a single lattice spacing. There is also reason to believe the discretization systematics are small [12,88,91] and, to the order we are working in the mixed-action effective field theory, they are

subleading. For these reasons, the continuum χPT extrapolation formula, presented in the previous section, is used.

The latest scale setting by the MILC Collaboration [2], as detailed in Ref. [74], is used; numerical results of Ref. [12] are first converted into r_1 units,[5] and then the MILC determination of $r_1(m_l^{\text{phy}}, m_s^{\text{phy}})$ is used to convert to physical units. Finally, extrapolations are performed as functions of the quark masses. The quark masses are not renormalization scheme or scale independent. However, at a fixed lattice spacing, the quark mass renormalization can be absorbed into the quantity B, where at leading order the Nambu-Goldstone boson masses are given by $m_{i,j}^2 = B(m_i + m_j)$. Lattice quark masses are then defined in physical units by

$$r_1^{\text{phy}}m_q^{\text{latt}} \equiv \frac{r_1}{a}(am_q + am^{\text{res}}), \qquad (25)$$

where m^{res} is the residual chiral-symmetry breaking present with the domain-wall lattice action at finite fifth dimensional extent [94]. These numerical values are collected in Table I.

To extrapolate the lattice results to the physical point, NLO χPT [95] is used to determine the values of m_q^{latt} which reproduce

$$m_\pi^{\text{phy}} \equiv 138 \text{ MeV}, \qquad m_K^{\text{phy}} \equiv 496 \text{ MeV}. \qquad (26)$$

It is interesting to note that, despite ignoring the issues of quark mass renormalization, this yields the values

$$m_{l,\text{phy}}^{\text{latt}} = 3.0(2) \text{ MeV}, \qquad m_{s,\text{phy}}^{\text{latt}} = 99(5) \text{ MeV}, \qquad (27)$$

which are remarkably similar to the proper lattice determination of the light and strange quark masses [9]. The NLO χPT formula provides a controlled and convergent description of both m_π and m_K over the full range of quark masses used (see Fig. 1).

---

[5]The length scale r_1 is determined with the heavy quark potential, defined such that $r_1^2 F(r_1) = -1$.

TABLE I.   The numerical lattice data from Ref. [12], converted to physical units. The uncertainties are the statistical and systematic uncertainties from Ref. [12] combined in quadrature. The strange quark mass is fixed to the largest am_q value in all calculations: $am_s^{\text{sea}} = 0.050$, $am_s^{\text{val}} = 0.081$.

β	6.76	6.76	6.79	6.81	6.83	6.85
am_l^{sea}	0.007	0.010	0.020	0.030	0.040	0.050
am_l^{val}	0.0081	0.0138	0.0313	0.0478	0.0644	0.081
am_l^{res}	0.00160(3)	0.00157(1)	0.00123(1)	0.00101(1)	0.00083(2)	0.00073(3)
m_q^{latt} [MeV]	16.8	26.6	58.0	88.8	121	155
m_π [MeV]	320(2)	389(2)	557(1)	685(2)	805(4)	905(2)
m_K [MeV]	640(2)	659(2)	726(1)	787(2)	852(4)	905(2)
$\frac{3}{2}R_1$ [MeV]	1285(6)	1315(6)	1454(6)	1556(12)	1698(13)	1769(9)
R_3 [MeV]	$-113(3)$	$-100(2)$	$-64(1)$	$-41(1)$	$-19(1)$	0
R_4 [MeV]	$-39(2)$	$-33(1)$	$-19(1)$	$-11(1)$	$-4.4(0.6)$	0
Δ_{GMO} [MeV]	5.6(2.3)	1.8(1.2)	0.18(48)	0.13(35)	0.13(0.09)	0

