

Lifshitz fermionic theories with $z=2$ anisotropic scalingH. Montani¹ and F. A. Schaposnik^{2,*}¹*Departamento de Ciencias Exactas y Naturales Unidad Académica Caleta Olivia Universidad Nacional de la Patagonia Austral, 9011 Caleta Olivia, Argentina*²*Departamento de Física, Universidad Nacional de La Plata, Instituto de Física La Plata-CONICET C.C.67, 1900 La Plata, Argentina*

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We construct fermionic Lagrangians with anisotropic scaling $z = 2$, the natural counterpart of the usual $z = 2$ Lifshitz field theories for scalar fields. We analyze the issue of chiral symmetry, construct the Noether axial currents and discuss the chiral anomaly giving explicit results for two-dimensional case. We also exploit the connection between detailed balance and the dynamics of Lifshitz theories to find different $z = 2$ fermionic Lagrangians and construct their supersymmetric extensions.

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I. INTRODUCTION

Lifshitz field theories exhibit an anisotropic scaling between space and time that breaks Lorentz symmetry. Introduced in the context of condensed matter phenomena [1–4], they are also actively investigated in connection with problems in gravity [5–7] and in particle physics (see [8] and reference therein).

The anisotropic gauge scaling for spatial coordinates x and time t takes the form

$$x \rightarrow bx, \quad t \rightarrow b^z t \quad (1)$$

where $b \in \mathbb{R}$ is the scaling factor and $z \in \mathbb{Z}$ the degree of anisotropy which in condensed matter systems is identified as a dynamical critical exponent. The $z=1$ case corresponds to the ordinary scaling law in the conformal group. In connection with this, it becomes natural to study, within the context of gauge/gravity correspondence, metrics with anisotropic scaling which could be dual to Lifshitz field theories in the same way as AdS metrics allow to establish duality leading to strongly coupled conformal field theories [9].

A prototype of a Lifshitz field theory is a $(d+1)$ -dimensional scalar theory with dynamics governed by the action

$$S = \frac{1}{2} \int d^d x dt ((\partial_t \phi)^2 - \kappa^2 \phi (-\nabla^2)^z \phi). \quad (2)$$

Here $\nabla^2 \equiv \partial_i \partial_i$ is the spatial Laplacian. The scaling dimensions are $[x^i] = -1$ and $[t] = -z$, with $[\phi] = (d-z)/2$ and $[\kappa] = z-2$.

Models with action (2), particularly in the $z=2$ case, interest condensed matter physicists because they exhibit a line of fixed points parametrized by κ and reproduce the phase diagrams of known materials [2]. Other $z \neq 1$ bosonic field theories have been discussed, and also their gravity duals have been investigated using the holographic

principle with a bulk geometry given (in Poincaré coordinates) by the metric

$$ds^2 = L^2 \left(-\frac{dt^2}{u^{2z}} + \frac{d\vec{x}^2}{u^2} + \frac{du^2}{u^2} \right) \quad (3)$$

which for $z=1$ becomes the usual AdS metric.

Defining fermionic Lifshitz theories is an open issue. Already in Refs. [3,4], dealing with conformal quantum critical points, the problem is addressed for the case of $z=2$ scaling by defining a ground state wave functional for the Hamiltonian of a $2+1$ system in terms of the $(1+1)$ Euclidean Dirac action, but the discussion in this work is restricted to a bosonized version of the theory. Self-interacting fermionic Hamiltonians leading to a $z=2$ dynamical critical exponent have been constructed in [10] in $1+1$ dimensions following an approach different to that based in theories defined as in (2). More recently fermion theories with $z=3$ have been studied in connection with its renormalizability, chiral anomaly and renormalization group flow [11–17].

In this work we undertake the construction of $z=2$ fermionic Lifshitz theory following a different way. Having in mind that the Dirac equation can be seen as the “square root” of the Klein-Gordon one, we first start from the bosonic action (2) and derive a γ -matrices algebra which naturally leads to a Lifshitz fermion model in $d+1$ space-time dimensions. In doing so, we introduce the square root of the Laplacian as the source of the anisotropic scaling. We consider the $z=2$ case which corresponds to $(-\nabla^2)^{1/2}$ but the power $z/2$ could be equally considered giving rise to theories with arbitrary even z . We shall also discuss the issue of chiral symmetry for massless fermions and consider the case in which they couple to an Abelian gauge field. We work out the corresponding classical Noether currents and we also compute the chiral anomaly in $1+1$ space-time dimensions which, as it is well-known, is closely related to bosonization, one of the issues of relevance in condensed matter applications of Lifshitz theories [3].

