"Short" spinning strings and structure of quantum $AdS_5 \times S^5$ spectrum

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Using information from the marginality conditions of vertex operators for the $AdS_5 \times S^5$ superstring, we determine the structure of the dependence of the energy of quantum string states on their conserved charges and the string tension $\sim\sqrt{\lambda}$. We consider states on the leading Regge trajectory in the flat space limit which carry one or two (equal) spins in AdS_5 or S^5 and an orbital momentum in S^5, with Konishi multiplet states being particular cases. We argue that the coefficients in the energy may be found by using a semiclassical expansion. By analyzing the examples of folded spinning strings in AdS_5 and S^5, as well as three cases of circular two-spin strings, we demonstrate the universality of transcendental (zeta-function) parts of few leading coefficients. We also show the consistency with target space supersymmetry with different states belonging to the same multiplet having the same nontrivial part of the energy. We suggest, in particular, that a rational coefficient (found by Basso for the folded string using Bethe Ansatz considerations and which, in general, is yet to be determined by a direct two-loop string calculation) should, in fact, be universal.

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I. INTRODUCTION AND SUMMARY

Recent progress in understanding the integrable system that should be computing the spectrum of the maximally supersymmetric example of AdS/CFT duality makes it important to further develop a detailed matching of the Bethe ansatz predictions with quantum $AdS_5 \times S^5$ string energies extracted from the perturbative string theory. While direct near-flat-space expansion of the quantum string theory determining the large tension ($T = \frac{\sqrt{\lambda}}{2\pi}$) expansion of quantum string energies with fixed quantum charges is still to be developed, here we shall follow the "semiclassical" approach suggested in Ref. [1] (see also Ref. [2]) and recently applied in Refs. [3–6] to demonstrate the matching of the numerical results of the thermodynamic Bethe ansatz (TBA) for the Konishi operator dimension interpolated from weak to strong coupling [7–9] with the perturbative string theory prediction for the corresponding string energy.

Our motivation is to further understand the structure of the dependence of the string energy on the string tension and its quantum numbers (spins) guided by the expected form of the string vertex operator marginality conditions [1,4] and recent progress on the Bethe ansatz side [10]. We shall consider several string states which belong (in the flat-space limit) to the leading Regge trajectory and for the lowest values of the spins or the lowest value of the string level represent states in the Konishi multiplet and discover the universality of some leading-order coefficients in the expansion of their energies.

A. General structure of the inverse tension expansion of the energy

Let us start with describing the general form of the dependence of the energy E of a string state on its quantum charges Q_i in the large string tension expansion ($\sqrt{\lambda} \gg 1$).[1] As follows from the structure of α' expansion of two-dimensional (2D) anomalous dimensions of the corresponding $AdS_5 \times S^5$ string vertex operators [11,12], the solution of the marginality condition should give $E = E(Q, \sqrt{\lambda})$ in the following general form [1,4]:

$$E^2 = 2\sqrt{\lambda}\sum_i a_i Q_i + \sum_{i,j} b_{ij} Q_i Q_j + \sum_i c_i Q_i + \frac{1}{\sqrt{\lambda}}\Big(\sum_{i,j,k} d_{ijk} Q_i Q_j Q_k + \sum_{i,j} e_{ij} Q_i Q_j + \sum_i f_i Q_i\Big) + O\Big(\frac{1}{(\sqrt{\lambda})^2}\Big), \quad (1.1)$$

where Q_i are supposed to be fixed in the limit $\sqrt{\lambda} \gg 1$. The highest power of charges in $\frac{1}{(\sqrt{\lambda})^n}$ term here is $n + 2$. This follows, e.g., from dimensional analysis, from the fact that higher-order terms in 2D anomalous dimension operator may contain higher derivative operators (e.g., E^2 comes from $SO(2,4)$ Casimir originating from Laplacian on AdS_5, etc.; see Ref. [12]) and also from the fact that, in any theory, an $(n + 1)$-loop Feynman graph renormalizing a (vertex) operator contains at most $(n + 2)$ Wick

[1]Examples of these charges discussed below are spins S_1, S_2 in AdS_5 and spins J_1, J_2, J_3 in S^5.

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contractions with fields in the (vertex) operator and thus contributes to its dimension terms like $Q^m/(\sqrt{\lambda})^n$ with $m \le n+2$.

More explicitly, if we consider a string state with an orbital momentum $J_3 \equiv J$ in S^5 and one extra oscillator number N (corresponding, e.g., to an intrinsic spin component due to an extended nature of the string) which determines the value of an effective string level then (1.1) is a consequence of the following 2D marginality condition[2]

$$0 = N + \frac{1}{2\sqrt{\lambda}}(-E^2 + J^2 + n_{02}N^2 + n_{11}N) + \frac{1}{2(\sqrt{\lambda})^2}(n_{01}NJ^2 + n_{03}N^3 + n_{12}N^2 + n_{21}N) + O\left(\frac{1}{(\sqrt{\lambda})^3}\right). \quad (1.2)$$

Including also some higher-order terms, the resulting expression for E^2 may be written as[3]

$$E^2 = 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N + \frac{1}{\sqrt{\lambda}}(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N) + \frac{1}{(\sqrt{\lambda})^2}(\tilde{n}_{11}J^2N + \tilde{n}_{02}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N) + \frac{1}{(\sqrt{\lambda})^3}(\tilde{n}_{01}J^4N + \tilde{n}_{21}J^2N + \tilde{n}_{12}J^2N^2 + n_{05}N^5 + \ldots) + \frac{1}{(\sqrt{\lambda})^4}(\tilde{n}_{11}J^4N + \ldots) + O\left(\frac{1}{(\sqrt{\lambda})^5}\right). \quad (1.3)$$

This expression follows under the assumption that in (1.2) E^2 enters only in the 1-loop $\frac{1}{\sqrt{\lambda}}$ term. On general grounds, as E may be thought of as a global charge analogous to J, one might wonder if (1.2) should also contain terms like $\frac{1}{(\sqrt{\lambda})^k}(E^{k+1} + \ldots + E^mN^n + \ldots)$. However, terms depending only on E (or on E and J) should be 2D scheme-dependent (like higher powers of Laplacian in 2D anomalous dimension operator) and would also contradict the BMN limit $E = J$ in the absence of other charges ($N=0$) leading to spurious $\frac{1}{\sqrt{\lambda}}$ dependent solutions of the marginality condition; they should thus be absent in a scheme preserving target space supersymmetry. Terms in (1.2) involving both E and N like $\frac{1}{(\sqrt{\lambda})^k}E^nN^m$ with $m+n \le k+1$, may be present, but in solving the marginality condition (1.2) for E in perturbative expansion in $\frac{1}{\sqrt{\lambda}}$ they cannot modify the leading-order solution $E^2 = 2\sqrt{\lambda}N + \ldots$, and their perturbative treatment leads just to redefinitions of coefficients already present in Eq. (1.3). Note also that the presence of the mixed terms J^kN^m terms reflects the fact that in curved space the center of mass and internal degrees of freedom do not in general decouple.

Expanding (1.3) in large $\sqrt{\lambda}$ for *fixed* N, J we get

$$E = \sqrt{2\sqrt{\lambda}N}\left[1 + \frac{A_1}{\sqrt{\lambda}} + \frac{A_2}{(\sqrt{\lambda})^2} + \frac{A_3}{(\sqrt{\lambda})^3} + O\left(\frac{1}{(\sqrt{\lambda})^4}\right)\right], \quad (1.4)$$

$$A_1 = \frac{1}{4N}J^2 + \frac{1}{4}(n_{02}N + n_{11}), \quad (1.5)$$

$$A_2 = -\frac{1}{2}A_1^2 + \frac{1}{4}(n_{01}J^2 + n_{03}N^2 + n_{12}N + n_{21}) = \frac{1}{4}\left[n_{21} - \frac{1}{8}n_{11}^2 + \left(n_{12} - \frac{1}{4}n_{11}n_{02}\right)N + \left(n_{03} - \frac{1}{8}n_{02}^2\right)N^2\right] + O(J^2), \quad (1.6)$$

$$A_3 = \frac{1}{128}[(n_{11}^3 - 8n_{11}n_{21} + 32n_{31}) + (3n_{02}n_{11}^2 - 8n_{11}n_{12} - 8n_{02}n_{21} + 32n_{22})N + (3n_{02}^2n_{11} - 8n_{03}n_{11} - 8n_{02}n_{12} + 32n_{13})N^2 + \ldots]. \quad (1.7)$$

Substituting particular values of N and J into (1.3) and (1.4) one can find the expansion of the corresponding quantum string state energy, i.e., the strong-coupling expansion of the dimension of the dual gauge theory operator. Note that the first two terms in the right-hand side of (1.3) have direct flat-space interpretation, so that N plays the role of string level and the spinning string states with maximal value of N for a given value on spin belong to the leading Regge trajectory. For example, $N = 0$ corresponds to massless (supergravity) states and $N = 2$ to states on the first excited string level which contains the Konishi long multiplet as its "floor" and also its "KK descendants" with higher values of J obtained by tensoring with the $[0, J, 0]$ representation [14]. The states in the Konishi multiplet that we will consider here correspond to $N = 2$, $J = 2$; see Refs. [1,4].

The goal is thus to determine the coefficients n_{km} in (1.3). To achieve this, one may use the observation [1,2] that a

[2]Here the $(-E^2 + J^2 + \ldots)$ term is the 1-loop correction to the 2D (anomalous) dimension, the next term is the 2-loop correction, etc., with all the terms at the same order in $\frac{1}{\sqrt{\lambda}}$ being here on the same footing. This expansion should emerge in the sigma model approach upon diagonalization of the 2D anomalous dimension matrix (as, e.g., in the NSR approach or in the context of a pure spinor approach like the one discussed in Ref. [13]). Here we ignore possible shifts of N and E by integers that depend on a choice of a reference vacuum state [in the bosonic string context the left-hand side of (1.2) should be equal to 2].

[3]Here the coefficient of J^2 in the first line should be 1 to be consistent with the BMN limit $N = 0$. Again, we assume that in general E and J may be redefined by possible constant shifts to be consistent with positions in a supermultiplet [e.g., $E(E-4) = J(J+4) + \ldots$ is equivalent to $(E-2)^2 = (J+2)^2 + \ldots$ for simplest point-like states]. This depends on a definition of string vacuum; see Ref. [4] for more details.

similar expansion of the string energy is also found by starting with a solitonic string carrying the same types of charges as the vertex operator representing a particular quantum string state and

(i) first performing the semiclassical expansion $\sqrt{\lambda} \gg 1$ for *fixed* charge densities $\mathcal{Q}_i = \frac{1}{\sqrt{\lambda}} Q_i$, i.e., $(\mathcal{N}, \mathcal{J}) = \frac{1}{\sqrt{\lambda}}(N, J)$, and then

(ii) expanding E in small values of $\mathcal{Q}_i$. Indeed, the limit $\mathcal{Q}_i = \frac{Q_i}{\sqrt{\lambda}} \to 0$ should correspond to taking $\sqrt{\lambda} \gg 1$ for fixed values of the quantum charges Q_i. Assuming that there is no order of limits problem, the same coefficients n_{km} should be found in these two different approaches.

Writing (1.3) in terms of $\mathcal{N}$, $\mathcal{J}$ as

$$\left(\frac{E}{\sqrt{\lambda}}\right)^2 = 2\mathcal{N} + \mathcal{J}^2 + n_{01}\mathcal{J}^2\mathcal{N} + n_{02}\mathcal{N}^2 + n_{03}\mathcal{N}^3 + n_{04}\mathcal{N}^4 + \tilde{n}_{01}\mathcal{J}^4\mathcal{N} + \tilde{n}_{02}\mathcal{J}^2\mathcal{N}^2 + \dots + \frac{1}{\sqrt{\lambda}}(n_{11}\mathcal{N} + \tilde{n}_{11}\mathcal{J}^2\mathcal{N} + \bar{n}_{11}\mathcal{J}^4\mathcal{N} + n_{12}\mathcal{N}^2 + \tilde{n}_{12}\mathcal{J}^2\mathcal{N}^2 + n_{13}\mathcal{N}^3 + \dots) + \frac{1}{(\sqrt{\lambda})^2}(n_{21}\mathcal{N} + \tilde{n}_{21}\mathcal{J}^2\mathcal{N} + n_{22}\mathcal{N}^2 + \dots) + O\left(\frac{1}{(\sqrt{\lambda})^3}\right), \tag{1.8}$$

one can then interpret the coefficient n_{km} in (1.3) as a k-loop contribution to a term scaling as $\mathcal{N}^m$ in the semiclassical expansion, i.e., n_{0m} can be extracted from the classical string energy, n_{1m}—from the 1-loop semiclassical correction, etc. Expanding E in (1.8) in small $\mathcal{N}$ for fixed $\mathcal{J}$ we get

$$\frac{E}{\sqrt{\lambda}} = \mathcal{J} + \left[\frac{\mathcal{N}}{\mathcal{J}}\left(1 + \frac{1}{2}n_{01}\mathcal{J}^2 + \frac{1}{2}\tilde{n}_{01}\mathcal{J}^4 + \dots\right) - \frac{\mathcal{N}^2}{2\mathcal{J}^3}\left(1 + (n_{01} - n_{02})\mathcal{J}^2 + \left(\tilde{n}_{01} - \tilde{n}_{02} + \frac{1}{4}n_{01}^2\right)\mathcal{J}^4 + \dots\right) + \dots\right] + \frac{1}{\sqrt{\lambda}}\left[\frac{\mathcal{N}}{2\mathcal{J}}(n_{11} + \tilde{n}_{11}\mathcal{J}^2 + \bar{n}_{11}\mathcal{J}^4 + \dots) + \frac{\mathcal{N}^2}{2\mathcal{J}^3}\left(-n_{11} + \left(n_{12} - \frac{1}{2}n_{01}n_{11} - \tilde{n}_{11}\right)\mathcal{J}^2 + \left(\tilde{n}_{12} - \bar{n}_{11} - \frac{1}{2}n_{01}\tilde{n}_{11} - \frac{1}{2}\tilde{n}_{01}n_{11}\right)\mathcal{J}^4 + \dots\right) + \frac{\mathcal{N}^3}{4\mathcal{J}^5}\left(3n_{11} + [3\tilde{n}_{11} - 2n_{12} + (3n_{01} - n_{02})n_{11}]\mathcal{J}^2 + \left[2(n_{13} - \tilde{n}_{12}) - n_{01}n_{12} + 3\bar{n}_{11} + \left(3\tilde{n}_{01} - \tilde{n}_{02} + \frac{3}{4}n_{01}^2\right)n_{11} + (3n_{01} - n_{02})\tilde{n}_{11}\right]\mathcal{J}^4 + \dots\right) + \dots\right] + \frac{1}{(\sqrt{\lambda})^2}\left[\frac{\mathcal{N}}{2\mathcal{J}}(n_{21} + \tilde{n}_{21}\mathcal{J}^2 + \dots) + \dots\right] + O\left(\frac{1}{(\sqrt{\lambda})^3}\right). \tag{1.9}$$

It should be noted that the quantum string sigma model loop (i.e., $\alpha' \sim \frac{1}{\sqrt{\lambda}} \ll 1$) expansion in (1.3) is, of course, different from the semiclassical loop expansion in (1.8): in (1.2) or (1.3) the first-order N term is classical, $J^2 + n_{02}N^2 + n_{11}N$ are 1-loop terms, etc., i.e., the coefficients n_{km}, in general, appear at different loop orders in the two expansions.[4] Note also that while each ℓ-loop term in (1.3) is a polynomial of *finite* degree, $(\ell + 1)$, in the charges, this does not, in general, apply to the semiclassical expansion (1.8) where each term may contain an infinite series of terms in the small $\mathcal{J}$, $\mathcal{N}$ expansion. To relate the two expansions, one would need to reorganize or even resum them.[5] For example, the classical string energy term in (1.8) receives contributions from all higher loop orders in (1.3), etc.[6]

Comparison of (1.9) or (1.4), (1.5), (1.6), and (1.7) to (1.3) shows that Eq. (1.3) for the square of the energy provides a much more "economical" description of the spectrum. Computing the semiclassical expansion (1.9) directly one finds, indeed, many relations between the

[4]Note that $n_{\ell 1}$ ($\ell = 1, 2, \dots$) are still ℓ-loop coefficients in both expansions.

[5]In particular, considering $\mathcal{J} \gg \mathcal{N}$ expansion will lead to inverse powers of $\mathcal{J}$ in the semiclassical expansion and thus will require a resummation to relate it to (1.3).

[6]Note also that "nonanalytic" terms [1] like $B_2, B_3, \dots$ in the large $\sqrt{\lambda}$ expansion of the energy $E = \sqrt{2\sqrt{\lambda}N}[1 + \frac{A_1}{\sqrt{\lambda}} + \frac{A_2}{(\sqrt{\lambda})^2} + \dots] + B_1 + \frac{B_2}{\sqrt{\lambda}} + \frac{B_3}{(\sqrt{\lambda})^2} + \dots$, which *a priori* could be present in the energy found by using semiclassical expansion, should not actually appear if this approach is consistent: they would lead to $\lambda^{1/4}$ dependent terms in E^2, i.e., $E^2 = 2\sqrt{\lambda}N + 2\sqrt{2N}[\frac{B_2}{\lambda^{1/4}} + \frac{B_3}{(\lambda^{1/4})^3} + \dots] + \dots$ which cannot be present in the standard sigma model perturbative computation of eigenvalues of 2D anomalous dimension matrix.

coefficients there in agreement with the general structure of E^2 in (1.3).

The expression for E^2 in (1.3) or in (1.8) may be formally organized as an expansion in small $\mathcal{N}$ which will then look like an expansion in powers of N:

$$E^2 = J^2 + h_1(\lambda, J)N + h_2(\lambda, J)N^2 + h_3(\lambda, J)N^3 + \ldots, \tag{1.10}$$

where for fixed J and large λ the coefficient functions h_k are given by

$$h_1 = 2\sqrt{\lambda} + n_{11} + \frac{n_{21}}{\sqrt{\lambda}} + \frac{n_{31}}{(\sqrt{\lambda})^2} + \ldots + J^2\left(\frac{n_{01}}{\sqrt{\lambda}} + \frac{\tilde{n}_{11}}{(\sqrt{\lambda})^2} + \frac{\tilde{n}_{21}}{(\sqrt{\lambda})^3} + \ldots\right) + \ldots, \tag{1.11}$$

$$h_2 = n_{02} + \frac{n_{12}}{\sqrt{\lambda}} + \frac{n_{22}}{(\sqrt{\lambda})^2} + \ldots + J^2\left(\frac{\tilde{n}_{02}}{(\sqrt{\lambda})^2} + \frac{\tilde{n}_{12}}{(\sqrt{\lambda})^3} + \ldots\right) + \ldots, \tag{1.12}$$

$$h_3 = \frac{n_{03}}{\sqrt{\lambda}} + \frac{n_{13}}{(\sqrt{\lambda})^2} + \ldots, \qquad h_4 = \frac{n_{03}}{(\sqrt{\lambda})^2} + \ldots. \tag{1.13}$$

The corresponding expansion of E in small $\mathcal{N}$ for fixed $\mathcal{J}$ is then

$$E = J + \frac{1}{2J}h_1(\lambda, J)N + \ldots, \tag{1.14}$$

i.e., $h_1(\lambda, J)$ may be called, following Ref. [10], a "slope" function. In Ref. [10] it was found exactly in the case of the folded string with spin S in AdS_5 (in this case $N = S$). While the coefficients in the "slope" function h_1 are expected, by analogy with the case in Ref. [10], to be rational (h_1 is determined [10] by the asymptotic Bethe ansatz and is also not sensitive to the phase) the coefficients in the next "curvature" function h_2 are already transcendental (as we shall discuss below n_{12} contains ζ_3, $\tilde{n}_{12}$ contains ζ_5, etc.) and h_2 is expected to be sensitive to "wrapping" corrections.

B. Summary of results for the coefficients

Below we shall consider the examples of "small" semiclassical spinning string states discussed in Refs. [1,4] that fall into the class of states described by (1.3), (1.8), and (1.9). They correspond to quantum string states with angular momentum J and few oscillator modes excited that are responsible for nonzero components of intrinsic spin. More specifically, we shall consider and compare the following solutions[7]: two folded string cases: (S, J) and (J', J) and three rigid two-spin circular string cases: $(J_1 = J_2 \equiv J', J)$, $(S_1 = S_2 \equiv S, J)$, and $(S = J_1 \equiv J', J)$. For lowest values of the winding numbers these represent (in the flat-space limit) states on the leading Regge trajectory with the string level being $N = S$ or $N = J$ in the folded one-spin cases and $N = 2J'$ or $N = 2S$ in the circular two-spin cases.

