

## Scheme transformations in the vicinity of an infrared fixed point

Thomas A. Ryttov<sup>1</sup> and Robert Shrock<sup>2</sup>

<sup>1</sup>*Jefferson Physical Laboratory, Physics Department, Harvard University, Cambridge, Massachusetts 02138, USA*

<sup>2</sup>*C. N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, New York 11794, USA*

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We analyze the effect of scheme transformations in the vicinity of an exact or approximate infrared fixed point in an asymptotically free gauge theory with fermions. We show that there is far less freedom in carrying out such scheme transformations in this case than at an ultraviolet fixed point. We construct a transformation from the  $\overline{\text{MS}}$  scheme to a scheme with a vanishing three-loop term in the  $\beta$  function and use this to assess the scheme dependence of an infrared fixed point in  $\text{SU}(N)$  theories with fermions. Implications for the anomalous dimension of the fermion bilinear operator are also discussed.

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The evolution of an asymptotically free gauge theory from the weakly coupled ultraviolet (UV) regime to the infrared (IR) regime is of fundamental interest. Here, we study this evolution for a theory with gauge group  $G$  and a given content of massless fermions. We focus mainly on vectorial gauge theories [1], but also remark on chiral gauge theories ( $\chi$ GT). The UV to IR evolution is determined by the renormalization group  $\beta$  function of the theory, which describes the dependence of  $g \equiv g(\mu)$ , the running gauge coupling, on the Euclidean momentum scale,  $\mu$ . We define  $\alpha = g^2/(4\pi)$ ,  $a = \alpha/(4\pi)$ , and  $\beta_\alpha \equiv d\alpha/dt$ , where  $t = \ln\mu$ . This has the series expansion

$$\beta_\alpha = -2\alpha \sum_{\ell=1}^{\infty} b_\ell a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \alpha^\ell, \quad (1)$$

where  $\bar{b}_\ell = b_\ell/(4\pi)^\ell$ . The coefficients  $b_1$  and  $b_2$  were calculated in Refs. [2,3], respectively. The asymptotic freedom (AF) property is the condition  $b_1 > 0$ , which we assume. As discussed further below, the  $b_\ell$  for  $\ell = 1, 2$  are independent of the scheme used for regularization and renormalization, while  $b_\ell$  with  $\ell \geq 3$  are scheme-dependent [4]. One scheme involves dimensional regularization [5] and minimal subtraction (MS) of the poles at dimension  $d = 4$  in the resultant Euler  $\Gamma$  functions [6]. The modified minimal subtraction ( $\overline{\text{MS}}$ ) scheme also subtracts certain related constants [7]. Calculations of  $b_3$  and  $b_4$  in the  $\overline{\text{MS}}$  scheme were given in Refs. [8,9]. Just as the calculation of  $b_1$  and demonstration that  $b_1 > 0$  was pivotal for the development of QCD, the computation of  $b_\ell$  for  $\ell = 2, 3, 4$  has been important in fits to  $\alpha_s(Q)$  [10]. In the vicinity of the UV fixed point (UVFP) at  $\alpha = 0$ , one can carry out a scheme transformation that renders three- and higher-loop terms zero [11]. Considerable work has been done on scheme (and related scale) transformations that reduce higher-order corrections in QCD [12].

Naively, one might think that there is a similarly great freedom in performing scheme transformations at an (exact or approximate) IR fixed point (IRFP). Here we show that, on the contrary, there is much less freedom in constructing

acceptable scheme transformations at an IRFP than at a UVFP, and we analyze constraints at an IRFP. We construct an example of a scheme transformation that satisfies these constraints, and we apply it to assess scheme-dependence of the value of an IRFP.