B. Large N_c and consistency of hyperon axial charges

One of the major failings in the application of $SU(3)$ heavy baryon χPT is a lack of consistency between the determination of the axial coupling constants, D, F, C and H when determined from the baryon spectrum [12,13] versus a direct calculation of the hyperon axial charges [75]. The direct lattice determination[6] yields values consistent with the phenomenological values [96], while the indirect determination from the baryon spectrum yields values consistent with zero. The small values of the axial couplings returned indicate the numerical results do not support evidence for the leading nonanalytic light quark mass dependence predicted in the spectrum. This problem is not unique to the $SU(3)$ heavy baryon χPT extrapolations, with large contributions from kaon and eta loops, but also observed in the $SU(2)$ extrapolation of the nucleon mass. As demonstrated in Refs. [12,16], for $m_\pi \gtrsim$ 300 MeV, there are large cancellations between the LO, NLO and NNLO contributions to the nucleon mass; in order to accommodate the large negative mass contribution occurring at NLO, the leading nonanalytic light quark mass dependence, there must be a compensating large but positive contribution from the LO and NNLO terms, signaling a breakdown of the perturbative expansion for these heavier pion masses. A similar and more severe situation occurs for the $SU(3)$ chiral expansion. As we shall demonstrate in the next sections, it may be we are simply asking the wrong questions of heavy baryon χPT.

1. Mass relation R_1

The mass relation R_1 is a flavor-singlet mass combination designed to isolate the M_0 contribution plus higher order chiral corrections. Starting with the lightest quark mass and including successively heavier values of m_q, there are four possible ranges of light quark masses which can be used to perform the chiral extrapolation analysis.

---

[6]In Ref. [75], g_A, $g_{\Sigma\Sigma}$ and $g_{\Xi\Xi}$ were computed which were used to infer the values of D and F.

Both LO and NLO analyses are performed over all these ranges of light quark masses. In the NLO analysis, the subleading in N_c axial coefficient is set to zero, $a_2 = 0$. Several choices of the parameter f are taken to explore systematics from higher orders in the chiral expansion:

$$
\begin{aligned}
(i) \quad & f = f_\pi(m_q), \\
(ii) \quad & f^2 = f_K(m_q)f_\pi(m_q), \\
(iii) \quad & f = f_K(m_q),
\end{aligned}
\tag{28}
$$

where $f_{\pi(K)}(m_q)$ is the value of the pion (kaon) decay constant calculated at the quark mass m_q. From the LO analysis, the following LECs are obtained:

$$
M_0[\text{LO}] = 903(20) \text{ MeV},
$$
$$
\left[b_1 + \frac{5}{18}b_2\right][\text{LO}] = -2.73(11).
\tag{29}
$$

Extrapolating to the physical values of the light and strange quark masses gives

$$
\frac{3}{2}R_1[\text{LO}] = 1124(18) \text{ MeV},
\tag{30}
$$

which is to be compared with $\frac{3}{2}R_1^{\text{phy}} = 1093$ MeV. Performing the NLO analysis, the LECs are determined to be

$$
M_0[\text{NLO}] = 899(40) \text{ MeV},
$$
$$
\left[b_1 + \frac{5}{18}b_2\right][\text{NLO}] = -3.26(70),
$$
$$
a_1[\text{NLO}] = 0.24(30),
\tag{31}
$$