For example, for $N = 2$ these represent states on the first excited string level. In this case all states with fixed J (i.e., on a fixed KK level [14]) should belong to a single long $PSU(2, 2|4)$ multiplet.[8] Furthermore, the string states with $N = 2$, $J = 2$ are dual to particular states in the Konishi multiplet on the gauge theory side [1,4].

As all operators in a given supermultiplet should have the same four-dimensional anomalous dimension, that means that the corresponding string states should have the same target space energy [up to constant integer or half-integer shifts reflecting their positions in the supermultiplet; such shifts are ignored in (1.3)], i.e., the expression for $E_{N=2}$ as a function of J and λ should be universal, with $E_{N=2}(J = 2, \lambda)$ being equal to the dimension of the Konishi multiplet.

As follows from (1.3), this expected universality of the $N = 2$ value of the energy for any J and $\sqrt{\lambda}$ imposes the following invariance constraints on the coefficients of states within a supermultiplet:

$$n_{01} = \text{inv}, \qquad 2n_{02} + n_{11} = \text{inv}, \qquad 4n_{03} + 2n_{12} + n_{21} = \text{inv}, \tag{1.15}$$

$$2\tilde{n}_{02} + \tilde{n}_{11} = \text{inv}, \qquad 8n_{04} + 4n_{13} + 2n_{22} + n_{31} = \text{inv}, \ldots \tag{1.16}$$

Note that these conditions relate different terms in the semiclassical loop expansion. Once the values of these coefficients are known at least for one state in the multiplet, then (1.15) and (1.16) constrain the coefficients for other states.

Explicitly, these universal coefficients enter $E_{N=2}$ in (1.3) and (1.4) as follows:

$$E_{N=2} = 2\sqrt[4]{\lambda}\left[1 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{(\sqrt{\lambda})^2} + \frac{a_3}{(\sqrt{\lambda})^3} + O\left(\frac{1}{(\sqrt{\lambda})^4}\right)\right], \tag{1.17}$$

$$a_1 = (A_1)_{N=2} = \frac{1}{8}J^2 + \frac{1}{4}(2n_{02} + n_{11}), \tag{1.18}$$

$$a_2 = (A_2)_{N=2} = -\frac{1}{2}a_1^2 + \frac{1}{4}n_{01}J^2 + \frac{1}{4}(4n_{03} + 2n_{12} + n_{21}), \tag{1.19}$$

[7] We shall use the following notation: S_1 and S_2 will stand for spins in AdS_5; $J_1 \equiv J'$ and J_2 will be spins in S^5 and $J_3 \equiv J$ will be orbital momentum in S^5.

[8] For example, the three circular string states in the flat-space limit are related by Lorentz transformations and thus belong to the same multiplet. This should remain so upon switching on the curvature.

$$a_3=(A_3)_{N=2}=-a_1a_2+\frac{1}{4}(2\tilde{n}_{02}+\tilde{n}_{11})J^2+\frac{1}{4}(8n_{04}+4n_{13}+2n_{22}+n_{31}). \quad (1.20)$$

$(a_k)_{J=2}$ are then the coefficients of the string coupling expansion of the dimension of the Konishi multiplet. a_1 thus depends on tree-level n_{02} and 1-loop n_{11} coefficients; a_2 depends on tree-level, extra 1-loop n_{12} and also 2-loop n_{21} coefficients; a_3 depends on tree-level, extra 1-loop $\tilde{n}_{11}$, n_{13}, extra 2-loop n_{22} and also 3-loop n_{31} coefficients, etc.

In general, the highest loop order ℓ coefficient $n_{\ell 1}$ in a_ℓ originates from the slope function h_1 in (1.11) and thus should be *rational* (as found for the (S, J) folded string state in Ref. [10]).[9] The subleading loop order coefficient $n_{\ell-1,2}$ (for $\ell > 1$) originating from h_2 in (1.11) should already be 'transcendental"—containing zeta function $\zeta(2\ell-1)\equiv\zeta_{2\ell-1}$. Also, $n_{\ell-2,3}$ (for $\ell > 2$) should contain $\zeta_{2\ell-1}$, etc. Then the highest transcendentality term in a_ℓ in (1.17) should contain $\zeta_{2\ell-1}$.

Indeed, as we shall see below the 1-loop coefficients n_{1k} obey this pattern: n_{12} contains ζ_3, n_{13} contains ζ_5, etc. What is unclear at the moment is if the 2-loop and higher coefficients in $h_2, h_3, \ldots$ (like $n_{22}, n_{32}, \ldots$) may contain other transcendental constants as well.[10] It would be important to carry out an explicit 2-loop computation of n_{22} to clarify this question.

It is interesting to note that the weak-coupling expansion of the anomalous dimension of the Konishi multiplet states also contains ζ_k constants at 4- and 5-loops (see, e.g., Ref. [17], and references therein) while the transcendentality origin of higher loop coefficients here again appears to be an open question (an answer should follow from an analytic solution of TBA equations at weak coupling [7,8]).

Let us now summarize what is known [1,3–6,10] and what will be found below about the coefficients $n_{\rm km}$, $\tilde{n}_{\rm km}$ in (1.3) using the semiclassical approach. We will try to identify the general universality patterns in the structure of these coefficients. First, in all cases

$$n_{01}=1, \qquad \tilde{n}_{01}=-\frac{1}{4}. \quad (1.21)$$

The universality of n_{01} is in agreement with (1.15). This follows from the universal form of the "near-BMN" expansion of the classical string energy:

$$E^2=J^2+2N\sqrt{\lambda+J^2}+\ldots$$
$$=J^2+N\Big(2\sqrt{\lambda}+\frac{1}{\sqrt{\lambda}}J^2-\frac{1}{4(\sqrt{\lambda})^3}J^4+\ldots\Big)+\ldots, \quad (1.22)$$

where we assumed that $N \ll J \ll \sqrt{\lambda}$. In other words, the first term in the semiclassical expansion of the slope function h_1 in (1.10) is universal: $h_1(\lambda, J)=2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2}+O(\mathcal{J})$.

The classical n_{02}, n_{03} and the leading 1-loop n_{11} coefficients are also rational [1,3]. We find that in all cases

$$2n_{02}+n_{11}=2, \quad (1.23)$$

verifying the first universality relation in (1.15). The value of $\tilde{n}_{11}$ is determined by the term linear in $\mathcal{N}$ in the 1-loop semiclassical energy computed for fixed $\mathcal{J}$ and small $\mathcal{N}$ and then expanded in small $\mathcal{J}$ [see (1.9)]. The results for the folded string [6,10] and the circular string results described below imply that in all cases

$$\tilde{n}_{11}=-n_{11}, \qquad \bar{n}_{11}=n_{11}. \quad (1.24)$$

More generally, these results imply the universality (for the states on the leading Regge trajectory) of the $\mathcal{J}$ dependence of the first two leading terms in the "slope" function h_1 in (1.10) expanded in the semiclassical limit $\sqrt{\lambda}\gg 1$ with $\mathcal{J}=\frac{J}{\sqrt{\lambda}}$ held fixed:

$$h_1=2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2}+\frac{n_{11}}{1+\mathcal{J}^2}+\frac{1}{\sqrt{\lambda}}[n_{21}+\tilde{n}_{21}\mathcal{J}^2+O(\mathcal{J}^4)]+O\Big(\frac{1}{(\sqrt{\lambda})^2}\Big). \quad (1.25)$$

We find also that the leading term in the semiclassical expansion of h_2 in (1.12) has the following general form:

$$h_2=n_{02}+\frac{\tilde{n}_{02}\mathcal{J}^2}{1+\mathcal{J}^2}+\frac{1}{\sqrt{\lambda}}[n_{12}+\tilde{n}_{12}\mathcal{J}^2+O(\mathcal{J}^4)]+O\Big(\frac{1}{(\sqrt{\lambda})^2}\Big). \quad (1.26)$$

Again, by inspection in all cases we observed, in agreement with first relation in (1.16) we find

$$2\tilde{n}_{02}+\tilde{n}_{11}=0, \quad (1.27)$$

so that [using (1.23) and (1.24)]

$$\tilde{n}_{02}=\frac{1}{2}n_{11}=1-n_{02}. \quad (1.28)$$

The 1-loop coefficient n_{12} in (1.3) and (1.26) contains a transcendental ζ_3 part. This was first observed in the small-spin expansion of the folded string [2,18] and pulsating string [19] energy, indicating also that higher-order 1-loop terms should contain ζ_5, etc. constants. The computation of n_{12} for the circular 2-spin string with $J_1=J_2\equiv J'$ $(N=2J')$ in Ref. [1] and for the folded spinning string

[9] In particular, for the (S, J) folded string state [10]: $n_{11}=-1$, $n_{21}=-\frac{1}{4}$, $n_{31}=-\frac{1}{4}$, $n_{41}=-\frac{25}{64}$, $n_{51}=-\frac{13}{16}$, $n_{61}=-\frac{1073}{512}$, etc.

[10] For example, the 2-loop and higher-order terms in the $\ln S$ coefficient of the large S limit of the folded string energy expanded in $\frac{1}{\sqrt{\lambda}}$ contain Dirichlet beta function constants $\mathrm{K}=\beta(2)$, etc. (as well as ζ_k) [15,16].

($N = S$) in Ref. [6] led to the exactly same coefficient of ζ_3 in n_{12}, suggesting its *universality*, i.e., that[11]

$$n_{12} = n'_{12} - 3\zeta_3, \tag{1.29}$$

where n'_{12} is a rational number depending on a particular string state on the leading Regge trajectory. The universality of the ζ_3 coefficient in (1.29) will be confirmed below also for the two other examples of the "small" circular string solutions: with two equal spins $S_1 = S_2$ in AdS_5; with one spin in AdS_5 and one spin $J_1 \equiv J'$ in S^5 with $S = J'$, $N = 2S$ (in Ref. [1] only n_{11} was computed in these cases).

As was found in Ref. [10] from the exact computation of the "slope" function h_1 in (1.10) for the "ground-state" (S, J) state in $sl(2)$ sector [corresponding to the folded (S, J) string], the 2-loop coefficient n_{21} is rational and given by[12]

$$n_{21} = -\frac{1}{4}. \tag{1.30}$$

In view of (1.15) and the observed universality of ζ_3 in n_{12} (1.29) the rationality of n_{21} should apply also to other states under consideration. Indeed, using the values of $n_{03} = -\frac{3}{8}$, $n'_{12} = \frac{3}{8}$ [6] and (1.30) [10] for the folded (S, J) string case the universality of the third combination in (1.15) translates into

$$4n_{03} + 2n'_{12} + n_{21} = -1. \tag{1.31}$$

Remarkably, as we shall find below, this constraint implies the same value (1.30) for the 2-loop coefficient n_{21} also for the folded (J', J), circular $(J_1 = J_2, J)$ and circular $(S_1 = S_2, J)$ strings. We thus suggest that this value $n_{21} = -\frac{1}{4}$, like the value of the ζ_3 coefficient in (1.29), should again be the same for all the states on the leading Regge trajectory.[13] This universality of n_{21} may help understand how to generalize the exact result of Ref. [10] for the function h_1 in (1.10) to states outside the $sl(2)$ sector. While the direct 2-loop computation of n_{21} is yet to be done for the circular string cases, the value (1.30) can be indirectly obtained from the knowledge of the 1-loop coefficients by using the expected universality of the subleading a_2 coefficient in the dimension of the Konishi state (1.19).

[11]The ζ_3 coefficient is no longer universal for an m-folded string [6] but has simple m^2 dependence (see also Sec. II B below for the corresponding circular string case).

[12]The simplicity of this coefficient may *a priori* be surprising as it should be given by some 2-loop world-sheet theory integral (with discrete sum over spatial momenta).

[13]The universality of this subleading coefficient in the slope function is supported by the fact that while n_{11} is sensitive to the curvature of subspace where string moves (i.e., it changes sign between the AdS_5 and S^5 cases) the 2-loop correction (determining, in particular, n_{21}) depends on the square of the curvature.

Note that in view of (1.29) and (1.31) the coefficients in the Konishi multiplet energy (1.17) take the following explicit form:

$$(a_1)_{J=2} = 1, \qquad (a_2)_{J=2} = \frac{1}{4} - \frac{3}{2}\zeta_3. \tag{1.32}$$

The universality of $(a_1)_{J=2} = 1$, i.e., the validity of (1.23) not only for the (S, J) folded [3] but also for the small circular string cases was already verified in Refs. [1,4].

Assuming the universality of the value of n_{21} in (1.30), we get from (1.31)

$$n'_{12} = -\frac{3}{8} - 2n_{03}. \tag{1.33}$$

We shall explicitly confirm this relation (and thus the $n_{21} = -\frac{1}{4}$ prediction) in Sec. II for the circular $J_1 = J_2$ and $S_1 = S_2$ cases. In the case of the circular $S = J'$ string one has $n_{03} = -\frac{1}{2}$ and then (1.33) implies $n'_{12} = \frac{5}{8}$. The direct computation of n'_{12} in this case will be discussed in Sec. II D and Appendix C. As it will be explained in Sec. II A, the result depends on a choice of a summation prescription over the fluctuation frequencies. One particular summation procedure discussed in Appendix C leads to $n'_{12} = \frac{11}{8}$. While so far we were unable to identify a prescription leading to the value $n'_{12} = \frac{5}{8}$ consistent with the universality of (1.30), we believe it should exist. Further support of the universality of n_{21} comes from the folded (J', J) string discussed in Appendix D where we show that in this case $n_{03} = \frac{1}{8}$ and $n'_{12} = -\frac{5}{8}$, in agreement with (1.33).

The 1-loop result for the (S, J) folded string in Ref. [6] [in Eq. (B5) there] and our present results for the circular and (J', J) folded string cases all lead also to the following universal expression for the coefficient $\tilde{n}_{12}$ in (1.3):

$$\tilde{n}_{12} = \tilde{n}'_{12} + 3\zeta_3 + \frac{15}{4}\zeta_5, \tag{1.34}$$

where $\tilde{n}'_{12}$ is a rational number depending on a particular state. Remarkably, like in the case of $n_{11} = -\tilde{n}_{11}$ in (1.24), the ζ_3 term here is the same as in n_{12} in (1.29), up to the sign. The coefficient $\tilde{n}_{12}$ contributes to a higher subleading term a_4 in the Konishi dimension (1.17).

The value of $\tilde{n}_{12}$ can be found from the coefficient of the $\frac{1}{2\sqrt{\lambda}}\mathcal{N}^2\mathcal{J}$ term in (1.9), i.e.,

$$\tilde{n}_{12} - \bar{n}_{11} - \frac{1}{2}(n_{01}\tilde{n}_{11} + \tilde{n}_{01}n_{11}) = \tilde{n}_{12} - \frac{3}{8}n_{11}, \tag{1.35}$$

where we used (1.21). For example, for the (S, J) folded string the result of Ref. [6] gives (1.34) with $\tilde{n}'_{12} = -\frac{27}{16}$.

The coefficient n_{13} can be found also by starting with solutions with $J = 0$, expanding in small $\mathcal{N}$ and comparing to (1.4) and (1.7) (see Sec. II): n_{13} is present in the N^2 term in A_3 in (1.7) which appears at one loop order in the semiclassical expansion (as $\frac{N^2}{(\sqrt{\lambda})^3} = \frac{\mathcal{N}^2}{\sqrt{\lambda}}$). Our 1-loop results for the circular strings ($N = 2J' = 2S$) imply that

$$n_{13} = n'_{13} + n''_{13}\zeta_3 + \frac{15}{4}\zeta_5, \tag{1.36}$$

where n'_{13} and n''_{13} are rational numbers. The coefficient of ζ_5 is again universal. In the semiclassical expansion of the energy at fixed $\mathcal{J}$ the coefficient n_{13} first appears in the $\frac{1}{4\sqrt{\lambda}}\frac{\mathcal{N}^3}{\mathcal{J}}$ term in (1.9), i.e., in the combination

$$2(n_{13} - \tilde{n}_{12}) - n_{01}n_{12} + 3\tilde{n}_{11} + (3\tilde{n}_{01} - \tilde{n}_{02} + \frac{3}{4}n_{01}^2)n_{11} + (3n_{01} - n_{02})\tilde{n}_{11} = (2n''_{13} - 3)\zeta_3 + 2n'_{13} - 2\tilde{n}'_{12} - n'_{12} - n_{11}^2 + n_{11}, \tag{1.37}$$

where we first used (1.21), (1.29), (1.30), and (1.28) and then (1.34) and (1.36). Note that ζ_5 terms cancel out in this combination. The absence of ζ_5 in the coefficient of $\mathcal{N}^3/\mathcal{J}$ term is seen in the expression for the 1-loop energy for the AdS_5 folded string in Ref. [6]; we will also find that the same is true for the folded string in S^5 and the three circular string examples. As for the ζ_3 term in (1.37) appearing in the coefficient of $\mathcal{N}^3/\mathcal{J}$ in (1.9), the result of Ref. [6] and our results described in Sec. II and Appendix D imply that it depends on a particular solution. Thus n''_{13} is not universal (we shall list its values for different solutions below). The results of Ref. [6] in the folded (S, J) string case lead to $n_{11} = -1$, $n'_{12} = \frac{3}{8}$, $n''_{13} = \frac{15}{4}$, $\tilde{n}'_{12} = -\frac{27}{16}$, and thus $n'_{13} = -\frac{9}{16}$.

We expect the 3-loop slope coefficient n_{31} to be rational for all states while the 2-loop coefficient n_{22} to contain only ζ_3 as its highest transcendentality part, i.e.,

$$n_{22} = n'_{22} + n''_{22}\zeta_3. \tag{1.38}$$

Then the universality of the combination $8n_{04} + 4n_{13} + 2n_{22} + n_{31}$ in (1.16) is consistent with the universality of the ζ_5 coefficient in (1.36). Thus the next-order coefficient a_3 in the first excited string level state energy (1.20) should contain a ζ_5 part.

Explicitly, as follows from the above discussion [cf. (1.27) and (1.36)] the coefficients in the energy (1.17) for the states on the first excited string level take the form:

$$a_1 = \frac{1}{8}J^2 + \frac{1}{2}, \tag{1.39}$$

$$a_2 = -\frac{1}{2}a_1^2 + \frac{1}{4}J^2 - \frac{1}{4} - \frac{3}{2}\zeta_3 = -\frac{1}{128}J^4 + \frac{3}{16}J^2 - \frac{3}{8} - \frac{3}{2}\zeta_3, \tag{1.40}$$

$$a_3 = -a_1a_2 + 2n_{04} + n_{13} + \frac{1}{2}n_{22} + \frac{1}{4}n_{31} = \frac{1}{4}a_1(2a_1^2 - J^2 + 1) + 2n_{04} + n'_{13} + \frac{1}{2}n'_{22} + \frac{1}{4}n_{31} + \left(\frac{3}{16}J^2 + \frac{3}{4} + n''_{13} + \frac{1}{2}n''_{22}\right)\zeta_3 + \frac{15}{4}\zeta_5. \tag{1.41}$$

The universality of a_3 implies that the coefficient of ζ_3 and thus $n''_{13} + \frac{1}{2}n''_{22}$ should have state-independent value. For the folded (S, J) string a_1, a_2 in (1.39) and (1.40) appeared in Refs. [3,6]. In this case the 3-loop coefficient n_{31} can be inferred from the exact expression (A2) for the "slope" h_1 in Ref. [10], i.e., $n_{31} = -\frac{1}{4}$. Using also that for folded string solution $n_{04} = \frac{31}{64}$ and the value for n_{13} in (1.36) given by $n_{13} = -\frac{9}{16} + \frac{15}{4}\zeta_3 + \frac{15}{4}\zeta_5$ (see Ref. [6] and (D28)) we conclude that for this state we should get

$$a_3 = \frac{1}{1024}(J^2 + 4)(J^4 - 24J^2 + 48) + \frac{11}{32} + \frac{1}{2}n'_{22} + \frac{1}{2}\left(\frac{3}{8}J^2 + 9 + n''_{22}\right)\zeta_3 + \frac{15}{4}\zeta_5. \tag{1.42}$$

To fix a_3 we thus need to know the 2-loop coefficient n_{22} in h_2 in (1.12). As the folded string is an elliptic solution, the required direct 2-loop string computation appears to be hard. It should be easier to find n_{22} for the rational circular $J_1 = J_2$ solution. In that case n_{31} should be again rational, while [see (2.22)] $n'_{13} = -\frac{3}{16}$, $n''_{13} = -\frac{3}{4}$ so that the coefficient of ζ_3 in a_3 is $\frac{1}{2}(\frac{3}{8}J^2 + n''_{22})$. The universality of this coefficient could be checked by an independent computation of n_{22} by another circular string, e.g., $S_1 = S_2$ one.