We first recall some background. In a non-Abelian gauge theory with no fermions or only a few fermions,  $b_2$  has the same positive sign as  $b_1$ , so  $\beta$  has no (perturbative) IR zero for  $\alpha \neq 0$  [13]. With a sufficient increase in the content of fermions,  $b_2$  reverses sign, while  $b_1$  is still positive, so the two-loop  $\beta$  function has a zero at

$$\alpha_{\text{IR},2\ell} = -\frac{4\pi b_1}{b_2}, \quad (2)$$

which is physical for  $b_2 < 0$ . This zero plays an important role in the UV to IR evolution of the theory [3,14]. If  $\alpha_{\text{IR},2\ell}$  is large enough, then, as  $\mu$  decreases through a scale denoted  $\Lambda$ , the gauge interaction grows strong enough to produce a bilinear fermion condensate in the most attractive channel with attendant spontaneous chiral symmetry breaking ( $S\chi$ SB) and dynamical generation of effective masses for the fermions involved [15]. In a one-gluon exchange approximation to the Dyson-Schwinger equation for the fermion propagator in a vectorial gauge theory, this occurs as  $\alpha$  increases through a value  $\alpha_{\text{cr}}$  given by  $\alpha_{\text{cr}} C_f \sim O(1)$  [16–18]. In a chiral gauge theory this breaks the gauge symmetry, while in the vectorial case, the most attractive channel is  $R \times \bar{R} \rightarrow 1$ , preserving the gauge symmetry [19]. Since the fermions that have gained dynamical masses are integrated out in the low-energy effective field theory below  $\Lambda$ , the  $\beta$  function changes, and the theory flows away from the original IRFP, which is thus only approximate. However, if  $\alpha_{\text{IR},2\ell}$  is sufficiently small, as is the case with a large enough (AF-preserving) fermion content, then the theory evolves from the UV to the IR without any  $S\chi$ SB. In this case the theory has an exact IRFP. For a given  $G$  and  $N_f$  (massless) fermions in a representation  $R$ , the critical value of  $N_f$  beyond which the theory flows to the IR conformal phase is denoted  $N_{f,\text{cr}}$ .

As  $N_f$  increases,  $\alpha_{\text{IR},2\ell}$  decreases, and  $N_{f,\text{cr}}$  is the value at which  $\alpha_{\text{IR},2\ell}$  decreases through  $\alpha_{\text{cr}}$ . Lattice simulations have been used to estimate  $N_{f,\text{cr}}$  [20].

Since  $\alpha_{\text{IR},2\ell}$  is  $\sim O(1)$ , especially in the quasi-conformal case where  $N_f \lesssim N_{f,\text{cr}}$ , there are significant corrections to the two-loop results from higher-loop terms in  $\beta$ . These motivate one to calculate these corrections to three- and four-loop order, and we have done this in the  $\overline{\text{MS}}$  scheme [21,22] (see also Ref. [23], which agrees with [21]). Because of the scheme-dependence of  $b_n$  for  $n \geq 3$ , the value of  $\alpha_{\text{IR},n\ell}$  calculated to finite order  $n \geq 3$  is scheme-dependent. It is important to assess this scheme dependence and the resultant uncertainties in the value of the (exact or approximate) IRFP. We address this task here. Besides its intrinsic field-theoretic interest, this is important for ongoing studies of quasi-conformal theories. These have a gauge coupling that gets large but runs slowly over a long interval of  $\mu$  [16], as occurs naturally due to an approximate IRFP [17]. Moreover, the UV to IR flow of a  $\chi$ GT and the associated sequential gauge symmetry breaking are important in certain approaches to physics beyond the standard model [24].

A scheme transformation (ST) is a map between  $\alpha$  and  $\alpha'$ . It will be convenient to write this as

$$a = a'f(a'). \quad (3)$$

To keep the UV properties the same, one requires  $f(0) = 1$ . We consider STs that are analytic about  $a = a' = 0$  [25] and hence can be expanded in the form

$$f(a') = 1 + \sum_{s=1}^{s_{\text{max}}} k_s (a')^s = 1 + \sum_{s=1}^{s_{\text{max}}} \bar{k}_s (\alpha')^s, \quad (4)$$

where the  $k_s$  are constants,  $\bar{k}_s = k_s/(4\pi)^s$ , and  $s_{\text{max}}$  may be finite or infinite. Hence, the Jacobian  $J = da/da'$  satisfies  $J = 1$  at  $a = a' = 0$ . We have

$$\beta_{\alpha'} \equiv \frac{d\alpha'}{dt} = \frac{d\alpha'}{d\alpha} \frac{d\alpha}{dt} = J^{-1} \beta_{\alpha}. \quad (5)$$