with a determination

$$
\frac{3}{2}R_1[\text{NLO}] = 1107(50) \text{ MeV}.
\tag{32}
$$

In Fig. 2, representative fits of R_1 from LO and NLO are displayed. One may take comfort in the consistent values of the LECs M_0 and $b_1 + \frac{5}{18}b_2$ between the LO and NLO
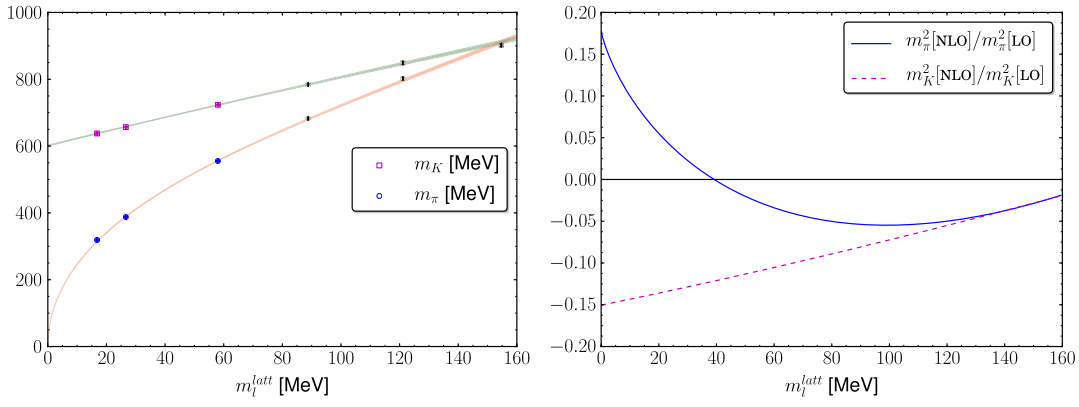
FIG. 1 (color online).   The results of $SU(3)$ χPT analysis of the pion and kaon masses. The left plot displays the fit result with the data and the right plot shows the size of the NLO contributions compared to LO. This particular fit is performed to just the lightest three values of the light quark mass, and it is clear from the plot that the fits including all six points yield consistent results.
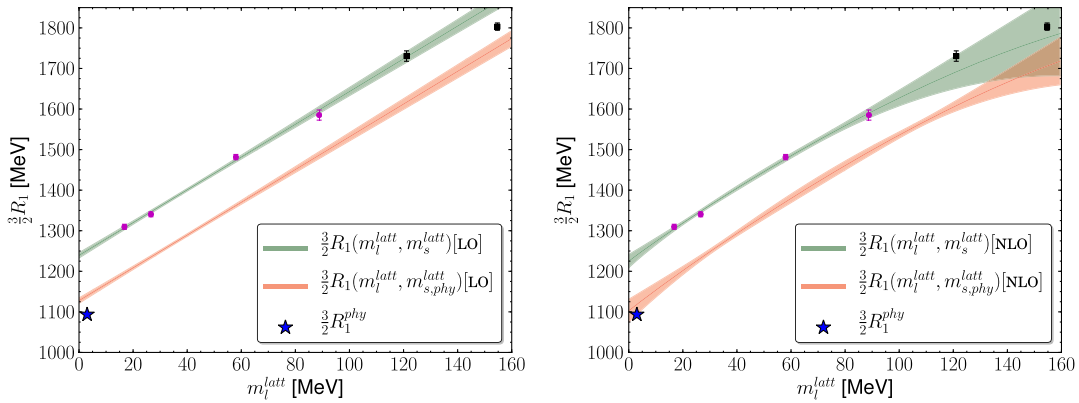

FIG. 2 (color online).   Representative fits to R_1 from LO (left) and NLO (right) HBχPT analysis. The blue star is the physical value, not used in the analysis. The upper error band results from a fit to the lightest four numerical data and the lower band is the result extrapolated to the physical value of the strange quark mass $m_s^{\mathrm{latt}} \rightarrow m_{s,\mathrm{phy}}^{\mathrm{latt}}$ [Eq. (27)].

analyses. However, this is not surprising given the small value of a_1 determined in the NLO analysis. This small value is consistent with no contributions from the NLO terms and inconsistent with the known phenomenological determination of the axial coupling. This is not surprising given the convergence issues observed in the $SU(2)$ extrapolation of the nucleon mass [12,16]. One is left to conclude that the $SU(3)$ heavy baryon χPT does not provide a controlled, convergent expansion for the mass combination R_1 for the range of quark masses used in this work and a value of a_1 consistent with phenomenology or direct lattice calculations of the baryon axial charges.

2. Mass relations R_3 and R_4

The relations R_3 and R_4 both receive leading contributions from flavor-octet mass operators, vanishing in both the $SU(3)$ vector as well as $SU(3)$ chiral limits. From these symmetries, the relations R_3 and R_4 are more sensitive to the nonanalytic light quark mass dependence occurring at NLO in the chiral expansion. As with the analysis of R_1,

three choices of the parameter f are taken to estimate higher order effects [Eq. (28)]. The LO expressions for R_3 and R_4, Eqs. (17) and (18) with $a_i = 0$, do not describe the numerical results well; it is clear higher order contributions are necessary for extrapolations of this data. At NLO, the analysis of R_3 and R_4 becomes correlated. The full covariance matrix is constructed as described in Ref. [37]. The numerical results of Ref. [12] are insufficient to constrain both the leading and subleading axial coefficients, and so the analysis is restricted to the set of LECs