It would be interesting also to extend the numerical TBA analysis in Ref. [9] to test the universal J dependence of a_3 and extract the value of n_{22} for the folded string state. The $J = 2, 3, 4$ data in Ref. [9] suggests that $n_{22} \sim -10$.

Let us now list the values of few leading coefficients n_{km}, $\tilde{n}_{km}$ for various folded and circular spinning strings adding question marks next to the values that were not yet derived directly but are conjectured to be true on the basis of the universality of (1.30) (see also table in Appendix E). For the folded strings with one spin N in AdS_5 or S^5 and an S^5 orbital momentum J one finds:

(i) folded string in AdS_5 with (S, J), $N = S$ [2,3,6,10]:

$$n_{01} = 1, \quad n_{02} = \frac{3}{2}, \quad n_{03} = -\frac{3}{8}, \quad n_{04} = \frac{31}{64}, \quad \tilde{n}_{02} = -\frac{1}{2}, \quad n_{11} = -1, \quad \tilde{n}_{11} = 1, \quad n'_{12} = \frac{3}{8}, \quad n''_{13} = \frac{15}{4}, \quad n_{21} = -\frac{1}{4}; \tag{1.43}$$

(ii) folded string in S^5 with (J', J), $N = J'$ [5,20]:

$$n_{01} = 1, \quad n_{02} = \frac{1}{2}, \quad n_{03} = \frac{1}{8}, \quad n_{04} = \frac{1}{64}, \quad \tilde{n}_{02} = \frac{1}{2}, \quad n_{11} = 1, \quad \tilde{n}_{11} = -1, \quad n'_{12} = -\frac{5}{8}, \quad n''_{13} = -\frac{3}{4}, \quad n_{21} = -\frac{1}{4}(?). \tag{1.44}$$

The value of n_{12} in (1.29) and (1.44) and $\tilde{n}_{11}$ will be determined below in Appendix D following the algebraic curve approach of Refs. [3,5,6]

For the circular strings with two equal spins in AdS_5 or S^5 and an S^5 momentum J one finds:

(iii) circular string with $(J_1 = J_2, J)$ $N = J_1 + J_2 = 2'$ [1,4] (see Sec. II B):

$$n_{01} = 1, \quad n_{02} = 0, \quad n_{03} = 0, \quad n_{04} = 0, \quad \tilde{n}_{02} = 1,$$
$$n_{11} = 2, \quad \tilde{n}_{11} = -2, \quad n'_{12} = \frac{3}{8}, \quad n''_{13} = -\frac{3}{4},$$
$$n_{21} = -\frac{1}{4}(?); \tag{1.45}$$

(iv) circular string with $(S_1 = S_2, J)$, $N = S_1 + S_2 = 2S$ [1,4] (see Sec. II C for $\tilde{n}_{11}$ and n'_{12}):

$$n_{01} = 1, \quad n_{02} = 2, \quad n_{03} = -1, \quad n_{04} = 2,$$
$$\tilde{n}_{02} = -1, \quad n_{11} = -2, \quad \tilde{n}_{11} = 2, \quad n'_{12} = \frac{13}{8},$$
$$n''_{13} = \frac{15}{4}, \quad n_{21} = -\frac{1}{4}(?); \tag{1.46}$$

(v) circular string with $(S = J', J)$, $N = S + J' = 2S$[14]:

$$n_{01} = 1, \quad n_{02} = 1, \quad n_{03} = -\frac{1}{2}, \quad n_{04} = \frac{3}{4},$$
$$\tilde{n}_{02} = 0, \quad n_{11} = 0, \quad \tilde{n}_{11} = 0,$$
$$n'_{12} = \frac{5}{8}(?), \quad n''_{13} = \frac{3}{2}, \quad n_{21} = -\frac{1}{4}(?). \tag{1.47}$$

It is useful also to add the corresponding expressions for the pulsating strings with N being the oscillation number (see Ref. [19], and references therein)[15]:

(vi) pulsating string in AdS_5:

$$n_{01} = 1, \quad n_{02} = \frac{5}{2}, \quad n_{03} = -\frac{13}{8},$$
$$n_{11} = -\tilde{n}_{11} = -3(?), \quad n'_{12} = \frac{23}{8}(?),$$
$$n_{21} = -\frac{1}{4}(?); \tag{1.48}$$

(vii) pulsating string in S^5:

$$n_{01} = 1, \quad n_{02} = -\frac{1}{2}, \quad n_{03} = -\frac{1}{8},$$
$$n_{11} = -\tilde{n}_{11} = 3(?), \quad n'_{12} = -\frac{1}{8}(?),$$
$$n_{21} = -\frac{1}{4}(?). \tag{1.49}$$

As discussed in Ref. [19], for $N = 2$ the pulsating strings should also represent states on the first excited string level, i.e., in particular (for $J = 2$) states from the Konishi multiplet. With the above values of n_{km} one indeed reproduces the coefficients in (1.32).

The rest of this paper is organized as follows. In the Sec. II we first comment on the general strategy of computing one-loop correction to the energy of classical solitons and then use it to evaluate the one-loop contributions to the energy of the three "small" circular spinning strings. The necessary characteristic polynomials are collected, in a factorized form, in Appendix B. While the solutions with two spins in AdS_5 or with two spins in S^5 yield coefficients n_{km} in line with the expectations and patterns outlined above, the rational terms in the result for the circular string solution with one spin in AdS_5 and one spin in S^5 are found to be ambiguous, depending on a choice of prescription for the summation of the characteristic frequencies. In Appendix C we compute the one-loop correction to the energy of the same small circular string solution using the algebraic curve approach and find a result consistent with a particular worldsheet summation prescription. In Appendix A we discuss the structure of the leading terms in the slope function h_1 [10] in the semiclassical expansion. The one-loop correction to the energy of folded string with spin in S^5 is found in Appendix D and E contains table with values of the leading coefficients discussed in this paper.

II. ONE-LOOP CORRECTION TO ENERGY OF "SMALL" CIRCULAR STRINGS

Below we shall revisit the semiclassical computation of 1-loop correction to energy of "small" semiclassical circular strings discussed in Refs. [1,4] with the aim to extend the expansion to next subleading order allowing one to extract the value of the coefficient n_{12} in (1.3) and (1.6), and thus n'_{12} in (1.29). In the case of the $J_1 = J_2$ string this

[14]Note that the values of all coefficients listed here are given by the mean average of the values for the $J_1 = J_2$ and $S_1 = S_2$ circular strings: symbolically, $n(SJ) = \frac{1}{2}[n(JJ) + n(SS)]$. An intuitive explanation for this may be that since we are considering a near-flat-space expansion certain leading coefficients should be given just by sums of independent contributions of oscillators in different dimensions. Then to leading order the AdS_5 and S^5 directions should contribute similarly in the near-flat expansion, modulo signs due to opposite sign of the curvature.

[15]To get the required 1-loop coefficients n_{11} it appears that one is to take the fermions in Ref. [19] with antiperiodic boundary conditions. The same applies to folded string cases discussed in Refs. [18,19]; this removes ln2 terms from n_{11} present in the periodic-fermion results of Refs. [2,18,19]; it remains to see that at the end one establishes the full agreement with the algebraic-curve computation of Ref. [3].

was already done in Ref. [1] but we will review this case as well for completeness.

A. General comments on computation of one-loop correction

We will be interested in computing 1-loop corrections to the energy of rigid circular spinning strings in $AdS_5 \times S^5$. While these solutions are among the simplest ones being stationary and leading to fluctuation Lagrangian with constant coefficients, this problem (addressed in the past, e.g., in Refs. [1,21–25]) turns out to be subtle. Expanding the string action near the solution and using a static gauge on fluctuations one ends up with a quadratic fluctuation operator $\Delta_2 = \mathrm{diag}(K_B, K_F)$ for 8 + 8 coupled bosonic+fermionic fluctuation modes. Equivalent result for Δ_2 (restricted to "physical" subspace) is found in the conformal gauge where 2 massless bosonic modes decouple and their contribution is cancelled against the conformal gauge ghost one. Since for all solutions we will consider the target-space time is proportional to the world-sheet one, $t = \kappa\tau$, the 1-loop correction to the target space energy can be found as

$$E_1 = \frac{1}{\kappa} E_{2\mathrm{D}}, \tag{2.1}$$

where $E_{2\mathrm{D}}$ is 1-loop correction to energy of the world-sheet theory on $R \times S^1 [\tau \in (-\frac{T}{2}, \frac{T}{2}), T \to \infty, \sigma \in (0, 2\pi)]$. Since in our case Δ_2 has constant coefficients, $E_{2\mathrm{D}}$ can be found as $\frac{1}{2T} \ln\det \Delta_2 = \frac{1}{2T} \ln \frac{\det K_F}{\det K_B}$. Even though $\ln\det\Delta_2$ is UV finite,[16] the computation of its finite part on 2D cylinder is potentially ambiguous—it may depend on how individual fluctuation modes are defined and how their contributions are combined together. One complication is that the space of bosonic fluctuations is multidimensional. Also, the lack of manifest Bose-Fermi 2D symmetry (like world-sheet supersymmetry in the NSR case) implies an extra ambiguity in choice of a consistent regularization. On general grounds, the choice of a prescription for computation of this quantum correction should be governed by the requirement of preservation of underlying symmetries of the theory (i.e., conserved charges, including "hidden" ones) which are "spontaneously broken" by a choice of a particular background we are expanding around. A practical implementation of this starting directly with the Green-Schwarz $AdS_5 \times S^5$ string action remains a nontrivial task.[17]

To give an example of possible ambiguities, consider a model where

$$E_{2\mathrm{D}} = \frac{1}{2} \sum_{r=1}^{h} c_r \sum_{p_1=-\infty}^{\infty} \int \frac{dp_0}{2\pi} \ln[(p_0 + a_r)^2 - (p_1 + k_r)^2 + m_r^2]. \tag{2.2}$$

Here p_1 is an integer momentum in S^1 direction and the sum rules $\sum_{r=1}^{h} c_r = 0$, $\sum_{i=1}^{h} c_r m_r^2 = 0$ ensure that $E_{2\mathrm{D}}$ is UV finite. The shifts a_i and (integer) k_r reflect particular choice of definitions of fluctuation modes. If one splits the sum over fluctuations into h separate 2D integrals and formally ignores the UV cutoffs in them one may shift the integration/summation variables so that to completely eliminate the dependence on a_r, k_r. However, if one first combines all the contributions into a single integrand the finite result will depend on a_r, k_r.

To evaluate similar 1-loop expressions, one may choose to diagonalize Δ_2 first to get its determinant over "flavor" indices as a product over roots of the corresponding characteristic polynomials, $P_{B,F}(p_0) =$"det"$K_{B,F} = \prod_i [p_0 - \omega_i^{(b,f)}(p_1)]$. One particular prescription for evaluating the resulting integral over p_0 is to first Wick-rotate it (which is equivalent to $i\epsilon$ prescription $p_0 \to p_0 - i\epsilon$).[18] Then performing the integral one gets a sum of absolute values of the characteristic frequencies

$$E_{2\mathrm{D(mod)}} = \frac{1}{4} \sum_{i=1}^{16} \sum_{p_1=-\infty}^{\infty} (|\omega_i^{(b)}(p_1)| - |\omega_i^{(f)}(p_1)|). \tag{2.3}$$

Alternatively, one may also treat the world-sheet theory expanded to quadratic order around the classical solution as a collection of infinitely many coupled harmonic oscillators (found by expanding the 2D fluctuation fields in Fourier series in σ) and evaluate the corresponding vacuum energy using the one-dimensional Hamiltonian (operator) quantization method. As was discussed in Ref. [24,27], upon a diagonalization of the mixing, the contribution of each normal mode to the energy will enter in the sum with a sign $s_i = \pm 1$ determined by a minor of the mixing matrix, i.e., in this case we get

$$E_{2\mathrm{D}(s)} = \frac{1}{4} \sum_{i=1}^{16} \sum_{p_1=-\infty}^{\infty} [s_{i,p_1}^{(b)} \omega_i^{(b)}(p_1) - s_{i,p_1}^{(b)} \omega_i^{(f)}(p_1)]. \tag{2.4}$$

While this expression is equivalent to (2.3) in some standard simple cases, this need not be true in general.[19] The computation in one-dimensional Hamiltonian quantization setting may be sensitive to low values of p_1 when sign of ω_i may fluctuate with p_1 and different treatments may correspond to different choices of oscillator vacuum for

[16] See Ref. [25,26] for discussions of the UV regularization of such determinants.

[17] Unfortunately, in more complicated 2-spin cases the integrability-based algebraic curve approach does not appear to help with the problem of ambiguities in the summation over the fluctuation modes.

[18] It is not clear *a priori* why the standard $i\epsilon$ prescription should be preferred given that 2D Lorentz invariance is broken by the background.

[19] The expression in (2.4) may be thought of also as a result of a generalized $i\epsilon$ prescription: $p_0 \to p_0 - i\tilde{s}_i\epsilon$, with $s_i\omega_i = \tilde{s}_i|\omega_i|$, $\tilde{s}_i^2 = 1$.

low (zero) modes. At the same time, the signs of sufficiently high mode number terms (i.e., with $|p_1| > n =$ finite number) cannot be sensitive to them. Indeed, since the mixing of modes is subleading (at most linear) in p_1 compared to the free kinetic term, the mixing can be ignored for large p_1; in particular,

$$|p_1| > n:\ s_{i,p_1}\omega_i(p_1) = |\omega_i(p_1)|. \tag{2.5}$$

Since the transcendental (ζ_3, ζ_5, etc.) terms that may appear in the expression for the 2D energy can originate solely from a summation over infinite range of p_1 (the sum over any finite set of modes can only produce a rational number), it follows that the *transcendental* parts of the 2D energy should be controlled by the $|p_1| \gg 1$ limit and thus should "not" depend on a sign prescription. Moreover, fluctuations with high mode numbers have large 2D energy and thus probe only short world-sheet distances.[20] Their contribution is thus less sensitive to details of the classical solution which is chosen as an expansion point for the the world-sheet action (they will, however, be sensitive to the "topological" features of the solution, such as winding number). We may then expect that at least some of the coefficients of the transcendental terms in E_1 should be *universal* within a given Regge trajectory (parametrized by values of spins with fixed values of windings). This explains, in particular, the universality of the ζ_3 term in (1.29) and of the ζ_5 terms in (1.34) and (1.36).

The choice of signs s_i may itself be sensitive to the definition of the fluctuation modes (related to shifts in fluctuation frequencies or choice of oscillator vacua that may also be different in different gauge choices). In general, one expects that the whole summation prescription should be determined by the requirement that the target space symmetry algebra is correctly realized on quantum string states. There are more practical physical conditions that are easier to verify, e.g., the vanishing of the one-loop correction to the energy in the limit in which all charges go to zero. The one-loop correction should also vanish in the limit in which the classical solution becomes supersymmetric (in cases where such limit exists),[21] e.g., one may require consistency with the BMN limit.

Another requirement one may impose is an analyticity in the smallest charge. Indeed, in the presence of a large charge one may expect that turning on another charge should be smooth; that is, the derivative of the energy with respect to the smallest charge evaluated at zero should not be singular. This translates into the absence in E_{2D} of fractional powers of small charges, Q^α with $\alpha < 1$. Such a requirement of the absence of "nonanalytic" terms (see Ref. [1]) turns out to be consistent with the structure of the energy (1.3) and (1.4) expected to follow from the marginality condition for the corresponding vertex operator.

[20]Classical scale invariance is broken by the background so this notion makes sense; "short distance" is measured with respect to the characteristic scale of the background which is set by the parameters of the solution.

[21]Such a requirement may seem inconsistent with the fact that the exact target space energy should contain a charge-independent term which describes the position of the corresponding state in a supersymmetry multiplet. However, from the perspective of a quantum string state, this constant term is governed by the fermionic zero mode content and should not be accessible semiclassically.

Circular string with spins $J_1 = J_2$ and orbital momentum J

We shall start with the "small" circular string in S^5 described by the following classical solution [1,4,21,23] ($t = \kappa\tau$, $X_k X_k = 1$):

$$\begin{aligned}
&X_1 + iX_2 = a e^{i(w\tau + m\sigma)}, \qquad X_3 + iX_4 = a e^{i(w\tau - m\sigma)},\\
&X_5 + iX_6 = \sqrt{1-2a^2}\, e^{i\nu\tau}\\
&\mathcal{E}_0^2 = \kappa^2 = \nu^2 + 4m^2a^2 = \nu^2 + \frac{4m^2\mathcal{J}'}{\sqrt{m^2+\nu^2}},\\
&w^2 = m^2 + \nu^2, \qquad \mathcal{J}' \equiv \mathcal{J}_1 = \mathcal{J}_2 = a^2 w,\\
&\mathcal{J} \equiv \mathcal{J}_3 = (1-2a^2)\nu, \qquad \nu = \frac{\mathcal{J}}{1 - \frac{2\mathcal{J}'}{\sqrt{m^2+\nu^2}}}.
\end{aligned} \tag{2.6}$$

In the limit $a \to 0$ this becomes a short string with small spin $\mathcal{J}'$. m is a winding number which is to be set to 1 to get a state on the leading Regge trajectory. For $\nu = 0$ the classical energy has the same expression as in flat-space, $\mathcal{E}_0 = 2\sqrt{m\mathcal{J}'}$. Expanding the classical string energy $E_0 = \sqrt{\lambda}\mathcal{E}_0$ for $\mathcal{J}' = \frac{J'}{\sqrt{\lambda}} \ll 1$, $\mathcal{J} = \frac{J}{\sqrt{\lambda}} \ll 1$ and assuming $\mathcal{J}^2 \ll \mathcal{J}'$ we get for $m = 1$

$$E_0 = 2\sqrt{\sqrt{\lambda}J'}\left[1 + \frac{1}{\sqrt{\lambda}}\frac{J^2}{8J'} - \frac{1}{(\sqrt{\lambda})^2}\left(\frac{J^4}{128J'^2} - \frac{J^2}{4}\right) + \dots\right]. \tag{2.7}$$

More generally, if we expand in small $\mathcal{J}'$ for fixed $\rho^2 = \mathcal{J}^2/(4m\mathcal{J}')$, we find

$$\begin{aligned}
E_0 = {}& 2\sqrt{1+\rho^2}\sqrt{m\sqrt{\lambda}J'}\left[1 + \frac{1}{m\sqrt{\lambda}}\frac{\rho^2 J'}{1+\rho^2}\right.\\
&\left. + \frac{1}{(m\sqrt{\lambda})^2}\frac{(4\rho^2 + \rho^4 - 2\rho^6)J'^2}{2(1+\rho^2)^2} + \dots\right],\\
\rho^2 = {}& \frac{J^2}{4m\sqrt{\lambda}J'}.
\end{aligned} \tag{2.8}$$

Expanding this further in the limit $\rho \to 0$ we get back to (2.7) for $m = 1$. An alternative expansion corresponding to $\mathcal{J}' \ll 1$ with fixed $\mathcal{J}$ (i.e., $\rho \gg 1$) gives [cf. (1.10), (1.22), and (1.26)]

$$E_0 = J + \frac{2}{J}\sqrt{m^2\lambda + J^2}J' - \frac{2m^2\lambda(m^2\lambda + 2J^2)}{J^3(m^2\lambda + J^2)}J'^2 + \ldots \tag{2.9}$$

It is useful to perform the one-loop calculation in terms of the two independent semiclassical parameters a and ν. We will first expand in small a at fixed ν and then expand in ν. An important feature of this expansion is that all 1-loop integrals are then regularized in the IR by a nonzero value of ν or $\mathcal{J}$ and therefore a^2 and thus the spin $\mathcal{J}'$ will appear in the 1-loop world-sheet energy only in integer powers, $E_{2D} = \sum_k f_k a^{2k}$. A further expansion in small $\mathcal{J}$ can then be carried out in the resulting coefficients.[22] Then

$$\begin{aligned} E_1 &= \frac{1}{\kappa}E_{2D} \\ &= \frac{1}{\kappa}[f_0(\nu,m) + f_1(\nu,m)a^2 + f_2(\nu,m)a^4 + \ldots] \\ &= e_0(\mathcal{J},m) + e_1(\mathcal{J},m)\mathcal{J}' + e_2(\mathcal{J},m)\mathcal{J}'^2 + \ldots \end{aligned} \tag{2.10}$$