This has the expansion

$$\beta_{\alpha'} = -2\alpha' \sum_{\ell=1}^{\infty} b'_{\ell} (\alpha')^{\ell} = -2\alpha' \sum_{\ell=1}^{\infty} \bar{b}'_{\ell} (\alpha')^{\ell}, \quad (6)$$

where  $\bar{b}'_{\ell} = b'_{\ell}/(4\pi)^{\ell}$ . Given the equality of Eqs. (5) and (6), one can solve for the  $b'_{\ell}$  in terms of the  $b_{\ell}$  and  $k_s$ . This leads to the well-known result that  $b'_{\ell} = b_{\ell}$  for  $\ell = 1, 2$  [4], i.e., that the one- and two-loop terms in  $\beta$  are scheme-independent [26]. We note that the scheme-invariance of  $b_2$  assumes that  $f(a')$  is gauge-invariant. This is evident from the fact that in the momentum subtraction (MOM) scheme,  $b_2$  is actually gauge-dependent [27] and is not equal to  $b_2$  in the  $\overline{\text{MS}}$  scheme. We restrict our analysis here to gauge-invariant STs and to schemes, such as  $\overline{\text{MS}}$ , where  $b_2$  is gauge-invariant.

In order to assess scheme-dependence of an IRFP, we have calculated the relations between the  $b'_{\ell}$  and  $b_{\ell}$  for higher  $\ell$ . For example, for  $\ell = 3, 4, 5$  we obtain

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1, \quad (7)$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1, \quad (8)$$

$$b'_5 = b_5 + 3k_1 b_4 + (2k_1^2 + k_2) b_3 + (-k_1^3 + 3k_1 k_2 - k_3) b_2 + (4k_1^4 - 11k_1^2 k_2 + 6k_1 k_3 + 4k_2^2 - 3k_4) b_1. \quad (9)$$

In general, in the coefficients of the terms  $b_n$  entering in the expression for  $b'_{\ell}$ , the sum of the subscripts of the  $k_s$ s is equal to  $\ell - n$  with  $1 \leq n \leq \ell - 1$ , and the products of the various  $k_s$ s correspond to certain partitions of  $\ell - n$ . A corollary is that the only  $k_s$ s that appear in the formula for  $b'_{\ell}$  are the  $k_s$ s with  $1 \leq s \leq \ell - 1$ . However, because of cancellations, in the expression for  $b'_{\ell}$  for even  $\ell$ , the coefficient of  $b_n$  does not contain all of the terms corresponding to the partitions of  $\ell - n$ . For example, in  $b'_2$ , there is no  $k_1 b_1$  term and in  $b'_4$ , the coefficient of  $b_2$  does not contain  $k_2$ .

In order to be physically acceptable, this transformation must satisfy several conditions,  $C_i$ . For finite  $s_{\text{max}}$ , Eq. (3) is an algebraic equation of degree  $s_{\text{max}} + 1$  for  $\alpha'$  in terms of  $\alpha$ . We require that at least one of the  $s_{\text{max}} + 1$  roots must satisfy these conditions. These are as follows:  $C_1$ : the ST must map a real positive  $\alpha$  to a real positive  $\alpha'$ , since a map taking  $\alpha > 0$  to  $\alpha' = 0$  would be singular, and a map taking  $\alpha > 0$  to a negative or complex  $\alpha'$  would violate the unitarity of the theory.  $C_2$ : the ST should not map a moderate value of  $\alpha$ , for which perturbation theory may be reliable, to a value of  $\alpha'$  that is so large that perturbation theory is unreliable.  $C_3$ :  $J$  should not vanish in the region of  $\alpha$  and  $\alpha'$  of interest, or else there would be a pole in Eq. (5). The existence of an IR zero of  $\beta$  is a scheme-independent property of an AF theory, depending (insofar as perturbation theory is reliable) only on the condition that  $b_2 < 0$ . Hence,  $C_4$ : an ST must satisfy the condition that  $\beta_{\alpha}$  has an IR zero if and only if  $\beta_{\alpha'}$  has an IR zero. These four conditions can always be satisfied by scheme transformations used to study the UVFP at  $\alpha = \alpha' = 0$  and hence in applications to perturbative QCD calculations, since the gauge coupling is small (e.g.,  $\alpha_s(m_Z) = 0.118$ ), and one can choose the  $k_s$  to have small magnitudes [28].