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Second, we present an approach based on the connection between the principle of detailed balance and the dynamics of Lifshitz theories stressed in Refs. [5–7]. The main ingredient in this approach is stochastic field theory quantization, a useful tool in the construction of Lifshitz field theory actions. Indeed, starting from the relativistic action for scalar fields in d Euclidean dimensions stochastic quantization leads to a $d + 1$ effective action with the fictitious time derivatives appearing quadratically and those of the d remaining derivatives quartically. Following this approach, we shall analyze the stochastic quantization of fermionic theories and construct various Dirac-Lifshitz actions with different anisotropy scaling z including $z = 2$.

The paper is organized as follows. We establish in Sec. II a connection between $z = 2$ bosonic and fermionic theories leading to a consistent $z = 2$ Dirac-Lifshitz action in $d + 1$ space-time dimensions. In Sec. III we discuss the issue of chiral symmetry finding the classically conserved axial currents and calculating the quantum chiral anomaly in the particular case of a $1 + 1$ dimensional space. Working within the stochastic quantization approach, we discuss in Sec. IV the construction of alternative $z = 2$ fermionic theories in $d + 1$ dimensions as well as their supersymmetric extension. Finally we present in Sec. V a discussion of our results.

II. CONNECTING BOSONIC AND FERMIONIC $z=2$ LIFSHITZ THEORIES

Consider a Lifshitz bosonic action of the form (2) in $d + 1$ Euclidean space-time with a $z = 2$ scaling

$$S_L = \int d^{d+1}x \left(\frac{1}{2} (\partial_0 \varphi)^2 + \frac{\kappa^2}{2} (\nabla^2 \varphi)^2 \right) \quad (4)$$

or

$$S = \frac{1}{2} \int d^{d+1}x \left(\frac{1}{2} (\partial_0 \varphi)^2 - \frac{\kappa^2}{2} (\partial_i \varphi) \nabla^2 (\partial_i \varphi) \right), \quad i = 1, 2, \dots, d. \quad (5)$$

It will be useful for our purposes to write action (5) in the form

$$S = \frac{1}{2} \int d^d x \partial_\mu \varphi G_{\mu\nu} (\nabla^2) \partial_\nu \varphi \quad (6)$$

with $G_{\mu\nu}$ defined as

$$(G_{\mu\nu}) = \begin{pmatrix} I_{1 \times 1} & 0_{1 \times n} \\ 0_{n \times 1} & -\kappa^2 \nabla^2 I_{n \times n} \end{pmatrix}. \quad (7)$$

The integral on the right-hand side of Eq. (6) can be regarded as a degenerate scalar product on the space of vector fields on M ,

$$(\cdot, \cdot)_M: \mathfrak{X}(M) \otimes \mathfrak{X}(M) \rightarrow \mathbb{R}$$

such that

$$(V(x), W(x))_M = \int d^d x (V_0 W_0 - \kappa^2 V_i \nabla^2 W_i). \quad (8)$$

The associated group of isometries can be obtained from the invariance relation

$$(\Lambda(V), \Lambda(W))_M = (V, W)_M$$

which explicitly reads

$$\Lambda_{0\mu} \Lambda_{0\nu} - \kappa^2 \Lambda_{i\mu} \Lambda_{i\nu} \nabla^2 = G_{\mu\nu} (\nabla^2). \quad (9)$$

Taking $\mu, \nu = 0$, we get

$$\Lambda_{i0} \Lambda_{i0} = 0, \quad \Lambda_{00}^2 = 1$$

which implies

$$\Lambda_{i0} = 0, \quad \Lambda_{00} = \pm 1$$

respectively. Alternatively, by taking $\mu = i$ and $\nu = j$ in Eq. (9), we determine the relations defining the remaining components, namely,

$$\Lambda_{0i} = 0 \quad \Lambda_{ki} \Lambda_{kj} = \delta_{ij}$$

implying that Λ_{ij} are the entries of an orthogonal matrix. In summary, the final form of Λ is

$$\Lambda_{00} = 1, \quad \Lambda_{0i} = \Lambda_{i0} = 0, \quad \Lambda_{ki} \Lambda_{kj} = \delta_{ij} \quad (10)$$

meaning that isometries of $G_{\mu\nu} (\nabla^2)$ reduce to rotations in the spatial sector and time reversal map.