Note that as the expansion of κ or the classical energy (2.9) contains inverse powers of $\mathcal{J}$, terms of higher-order in $\mathcal{J}^{-1}$ in f_i contribute to terms of lower order in the corresponding expansion of e_i. Note also that in view of (2.1) we have

$$\begin{aligned} E^2 &= E_0^2 + 2\sqrt{\lambda}E_{2D} + \ldots \\ &= E_0^2 + 2\sqrt{\lambda}[f_0(\nu,m) + f_1(\nu,m)a^2 \\ &\quad + f_2(\nu,m)a^4 + \ldots] + \ldots \end{aligned} \tag{2.11}$$

To compute the 1-loop energy E_{2D} we need the quadratic fluctuation operators $K_{B,F}$ or the corresponding bosonic and fermionic characteristic polynomials. They can be extracted from Ref. [22] and are listed in Appendix B. As discussed in the previous subsection, we need also to choose an appropriate definition of $\ln\frac{\det K_B}{\det K_F}$ or a quantization scheme in the Hamiltonian approach. Since in the present case the characteristic polynomials depend on p_0 only through p_0^2, for each mode number p_1 we have a positive and a negative root which are equal in absolute value. In the Hamiltonian approach it is then natural to define the vacuum energy as a graded sum of the positive roots [cf. (2.4)]. Such a prescription gives the same result as the path integral approach with the "standard" $i\epsilon$ prescription leading to (2.3). We then find that the one-loop correction to the energy vanishes in the limit $J' \to 0$. This is a required feature since for $J' = 0$ ($a = 0$) and $J' = 0$ the solution (2.6)) reduces to a BMN geodesic.[23]

Let us summarize the results for the 1-loop coefficients (2.10) in the $\mathcal{J}' \ll \mathcal{J} \ll 1$ expansion. Expanding E_{2D} first in a at fixed ν and then expanding the result in small ν we find for the coefficients f_k in (2.10) (for $m = 1$):

$$\begin{aligned} &f_0(\nu,1) = 0, \qquad f_1(\nu,1) = 2 - \nu^2 + \frac{3}{4}\nu^4 + \mathcal{O}(\nu^6), \\ &\qquad f_2(\nu,1) = -\frac{3}{4} - 6\zeta_3 + \mathcal{O}(\nu^2). \end{aligned} \tag{2.12}$$

Then $e_0(\mathcal{J},1) = 0$ and

$$\begin{aligned} &e_1(\mathcal{J},1) = \frac{2}{\mathcal{J}} - 2\mathcal{J} + \mathcal{O}(\mathcal{J}^3), \\ &e_2(\mathcal{J},1) = -\frac{4}{\mathcal{J}^3} + \frac{2}{\mathcal{J}}\left(\frac{5}{8} - 3\zeta_3\right) + \mathcal{O}(\mathcal{J}). \end{aligned} \tag{2.13}$$

Comparing this with the general expression for the energy (1.9) (here $\mathcal{N} = 2\mathcal{J}'$) we conclude that the resulting values of n_{11}, n_{12}, n'_{12}, $\tilde{n}_{11}$ are as given in (1.29) and (1.45). The values of n_{11} and n_{12} were already found in Ref. [1].

We can also find the exact dependence of f_1 and e_1 on $\mathcal{J}$:

$$f_1(\nu,1) = \frac{2}{\sqrt{1+\nu^2}}, \quad e_1(\mathcal{J},1) = \frac{2}{\mathcal{J}(1+\mathcal{J}^2)}. \tag{2.14}$$

Then the coefficient of $\mathcal{J}'$ in the energy, i.e., the semiclassical expansion for the corresponding circular string analog of the "slope" [10] function is [see (1.10) and (1.14)]

$$h_1 = 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} + \frac{n_{11}}{1+\mathcal{J}^2} + \ldots, \quad n_{11} = 2. \tag{2.15}$$

Together with a similar expression found in the (S, J) folded string case [6,10] this provides an evidence of the universality of the general expression in (1.25).

Note that when formally expanded in large $\mathcal{J}$, the function h_1 in (2.15) takes the following form: $h_1 = 2J + \frac{\lambda}{J}(1 + \frac{2}{J} + \ldots) + \ldots$ Here the $\frac{2}{J}$ term is different by a factor of 2 from the result for the leading 1-loop finite size correction found in Ref. [28]. This disagreement should not, however, be surprising as the two expansions are derived in different limits (see also Appendix A). In the present case, relevant for "short" strings, we assumed that $\mathcal{J}' \ll 1$ and $\mathcal{J}$ is fixed. In contrast, the finite size

[22]Note that for fixed $\mathcal{J}$ the small $\mathcal{J}'$ expansions of a and κ [over which we are to divide E_{2D} to get E_1 in (2.1)] are given by

$$\begin{aligned} a &= \frac{\mathcal{J}'^{1/2}}{(\mathcal{J}^2+m^2)^{1/4}} - \frac{\mathcal{J}'^{3/2}\mathcal{J}^2}{(\mathcal{J}^2+m^2)^{7/4}} + \mathcal{O}(\mathcal{J}'^{5/2}), \\ \kappa &= \mathcal{J} + \frac{2\mathcal{J}'\sqrt{\mathcal{J}^2+m^2}}{\mathcal{J}} - \frac{2\mathcal{J}'^2m^2(2\mathcal{J}^2+m^2)}{\mathcal{J}^2(\mathcal{J}^2+m^2)} + \mathcal{O}(\mathcal{J}'^3). \end{aligned}$$

[23]Let us note that to carry out the calculation in a path integral approach in the case of $\mathcal{J} = 0$ one should write the p_0 integral as $\int dp_0 \ln\frac{\det K_B}{\det K_F} = -\int dp_0 p_0 \frac{d}{dp_0}\ln\frac{\det K_B}{\det K_F}$. This integration by parts step here is legal as $\ln\frac{\det K_B}{\det K_F}$ vanishes fast enough at infinity. The resulting rational function may then be expanded in $\mathcal{J}'$ and integrated without a difficulty.

correction calculation of Ref. [28] assumed the standard "fast string" limit of $\mathcal{J}' \gg 1$, $\mathcal{J} \gg 1$ with $\frac{\mathcal{J}'}{\mathcal{J}}$ being fixed and then taken to be small.[24]

Let us now present the results for the 1-loop coefficients $f_k(\nu, m)$ in (2.10) in the case of higher winding numbers $m \geq 1$ (i.e., for states on subleading Regge trajectories):[25]

$$\begin{array}{cccc} & f_0 & f_1 & f_2 \\ m=1 & 0 & 2-\nu^2+\mathcal{O}(\nu^4) & -\frac{3}{4}-6\times 1^4\times\zeta_3+\mathcal{O}(\nu^2) \\ m=2 & 0 & 20-\frac{17}{2}\nu^2+\mathcal{O}(\nu^4) & -\frac{89}{6}-6\times 2^4\times\zeta_3+\mathcal{O}(\nu^2) \\ m=3 & 0 & 60-\frac{247}{12}\nu^2+\mathcal{O}(\nu^4) & -\frac{3357}{40}-6\times 3^4\times\zeta_3+\mathcal{O}(\nu^2) \\ m=4 & 0 & \frac{376}{3}-\frac{4043}{108}\nu^2+\mathcal{O}(\nu^4) & -\frac{263939}{945}-6\times 4^4\times\zeta_3+\mathcal{O}(\nu^2) \end{array} \tag{2.16}$$

Simple inspection shows that the coefficient of ζ_3 in f_2 grows like m^4. This dependence is changed, however, after we express the parameters of the solution in terms of the spins, using, in particular, the relation $a^2 = m^{-1}\mathcal{J}' + \mathcal{O}(\mathcal{J}^2)$. The coefficients $e_k(\mathcal{J}, m)$ in (2.10) are then found to be:

$$\begin{array}{cccc} & e_0 & e_1 & e_2 \\ m=1 & 0 & \frac{2}{\mathcal{J}}-2\mathcal{J}+\mathcal{O}(\mathcal{J}^3) & -\frac{4}{\mathcal{J}^3}+\frac{2}{\mathcal{J}}\left(\frac{5}{8}-3\times 1^2\times\zeta_3\right)+\mathcal{O}(\mathcal{J}) \\ m=2 & 0 & \frac{10}{\mathcal{J}}-\frac{11}{2}\mathcal{J}+\mathcal{O}(\mathcal{J}^3) & -\frac{40}{\mathcal{J}^3}+\frac{2}{\mathcal{J}}\left(\frac{319}{48}-3\times 2^2\times\zeta_3\right)+\mathcal{O}(\mathcal{J}) \\ m=3 & 0 & \frac{20}{\mathcal{J}}-\frac{287}{36}\mathcal{J}+\mathcal{O}(\mathcal{J}^3) & -\frac{120}{\mathcal{J}^3}+\frac{2}{\mathcal{J}}\left(\frac{3821}{240}-3\times 3^2\times\zeta_3\right)+\mathcal{O}(\mathcal{J}) \\ m=4 & 0 & \frac{94}{3\mathcal{J}}-\frac{2233}{216}\mathcal{J}+\mathcal{O}(\mathcal{J}^3) & -\frac{752}{3\mathcal{J}^3}+\frac{2}{\mathcal{J}}\left(\frac{289367}{10080}-3\times 4^2\times\zeta_3\right)+\mathcal{O}(\mathcal{J}) \end{array} \tag{2.17}$$

As in the folded string case [6], the coefficient of ζ_3 in e_2 grows like m^2, supporting the above argument for the universality of the transcendental terms.[26]

It is possible to find higher-orders in the small $\mathcal{N} = 2\mathcal{J}'$ expansion of the one-loop correction (2.10) to the energy:

$$\begin{aligned} E_1 = {}& \left(\frac{1}{\mathcal{J}} - \mathcal{J} + \mathcal{J}^3 + \ldots\right)\mathcal{N} + \left[-\frac{1}{\mathcal{J}^3} + \left(\frac{5}{16} - \frac{3}{2}\zeta_3\right)\frac{1}{\mathcal{J}} - \left(\frac{69}{32} - \frac{3}{2}\zeta_3 - \frac{15}{8}\zeta_5\right)\mathcal{J} \right. \\ & \left. - \left(\frac{655}{128} + \frac{25}{16}\zeta_3 + \frac{15}{8}\zeta_5 + \frac{35}{16}\zeta_7\right)\mathcal{J}^3 + \ldots\right]\mathcal{N}^2 + \left[\frac{3}{2\mathcal{J}^5} + \left(\frac{3}{16} + \frac{3}{2}\zeta_3\right)\frac{1}{\mathcal{J}^3} + \left(\frac{41}{32} - \frac{9}{8}\zeta_3\right)\frac{1}{\mathcal{J}} \right. \\ & \left. - \left(\frac{175}{32} - \frac{33}{8}\zeta_3 - \frac{25}{8}\zeta_5 + \frac{35}{16}\zeta_7\right)\mathcal{J} + \ldots\right]\mathcal{N}^3 + \ldots \end{aligned} \tag{2.18}$$

We notice that through $\mathcal{O}(\mathcal{N}^2)$ order all the transcendental terms are the same as in the case of the folded string in AdS_5 [6]; we will find them also to be the same for other two circular string solutions and the folded string in S^5. Comparing to the general expansion in (1.9) where the corresponding coefficient is in (1.37) we find then the values of $\tilde{n}_{12}$, n_{13} quoted in (1.34), (1.36) with $\tilde{n}'_{12} = -\frac{57}{16}$, $n''_{13} = -\frac{3}{4}$, and $n'_{13} = -\frac{3}{16}$.

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[24]Let us recall the distinction between the "small" and "large" circular 2-spin solutions [21,22]. The distinction is sharp at $\mathcal{J} \equiv \mathcal{J}_3 = 0$: (i) the solution is "small" if $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}'$ is such that $\mathcal{J}' < \frac{1}{2}$ (here $\mathcal{J} = 0$ since $\nu = 0$; this solution is stable); (ii) the solution is "large" if $\mathcal{J}' > \frac{1}{2}$—(here $\mathcal{J} = 0$ since $a^2 = \frac{1}{2}$; this solution is unstable). For nonzero $\mathcal{J}$ the "small" solution may be defined by requiring that $\mathcal{J}^2 \ll \mathcal{J}'$; then its classical energy still starts with $\sqrt{4\mathcal{J}'}$ and thus scales as $\lambda^{1/4}$ for fixed J'. The "large" solution is the one with $\mathcal{J} \sim \mathcal{J}'$ and $\mathcal{J} \gg 1$ so that $\mathcal{E}_0 = \mathcal{J} + 2\mathcal{J}' + \frac{1}{\mathcal{J}}\epsilon(\frac{J'}{J}) + \ldots$ It is stable if $\mathcal{J}' < \frac{3}{2}\mathcal{J}$. While the "small' and "large" cases are smoothly connected for the folded spinning string, that does not apply to the circular 2-spin case as the two expansions have different origins ($a \to 0$ and $a \to \frac{1}{\sqrt{2}}$).

[25]The Green-Schwarz fermions here are taken to be periodic for any m (see Ref. [29]).

[26]An interesting open question is how the quantum string states corresponding to folded and circular spinning strings with $m > 1$ fit into supermultiplets at higher excited string levels. Note, however, that the pattern of the $\frac{1}{\mathcal{J}^3}$ terms in e_2 in (2.17) appears to be different from the one in Ref. [6].

Let us now present the result for the 1-loop correction to the energy in the limit of small $\mathcal{J}'$ and fixed $\rho^2 = \frac{\mathcal{J}^2}{4m\sqrt{\lambda}\mathcal{J}'}$. At fixed ρ and $\mathcal{J}' \ll 1$ the relation between the parameters of the solution and the charges is

$$\begin{aligned}\nu &= 2\rho\sqrt{m\mathcal{J}'}\left[1 + \frac{2\mathcal{J}'}{m} - \frac{4\mathcal{J}'^2(\rho^2-1)}{m^2} + \mathcal{O}(\mathcal{J}'^3)\right],\\ \kappa &= 2\sqrt{m\mathcal{J}'}\sqrt{1+\rho^2}\left[1 + \frac{\mathcal{J}'\rho^2}{m(1+\rho^2)}\right.\\ &\quad\left. + \frac{\mathcal{J}'^2\rho^2(4+\rho^2-2\rho^4)}{2m^2(1+\rho^2)^2} + \mathcal{O}(\mathcal{J}'^3)\right],\\ a^2 &= \frac{\mathcal{J}'^2}{m}\left[1 - \frac{2\mathcal{J}'\rho^2}{m} + \frac{\mathcal{J}'^2(6\rho^4 - 8\rho^2)}{m^2} + \mathcal{O}(\mathcal{J}'^3)\right].\end{aligned} \tag{2.19}$$

We may use these expressions and f_k in (2.10) given in (2.16) to find the fixed-ρ expansion of E_1. Indeed, since $a^2 \propto \mathcal{J}'^2$ contains only positive powers of $\mathcal{J}'$ while κ and ν do not contain inverse powers of $\mathcal{J}'$, higher-orders in the small a and small ν expansion cannot affect lower orders. For $m = 1$ we then find

$$\begin{aligned}E_1 &= \frac{\sqrt{\mathcal{J}'}}{\sqrt{1+\rho^2}}\left[1 + \left(-\frac{3+43\rho^2+32\rho^4}{8(1+\rho^2)} - 3\zeta_3\right)\mathcal{J}'\right.\\ &\quad\left. + \mathcal{O}(\mathcal{J}'^2)\right].\end{aligned} \tag{2.20}$$

Taking the limit $\rho \to 0$ we may read off the value of the coefficient n_{12} in (1.4) and (1.6) (here $n_{02} = 0$)

$$n_{12} = -\frac{3}{8} - 3\zeta_3, \tag{2.21}$$

which is in agreement with (1.29) and (1.45).

It is possible also to determine the transcendental part of the next terms in the small J' expansion of the one-loop energy directly at $J = 0$, extending the $\rho = 0$ limit of the expression in (2.20) and showing that this limit can be safely taken in that equation:

$$\begin{aligned}(E_1)_{\mathcal{J}_1=\mathcal{J}_2=\mathcal{J}',\mathcal{J}=0} &= \sqrt{\mathcal{J}'}\left[1 + \left(-\frac{3}{8} - 3\zeta_3\right)\mathcal{J}'\right.\\ &\quad + 2\left(-\frac{3}{16} - \frac{3}{4}\zeta_3 + \frac{15}{4}\zeta_5\right)\mathcal{J}'^2\\ &\quad\left. + \mathcal{O}(\mathcal{J}'^3)\right].\end{aligned} \tag{2.22}$$

Comparing to (1.7) (where the transcendental part of the N^2 term is contained in $n_{13} - \frac{1}{4}n_{02}n_{12}$) we find the value of n_{13} to be in agreement with (1.36) again with $n'_{13} = -\frac{3}{16}$ and $n''_{13} = -\frac{3}{4}$ (here $n_{02} = 0$).

C. Circular string with spins $S_1 = S_2$ and orbital momentum J

Let us now consider the small string with 2 equal spins in AdS_5 orbiting big circle in S^5 [1,4,21,23] ($Y_0^2 + Y_5^2 - Y_m Y_m = 1$):

$$\begin{aligned}&Y_0 + iY_5 = \sqrt{1+2r^2}e^{i\kappa\tau}, \qquad Y_1 + iY_2 = re^{i(w\tau+m\sigma)},\\ &Y_3 + iY_4 = re^{i(w\tau-m\sigma)}, \qquad X_1 + iX_2 = e^{i\nu\tau},\\ &w^2 = \kappa^2 + m^2, \qquad \kappa^2 = 4m^2r^2 + \nu^2,\\ &\mathcal{E}_0 = (1+2r^2)\kappa = \kappa + \frac{2\kappa\mathcal{S}}{\sqrt{m^2+\kappa^2}},\\ &\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2 = r^2 w, \qquad \mathcal{J} = \nu.\end{aligned} \tag{2.23}$$

Short string limit corresponds to $r \to 0$ when the solution approaches its flat-space limit (for $\nu = 0$). The parameter κ determined from the conformal gauge condition may be written as

$$\kappa^2 = \frac{4m^2}{\sqrt{m^2+\kappa^2}}\mathcal{S} + \mathcal{J}^2. \tag{2.24}$$

Below we shall consider the case of $m = 1$. For small $\mathcal{S}$ and small $\mathcal{J}$ we get the following "short" string expansion of the classical energy ($E_0 = \sqrt{\lambda}\mathcal{E}_0$):

$$\mathcal{E}_0 = 2\sqrt{\mathcal{S}}\left(1 + \mathcal{S} + \frac{\mathcal{J}^2}{8\mathcal{S}} + \ldots\right). \tag{2.25}$$

In the limit of small $\mathcal{S}$ with fixed $\mathcal{J}$ we get

$$\mathcal{E}_0 = \mathcal{J} + \frac{2}{\mathcal{J}}\sqrt{1+\mathcal{J}^2}\mathcal{S} - \frac{2\mathcal{S}^2}{\mathcal{J}^3(1+\mathcal{J}^2)} + \mathcal{O}(\mathcal{S}^3). \tag{2.26}$$

At small $\mathcal{S}$ with fixed $\rho^2 = \frac{\mathcal{J}^2}{4\mathcal{S}}$ we find instead

$$\begin{aligned}\mathcal{E}_0 &= \sqrt{\mathcal{S}}\left[\left(-\frac{1}{4\rho^3} + \frac{1}{\rho} + 2\rho\right) - \left(\frac{1}{2\rho^3} - \frac{1}{\rho} - 2\rho\right)\mathcal{S}\right.\\ &\quad\left. + \left(\frac{1}{\rho^3} - \frac{5}{\rho} - 4\rho - 2\rho^3\right)\mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3)\right].\end{aligned} \tag{2.27}$$