$$\lambda = (b_1, b_2, a_1), \qquad (33)$$

with $a_2 = 0$. From the NLO analysis, the LECs are determined to be

$$b_1[\mathrm{NLO}] = -6.6(5),$$
$$b_2[\mathrm{NLO}] = 4.3(4), \qquad (34)$$
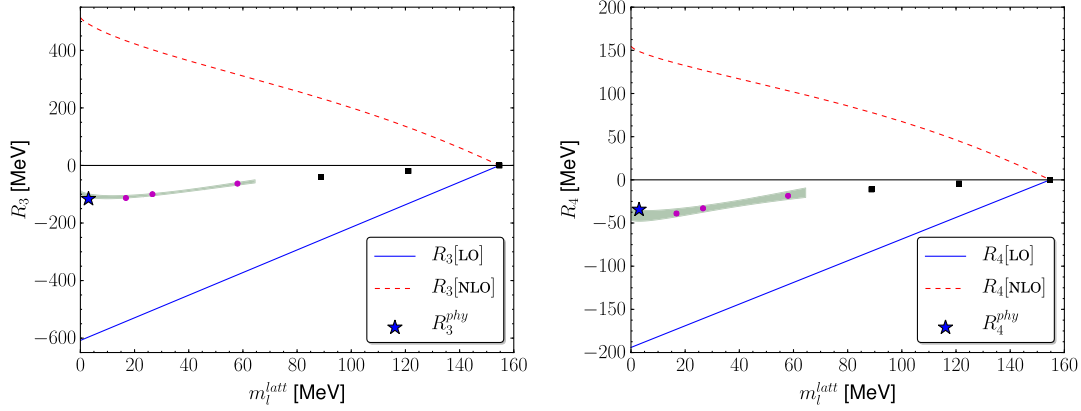$$a_1[\mathrm{NLO}] = 1.4(1).$$

FIG. 3 (color online). The LO and NLO contributions to R_3 (left) and R_4 (right). A (blue) star is used to denote the physical values, not included in the analysis. The particular fit displayed is a combined analysis of R_3 and R_4 to the data at the lightest three values of m_l^{latt}.

Using the leading large N_c relations with $a_2 = 0$ in Eq. (11), this corresponds to

$$D = 0.70(5), \qquad F = 0.47(3),$$
$$C = -1.4(1), \qquad H = -2.1(2). \qquad (35)$$

The significance of this is prominent; the large value of the axial coupling is strong evidence for the presence of the nonanalytic light quark mass dependence in these mass relations. Further, this is the first time an analysis of the baryon spectrum has returned values of the axial couplings consistent with phenomenology.[7]

However, caution is in order. Examining the resulting contributions to R_3 and R_4 from LO and NLO separately, one observes a delicate cancellation between the different contributions (see Fig. 3). Further studies are needed with more numerical data sufficient to also constrain the parameter a_2 as well as the NNLO contributions.

C. Gell-Mann–Okubo relation

The leading contribution to the Gell-Mann–Okubo relation is from a flavor-**27**, which in HBχPT comes from the leading nonanalytic light quark mass dependence [Eq. (21)]. For this reason, it is a particularly interesting mass relation to study, as has been done in Refs. [12,62]. In this work, the analysis is taken further. First, it is demonstrated that the numerical results are inconsistent with a Taylor expansion about the $SU(3)$ vector limit. Second, a NNLO analysis is performed for the first

---

[7]Finding values of the axial couplings consistent with phenomenology has not just been a challenge for lattice QCD, but also observed in large N_c χPT analysis of the experimentally measured baryon magnetic moments [97,98]. It is also interesting to note that while the $SU(3)$ chiral expansion for the baryon spectrum is not convergent, it was found that the volume dependence of the octet baryon masses is consistent with $SU(3)$ HBχPT. Analysis of the volume dependence yielded a large value of $g_{\pi N\Delta}(C)$ with g_A fixed to its physical value [99].

and it is demonstrated, at this order, HBχPT naturally accommodates the strong light quark mass dependence observed, and is dominated by the nonanalytic contributions.

In Fig. 4, four plots are displayed. In each plot, the (red) star is the physical value, which is not used in the analysis. The (blue) circles are numerical data points included in the analysis and the (gray) squares are not included. The filled (blue) band is the one-sigma confidence interval. The first plot (upper left) is the result of a NLO analysis of the GMO formula, allowing the axial coupling to be determined from the data, resulting in a small, but nonzero, value for a_1. For comparison, the upper right plot displays the predicted NLO results for the GMO relation for $a_1 = 1.4(1)$. This highlights tension between the NLO analysis/prediction and the numerical data.