However, we stress that these conditions are not automatically satisfied, and are significant constraints, in the analysis of an (exact or approximate) IRFP. To show this, we first exhibit an apparently reasonable ST that satisfies  $C_1$  and  $C_3$  but fails  $C_2$  and  $C_4$ . This is the map (with  $s_{\text{max}} = \infty$ ) [29]

$$\alpha = \tanh(\alpha'), \quad (10)$$

with the inverse  $\alpha' = (1/2) \ln[(1 + \alpha)/(1 - \alpha)]$  and Jacobian  $J = 1/\cosh^2(\alpha')$ . This ST is acceptable at a

UVFP. But at an IRFP, it can easily happen that  $\alpha_{\text{IR},2\ell} > 1$ , in which case this ST yields a complex, unphysical  $\alpha'$ . For example (see Table III in Ref. [21]), for  $G = \text{SU}(2)$  with  $N_f = 8$  fermions in the fundamental representation,  $\alpha_{\text{IR},2\ell} = 1.26$  and for  $\text{SU}(3)$  with  $N_f = 11$ ,  $\alpha_{\text{IR},2\ell} = 1.23$ .

To exhibit another type of pathology that can arise at an IRFP, but not a UVFP, consider an ST with  $s_{\text{max}} = 2$  and, for simplicity,  $k_1 = 0$ , viz.,

$$a = a'[1 + k_2(a')^2], \quad (11)$$

with a moderate value of  $|k_2|$ . This is a cubic equation for  $a'$  in terms of  $a$ , and, by continuity arguments, in the vicinity of the UVFP, it is guaranteed that this cubic yields a root that satisfies  $C_1 - C_4$ . But the situation is different at an IRFP. Consider sufficiently large  $N_f$  that  $b_2 < 0$ , so there is a two-loop zero of  $\beta$ , at the value (2). For a given  $G$  and  $R$ , as  $N_f$  increases from 0,  $b_2$  decreases through positive values and vanishes, becoming negative, as  $N_f$  increases through the value  $N_{f,b2z} = 17C_A^2/[2T_f(5C_A + 3C_f)]$  (which is always less than the value  $N_{f,b1z} = 11C_A/(4T_f)$  at which  $b_1$  turns negative and AF is lost) [30,31]. The two-loop IR zero of  $\beta$  is thus present for  $N_f$  in the interval  $I$  defined by  $N_{f,b2z} < N_f < N_{f,b1z}$ . Now with  $N_f \in I$ , let us investigate the ST (11). The condition  $b'_3 = 0$  is then a linear equation for  $k_2$ , with the solution  $k_2 = b_3/b_1$ . To guarantee that this ST satisfies  $C_1$ , we require  $1 + k_2(a')^2 > 0$ , i.e.,  $1 + (b_3/b_1)(a')^2 > 0$ . This must be satisfied, in particular, in the vicinity of the two-loop IR zero of  $\beta$ , so substituting the (scheme-independent)  $a_{\text{IR},2\ell} = a'_{\text{IR},2\ell} = -b_1/b_2$  from Eq. (2), we obtain the inequality

$$1 + \frac{b_1 b_3}{b_2^2} > 0. \quad (12)$$

But this inequality is not, in general, satisfied. This can be seen by substituting explicit values of  $b_\ell$  from Table I of Ref. [21] for  $G = \text{SU}(N)$  and  $N_f$  fermions in the fundamental representation, for example.