As in the previous $J_1 = J_2$ case it is convenient to carry out the 1-loop calculation in terms of ν and r and then evaluate the result in the two limits: (i) small $\mathcal{S}$ with fixed $\mathcal{J}$ or (ii) small $\mathcal{S}$ with fixed ρ. As in (2.10) the 1-loop correction to the energy may be written as

$$\begin{aligned}E_1 &= \frac{1}{\kappa}E_{2D}\\ &= \frac{1}{\kappa}[f_0(\nu,m) + f_1(\nu,m)r^2 + f_2(\nu,m)r^4 + \ldots]\\ &= e_0(\mathcal{J},m) + e_1(\mathcal{J},m)\mathcal{S} + e_2(\mathcal{J},m)\mathcal{S}^2 + \ldots\end{aligned} \tag{2.28}$$

Using the expressions for the characteristic polynomials in Appendix B[27] and the "standard" choice of summation prescription (2.3) in which we keep unspecified the signs of the terms that vanish in the $r^2 \sim \mathcal{S} \to 0$ limit, we found that expanding first in r and then in ν the expansion of the world-sheet energy E_{2D} in (2.28) contains the following terms:

$$E_{2D} = E_{2D\text{low}} + E_{2D\text{high}},$$
$$E_{2D\text{low}} = \left[-\frac{q}{\nu} - \frac{7}{3} + \frac{235}{216}\nu^2 + \mathcal{O}(\nu^4)\right]r^2 + \left[\frac{q}{\nu^3} - \frac{1565}{432} + \mathcal{O}(\nu^2)\right]r^4 + \mathcal{O}(r^6), \quad (2.29)$$

$$E_{2D\text{high}} = \left[\frac{1}{3} - \frac{19}{216}\nu^2 + \mathcal{O}(\nu^4)\right]r^2 + \left[\frac{2969}{432} - 6\zeta_3 + \mathcal{O}(\nu^2)\right]r^4 + \mathcal{O}(r^6). \quad (2.30)$$

We split the result into the contribution of few "low" modes ($p_1 = 0, \pm 1, \pm 2$) and the rest of "higher" modes. The coefficient q of the singular in $\nu \to 0$ contributions depends on the signs s_{p_1} of low fermionic frequencies which vanish at $r = 0$ for $p_1 = \pm 1$, i.e., $q = 2 + s_1 + s_{-1}$. There is thus a choice of a sign prescription that ensures the absence of unwelcome singular terms in ν. The natural value for this coefficient is $q = 0$ as the complete two-dimensional energy of the solution, whose 1-loop part is E_{2D} above, is the right-hand side of Eq. (1.2) and is therefore expected to contain only even powers of $\mathcal{J} = \nu$. Setting thus $q = 0$, the resulting values of the coefficients f_k in (2.28) are

$$f_0(\nu, 1) = 0, \qquad f_1(\nu, 1) = -2 + \nu^2 + \mathcal{O}(\nu^4),$$
$$f_2(\nu, 1) = \frac{13}{4} - 6\zeta_3 + \mathcal{O}(\nu^2). \quad (2.31)$$

Using that $\nu = \mathcal{J}$ and

$$r^2 = \frac{\mathcal{S}}{\sqrt{1+\mathcal{J}^2}} - \frac{2\mathcal{S}^2}{(1+\mathcal{J}^2)^2} + \ldots,$$
$$\kappa = \mathcal{J} + \frac{2}{\mathcal{J}\sqrt{1+\mathcal{J}^2}}\mathcal{S} - \frac{2(1+3\mathcal{J}^2)}{\mathcal{J}^3(1+\mathcal{J}^2)^2}\mathcal{S}^2 + \ldots, \quad (2.32)$$

it follows that e_k in (2.28) are given by

$$e_0 = 0, \qquad e_1 = -\frac{2}{\mathcal{J}} + 2\mathcal{J} + \mathcal{O}(\mathcal{J}^3),$$
$$e_2 = \frac{4}{\mathcal{J}^3} + \frac{2}{\mathcal{J}}\left(\frac{5}{8} - 3\zeta_3\right) + \mathcal{O}(\mathcal{J}). \quad (2.33)$$

[27]They can be obtained from those in the $J_1 = J_2$ case as the two solutions are related by an analytic continuation effectively interchanging the AdS_5 and S^5 parts, $a^2 \to -r^2$, $\kappa \to \nu$, etc.

Comparing to (1.9) (here $\mathcal{N} = 2\mathcal{S}$) we find, in agreement with (1.29) and (1.46), that in the present case $n_{01} = 1$, $n_{02} = 2$, $n_{11} = -2$, $\tilde{n}_{11} = 2$, and

$$n_{12} = \frac{13}{8} - 3\zeta_3. \quad (2.34)$$

The value of n_{11} was previously found in Ref. [1]. The value $n'_{12} = \frac{13}{8}$ is the expected one, i.e., is in agreement with (1.33), implying the universality of the value of the energy for the corresponding (Konishi-multiplet) state with $J = S = 2$ on the lowest massive string level.

As in (2.18) we may determine the transcendental part of the higher-order terms in the small $\mathcal{S}$ expansion of the energy ($\mathcal{N} = 2\mathcal{S}$)[28]:

$$E_1 = \left(-\frac{1}{\mathcal{J}} + \mathcal{J} - \mathcal{J}^3 + \ldots\right)\mathcal{N} + \left[\frac{1}{\mathcal{J}^3} + \left(\frac{5}{16} - \frac{3}{2}\zeta_3\right)\frac{1}{\mathcal{J}} + \left(-\frac{93}{32} + \frac{3}{2}\zeta_3 + \frac{15}{8}\zeta_5\right)\mathcal{J} + \ldots\right]\mathcal{N}^2 + \left[-\frac{3}{2\mathcal{J}^5} + \left(\frac{3}{2}\zeta_3 - \frac{3}{16}\right)\frac{1}{\mathcal{J}^3} + \left(\frac{9}{8}\zeta_3 - \frac{41}{32}\right)\frac{1}{\mathcal{J}} + \left(\frac{363}{32} - \frac{43}{8}\zeta_3 - 5\zeta_5 - \frac{35}{16}\zeta_7\right)\mathcal{J} + \ldots\right]\mathcal{N}^3 + \ldots \quad (2.35)$$

Comparing to (1.9) and (1.37) the $\mathcal{O}(\mathcal{J}\mathcal{N}^2)$ term here gives the value of $\tilde{n}_{12}$ in (1.34) with $\tilde{n}'_{12} = -\frac{105}{16}$. Together with the absence of ζ_5 at $\mathcal{O}(\mathcal{N}^3/\mathcal{J})$, this determines n_{13} as quoted in Eq. (1.36) with $n'_{13} = -\frac{85}{16}$ and $n''_{13} = \frac{15}{4}$.

Next, let us mention the case of small $\mathcal{S}$ expansion for fixed $\rho^2 = \frac{\mathcal{J}^2}{4\mathcal{S}}$. Since the expressions in (2.32) contain the exact $\mathcal{J}$ dependence, we may get the corresponding E_1 from E_{2D} in (2.28) and (2.31) [cf. (2.20)]

$$E_1 = \frac{\sqrt{\mathcal{S}}}{\sqrt{1+\rho^2}}\left[-1 + \left(\frac{21}{8} + 4\rho^2 - 3\zeta_3\right)\mathcal{S} + \mathcal{O}(\mathcal{S}^2)\right]. \quad (2.36)$$

Taking the limit $\rho \to 0$ we may read off again the value of the coefficient n_{12} in (1.4), (1.6), and (2.34).[29]

Summing up the small ν expansion of the function $f_1(\nu, 1)$ in (2.31) we may find the exact form of $e_1(\mathcal{J}, 1)$ in (2.33):

$$f_1(\nu, 1) = -\frac{2}{\sqrt{1+\nu^2}}, \qquad e_1(\mathcal{J}, 1) = -\frac{2}{\mathcal{J}(1+\mathcal{J}^2)}. \quad (2.37)$$

[28]It is interesting to mention that, in a small ν expansion of the coefficient $f_2(\nu, 1)$ in E_{2D}, at $\mathcal{O}(\nu^0)$ there is only ζ_3 term and at $\mathcal{O}(\nu^2)$ there is only ζ_5 for both $J_1 = J_2$ and $S_1 = S_2$ cases. This implies that ζ_3 in $\tilde{n}_{12}$ has the same origin as ζ_3 in n_{12}: the only difference in its coefficient comes from the expansion of $\frac{a^4}{\kappa}$ vs. $\frac{r^4}{\kappa}$.

[29]Note that in (2.36) we have the following combination: $n'_{12} - \frac{1}{4}n_{11}n_{02} = \frac{13}{8} + 1 = \frac{21}{8}$.

These expressions are just negative of the corresponding functions in the $J_1 = J_2$ case in (2.14), in agreement with the general expression (1.25) and the opposite signs of the n_{11} coefficients in (1.45) and (1.46).

One may also perform the computation of E_1 by setting $\mathcal{J} = 0$ directly from the start.[30] While similarly to the $J_1 = J_2$ string case the characteristic polynomials here depend only on p_0^2 and thus for each mode number there are two roots equal in absolute value and opposite in sign, a sign prescription similar to that of the $J_1 = J_2$ case in which the one-loop energy is given by the graded sum of the positive roots of the characteristic polynomial (2.3) leads to an unwanted feature: a nonzero value for $E_{2\text{D}}$ in the $\mathcal{S} \to 0$ limit (see also Eq. (3.35) in Ref. [1]). As discussed in Appendix A of Ref. [1], this constant term may be removed by a specific reorganization of modes together with a change of integration variables, leading to a cancellation of the problematic term at the level of the p_0 integrand (so that a specific $i\epsilon$ prescription was not necessary). The same result may be obtained by adjusting the sign of just one root of each of the two fermionic characteristic polynomials F_1 and F_2 which for $p_1 = \pm 1$ scale as $\sqrt{\mathcal{S}}$ in the limit $\mathcal{S} \to 0$: their signs should be such that their contribution adds up to zero.[31] Then the "low" modes with $p_1 = 0, \pm 1, \pm 2$ contribute to the sum over the roots of the characteristic polynomial as

$$E_1 = \frac{1}{\kappa}(E_{2\text{Dlow}} + E_{2\text{Dhigh}}),$$
$$E_{2\text{Dlow}} = -\frac{7r^2}{3} - \frac{1565 r^4}{432} + \mathcal{O}(r^6), \qquad (2.38)$$

$$E_{2\text{Dhigh}} = \sum_{p_1=3}^{\infty}\Big[-\frac{4}{p_1(1-p_1^2)}r^2 + \frac{4 - 17p_1^2 + 137p_1^4 - 40p_1^6}{p_1^3(4-p_1^2)(1-p_1^2)^3}r^4 + \mathcal{O}(r^6)\Big]. \qquad (2.39)$$

Using that $E_{2\text{Dhigh}} = \frac{2}{3}r^2 + (\frac{2969}{216} - 12\zeta_3)r^4 + \mathcal{O}(r^6)$ we find

$$E_1 = \sqrt{\mathcal{S}}\Big[-1 + \Big(\frac{21}{8} - 3\zeta_3\Big)\mathcal{S} + \mathcal{O}(\mathcal{S}^2)\Big], \qquad (2.40)$$

which is the same as the $\rho = 0$ limit of (2.36).

It is possible also to find the analog of (2.22), i.e., to determine the transcendental part of the next terms in the expansion of the one-loop energy of the $S_1 = S_2$ string at $J = 0$, extending (2.40) to next order:

$$(E_1)_{\mathcal{S}_1=\mathcal{S}_2=\mathcal{S},\mathcal{J}=0} = \sqrt{\mathcal{S}}\Big[-1 + \Big(\frac{21}{8} - 3\zeta_3\Big)\mathcal{S} + 2\Big(-\frac{59}{8} + \frac{21}{4}\zeta_3 + \frac{15}{4}\zeta_5\Big)\mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3)\Big]. \qquad (2.41)$$

Comparing to (1.7) we conclude that the highest transcendental coefficient ζ_5 at the next order is again universal, leading to the expression for n_{13} in (1.36) again with $n'_{13} = -\frac{85}{16}$ and $n''_{13} = \frac{15}{4}$.

D. Circular string with spins $S = J'$ and orbital momentum J

The "mixed" $\text{AdS}_5 \times S^5$ circular solution is described by (we set the two windings equal to 1)

$$Y_0 + iY_5 = \sqrt{1+r^2}e^{i\kappa\tau}, \qquad Y_1 + iY_2 = re^{i(w\tau+\sigma)},$$
$$w^2 = \kappa^2 + 1, \qquad X_1 + iX_2 = ae^{i(w'\tau-\sigma)},$$
$$X_3 + iX_4 = \sqrt{1-a^2}e^{i\nu\tau}, \qquad w'^2 = \nu^2 + 1, \qquad (2.42)$$

$$\kappa^2 - \nu^2 = 2r^2 + 2a^2, \qquad r^2 w = a^2 w',$$
$$\mathcal{E}_0 = \kappa(1+r^2), \quad \mathcal{S} = r^2 w = a^2 w' = \mathcal{J}', \quad \mathcal{J} = (1-a^2)\nu. \qquad (2.43)$$

Note that this solution is "self-dual" under the analytic continuation interchanging AdS_5 and S^5 parts: $\kappa \leftrightarrow \nu$, $r \leftrightarrow ia$, $w \leftrightarrow -w'$. The parameters κ and ν may be expressed in terms of the spins by solving the equations

$$\kappa^2 - \nu^2 = \frac{2\mathcal{S}}{\sqrt{1+\kappa^2}} + \frac{2\mathcal{S}}{\sqrt{1+\nu^2}},$$
$$\mathcal{J}_2 = \nu - \frac{\nu\mathcal{S}}{\sqrt{1+\nu^2}}. \qquad (2.44)$$

The classical energy has the following expansions:

$$(\mathcal{E}_0)_{\mathcal{S}\ll 1,\mathcal{J}=\text{fixed}} = \mathcal{J} + \frac{2}{\mathcal{J}}\sqrt{1+\mathcal{J}^2}\mathcal{S} - \frac{2}{\mathcal{J}^3}\mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3), \qquad (2.45)$$

$$(\mathcal{E}_0)_{\mathcal{S}\ll 1,\rho^2=\frac{\mathcal{J}^2}{4\mathcal{S}}=\text{fixed}} = 2\sqrt{1+\rho^2}\sqrt{\mathcal{S}}\Big[1 + \frac{1+2\rho^2}{2(1+\rho^2)}\mathcal{S} - \frac{5 + 8\rho^2 + 12\rho^4 + 8\rho^6}{8(1+\rho^2)^2}\mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3)\Big], \qquad (2.46)$$

$$(\mathcal{E}_0)_{\mathcal{J}^2\ll\mathcal{S}\ll 1} = 2\sqrt{\mathcal{S}}\Big(1 + \frac{1}{2}\mathcal{S} + \frac{\mathcal{J}^2}{8\mathcal{S}} + \dots\Big). \qquad (2.47)$$

As in the previous cases we shall carry out the 1-loop computation in terms of the parameters ν and r and then evaluate the result in the small $\mathcal{S}$ limit with fixed $\mathcal{J}$ or fixed

[30] For $\mathcal{J} = \nu = 0$ one has $\kappa = 2r = 2\sqrt{\mathcal{S}} - 2\mathcal{S}^{3/2} + 9\mathcal{S}^{5/2} + \mathcal{O}(\mathcal{S}^{7/2})$, etc.

[31] Interestingly, the only effect of this choice is to remove the problematic term and thus to restore the expected $\mathcal{S} \to 0$ limit (all related higher integer powers of $\mathcal{S}$ are simultaneously removed).

$\rho^2 = \frac{\mathcal{J}^2}{4\mathcal{S}}$, i.e., we will define f_k and e_k as in (2.28). We will need the following small $\mathcal{S}$ expansions of the parameters:

$$\kappa = \mathcal{J} + \frac{2+\mathcal{J}^2}{\mathcal{J}\sqrt{1+\mathcal{J}^2}}\mathcal{S} - \frac{2+6\mathcal{J}^2+3\mathcal{J}^4}{\mathcal{J}^3(1+\mathcal{J}^2)^2}\mathcal{S}^2 + \ldots,$$
$$r = \frac{\mathcal{S}^{1/2}}{(1+\mathcal{J}^2)^{1/4}} - \frac{2+\mathcal{J}^2}{2(1+\mathcal{J}^2)^{7/4}}\mathcal{S}^{3/2} + \ldots,$$
$$\nu = \mathcal{J} + \frac{\mathcal{J}\mathcal{S}}{\sqrt{1+\mathcal{J}^2}} + \frac{\mathcal{J}\mathcal{S}^2}{(1+\mathcal{J}^2)^2} + \ldots \quad (2.48)$$

The corresponding characteristic polynomials are given in Appendix B. The summation prescription in (2.4) may be fixed as follows. All frequencies which are nonzero in the BMN limit ($r \to 0$) are summed with uniform signs such that at $p_1 \gg 1$ they contribute positively to the energy (this guarantees the vanishing of 1-loop correction to the BMN vacuum state). The signs of some remaining frequencies are fixed by requiring the absence of $\frac{r^2}{\nu}$ terms in the frequency sum. Few other signs are fixed by requiring that all $\mathcal{O}(r^2)$ terms vanish (such terms are expected to cancel due to opposite curvatures of AdS_5 and S^5). Then as in (2.29), (2.30), and (2.38) we may split the contribution of modes with $p_1 = -2, \ldots, 2$ from that of the higher ones

$$E_{\rm 2Dlow} = \left[\frac{1}{2}\left(-\frac{961}{72} - \frac{9}{8}u\right) + \frac{1}{2}\left(\frac{141337}{10368} + \frac{21}{16}u\right)\nu^2 + \mathcal{O}(\nu^4)\right]r^4 + \mathcal{O}(r^6), \quad (2.49)$$

$$E_{\rm 2Dhigh} = \sum_{p_1=3}^{\infty}\left[\frac{4p_1^4+11p_1^2-3}{p_1^3(1-p_1^2)^3} + \mathcal{O}(\nu^2)\right]r^4 + \mathcal{O}(r^6) = \left[\frac{1033}{144} - 6\zeta_3 + \mathcal{O}(\nu^2)\right]r^4 + \mathcal{O}(r^6). \quad (2.50)$$

Here the parameter u represents the still unfixed sum of 4 bosonic $p_1 = \pm 2$ frequency signs; it can take values $u = -4, -2, 0, 2, 4$. Then f_k in the analog of (2.28) are

$$f_0(\nu, 1) = 0, \qquad f_1(\nu, 1) = 0,$$
$$f_2(\nu, 1) = \frac{1}{2} - \frac{9u}{16} - 6\zeta_3 + \mathcal{O}(\nu^2). \quad (2.51)$$

Expanding E_1 first in small $\mathcal{S}$ at fixed $\mathcal{J}$ and then in small $\mathcal{J}$ we get

$$E_1 = \frac{1}{\kappa}E_{\rm 2D} = \left[\frac{2n_{12}}{\mathcal{J}} + \mathcal{O}(\mathcal{J})\right]\mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3), \quad (2.52)$$

$$n_{12} = n'_{12} - 3\zeta_3, \qquad n'_{12} = \frac{8-9u}{32}. \quad (2.53)$$

This gives $n'_{12} = (\frac{11}{8}, \frac{13}{16}, \frac{1}{4})$ for $u = (-4, -2, 0)$. The choice of $n'_{12} = \frac{11}{8}$ appears to be preferred in the algebraic curve approach that we discuss in Appendix C. None of these choices leads to $n'_{12} = \frac{5}{8}$ consistent with the universality of (1.30) observed for *four* other (two folded and two circular) examples of the solutions. This suggests that a consistent summation prescription in this $S = J'$ case is yet to be identified.

Expanding in $\mathcal{S}$ for fixed ρ when

$$\nu = 2\rho\mathcal{S}^{1/2} + 2\rho\mathcal{S}^{3/2} + \mathcal{O}(\mathcal{S}^{5/2}),$$
$$\kappa = 2\sqrt{1+\rho^2}\mathcal{S}^{1/2} - \frac{\mathcal{S}^{3/2}}{\sqrt{1+\rho^2}} + \mathcal{O}(\mathcal{S}^{5/2}), \quad (2.54)$$

we get [cf. (2.20) and (2.36)]

$$E_1 = \frac{\sqrt{\mathcal{S}}}{\sqrt{1+\rho^2}}[(n'_{12} - 3\zeta_3)\mathcal{S} + \mathcal{O}(\mathcal{S}^2)], \quad (2.55)$$

where n'_{12} is the same as in (2.53).