Close to the $SU(3)$ vector limit, the GMO relation can be described by a Taylor expansion in $m_s - m_l$,

$$\Delta_{\text{GMO}}^V = d_2(m_s - m_l)^2 + d_3(m_s - m_l)^3 + \cdots. \qquad (36)$$

The leading term proportional to $(m_s - m_l)$ must vanish as it transforms as a flavor-**8**. The $(m_s - m_l)^2$ contribution is equivalent to a NNLO contribution from HBχPT and the $(m_s - m_l)^3$ contribution is equivalent to a NNNNLO HBχPT contribution. The lower left plot displays a fit to the numerical data to these first two nonvanishing contributions in the Taylor expansion, which do not give rise to the observed steep rise as $m_l^{\text{latt}} \to 0$. Inclusion of the $(m_s - m_l)^4$ term provides a fit which agrees with the data and the physical value but with a severely nonconverging expansion for $m_l^{\text{latt}} < 100$ MeV.

Finally, the NNLO analysis is displayed, using Eqs. (21) and (22) with $a_2 = 0$, and taking a_1 from the determination from R_3 and R_4 (bottom right). Only the NNLO analysis is consistent with the values of the numerical data over the full range of light quark masses, in particular, the steep rise observed as $m_l^{\text{latt}} \to 0$, as well as
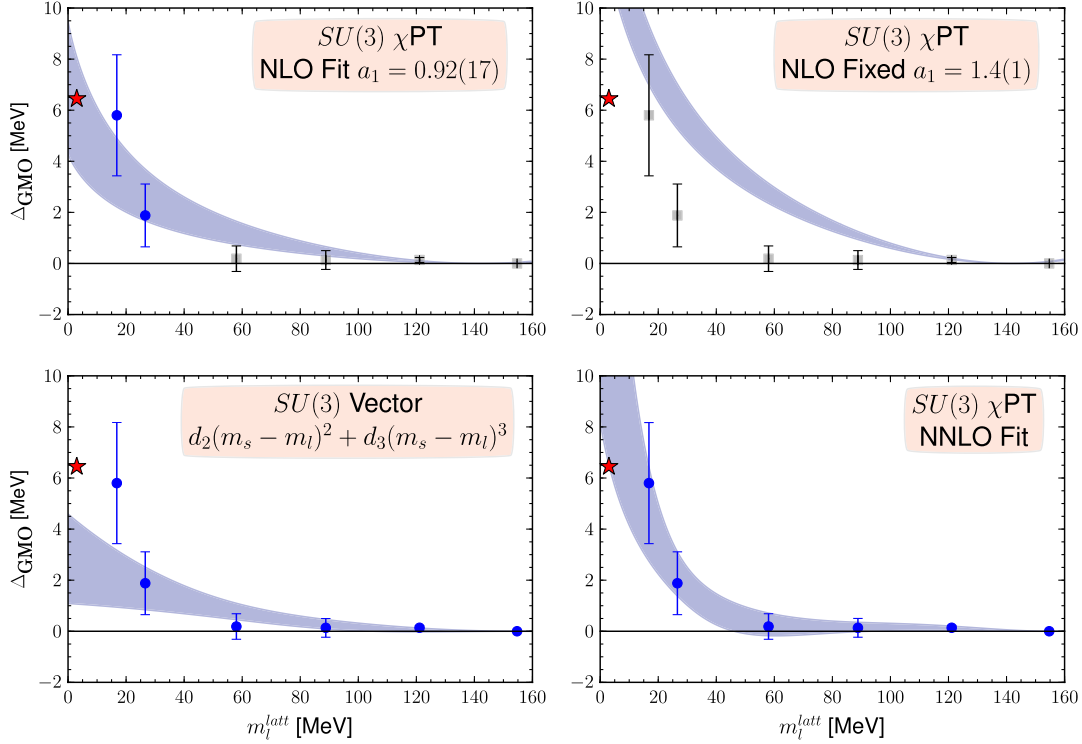
FIG. 4 (color online).   GMO mass splitting plotted as a function of m_l^{latt}. The star is the Particle Data Group point, not included in the analysis. The various fits are described in the text. In a given plot, the (blue) circles denote results included in the analysis while the (gray) boxes are excluded.
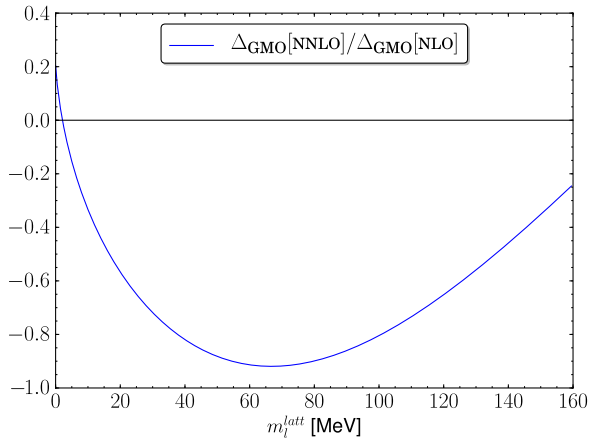