We proceed to construct and study an ST that does satisfy our constraints and provides a measure of the scheme dependence of the value of the IR zero of  $\beta$  that we calculated in Ref. [21] up to four-loop order in the  $\overline{\text{MS}}$  scheme. We assume  $N_f \in I$ , so a two-loop IR zero of  $\beta$  exists. Starting in this  $\overline{\text{MS}}$  scheme, we construct an ST with  $s_{\text{max}} = 1$  that yields  $b'_3 = 0$ . Equation (3) reads  $a = a'(1 + k_1 a')$ . Solving this for  $a'$ , or equivalently,  $\alpha'$ , we have, formally, two solutions,

$$\alpha'_\pm = \frac{1}{2\bar{k}_1} \left( -1 \pm \sqrt{1 + 4\bar{k}_1 \alpha} \right). \quad (13)$$

Only  $\alpha'_+$  is acceptable, since only this solution has  $\alpha \rightarrow \alpha'$  as  $\alpha \rightarrow 0$ . For  $\alpha'$  to be real, it is necessary that  $\bar{k}_1 > -1/(4\alpha)$ . Solving the equation  $b'_3 = 0$  for  $k_1$ , we get, formally, two solutions,

$$k_{1p}, k_{1m} = \frac{1}{2b_1} [-b_2 \pm \sqrt{b_2^2 - 4b_1 b_3}], \quad (14)$$

where  $(p, m)$  refer to  $\pm$ . We will focus on  $G = \text{SU}(N)$  with fermions in the fundamental and adjoint representation. The discriminant  $b_2^2 - 4b_1 b_3 > 0$  satisfies the requirement of being non-negative here. The solution  $k_{1m}$  must be discarded because it leads to  $\alpha$  and  $\alpha'$  having opposite signs for some  $N_f \in I$ . We thus choose the solution  $k_{1p}$ . We denote this as the  $S_1$  scheme, i.e.,

$$S_1: \quad a = a'(1 + k_{1p} a'). \quad (15)$$

By construction, since  $b'_3 = 0$  in this scheme, the three-loop zero of  $\beta_{\alpha'}$  is equal to the two-loop zero,  $\alpha'_{\text{IR},3\ell} = \alpha'_{\text{IR},2\ell} = \alpha_{\text{IR},2\ell} = -4\pi b_1/b_2$  [26]. At the four-loop level, the IR zero is given by the physical (smallest positive) solution of the cubic  $b_1 + b_2 a' + b'_4(a')^3 = 0$ , with  $b'_4$  given by Eq. (8) with  $k_1 = k_{1p}$  and  $k_2 = k_3 = 0$ .

We have calculated the resultant  $\alpha'_{\text{IR},n\ell}$  in the  $S_1$  scheme up to  $(n = 4)$ -loop level. In Table I we list values of the  $n$ -loop IR zero,  $\alpha'_{\text{IR},n\ell}$  for  $n = 2, 3, 4$  for relevant  $N_f$ , with fermions in the fundamental representation and several values of  $N$ . For comparison we also include the values of  $\alpha_{\text{IR},n\ell}$  for  $n = 3, 4$  in the  $\overline{\text{MS}}$  scheme from Ref. [21]. We

TABLE I. Values of the IR zeros of  $\beta_\alpha$  in the  $\overline{\text{MS}}$  scheme and  $\beta_{\alpha'}$  in the  $S_1$  scheme, for an  $\text{SU}(N)$  theory with  $N_f$  fermions in the fundamental representation, for  $N = 2, 3, 4$ , calculated to  $n$ -loop order and denoted as  $\alpha_{\text{IR},n\ell,\overline{\text{MS}}}$  and  $\alpha'_{\text{IR},n\ell}$ . Here,  $\alpha_{\text{IR},2\ell,\overline{\text{MS}}} = \alpha'_{\text{IR},2\ell}$  is scheme-independent, so we denote it simply as  $\alpha_{\text{IR},2\ell}$ . In the  $S_1$  scheme,  $\alpha'_{\text{IR},3\ell} = \alpha'_{\text{IR},2\ell} = \alpha_{\text{IR},2\ell}$ .