Similarly to the $J_1 = J_2$ and $S_1 = S_2$ cases in (2.18) and (2.35), the transcendental parts of the higher terms in the small $\mathcal{S}$ expansion of E_1 here are found to be ($\mathcal{N} = 2\mathcal{S}$)

$$E_1 = \left[\left(\frac{n'_{12}}{2} - \frac{3}{2}\zeta_3\right)\frac{1}{\mathcal{J}} + \left(q_1 + \frac{3}{2}\zeta_3 + \frac{15}{8}\zeta_5\right)\mathcal{J} + \ldots\right]\mathcal{N}^2 + \left[\frac{q_2}{\mathcal{J}^5} + \left(q_3 + \frac{3}{2}\zeta_3\right)\frac{1}{\mathcal{J}^3} + \frac{q_4}{\mathcal{J}} + \left(q_5 - \frac{5}{8}\zeta_3 - \frac{15}{16}\zeta_5 - \frac{35}{16}\zeta_7\right)\mathcal{J} + \ldots\right]\mathcal{N}^3 + \ldots, \quad (2.56)$$

where q_k are rational numbers. The coefficient of $\mathcal{J}\mathcal{N}^2$ term again leads to the same universal value of $\tilde{n}_{12}$ in (1.34) with $\tilde{n}'_{12} = 2q_1$. At $\mathcal{O}(\mathcal{N}^3)$ we should find that $q_2 = \frac{3}{4}n_{11} = 0$ and that $q_3 = -\frac{1}{2}n'_{12}$. The absence of ζ_5 in $\mathcal{N}^3/\mathcal{J}$ term confirms again the universality of ζ_5 in n_{13} in (1.36), the absence of ζ_3 implies that $n''_{13} = \frac{3}{2}$ and the rational term fixes $n'_{13} = 2(q_1 + q_4) + \frac{1}{2}n'_{12}$.

It is also possible to determine unambiguously the transcendental part of E_1 in the small $\mathcal{S}$ expansion at $J = 0$ [cf. (2.22) and (2.41)]

$$(E_1)_{\mathcal{S}=\mathcal{J}', \mathcal{J}=0} = \sqrt{\mathcal{S}}\left[(n'_{12} - 3\zeta_3)\mathcal{S} + 2\left(k_3 + \frac{9}{4}\zeta_3 + \frac{15}{4}\zeta_5\right)\mathcal{S}^2 + \ldots\right], \quad (2.57)$$

where k_3 is a rational number. This again leads to n_{13} in Eq. (1.36) with $n'_{13} = k_3 + \frac{1}{4}n'_{12}$ and $n''_{13} = \frac{3}{2}$ (here $n_{02} = 1$). Consistency of the two values for n'_{13} requires then that $k_3 = 2(q_1 + q_4) + \frac{1}{4}n'_{12}$.

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APPENDIX A: COMMENTS ON SMALL AND LARGE $\mathcal{J}$ EXPANSIONS OF $h_1(\lambda, J)$ IN EQ. (1.10)

Let us comment on the exact expression for the slope function $h_1(\lambda, J)$ in (1.10) proposed in Ref. [10] in the case of the folded spinning string state in the $sl(2)$ sector and its possible generalizations for other string states. One motivation to try understand the structure of h_1 better is that it determines, in particular, the value of the 2-loop coefficient n_{21} in (1.11) that is still to be derived by a direct world-sheet computation.

It was suggested in Ref. [10] that the exact form of h_1 function in the energy (dimension) (1.10) of the $sl(2)$ sector ground state corresponding in the semiclassical limit to the (S, J) folded string in AdS_5 is given by

$$h_1 = 2\sqrt{\lambda}\frac{d}{d\sqrt{\lambda}}\ln I_J(\sqrt{\lambda}) \tag{A1}$$

$$= 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} - \frac{1}{1+\mathcal{J}^2} - \frac{\frac{1}{4}-\mathcal{J}^2}{\sqrt{\lambda}(1+\mathcal{J}^2)^{5/2}} - \frac{\frac{1}{4}-\frac{5}{2}\mathcal{J}^2+\mathcal{J}^4}{(\sqrt{\lambda})^2(1+\mathcal{J}^2)^4} + \dots \tag{A2}$$

$$= 2\sqrt{\lambda+J^2} - \frac{\lambda}{\lambda+J^2} - \frac{\lambda(\frac{1}{4}\lambda - J^2)}{(\lambda+J^2)^{5/2}} - \frac{\lambda(\frac{1}{4}\lambda^2 - \frac{5}{2}\lambda J^2 + J^4)}{(\lambda+\mathcal{J}^2)^4} + \dots, \tag{A3}$$

where I_J is the modified Bessel function and $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$. The second line corresponds to the string semiclassical expansion: $\lambda \gg 1$ for fixed $\mathcal{J}$; the first term in it is the classical string contribution, the second is 1-loop term, the third is 2-loop one, etc. The third line is found by rewriting the semiclassical result back in terms of J.

Starting with $E = \sqrt{J^2 + h_1(\lambda, J)N + \dots}$ in (1.10) and expanding it in semiclassical regime with fixed $\mathcal{J}$ and small $\mathcal{N}$ we get

$$E = J + \frac{N}{2J}h_1(\lambda, J) + \dots = J + \frac{N}{2J}\left[2\sqrt{\lambda+J^2} - \frac{\lambda}{\lambda+J^2} + \dots\right] + \dots \tag{A4}$$

The 1-loop term $-\frac{\mathcal{N}}{2\mathcal{J}}\frac{1}{1+\mathcal{J}^2}$ was found directly in the semiclassical limit in Ref. [6]. As was mentioned in Sec. I, this term is universal, i.e. found also for other semiclassical states [see (1.25)]. This expression can be expanded in several different limits and interpolates between some previously known results. If we assume that $\mathcal{J} \gg 1$, i.e., $J^2 \gg \lambda$, then we get from (A2)

$$h_1 = 2J + \frac{\lambda}{J}\left(1 - \frac{1}{J} + \frac{1}{J^2} + \dots\right) + \dots, \tag{A5}$$

implying that the expansion of E in the large J, small $\frac{\mathcal{N}}{\mathcal{J}}$ limit is

$$E = J + N + \frac{\lambda}{2J^2}N\left(1 - \frac{1}{J} + \frac{1}{J^2} + \dots\right) + O\left(\left(\frac{N}{J}\right)^2\right). \tag{A6}$$

This matches the known tree level plus 1-loop result in string semiclassical expansion.[32] Notice that in $(1 - \frac{1}{J} + \frac{1}{J^2} + \dots)$ in (A5) the string 1-loop term $-\frac{1}{J}$ came from the $-\frac{\lambda}{\lambda+J^2}$ term in (A2) while the string 2-loop term $+\frac{1}{J^2}$ came from the $-\frac{\lambda(\frac{1}{4}\lambda-J^2)}{(\lambda+J^2)^{5/2}}$ term in (A2).

These two leading terms are, in fact, protected, i.e. are the same as on the 1-loop gauge theory (spin chain) side where the $\frac{1}{J}$ term is the leading finite size correction [31]. The structure $(1 - \frac{1}{J})$ of the leading correction appears to be universal: it is found also for the circular (S, J) string [24,31].[33] This is consistent with the relations (1.24) and (1.25). The linear in $\frac{N}{J}$ term comes only from the zero-mode contribution on the string side or only from the non-anomalous finite-size correction on the 1-loop gauge theory side. The next $\frac{1}{J^2}$ correction (1-loop on gauge theory side and 2-loop on the semiclassical string theory side) which should again be protected was computed on the spin chain side in Ref. [32] (to all orders in $\frac{N}{J}$).[34]

If instead we consider the opposite limit of $\mathcal{J} \ll 1$, i.e. $J^2 \ll \lambda$, then we get from (1.6) [10]

$$h_1 = 2\sqrt{\lambda} - 1 - \frac{\frac{1}{4}-J^2}{\sqrt{\lambda}} - \frac{\frac{1}{4}-J^2}{(\sqrt{\lambda})^2} - \frac{\frac{25}{64}-\frac{13}{8}J^2+\frac{1}{4}J^4}{(\sqrt{\lambda})^3} + \dots, \tag{A7}$$

implying the values $n_{11} = -1$, $\tilde{n}_{11} = 1$ and $n_{21} = -\frac{1}{4}$ in (1.11) and (1.43). This value for the 1-loop coefficient n_{11} in the small $\mathcal{S}$ semiclassical expansion (matching the one

[32]For folded string the $\frac{1}{J}$ term was found in Appendix D of Ref. [30].

[33]To see that there is no linear in $S/J \equiv N/J$ term in the "anomalous" part of the 1-loop correction $E_{\rm anom} = \frac{\lambda}{2J^2} \times (\sum_{n=1}^{\infty}[n\sqrt{n^2+4M^2} - n^2 - 2M^2]$ where $M^2 = \frac{S}{J}(1+\frac{S}{J})$ one needs to differentiate this over M^2 (the first derivative vanishes).

[34]As we have checked explicitly from the results in the Appendix of Ref. [32], the same subleading $1/J^2$ finite-size term as in (A6) appears also for the circular (S, J) string state in the $sl(2)$ sector (here J is the momentum along the circle in S^5 which the string is wound on). This suggests the universality of the terms given explicitly in (A6) in the $sl(2)$ sector.

directly computed using the algebraic curve approach in Ref. [3]) has the same origin in (A2) as the $\frac{1}{J}$ string term in (A6): both come from two different limits of the the 1-loop semiclassical term $-\frac{1}{1+\mathcal{J}^2}$ there. This confirms that this term should not be sensitive to wrapping (Luscher) corrections, being at the same time the origin of a finite-size (and even nonanomalous) term at large J. This also suggests that, like the coefficient of the $-\frac{1}{J}$ term, n_{11} may be coming only from the zero-mode contributions in the near folded-string expansion. This supports the claim [10] that $h_1(\lambda, J)$ has its origin just in the asymptotic Bethe ansatz and is not even sensitive to the string phase.

One may expect to find similar expressions for the corresponding (J', J) folded string state in the $su(2)$ sector. Indeed, the folded string in S^5 is related to its AdS_5 counterpart by an analytic continuation [33], implying (up to signs) $(E, S; J) \rightarrow (E; J', J)$, $E = -J$, $S = J'$, $J = -E$. In this case $N = J'$ so we may expect to get similar relations as above up to some sign changes, i.e.,[35]

$$E^2 = J^2 + h_1(\lambda, J)J' + \ldots,$$
$$h_1 = 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} + \frac{1}{1+\mathcal{J}^2} + \ldots \qquad \text{(A8)}$$

Changing the of sign of the subleading term in (A8) compared to (A2) has two implications: the signs of n_{11}, of $\tilde{n}_{11}$ and of the leading $\frac{1}{J}$ term also change. Now $n_{11} = 1 = -\tilde{n}_{11}$ as in (1.44) in agreement with Refs. [1,5] (see also Appendix D).[36] For large $\mathcal{J}$ we get

$$E = J + J' + \frac{\lambda}{2J^2}J'\left(1 + \frac{1}{J} + \frac{1}{J^2} + \ldots\right) + \ldots, \qquad \text{(A9)}$$

where the $(1 + \frac{1}{J})$ term is in agreement with the result for the finite size corrections from the spin chain and the string sides (cf. Eqs. (7.33, 7.34) in Ref. [30]). As in the $sl(2)$ sector case in (A2) and (A5), the subleading term $\frac{1}{J^2}$ in (A9) should originate from the next (string 2-loop) term in h_1 in (A8). The coefficient of this $\frac{1}{J^2}$ term should be universal in the $su(2)$ sector, i.e., the same also as for the circular string. Indeed, for the circular string in the $su(2)$ sector we get (A9) with the same terms in the bracket $(1 + \frac{1}{J} + \frac{1}{J^2} + \ldots)$, as one can see from Ref. [34] (these terms come from nonanomalous finite size contribution only). Such a correction in the near-BMN expansion was found also in Ref. [30]. It came out the same from the Bethe ansatz and the Landau-Lifshitz approach, so it should be a protected one.[37] Direct check of the universality of the $\frac{1}{J^2}$ term requires a 2-loop computation on the string side. The knowledge of this $\frac{1}{J^2}$ term provides *a priori* only a weak constraint on a possible next term in the expansion of h_1 in (A8), but there is a natural guess: the direct analog of the $-\frac{\lambda(\frac{1}{4}\lambda - J^2)}{(\lambda+J^2)^{5/2}}$ term in (A3) reproduces both the $\frac{1}{J^2}$ term and the expected universal value of n_{21} in (1.30) [see (1.44)].

In the case of "small" circular strings with 2 internal spins we again find

$$h_1 = 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} + \frac{n_{11}}{1+\mathcal{J}^2} + \ldots, \qquad \text{(A10)}$$

where, e.g., for $(J_1 = J_2 = J', J)$ case $N = J_1 + J_2 = 2J'$ and $n_{11} = 2 = -\tilde{n}_{11}$ [see (1.45)]. Indeed, according to (2.37), in this case

$$E_1 = \frac{\mathcal{N}}{\mathcal{J}(1+\mathcal{J}^2)} + O(\mathcal{N}^2),$$
$$h_1(J \gg \sqrt{\lambda}) = 2J\left[1 + \frac{\lambda}{2J^2}(1 + \frac{n_{11}}{J}) + \ldots\right]. \qquad \text{(A11)}$$

The term $1 + \frac{n_{11}}{J}$ with $n_{11} = 2$ here appears to be in contradiction with the form of the finite size correction—$(1 + \frac{1}{J})$ times the classical $\frac{\lambda}{2J^2}N$ term—found earlier [28,31].[38] As already mentioned below Eq. (2.15) this is not really a disagreement as, in the *2-spin case*, the two expressions are derived in different limits: here we have $\mathcal{J}' \ll 1$ for fixed $\mathcal{J}$, while in the standard discussions of finite-size corrections in the thermodynamic limit one first assumes $\mathcal{J}' \gg 1$, $\mathcal{J} \gg 1$, with $\frac{\mathcal{J}'}{\mathcal{J}}$ = fixed, and then may expand in $\frac{\mathcal{J}'}{\mathcal{J}}$.

APPENDIX B: CHARACTERISTIC POLYNOMIALS FOR CIRCULAR STRING FLUCTUATION FREQUENCIES

Rigid circular strings with two equal spins and orbital momentum J in S^5 discussed in this paper are homogeneous solutions for which the quadratic fluctuation operator has constant coefficients. In Fourier transformed form this is a matrix depending on 2D momenta (p_0, p_1) [with p_1 being integer as $\sigma \in (0, 2\pi)$] whose determinant is thus

[35]Note that this analytic continuation is not useful if J is fixed, while $E \sim \lambda^{1/4} \gg 1$ so there is no way of interchanging E and J. It still works at large $\mathcal{J}$ and thus large $\mathcal{E}$ and explains why the sign of first finite size correction changes: $E = J + \frac{\lambda N}{2J^2} \times (1 - J^{-1} + J^{-2})$ translates into $J = E - \frac{\lambda N}{2E^2}(1 + E^{-1} + E^{-2})$ and then using that $E = J + \ldots$ we get the required result.

[36]The change of sign of the leading 1-loop string correction can be attributed to the change in sign of the curvature between AdS_5 and S^5 [1].

[37]The fact that it comes out of the Landau-Lifshitz approach means that one does not need the full superstring computation to reproduce it, provided one regularizes properly (in addition, only zero modes are expected to contribute to this term).

[38]This structure from expansion of Eq. (2.23) in Ref. [31] to linear order in $\mathcal{N}$: again only the analytic spin chain side part or 0-mode string side part is contributing to it. It appears that the analytic finite size correction to the linear in $\mathcal{N}$ term is universal: $1 + \frac{1}{L}$ in compact [su(2), etc.] sector and $1 - \frac{1}{L}$ in non-compact [sl(2), etc.] sector. Here $L = J + N$ is total length, its difference from J is irrelevant to leading order in N. The sign difference is due to the analytic continuation between the sectors.

a finite-order polynomial in (p_0, p_1). The roots of this characteristic polynomial determine the fluctuation frequencies $p_0 = \omega(p_1)$ that appear in the 1-loop correction to 2D energy [see (2.3) or (2.4)]. While we focused on the solutions with unit winding number, $m = 1$, a nontrivial value of m may be introduced in the characteristic equations for all three circular string solutions through the formal rescaling

$$p_0 \to \frac{p_0}{m}, \qquad p_0 \to \frac{p_1}{m}, \qquad \kappa \to \frac{\kappa}{m}, \qquad \nu \to \frac{\nu}{m},$$
$$w \to \frac{w}{m}, \qquad w' \to \frac{w'}{m}, \qquad r \to r, \qquad a \to a.$$

This rescaling may be identified in the classical solutions (2.6), (2.23), and (2.42).

1. $J_1 = J_2$ string

The characteristic polynomials for this circular string have been derived in Refs. [21,22]. The AdS_5 fluctuations have the standard BMN type form with mass κ [expressed in terms of the other independent parameters a and ν; see (2.6)] while the characteristic polynomial for the S^5 part is more complicated. Explicitly,

$$B_8^{AdS_5} = (-p_0^2 + p_1^2 + \nu^2 + 4m^2a^2)^4, \tag{B1}$$

$$\begin{aligned} B_8^{S^5} = &[(p_0^2 - p_1^2)^2 - 4\nu^2 p_0^2]^2 - 16(2a^2-1)m^4(p_0^2 - p_1^2)^2 \\ &+ 8m^2[(a^2-1)(p_0^2-p_1^2)^2(p_0^2+p_1^2) \\ &- 4\nu^2 p_0^2[(a^2-1)p_0^2 + (1-3a^2)p_1^2]]. \end{aligned} \tag{B2}$$

As discussed in Refs. [21,22], the determinant of the fermionic quadratic operator is the square of the determinant of an operator expressed solely in terms of six-dimensional Dirac matrices. We note here that, due to the chirality of six-dimensional spinors, this determinant (over spinor indices) further factorizes:

$$\det K_f^{10D} = (\det K_f^{6D})^2, \qquad \det K_f^{6D} = F_1 F_2, \tag{B3}$$

where $F_{1,2}$ are the corresponding fermionic characteristic polynomials

$$\begin{aligned} F_1 = &(p_0^2 - p_1^2)^2 + p_0^2[\nu(-\sqrt{4a^2m^2+\nu^2} - 3\nu) \\ &- 2(a^2+1)m^2] + p_1^2[\nu(\nu - \sqrt{4a^2m^2+\nu^2}) \\ &+ (6a^2-2)m^2] + (a^2-1)^2 m^4 + m^2\nu[\nu + (a^2-1) \\ &\times \sqrt{4a^2m^2+\nu^2}] + \frac{1}{2}\nu^3(\nu - \sqrt{4a^2m^2+\nu^2}), \end{aligned} \tag{B4}$$

$$\begin{aligned} F_2 = &(p_0^2 - p_1^2)^2 + p_0^2[\nu(\sqrt{4a^2m^2+\nu^2} - 3\nu) - 2(a^2+1)m^2] \\ &+ p_1^2[\nu(\nu + \sqrt{4a^2m^2+\nu^2}) + (6a^2-2)m^2] \\ &+ (a^2-1)^2 m^4 + m^2\nu[\nu - (a^2-1)\sqrt{4a^2m^2+\nu^2}] \\ &+ \frac{1}{2}\nu^3(\nu + \sqrt{4a^2m^2+\nu^2}). \end{aligned} \tag{B5}$$

Using the relations between the parameters of the solution, one can check that the product F_1F_2 reproduces the fermionic characteristic polynomial in Ref. [22].

2. $S_1 = S_2$ string

As was mentioned in Sec. II, this solution may be obtained from the $J_1 = J_2$, J by the analytic continuation

$$\kappa \leftrightarrow \nu, \qquad a^2 \leftrightarrow -r^2. \tag{B6}$$

This observation may be used to find the corresponding characteristic polynomials from their $J_1 = J_2$ counterparts. The bosonic ones are then

$$\begin{aligned} B_8^{AdS_5} = &[(p_0^2 - p_1^2)^2 - 4\kappa^2 p_0^2]^2 - 16(2r^2-1)m^4(p_0^2-p_1^2)^2 \\ &+ 8m^2[(r^2-1)(p_0^2-p_1^2)^2(p_0^2+p_1^2) \\ &- 4\kappa^2 p_0^2[(r^2-1)p_0^2 + (1-3r^2)p_1^2]], \end{aligned} \tag{B7}$$

$$B_8^{S^5} = (-p_0^2 + p_1^2 + \nu^2)^4. \tag{B8}$$

The fermionic determinant has factorization property similar to that in the $J_1 = J_2$, J solution (B3) with

$$\begin{aligned} F_1 = &(p_0^2 - p_1^2)^2 + p_0^2[-\kappa(\nu + 3\kappa) - 2(-r^2+1)m^2] \\ &+ p_1^2[\kappa(\kappa - \nu) - 2(3r^2+1)m^2] + (r^2+1)^2 m^4 \\ &+ m^2\kappa[\kappa - (r^2+1)\nu] + \frac{1}{2}\kappa^3(\kappa - \nu), \end{aligned} \tag{B9}$$

$$\begin{aligned} F_2 = &(p_0^2 - p_1^2)^2 + p_0^2[\kappa(\nu - 3\kappa) - 2(-r^2+1)m^2] \\ &+ p_1^2[\nu(\nu + \kappa) - 2(3r^2+1)m^2] + (r^2+1)^2 m^4 \\ &+ m^2\kappa[\kappa + (r^2+1)\nu] + \frac{1}{2}\kappa^3(\kappa + \nu). \end{aligned} \tag{B10}$$

Upon setting $\nu = 0$ we may recover the characteristic polynomials in Ref. [35].