One can extend this framework to the case of fermionic theories in a natural way by defining ‘‘local’’ gamma matrices Γ satisfying

$$\{\Gamma_\mu, \Gamma_\nu\} = 2G_{\mu\nu}.$$

An appropriate choice for these gamma matrices is

$$\Gamma_i = \kappa (-\nabla^2)^{1/2} \gamma_i, \quad \Gamma_0 = \gamma_0, \quad i = 1, 2, \dots, d$$

where $(\gamma_a) \equiv (\gamma_0, \gamma_i)$ satisfy the ordinary Euclidean-space Clifford algebra

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}, \quad a = 0, 1, \dots, d$$

and $(-\nabla^2)^{1/2}$ is the nonlocal operator which can be defined as in [18] (see also Refs. [19–22]).

$$(-\nabla^2)^{1/2} f(x) = P.V. \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+1}} d^d y. \quad (11)$$

Defined in this way the square root of the Laplacian is an elliptic pseudodifferential operator of order 1. However, a better insight of the nonlocal operator $(-\nabla^2)^{1/2}$ is attained in the context of the harmonic extensions of a compactly supported function on \mathbb{R}^n to $\mathbb{R}^n \times (0, \infty)$, as in Ref. [18], where it appears as a map from the Dirichlet-type data problem to the Neumann-type data one. Moreover, it can be shown that it is self-adjoint and positive-definite (see [23,24] and references therein). In the Appendix we give a brief description of this approach and some properties of

this operator. From now on, we shall use a more compact notation for the square root of the Laplacian naming it by T

$$T := (-\nabla^2)^{1/2}. \quad (12)$$

For further developments, it is useful to work out the action of the operator T on imaginary exponentials. From the identity

$$-\nabla^2 \exp(ikx) = |k|^2 \exp(ikx) \quad (13)$$

and the natural definition of the square root Laplacian it can be established that

$$T \exp(ikx) = |k| \exp(ikx) \quad (14)$$

(see the Appendix for a proof of this result). Let us point out that, strictly speaking, since the operator symbol $|k|$ is not smooth at the origin, one needs to define it in a distributional sense [18], so that an appropriate cutoff function brings its Schwartz kernel in the class of hypoelliptic operators. It is in this way that the square root of the Laplacian can be defined as a regular elliptical operator. In brief, the procedure is as follows (for a detailed treatment see for example [25]). One writes $|k|$ in the form

$$|k| = \chi(k)|k| + (1 - \chi(k))|k|$$

where χ is smooth, compactly supported and equals 1 near $k = 0$. By applying the inverse Fourier transform, one gets a properly supported operator (an elliptic pseudodifferential operator of order 1) plus a smoothing operator (i.e., one with a smooth Schwartz kernel), the sum composing a regular operator.

In summary, working in the sense of distributions and using a smoothing function the problem at $k = 0$ is overcome in the same way as when distribution theory is used in Lorentz invariant quantum field theories. Moreover, we shall see below that this problem has no consequence in our calculations of the anomaly since continuity of the symbol at the origin is sufficient and the required regularization deals, as usual, with the ultraviolet problem.

We shall now propose a vierbein-like connection between Γ and γ

$$(E_\mu^a) = \begin{pmatrix} I & 0 \\ 0 & \kappa T I \end{pmatrix} \delta(x-y), \quad (E_a^\mu) = \begin{pmatrix} I & 0 \\ 0 & -\frac{1}{\kappa} T I \end{pmatrix} \delta(x-y) \quad (15)$$

lowering and raising indices with $G_{\mu\nu}$ and δ_{ab} so that

$$(E_\mu^a E_{a\nu}) = (G_{\mu\nu}), \quad (E_a^\nu E_{\nu b}) = (\delta_{ab})$$

with products including an integration over space-time.

This set of matrices $\{\Gamma_\mu\}$ naturally induces a definition of a Dirac-Lifshitz equation in the form

$$(i\Gamma_\mu \partial_\mu - m)\psi = 0. \quad (16)$$

Here m has dimensions of mass squared. In terms of the ordinary γ_μ matrices Eq. (16) reads

$$(i\gamma_0 \partial_0 + i\kappa T(\gamma_i \partial_i) - m)\psi = 0. \quad (17)$$

This is nothing but the Euler-Lagrange equation for action

$$S_{\text{DL}} = \int d^{d+1}x \bar{\psi} (i\gamma_0 \partial_0 + i\kappa T \gamma_i \partial_i - m) \psi. \quad (18)$$