FIG. 5 (color online).   The ratio of NNLO to NLO contributions to the GMO relation from the NNLO analysis.

the value of the axial coupling a_1 determined from phenomenology. However, as in the case of R_3 and R_4, the convergence is not great. The NNLO contributions are smaller than the NLO contributions over the full range of quark masses but are as large as 90% in magnitude compared with the NLO contributions: see Fig. 5. Despite this issue, the success of the NNLO analysis is further evidence for the presence of nonanalytic light quark mass dependence in the baryon spectrum.

VI. CONCLUSIONS

In this article, we have presented the first substantial evidence for nonanalytic light quark mass contributions to the baryon spectrum. This was achieved by comparing the predictions from heavy baryon χPT, combined with the large N_c expansion to relatively high statistics lattice computations of the octet and decuplet spectrum. The numerical results available [12] allowed for a detailed comparison of the mass relations R_1, R_3 and R_4 [43] as well as the Gell-Mann–Okubo relation. It was demonstrated that the mass relation R_1 does not support large values of the axial couplings, signaling a failed convergence of $SU(3)$ heavy baryon χPT for this quantity. An analysis of mass relations R_3 and R_4 provided, for the first time, values of the axial couplings which are consistent with the phenomenological determination, signaling significant contributions from nonanalytic light quark mass dependence in R_3 and R_4. At leading order in the large N_c expansion, it was found

$$D = 0.70(5), \qquad F = 0.47(3),$$
$$C = -1.4(1), \qquad H = -2.1(2).$$

It was further demonstrated that the Gell-Mann–Okubo relation is inconsistent with the first two nonvanishing terms in a Taylor expansion about the $SU(3)$ vector limit, and that the steep rise in the numerical data, observed as $m_l^{\text{latt}} \to 0$, can only be described by the NNLO heavy

baryon χPT formula which is dominated by chiral loop contributions. Taken together, these observations indicate the first significant evidence for the presence of nonanalytic light quark mass dependence in the baryon spectrum.

This is not the definitive work however. There are several known systematics which were not addressed in the present article, and require future, more precise lattice results:

   (i) the numerical data used [12] exist at only a single lattice spacing,

   (ii) a continuum χPT analysis was performed,

   (iii) there may be contamination from finite volume effects [99],

   (iv) the convergence issues need further examination,

   (v) more precise numerical results are needed to explore mass relations R_5–R_8 which are more sensitive to nonanalytic light quark mass dependence,

   (vi) results with smaller values of the light quark mass are desirable,

   (vii) the strange quark mass used in this work is known to be $\sim$25% to large [100].

We regard the most severe of these systematics the convergence issues observed in R_3, R_4 and the GMO relation. While this work is very promising, the analysis needs to be carried out to at least one higher order in both the chiral and large N_c expansions. A further order in the large N_c expansion is needed to determine the subleading axial coupling a_2, allowing for a more detailed comparison with the phenomenologically determined axial couplings, D, F, H and C. Taking the analysis to one higher order in the chiral expansion is crucial to test whether the observed convergence improves or not, and also to test whether the large values of the axial couplings will persist. Unfortunately, exploring these systematics, as well as all mentioned above, is beyond the scope of this work, as it requires a larger set of statistically precise numerical lattice QCD results, which do not yet exist. There are currently some lattice calculations underway which may be able to explore these mass relations in more detail. In particular, the new strategy presented in Refs. [101,102], where the sum of the quark masses is held fixed, $m_u + m_d + m_s = \bar{m}$, for a range of light and strange quark masses, proves very promising for comparing with predictions from χPT. Further, the strategy is not limited to the spectrum, with similar relations having been recently determined for the baryon magnetic moments [103].

ACKNOWLEDGMENTS

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