3. $S = J'$ string

Here the AdS_5 bosonic characteristic polynomial can be directly extracted from Ref. [24] (from the expression found before using the conformal gauge constraint).[39] Then its S^5 counterpart can be found by using the "self-duality" property of the solution (2.42) under

$$\kappa \leftrightarrow \nu, \qquad r \leftrightarrow ia, \qquad w \leftrightarrow -w'. \tag{B11}$$

We end up with

$$\begin{aligned} B_8^{AdS_5} = &(-p_0^2 + p_1^2 + w^2 - m^2)^2 \\ &\times [(p_0^2 - p_1^2)^2 - 4m^2p_1^2(1+r^2) + 8mp_0p_1(1+r^2)w \\ &- 4p_0^2[-\kappa^2r^2 + (1+r^2)w^2]], \end{aligned} \tag{B12}$$

[39] One can check directly that the massless mode decouples in the characteristic polynomial for three coupled AdS_5 fluctuation modes that follows from the fluctuation Lagrangian in Eq. (4.13) in Ref. [24].

$$B_8^{S^5} = (-p_0^2 + p_1^2 + w'^2 - m^2)^2 \times [(p_0^2 - p_1^2)^2 - 4m^2 p_1^2(1-a^2) - 8mp_0p_1(1-a^2)w' - 4p_0^2[-\nu^2a^2 + (1-a^2)w'^2]]. \quad \text{(B13)}$$

As in the previous cases here the fermionic operator can be put into a block-diagonal form where each block may be written in terms of the six-dimensional Dirac matrices. While the two blocks are not identical, parity invariance requires that their determinants are the same. The fact that six-dimensional spinors are chiral implies that the determinant of each block further factorizes as in (B3), where now

$$F_1 = (p_0^2 - p_1^2)^2 + 2mp_0p_1\left[2a^2\left(w' + \frac{\kappa\nu}{w}\right) + (w - w')\right] + p_1^2[-\kappa\nu + 3\nu^2 + (w - 2w')(w + w')] - p_0^2[\kappa\nu + \nu^2 + w(w + w')] + \frac{1}{4}[-2\kappa\nu[w'(w + w') - \nu^2] + 2\nu^4 + \nu^2(w - 3w')(w + w') + w'^2(w + w')^2], \quad \text{(B14)}$$

$$F_2 = (p_0^2 - p_1^2)^2 + 2mp_0p_1\left[2a^2\left(w' - \frac{\kappa\nu}{w}\right) + (w - w')\right] + p_1^2[\kappa\nu + 3\nu^2 + (w - 2w')(w + w')] - p_0^2[-\kappa\nu + \nu^2 + w(w + w')] + \frac{1}{4}[2\kappa\nu[w'(w + w') - \nu^2] + 2\nu^4 + \nu^2(w - 3w')(w + w') + w'^2(w + w')^2]. \quad \text{(B15)}$$

Let us comment on derivation of these expressions [that reduce to the ones in Ref. [24] for $a = 1$ in (2.42)]. In the κ-symmetry gauge $\theta_1 = \theta_2$ the quadratic part of the fermionic Lagrangian is (see, e.g., Ref. [22,24], and references therein)

$$L = -2i\eta^{\alpha\beta}e^A_\alpha\bar{\theta}\Gamma^A\mathcal{D}_\beta\theta - \epsilon^{\alpha\beta}\bar{\theta}\Gamma_A\Gamma_*\Gamma_B\theta e^A_\alpha e^B_\beta, \quad \text{(B16)}$$

where $\mathcal{D} = d + \frac{1}{4}\omega^{AB}\Gamma_{AB}$ is the usual spinor covariant derivative. For the solution (2.42) the 2D projected combinations $e^A_\alpha\Gamma_A$ and $\omega^{AB}_\alpha\Gamma_{AB}$ are

$$e^A_0\Gamma_A = \Gamma_0\sqrt{1+r^2}\kappa + \Gamma_4 rw + \Gamma_5\sqrt{1-a^2}\nu + \Gamma_9 aw',$$
$$e^A_1\Gamma_A = m(\Gamma_4 r - \Gamma_9 a),$$
$$\omega^{AB}_0\Gamma_{AB} = 2\kappa r\Gamma_{01} - 2(\sqrt{1+r^2}w\Gamma_{14} + a\nu\Gamma_{56} + \sqrt{1-a^2}w'\Gamma_{69}),$$
$$\omega^{AB}_1\Gamma_{AB} = m(-2\sqrt{1+r^2}\Gamma_{14} + 2\sqrt{1-a^2}\Gamma_{69}), \quad \text{(B17)}$$

where Γ_A are the ten-dimensional (10D) Dirac matrices; one should project the quadratic operator onto its chiral part thus rendering it a 16×16 matrix. To evaluate the determinant of the quadratic fermionic operator we first notice that the matrices Γ_2 and Γ_3 in $\Gamma_* = i\Gamma_{01234}$ in (B16) do not appear elsewhere in the quadratic operator. The product Γ_{23} may therefore be diagonalized; its diagonal entries are $\pm i$. In this representation the quadratic operator is block-diagonal and each block may be obtained from (B16) and (B17) by using for Γ_i and Γ_{ij} the $d = 6$ Dirac matrices and $\Gamma_* = \pm\Gamma_{014}$. Since the sign of Γ_* affects only the sign of the Wess-Zumino term which can also be changed by parity transformations, the determinants of the two blocks are equal and thus the 10D determinant is a perfect square, as in the first equation in (B3). Since the six-dimensional (6D) spinors are chiral, there exists a representation of the 6D Dirac matrices in which each block of the quadratic operator is itself block-diagonal. Thus, the determinant of each block further factorizes; each block is only a 4×4 matrix and its determinant can be easily evaluated leading to the two factors F_1 and F_2 in Eq. (B3) given by (B14) and (B15).

In Sec. II D we discussed the small r expansion of the energy of the $S = J'$ string with angular momentum J. For this purpose, we need that

$$a = r\sqrt{1 + \frac{2r^2}{1+\nu^2}}, \quad \kappa = \sqrt{\nu^2 + 4r^2 + \frac{4r^4}{1+\nu^2}}, \quad \text{(B18)}$$

$$w = \sqrt{1 + \nu^2 + 4r^2 + \frac{4r^4}{1+\nu^2}}, \quad w' = \sqrt{1+\nu^2}. \quad \text{(B19)}$$

Plugging these expressions in F_1 and F_2 and dividing by a factor of r^4 we find that

$$F_{1,2} = c_0^{(1,2)} + c_2^{(1,2)}r^2 + c_4^{(1,2)}r^4 + \ldots, \quad \text{(B20)}$$

with

$$c_0^{(1)} = c_0^{(2)} = (\nu^2 - p_0^2 + p_1^2 - 2p_1 + 1) \times (\nu^2 - p_0^2 + p_1^2 + 2p_1 + 1)$$
$$c_2^{(1)} = \frac{8}{(1+\nu^2)^{3/2}}[\sqrt{\nu^2+1}(-2p_0^2(4\nu^2 + p_1^2 + 3) + p_0^4 + (p_1^2 - 1)^2) + 4(\nu^2+1)^2p_0p_1], \quad \text{(B21)}$$

$$c_2^{(2)} = \frac{8}{(1+\nu^2)^{3/2}}[4(\nu^2+1)p_0p_1 + \sqrt{\nu^2+1}(3\nu^4 + \nu^2(-4p_0^2 + 4p_1^2 + 6) + p_0^4 - 2p_0^2(p_1^2+2) + p_1^4 + 3)], \quad \text{(B22)}$$

$$c_4^{(1)} = \frac{4}{\nu^2(1+\nu^2)^{5/2}}[32(\nu^3+\nu)^2p_0p_1 + \sqrt{\nu^2+1}(4\nu^4(p_1^2 - 6p_0^2) + \nu^2(p_0^4 - 2p_0^2(p_1^2+10) + p_1^4 + 4p_1^2 + 3) + 2(p_0^2 + p_1^2 + 1))], \quad \text{(B23)}$$

$$c_4^{(2)} = \frac{4}{\nu^2(1+\nu^2)^{5/2}}[16(\nu^2+1)\nu^2 p_0 p_1 + \sqrt{\nu^2+1}(13\nu^6 + \nu^4(-14p_0^2 + 14p_1^2 + 24) + \nu^2(p_0^4 - 2p_0^2(p_1^2+8) + p_1^4 + 8p_1^2 + 9) - 2(p_0^2 + p_1^2 + 1))]. \tag{B24}$$

It is not difficult to construct higher-order terms in the small r expansion at fixed ν.

APPENDIX C: ONE-LOOP ENERGY OF $S = J'$ CIRCULAR STRING FROM THE ALGEBRAIC CURVE APPROACH

Here we shall revisit the computation of the 1-loop correction to the energy of the $S = J'$ circular string solution (2.42) discussed in Sec. II D using the algebraic curve approach [25,36] to determine the fluctuation frequencies.

In order to focus on a near flat-space expansion in the short string limit we will consider the limit $\mathcal{S} \to 0$ for fixed ϱ

$$\varrho = \frac{\nu}{2\sqrt{\mathcal{S}}}. \tag{C1}$$

In Sec. II D in (2.46) we used instead

$$\rho = \frac{\mathcal{J}}{2\sqrt{\mathcal{S}}} = \varrho\Big(1 - \frac{\mathcal{S}}{\sqrt{1+4\varrho^2\mathcal{S}}}\Big). \tag{C2}$$

Note also that

$$\mathcal{S} = \frac{w^2\sqrt{1+2\varrho^2 w + \varrho^4 w^4} - \omega(1 + 2\varrho^2 w - \varrho^2 w^3)}{2(1+2\varrho^2 w)^2}. \tag{C3}$$

1. Quasimomenta

The quasimomenta can be obtained by explicit diagonalization of the monodromy matrix [25]; for the S^5 part the basic single cut quasimomenta vanishing at infinity are determined by

$$\tilde{p}(x) = -\pi + \pi\frac{x - \tilde{x}_1}{x^2 - 1}\sqrt{(x-\tilde{x}_2)(x-\bar{\tilde{x}}_2)}, \tag{C4}$$

where the two roots $\tilde{x}_1$, $\tilde{x}_2$ are given by

$$\tilde{x}_1 = -\frac{1}{2\varrho\sqrt{\mathcal{S}} + \sqrt{1+4\varrho^2\mathcal{S}}}, \qquad \tilde{x}_2 = \frac{(\sqrt{1+4\varrho^2\mathcal{S}} + 2\varrho\sqrt{\mathcal{S}})(\sqrt{1+4\varrho^2\mathcal{S}} + 2i\sqrt{\mathcal{S}(\sqrt{1+4\varrho^2\mathcal{S}} - \mathcal{S})} - 2\mathcal{S})}{\sqrt{1+4\varrho^2\mathcal{S}}}. \tag{C5}$$

The four S^5 quasimomenta can be identified looking at the asymptotic $x \to \infty$ behavior of $\tilde{p}(x)$ and $\tilde{p}(x^{-1})$, which is related to the conserved global charges:

$$\frac{x}{2\pi}\tilde{p}(x) \to \mathcal{S} - \mathcal{J} + \ldots, \qquad \frac{x}{2\pi}\tilde{p}(x^{-1}) \to -1 - \mathcal{S} - \mathcal{J} + \ldots \tag{C6}$$

Hence, we can identify

$$p_{\tilde{1}}(x) = -2\pi - \tilde{p}(x^{-1}), \qquad p_{\tilde{2}}(x) = \tilde{p}(x), \qquad p_{\tilde{3}}(x) = -\tilde{p}_2(x), \qquad p_{\tilde{4}}(x) = -\tilde{p}_1(x). \tag{C7}$$

For the AdS_5 quasimomenta, the basic function is given by

$$\hat{p}(x) = \pi\frac{x - \hat{x}_3}{x^2 - 1}(\sqrt{x - \hat{x}_1}\sqrt{x - \hat{x}_2} - 1), \tag{C8}$$

where the $\hat{x}_i$ are

$$\hat{x}_1 = (\hat{x}_2\hat{x}_3^2)^{-1}, \qquad \hat{x}_2 = -\frac{2\mathcal{S} + w - 2\sqrt{\mathcal{S}(\mathcal{S}+w)}}{w(w - \sqrt{w^2-1})}, \qquad \hat{x}_3 = w - \sqrt{w^2 - 1}. \tag{C9}$$

Again, comparing with the asymptotic, the identification of the quasimomenta goes as follows:

$$p_{\hat{1}}(x) = -\hat{p}(x^{-1}), \qquad p_{\hat{2}}(x) = \hat{p}(x), \qquad p_{\hat{3}}(x) = -\hat{p}(x), \qquad p_{\hat{4}}(x) = \hat{p}(x^{-1}). \tag{178}$$

2. Off-shell frequencies

Due to the symmetry of the circular string solution, all the fluctuation energies can be conveniently written as combinations of only two independent functions $\Omega_A(x) = \Omega^{\hat{2}\hat{3}}(x)$ and $\Omega_S(x) = \Omega^{\tilde{2}\tilde{3}}(x)$ [36]:

$$
\begin{aligned}
\Omega_{B_1}(x) &= \Omega^{\tilde{1}\tilde{4}}(x) = -\Omega_S(x^{-1}) + \Omega_S(0),\\
\Omega_{B_2}(x) &= \Omega^{\tilde{2}\tilde{4}}(x) = \Omega^{\tilde{1}\tilde{3}}(x)\\
&= \frac{1}{2}[\Omega_S(x) - \Omega_S(x^{-1}) + \Omega_S(0)],\\
\Omega_{B_3}(x) &= \Omega^{\hat{1}\hat{4}}(x) = -\Omega_A(x^{-1}) - 2,\\
\Omega_{B_4}(x) &= \Omega^{\hat{2}\hat{4}}(x) = \Omega^{\hat{1}\hat{3}}(x) = \frac{1}{2}[\Omega_A(x) - \Omega_A(x^{-1})] - 1,\\
\Omega_{F_1}(x) &= \Omega^{\hat{2}\tilde{4}}(x) = \Omega^{\tilde{1}\hat{3}}(x)\\
&= \frac{1}{2}[\Omega_A(x) - \Omega_S(x^{-1}) + \Omega_S(0)],\\
\Omega_{F_2}(x) &= \Omega^{\tilde{2}\hat{4}}(x) = \Omega^{\hat{1}\tilde{3}}(x) = \frac{1}{2}[\Omega_S(x) - \Omega_A(x^{-1})] - 1,\\
\Omega_{F_3}(x) &= \Omega^{\tilde{1}\hat{4}}(x) = \Omega^{\hat{1}\tilde{4}}(x)\\
&= \frac{1}{2}[-\Omega_S(x) - \Omega_A(x^{-1}) + \Omega_S(0)] - 1,\\
\Omega_{F_4}(x) &= \Omega^{\hat{2}\tilde{3}}(x) = \Omega^{\tilde{2}\hat{3}}(x) = \frac{1}{2}[\Omega_A(x) - \Omega_A(x)]. \qquad \text{(C11)}
\end{aligned}
$$

Following [36], the two functions $\Omega_A(x)$ and $\Omega_S(x)$ can be uniquely fixed imposing the correct analytical and asymptotic properties for the perturbed quasimomenta $p + \delta p$:

$$
\begin{aligned}
\Omega_S(x) &= \Omega^{\tilde{2}\tilde{3}}(x) = \frac{\hat{f}(1)}{\tilde{f}(1)}\left(\frac{\tilde{f}(x)}{x-1} - 1\right) + \frac{\hat{f}(-1)}{\tilde{f}(-1)}\left(\frac{\tilde{f}(x)}{x+1} - 1\right),\\
\Omega_A(x) &= \Omega^{\hat{2}\hat{3}}(x) = 2\left(\frac{x}{x^2-1}\hat{f}(x) - 1\right), \qquad \text{(C12)}
\end{aligned}
$$

where the two functions $\hat{f}(x)$ and $\tilde{f}(x)$ are defined as

$$
\begin{aligned}
\tilde{f}(x) &= \sqrt{(x - \tilde{x}_2)(x - \bar{\tilde{x}}_2)},\\
\hat{f}(x) &= \sqrt{(x - \hat{x}_1)(x - \hat{x}_2)}, \qquad \text{(C13)}
\end{aligned}
$$

with a suitable choice of the cuts.

3. One-loop energy

Given the above set of off-shell frequencies $\Omega_I = \Omega^{i,j}$, $I \in \{A, S, B_{1,2,3,4}, F_{1,2,3,4}\}$, the corresponding physical on-shell fluctuations energies associated to the (i, j) excitations with mode number n, are given by

$$\omega_I^{(n)} = \omega_{i,j}^{(n)} = \Omega^{i,j}(x_n^{i,j}), \qquad \text{(C14)}$$

where, for any pair (i, j), $x_n^{i,j}$ is determined as the solution of the equation

$$p_i(x_n^{i,j}) - p_j(x_n^{i,j}) = 2\pi n. \qquad \text{(C15)}$$

The one-loop correction to the energy can be obtained as a sum over n and polarizations[40]

$$E_1 = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\sum_{i,j}(-1)^{F_{i,j}}\omega_{i,j}^{(n)}. \qquad \text{(C16)}$$

This sum is sensitive to integer shifts in the labeling of the frequencies $n \to n + \delta$; following [25] here we propose to use the following choice:

$$
\begin{aligned}
E_1 = \frac{1}{2}\sum_{n=-\infty}^{+\infty}[&\omega_S^{(n-1)} + \omega_A^{(n-1)} + \omega_{B_1}^{(n-1)} + \omega_{B_2}^{(n-1)}\\
&+ \omega_{B_3}^{(n+1)} + \omega_{B_4}^{(n)} - 2\omega_{F_1}^{(n-1)} - 2\omega_{F_2}^{(n)}\\
&- 2\omega_{F_3}^{(n)} - 2\omega_{F_4}^{(n-1)}]. \qquad \text{(C17)}
\end{aligned}
$$

Then the final result in the short string limit has the same form as in (2.55)

$$E_1 = \frac{\frac{11}{8} - 3\zeta_3}{\sqrt{\varrho^2 + 1}}\mathcal{S}^{3/2} + \mathcal{O}(\mathcal{S}^2), \qquad \text{(C18)}$$

corresponding to the rational part of n_{12} in (1.29) and (2.53) being

$$n'_{12} = \frac{11}{8}. \qquad \text{(C19)}$$

The prescription (C17) thus does not lead to the preferred choice $n'_{12} = \frac{5}{8}$ consistent with the universal value (1.30) of the 2-loop coefficient n_{21}. The value in (C19) together with universality of Konishi dimension implying Eq. (1.31) then leads to $n_{21} = -\frac{7}{4}$ ($n_{03} = -\frac{1}{2}$).