$N$	$N_f$	$\alpha_{\text{IR},2\ell}$	$\alpha_{\text{IR},3\ell,\overline{\text{MS}}}$	$\alpha_{\text{IR},4\ell,\overline{\text{MS}}}$	$\alpha'_{\text{IR},4\ell}$
2	7	2.83	1.05	1.21	0.640
2	8	1.26	0.688	0.760	0.405
2	9	0.595	0.418	0.444	0.2385
2	10	0.231	0.196	0.200	0.109
3	10	2.21	0.764	0.815	0.463
3	11	1.23	0.578	0.626	0.344
3	12	0.754	0.435	0.470	0.254
3	13	0.468	0.317	0.337	0.181
3	14	0.278	0.215	0.224	0.121
3	15	0.143	0.123	0.126	0.068
3	16	0.0416	0.0397	0.0398	0.0215
4	13	1.85	0.604	0.628	0.365
4	14	1.16	0.489	0.521	0.293
4	15	0.783	0.397	0.428	0.235
4	16	0.546	0.320	0.345	0.187
4	17	0.384	0.254	0.271	0.146
4	18	0.266	0.194	0.205	0.110
4	19	0.175	0.140	0.145	0.0785
4	20	0.105	0.091	0.092	0.050
4	21	0.0472	0.044	0.044	0.023

TABLE II. Values as in Table I, but for  $N_f = 2$  fermions in the adjoint representation of  $SU(N)$ .

$N$	$\alpha_{\text{IR},2\ell,\text{adj}}$	$\alpha_{\text{IR},3\ell,\text{adj},\overline{\text{MS}}}$	$\alpha_{\text{IR},4\ell,\text{adj},\overline{\text{MS}}}$	$\alpha'_{\text{IR},4\ell,\text{adj}}$
2	0.628	0.459	0.450	0.258
3	0.419	0.306	0.308	0.173
4	0.314	0.2295	0.234	0.130

have carried out the analogous calculations for fermions in the adjoint representation of  $SU(N)$ . Here,  $N_{f,b1z} = 11/4$  and  $N_{f,b2z} = 17/16$ , so the only physical, integer value of  $N_f \in I$  is  $N_f = 2$ .  $SU(2)$  models with  $N_f = 2$  adjoint fermions have been of recent interest [32]. We list our results in Table II. For both of these cases, we find that  $\alpha'_{\text{IR},3\ell} > \alpha_{\text{IR},3\ell,\overline{\text{MS}}}$  and  $\alpha'_{\text{IR},4\ell} < \alpha_{\text{IR},4\ell,\overline{\text{MS}}}$ .

The anomalous dimension  $\gamma_m$  describes the scaling of a fermion bilinear and the running of a dynamically generated fermion mass in the phase with  $S\chi\text{SB}$ . It plays an important role in technicolor theories, via the renormalization group factor  $\eta = \exp[\int dt \gamma_m(\alpha(t))]$  that can enhance dynamically generated standard-model fermion masses. In the (conformal) non-Abelian Coulomb phase, the IR zero of  $\beta$  is exact, although a calculation of it to a finite-order in perturbation theory is only approximate, and  $\gamma_m$  evaluated at this IRFP is exact. In the phase with  $S\chi\text{SB}$ , where an IRFP, if it exists, is only approximate,  $\gamma_m$  is an effective quantity describing the running of a dynamically generated fermion mass for the evolution of the theory near this approximate IRFP. In Ref. [21] we evaluated  $\gamma_m$  to three- and four-loop order at the IR zero of  $\beta$  calculated to the same order and showed that these higher-loop results were somewhat smaller than the two-loop evaluation. In both the conformal and nonconformal phases it is important to

assess the scheme-dependence of  $\gamma_m$  when calculated to finite order.  $\gamma_m$  is defined as  $\gamma_m = d \ln Z_m / dt$ , where  $Z_m$  is the corresponding renormalization constant. This has the expansion  $\gamma_m = \sum_{\ell=1}^{\infty} \bar{c}_\ell \alpha^\ell$  with  $\bar{c}_\ell$  calculated up to  $\ell = 4$  order in the  $\overline{\text{MS}}$  scheme [33]. Under the general ST (3),  $c_1$  is invariant, while the  $c_\ell$  with  $\ell \geq 2$  change. With  $Z_m(\alpha) = Z'_m(\alpha') F_m(\alpha')$ ,

$$\gamma_m(\alpha) = \gamma'_m(\alpha') + \frac{d\alpha'}{dt} \frac{d \ln F_m}{d\alpha'} = \gamma'_m(\alpha') + \beta_{\alpha'} \frac{d \ln F_m}{d\alpha'}. \quad (16)$$

Hence, at a zero of  $\beta_{\alpha'}$ ,  $\gamma_m(\alpha) = \gamma'_m(\alpha')$  [4]. Although  $\gamma_m$  calculated to all orders is invariant under an ST at a zero of  $\beta$ , in particular an exact IRFP, our present results with the  $\overline{\text{MS}}$  and  $S_1$  schemes show that  $\alpha_{\text{IR},n\ell}$  and  $\gamma_{m,n\ell}(\alpha_{\text{IR},n\ell})$  still exhibit significant scheme-dependence up to  $(n = 4)$ -loop order. This is understandable, since the relevant IRFP occurs at  $\alpha \sim O(1)$ .

It is also of interest to consider STs that are not designed to render any  $b'_\ell = 0$ . Accordingly, we have also done calculations with one-parameter STs having  $s_{\text{max}} = \infty$  and exactly known inverses, such as

$$a = \frac{\tanh(ra')}{r}, \quad (17)$$

and  $a = (1/r) \sinh(ra')$ , where  $r$  is a positive constant. For these, we can vary the effect of the transformation by varying  $r$  from  $r \ll 1$  to values  $r \geq 1$ . These STs provide a further measure of the scheme-dependence of an IRFP [34].

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- [25] One can generalize this to certain STs with  $f(a')$  functions that are finite but nonanalytic at  $a' = 0$ , such as  $f(a') = [1 + \sum_{s=1}^{s_{\max}} k_s (a')^s][1 + \kappa e^{-\nu/a'}]$ , where  $\kappa$  and  $\nu$  are (real) constants and  $\nu > 0$ . Equations (7)–(9) and the analogous relations for higher- $s$  continue to hold for these scheme transformations.
- [26] Hence, if there is an IR zero in the two-loop  $\beta_\alpha$ , at  $\alpha_{\text{IR},2\ell}$  given by (2), then there is also an IR zero in the two-loop  $\beta_{\alpha'}$  at the same value of  $\alpha'$ . This is consistent with the fact that (3) in general maps  $a' = -b_1/b_2$  to  $a \neq -b_1/b_2$ , since (3) is an exact result, whereas the equality of two-loop IR zero values holds for the truncations of  $\beta$  and  $\beta'$  to two-loop order.
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- [29] Equation (10) is the special case of Eq. (17) with  $r = 4\pi$ . For  $r \simeq 1$ , we find that the ST (17) can satisfy  $C_1 - C_4$  at an IRFP.
- [30] The Casimir invariants  $C_R$  and  $T_R$  are defined as  $\sum_a \sum_j \mathcal{D}_R(T_a)_{ij} \mathcal{D}_R(T_a)_{jk} = C_R \delta_{ik}$  and  $\sum_{i,j} \mathcal{D}_R(T_a)_{ij} \times \mathcal{D}_R(T_b)_{ji} = T_R \delta_{ab}$ , where  $R$  is the representation and  $T_a$  are the generators of  $G$ , so that for  $\text{SU}(N_c)$ ,  $C_A = N_c$  for the adjoint ( $A$ ) and  $T_{\text{fund}} = 1/2$  for the fundamental representation, etc.  $C_f$  denotes  $C_R$  for the fermion representation.
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