Making a natural guess about the structure of the leading term in the 2-loop correction to the slope function, we then get

$$
\begin{aligned}
E &= E_0 + E_1 + E_2 + \ldots\\
&= 2\sqrt{1+\varrho^2}\sqrt{\lambda}\sqrt{\mathcal{S}}\Big[1 + \frac{1}{2(\varrho^2+1)}\mathcal{S}\\
&\quad + \frac{8\varrho^6 - 4\varrho^4 - 16\varrho^2 - 5}{8(\varrho^2+1)^2}\mathcal{S}^2 + \ldots\Big]\\
&\quad + \frac{n'_{12} - 3\zeta_3}{\sqrt{\varrho^2+1}}\mathcal{S}^{3/2} + \ldots + \frac{1}{\sqrt{\lambda}}\frac{n_{21}}{(\varrho^2+1)^{3/2}}\sqrt{\mathcal{S}} + \ldots
\end{aligned}
\qquad \text{(C20)}
$$

APPENDIX D: ONE-LOOP ENERGY OF THE (J', J) FOLDED STRING FROM THE ALGEBRAIC CURVE APPROACH

Here we shall derive the 1-loop coefficients in (1.44) in the small spin expansion of the energy of a folded string with spin $J_1 = J'$ and orbital momentum $J_3 = J$

[40] In the algebraic curve formalism, the on-shell energies $\omega_{i,j}^{(n)}$ enter directly E_1 and do not require $1/\kappa$ factors.

representing a state in the $su(2)$ sector on the dual gauge theory side. This will a direct counterpart of the computation done for the (S, J) folded string in Ref. [6].

1. Quasimomenta

The classical solution [37] for the folded string with spin J' and orbital momentum J in S^5 is related to the folded string with spin S in AdS_5 and orbital momentum J in S^5 by an analytical continuation [33] implying a relation between the string profiles and the global conserved charges

$$(E; J', J) \to (-J; S, -E). \tag{D1}$$

In the algebraic curve approach the quasimomenta for the (J', J) string can then be obtained by an analytical continuation of the quasimomenta for the (S, J) string given in Ref. [3]. According to [5], the S^5 quasimomentum $p_{\tilde{2}}$ as a function of the branch points is expressed in terms of the elliptic functions:

$$\begin{aligned} p_{\tilde{2}}(x) = {} & \pi - i2\pi\mathcal{E}_0\left(\frac{a}{a^2-1} - \frac{x}{x^2-1}\right) \\ & \times \sqrt{\frac{b}{a}\frac{a^2-1}{b^2-1}}\sqrt{\frac{|a|-ia}{|a|-i\bar{a}}\frac{\bar{a}-x}{a-x}}\sqrt{\frac{a}{\bar{a}}\frac{|a|-i\bar{a}}{|a|-ia}\frac{\bar{a}+x}{a+x}} \\ & - \frac{8\pi ab\mathcal{J}'}{(b-a)(ab+1)}\mathbf{F}_1(x) - \frac{2\pi\mathcal{E}_0(a-b)}{\sqrt{(a^2-1)(b^2-1)}}\mathbf{F}_2(x), \end{aligned} \tag{D2}$$

$$\mathbf{F}_1(x) = i\mathbb{F}\left(i\sinh^{-1}\sqrt{-\frac{a-b}{a+b}\frac{a-x}{a+x}}, \frac{(a+b)^2}{(a-b)^2}\right), \tag{D3}$$

$$\mathbf{F}_2(x) = i\mathbb{E}\left(i\sinh^{-1}\sqrt{-\frac{a-b}{a+b}\frac{a-x}{a+x}}, \frac{(a+b)^2}{(a-b)^2}\right), \tag{D4}$$

where $\mathrm{Re}(a), \mathrm{Im}(a) > 0$, $b = -\bar{a}$ and

$$\begin{aligned} \mathcal{J} &= \frac{1}{2\pi}\frac{ab-1}{ab}\left[b\mathbb{E}\left(1-\frac{a^2}{b^2}\right) + a\mathbb{K}\left(1-\frac{a^2}{b^2}\right)\right], \\ \mathcal{J}' &= -\frac{1}{2\pi}\frac{ab+1}{ab}\left[b\mathbb{E}\left(1-\frac{a^2}{b^2}\right) - a\mathbb{K}\left(1-\frac{a^2}{b^2}\right)\right], \\ \mathcal{E}_0 &= -\frac{1}{\pi b}\sqrt{(a^2-1)(b^2-1)}\mathbb{K}\left(1-\frac{a^2}{b^2}\right). \end{aligned} \tag{D5}$$

The inversion symmetry provides the other sphere quasimomenta through the relations

$$p_{\tilde{2}}(x) = -p_{\tilde{3}}(x) = -p_{\tilde{1}}(x^{-1}) = p_{\tilde{4}}(x^{-1}). \tag{D6}$$

Since the motion in the AdS_5 part is trivial, the corresponding quasimomenta are simply

$$p_{\hat{1},\hat{2}}(x) = -p_{\hat{3},\hat{4}}(x) = 2\pi\mathcal{E}_0\frac{x}{x^2-1}. \tag{D7}$$

2. Off-shell frequencies

The symmetry of the solution allows to express all the off-shell fluctuation frequencies as combinations of only two independent functions [5]:

$$\Omega_A(x) = \frac{2}{x^2-1}\left(1 + x\frac{f(1)-f(-1)}{f(1)+f(-1)}\right), \tag{D8}$$

$$\Omega_S(x) = \frac{4}{f(1)+f(-1)}\left(\frac{f(x)}{x^2-1} - 1\right), \tag{D9}$$

where $(f(x))^2 = (x-a)(x-\bar{a})(x-b)(x-\bar{b})$. The complete list of the frequencies is given by

$$\begin{aligned} &\Omega^{\tilde{2}\tilde{3}}(x) = \Omega_S(x), \qquad \Omega^{\hat{2}\hat{3}}(x) = \Omega_A(x), \\ &\Omega^{\tilde{1}\tilde{4}}(x) = -\Omega_S(x^{-1}) + \Omega_S(0), \\ &\Omega^{\tilde{2}\tilde{4}}(x) = \Omega^{\tilde{1}\tilde{3}}(x) = \frac{1}{2}[\Omega_S(x) - \Omega_S(x^{-1}) + \Omega_S(0)], \\ &\Omega^{\hat{1}\hat{4}}(x) = \Omega^{\hat{2}\hat{4}}(x) = \Omega^{\hat{1}\hat{3}}(x) = \Omega^{\hat{2}\hat{3}}(x), \\ &\Omega^{\hat{2}\tilde{4}}(x) = \Omega^{\tilde{1}\hat{3}}(x) = \Omega^{\tilde{1}\hat{4}}(x) = \Omega^{\hat{1}\tilde{4}}(x) \\ &\qquad = \frac{1}{2}[\Omega_A(x) - \Omega_S(x^{-1}) + \Omega_S(0)], \\ &\Omega^{\tilde{2}\hat{4}}(x) = \Omega^{\hat{1}\tilde{3}}(x) = \Omega^{\hat{2}\tilde{3}}(x) = \Omega^{\tilde{2}\hat{3}}(x) \\ &\qquad = \frac{1}{2}[\Omega_S(x) + \Omega_A(x)]. \end{aligned} \tag{D10}$$

The off-shell frequencies provide the fluctuation energies when evaluated on the solutions of the equations:

$$p_i(x_n^{i,j}) - p_j(x_n^{i,j}) = 2\pi n. \tag{D11}$$

3. One-loop correction to the energy

We have computed the one-loop energy correction E_1 in the two limits. The first one is motivated by the analysis in Ref. [38] and is defined as

$$\mathcal{J}' \to 0, \qquad \mathrm{t} \equiv \frac{\mathcal{J}}{\sqrt{2\mathcal{J}'}} = \text{fixed}. \tag{D12}$$

In this limit, the classical energy is given by

$$\begin{aligned} \frac{\mathcal{E}_0}{\sqrt{2\mathcal{J}'}} &= \sqrt{\mathrm{t}^2+1} + \frac{4\mathrm{t}^2+1}{8\sqrt{\mathrm{t}^2+1}}\mathcal{J}' \\ &\quad + \frac{-32\mathrm{t}^6 - 16\mathrm{t}^4 + 28\mathrm{t}^2 + 3}{128(\mathrm{t}^2+1)^{3/2}}\mathcal{J}'^2 + \dots \end{aligned} \tag{D13}$$

For the one-loop correction, we find

$$\begin{aligned} E_1 &= \sum_{p\geq 0} a_p(\mathrm{t})(\mathcal{J}')^{p+(1/2)} \\ &= a_0(\mathrm{t})(\mathcal{J}')^{1/2} + a_1(\mathrm{t})(\mathcal{J}')^{3/2} + \dots, \end{aligned} \tag{D14}$$

$$a_0(\mathrm{t}) = \frac{1}{2\sqrt{2(\mathrm{t}^2+1)}},$$
$$a_1(\mathrm{t}) = -\frac{16\mathrm{t}^4+25\mathrm{t}^2+6}{8[2(\mathrm{t}^2+1)]^{3/2}} - \frac{3}{2\sqrt{2(\mathrm{t}^2+1)}}\zeta_3. \qquad \text{(D15)}$$

Adding the classical energy and re-expanding at large λ for fixed J', J, this gives

$$E^2 = 2\sqrt{\lambda}J' + \frac{1}{2}J'^2 + J' + J^2 + \frac{1}{\sqrt{\lambda}}\Big[\frac{1}{8}J'^3 + J'J^2 + \Big(-\frac{5}{8} - 3\zeta_3\Big)J'^2 + \frac{1}{8}J'\Big] + \dots, \qquad \text{(D16)}$$

leading to the values of the coefficients n_{ij} in (1.44). The resulting value

$$n'_{12} = -\frac{5}{8} \qquad \text{(D17)}$$

is perfectly consistent with the universality of the two-loop coefficient n_{21} in (1.30), i.e., as follows from (1.31),

$$n_{21} = -\frac{1}{4}. \qquad \text{(D18)}$$

As in Ref. [38], expanding E_1 at large t we recover the expansion in small $\mathcal{J}'$ for fixed small $\mathcal{J}$:

$$E_1 = \Big(\frac{1}{2\mathcal{J}} - \frac{1}{2}\mathcal{J} + \dots\Big)\mathcal{J}' + \Big(-\frac{1}{2\mathcal{J}^3} + \frac{-\frac{1}{8}-3\zeta_3}{2\mathcal{J}} + \dots\Big)\mathcal{J}'^2 + \Big(\frac{3}{4\mathcal{J}^5} + \frac{\frac{3}{8}+3\zeta_3}{2\mathcal{J}^3} + \dots\Big)\mathcal{J}'^3 + \Big(-\frac{5}{4\mathcal{J}^7} + \frac{-\frac{23}{8}-9\zeta_3}{4\mathcal{J}^5} + \dots\Big)\mathcal{J}'^4 + \dots \qquad \text{(D19)}$$

The second limit is

$$\mathcal{J}' \to 0, \qquad \mathcal{J} = \text{fixed}. \qquad \text{(D20)}$$

In this limit, the classical energy reads[41]

$$\mathcal{E}_0 = \mathcal{J} + \frac{\sqrt{\mathcal{J}^2+1}}{\mathcal{J}}\mathcal{J}' - \frac{3\mathcal{J}^2+2}{4\mathcal{J}^3(\mathcal{J}^2+1)}\mathcal{J}'^2 + \frac{15\mathcal{J}^6+33\mathcal{J}^4+28\mathcal{J}^2+8}{16\mathcal{J}^5(\mathcal{J}^2+1)^{5/2}}\mathcal{J}'^3 + \dots \qquad \text{(D21)}$$

For the one loop correction, we find

$$E_1 = e_1(\mathcal{J})\mathcal{J}' + e_2(\mathcal{J})\mathcal{J}'^2 + e_3(\mathcal{J})\mathcal{J}'^2 + \dots, \qquad \text{(D22)}$$

and, at order $\mathcal{J}'^2$,

$$E_1 = \frac{\mathcal{J}'}{2\mathcal{J}(1+\mathcal{J}^2)} + \Big[\frac{-21\mathcal{J}^4-29\mathcal{J}^2+1}{16\mathcal{J}^3(\mathcal{J}^2+1)^{5/2}} - \sum_{n=2}^{\infty}\frac{n^2(\mathcal{J}^2+2n^2-1)}{\mathcal{J}^3(n^2-1)^2(\mathcal{J}^2+n^2)^{3/2}}\Big]\mathcal{J}'^2 + \dots \qquad \text{(D23)}$$

This expression is very similar to the one for the (S, J) folded string found in Ref. [6]:

$$E_1^{(\mathcal{S},\mathcal{J})} = -\frac{\mathcal{S}}{2\mathcal{J}(1+\mathcal{J}^2)} + \Big[\frac{3\mathcal{J}^4+11\mathcal{J}^2+17}{16\mathcal{J}^3(\mathcal{J}^2+1)^{5/2}} - \sum_{n=2}^{\infty}\frac{n^2(\mathcal{J}^2+2n^2-1)}{\mathcal{J}^3(n^2-1)^2(\mathcal{J}^2+n^2)^{3/2}}\Big]\mathcal{S}^2 + \dots \qquad \text{(D24)}$$

The only differences are in the sign of the first term [i.e., the sign of the 1-loop term in the "slope" function (1.25)] and in the coefficients of the contributions of low modes in the second term.

Extending the calculation to the order $\mathcal{J}'^3$ we find the following correction to E_1:

$$e_3(\mathcal{J}) = \frac{150\mathcal{J}^8+456\mathcal{J}^6+202\mathcal{J}^4+8\mathcal{J}^2-27}{64\mathcal{J}^5(\mathcal{J}^2+1)^4} + \sum_{n=2}^{\infty}\frac{1}{2\mathcal{J}^5(\mathcal{J}^2+1)^{3/2}(n^2-1)^4(\mathcal{J}^2+n^2)^{5/2}} \times[(8\mathcal{J}^4+17\mathcal{J}^2+10)n^{10} + 2(10\mathcal{J}^6+9\mathcal{J}^4 - 13\mathcal{J}^2-14)n^8 + 2(3\mathcal{J}^8-19\mathcal{J}^6-43\mathcal{J}^4 - 17\mathcal{J}^2+7)n^6 - 2(6\mathcal{J}^8+2\mathcal{J}^6-13\mathcal{J}^4 - 9\mathcal{J}^2+2)n^4 - \mathcal{J}^2(2(\mathcal{J}^4+5\mathcal{J}^2+7)\mathcal{J}^2+7)n^2]. \qquad \text{(D25)}$$

Expanding the coefficients of each power of $\mathcal{J}'$ in (D23) in small $\mathcal{J}$ we get explicitly [here $\mathcal{N} = J'$; cf. (2.18), (2.35), and (2.56)]

$$E_1 = \Big(\frac{1}{2\mathcal{J}} - \frac{\mathcal{J}}{2} + \frac{\mathcal{J}^3}{2} + \dots\Big)\mathcal{J}' + \Big[-\frac{1}{2\mathcal{J}^3} + \frac{1}{\mathcal{J}}\Big(-\frac{1}{16} - \frac{3}{2}\zeta_3\Big) + \mathcal{J}\Big(-\frac{9}{32} + \frac{3}{2}\zeta_3 + \frac{15}{8}\zeta_5\Big) + \mathcal{J}^3\Big(\frac{125}{128} - \frac{25}{16}\zeta_3 - \frac{15}{8}\zeta_5 - \frac{35}{16}\zeta_7\Big) + \dots\Big]\mathcal{J}'^2 + \Big[\frac{3}{4\mathcal{J}^5} + \frac{1}{\mathcal{J}^3}\Big(\frac{3}{16} + \frac{3}{2}\zeta_3\Big) + \frac{1}{\mathcal{J}}\Big(\frac{1}{32} - \frac{9}{8}\zeta_3\Big) + \mathcal{J}\Big(\frac{1}{8} + 3\zeta_3 + \frac{35}{16}\zeta_5 - \frac{35}{16}\zeta_7\Big) + \cdots\Big]\mathcal{J}'^3 + \dots \qquad \text{(D26)}$$

This is in perfect agreement with the expansion (D19) found in the case of fixed $\mathrm{t} = \frac{\mathcal{J}}{\sqrt{2\mathcal{J}'}}$.

From this expansion one extracts, in particular, the following values [cf., (1.9), (1.35), and (1.37)]:

[41] Equivalently, $\mathcal{E}_0^2 = \mathcal{J}^2 + 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2}\mathcal{J}' + \frac{1+2\mathcal{J}^2}{2(1+\mathcal{J}^2)}\mathcal{J}'^2 + \dots$ For comparison, in the (S, J) folded string case $\mathcal{E}_0^2 = \mathcal{J}^2 + 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2}\mathcal{S} + \frac{3+2\mathcal{J}^2}{2(1+\mathcal{J}^2)}\mathcal{S}^2 + \dots$

TABLE I. Summary of coefficients in Eq. (1.3).

n_{ij}	(S, J)	(J', J)	$(J_1 = J_2, J)$	$(S_1 = S_2, J)$	$(S = J', J)$
n_{01}	1	1	1	1	1
$\tilde{n}_{01}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
n_{02}	$\frac{3}{2}$	$\frac{1}{2}$	0	2	1
$\tilde{n}_{02}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	0
n_{03}	$-\frac{3}{8}$	$\frac{1}{8}$	0	-1	$-\frac{1}{2}$
n_{04}	$\frac{31}{64}$	$\frac{1}{64}$	0	2	$\frac{3}{4}$
n_{11}	-1	1	2	-2	0
$\tilde{n}_{11}$	1	-1	-2	2	0
$\bar{n}_{11}$	-1	1	2	-2	0
n'_{12}	$\frac{3}{8}$	$-\frac{5}{8}$	$-\frac{3}{8}$	$\frac{13}{8}$	$\frac{5}{8}$(?)
$\tilde{n}'_{12}$	$-\frac{27}{16}$	$-\frac{3}{16}$	$-\frac{57}{16}$	$-\frac{105}{16}$	-
n'_{13}	$-\frac{9}{16}$	$-\frac{7}{16}$	$-\frac{3}{16}$	$-\frac{85}{16}$	-
n''_{13}	$\frac{15}{4}$	$-\frac{3}{4}$	$-\frac{3}{4}$	$\frac{15}{4}$	$\frac{3}{2}$
n_{21}	$-\frac{1}{4}$	$-\frac{1}{4}$(?)	$-\frac{1}{4}$(?)	$-\frac{1}{4}$(?)	$-\frac{1}{4}$(?)

$$n_{12} = -\frac{5}{8} - 3\zeta_3, \qquad \tilde{n}_{12} = -\frac{3}{16} + 3\zeta_3 + \frac{15}{4}\zeta_5,$$
$$n_{13} = -\frac{7}{16} - \frac{3}{4}\zeta_3 + \frac{15}{4}\zeta_5. \tag{D27}$$

For comparison, the corresponding values for the (S, J) folded string that follow from the analog of (2.6) in Ref. [6] are

$$n_{12} = \frac{3}{8} - 3\zeta_3, \qquad \tilde{n}_{12} = -\frac{27}{16} + 3\zeta_3 + \frac{15}{4}\zeta_5,$$
$$n_{13} = -\frac{9}{16} + \frac{15}{4}\zeta_3 + \frac{15}{4}\zeta_5. \tag{D28}$$

The value of $n''_{13} = -\frac{3}{4}$ in (1.36) for the folded (J', J) string in (D27) is the same as for the $J_1 = J_2$ circular string found in sect II B; $n''_{13} = \frac{15}{4}$ for the folded (S, J) string in (D28) is the same as for the $S_1 = S_2$ circular string found in Sec. II C.

Similarly to the cases of the (S, J) folded string [6] and the circular strings discussed in Sec. II, the coefficient of $\mathcal{J}'^3/\mathcal{J}$ in (D26) does not contain ζ_5, supporting the universality of the transcendental terms in $\tilde{n}_{12}$ in (1.34) and of the ζ_5 term in n_{13} in (1.36). Note also that the highest transcendentality ζ_7 term in the coefficient of $\mathcal{J}\mathcal{J}'^3$ in (D26) is also universal, i.e., has the same value $(-35/16)$ as in Ref. [6] and in all circular string cases [cf., (2.18), (2.35), and (2.56)].

APPENDIX E: SUMMARY OF COFFICIENTS

Here we summarize the known values of the leading coefficients in E^2 in (1.3) for two single-spin folded and three equal-spin circular solutions. We omitted the values of $\tilde{n}'_{12}$, n'_{13} for the circular $S = J'$ solution that appear to be scheme-dependent (see Sec. II D). We added question marks to the values that were not computed directly but are expected on the basis of universality of the Konishi multiplet dimension. Let us recall the definitions of $n'_{\rm km}$, $n''_{\rm km}$ as rational coefficients in n_{12}, $\tilde{n}_{12}$, n_{13}:

$$n_{12} = n'_{12} - 3\zeta_3, \qquad \tilde{n}_{12} = \tilde{n}'_{12} + 3\zeta_3 + \frac{15}{4}\zeta_5,$$
$$n_{13} = n'_{13} + n''_{13}\zeta_3 + \frac{15}{4}\zeta_5.$$

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