These Dirac-Lifshitz fields share the anisotropic gauge scaling (1) for $z = 2$. From this expressions we define the Dirac-Lifshitz operator

$$\gamma_\mu D_\mu = i\gamma_0 \partial_0 + i\kappa T \gamma_i \partial_i - m. \quad (19)$$

It is interesting to note that, as in the ordinary Dirac equation case, one can proceed to “square” the Dirac-Lifshitz equation looking for a resulting Klein-Gordon-like equation. Indeed, multiplying (16) equation by the operator $(i\Gamma_\mu \partial_\mu + m)$ and using the identity (see Appendix

$$T \partial_i = \partial_i T \quad (20)$$

we get the usual $z = 2$ Lifshitz equation for scalar fields, subject of many recent investigations

$$(\partial_0^2 - \kappa^2 (\nabla^2)^2 - m^2)\psi = 0. \quad (21)$$

This is precisely the equation of motion arising from the $z = 2$ bosonic action (4) in the $m = 0$ case and in this respect Lagrangian (18) seems to be a natural candidate to study Lifshitz fermionic models in connection with the problems of the conformal quantum critical points [3].

III. CHIRAL SYMMETRY AND THE $z=2$ DIRAC-LIFSHITZ LAGRANGIAN

The issue of chiral symmetry and the quantum anomaly of $z = 3$ Lifshitz theories with massless fermions coupled to a gauge field in $3 + 1$ space-time dimensions has been thoroughly studied in [12,15,16]. Using the path-integral framework the resulting anomaly \mathcal{A} was shown to be identical to the usual relativistic calculation and this is likely to be related to its topological character in the sense that \mathcal{A} is metric independent and its integral—the topological charge—does not depend on local details but only on global properties. The calculation for the $z = 2$ Lifshitz defined in the preceding section seems more involved since it implies handling the square root of the Dirac operator both in the action and as the natural regulator in a sensible definition of the path-integral measure. We shall now discuss this issue.

A. The free massless fermion theory

In the $m = 0$ case the free fermion Lifshitz action (18) is invariant under chiral rotations

$$\psi \rightarrow \exp(\gamma_5 \epsilon) \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \exp(\gamma_5 \epsilon). \quad (22)$$

There should then be, at the classical level, a continuity equation for the chiral current and an associated conserved chiral charge. To obtain their explicit form one can proceed as follows. One promotes the global change to a local (infinitesimal for simplicity) one

$$\begin{aligned}\psi &\rightarrow \psi' = \exp(\epsilon(x)\gamma_5)\psi \approx (1 + \epsilon(x)\gamma_5)\psi \\ \bar{\psi} &\rightarrow \bar{\psi}' = \bar{\psi} \exp(\epsilon(x)\gamma_5) \approx \bar{\psi}(1 + \epsilon(x)\gamma_5)\end{aligned}\quad (23)$$

under which the action should necessarily change,

$$\begin{aligned}\delta S_{\text{DL}} &= \int d^{d+1}x \bar{\psi}'(1 + \gamma_5 \epsilon(x))(i\gamma^0 \partial_0 + i\kappa T \gamma^i \partial_i) \\ &\quad \times (1 + \gamma_5 \epsilon(x))\psi - S_{\text{DL}}.\end{aligned}$$

After some work one finds

$$\begin{aligned}\delta S_{\text{DL}} &= \int d^{d+1}x i \bar{\psi} \gamma^1 \gamma_5 \psi \partial_0 \epsilon + \int d^{d+1}x i \kappa \bar{\psi} T \gamma^i \gamma_5 \psi \partial_i \epsilon \\ &= -i \int d^{d+1}x \bar{\psi} (\gamma^0 \gamma_5 \psi \partial_0 + \kappa T \gamma^i \gamma_5) \psi \epsilon.\end{aligned}\quad (24)$$

For constant ϵ this term should be absent and then we get a continuity equation and the explicit form of the chiral current

$$\partial_\mu j_5^\mu = 0 \quad j_5^0 = \bar{\psi} \gamma^0 \gamma_5 \psi, \quad j_5^i = \kappa \bar{\psi} \gamma^i \gamma_5 T \psi. \quad (25)$$

As it was to be expected, the chiral charge coincides with the usual (Dirac theory) one,

$$Q_5 = \int d^d x \bar{\psi} \gamma^0 \gamma_5 \psi = \int d^d x \psi^\dagger \gamma_5 \psi. \quad (26)$$

There is of course a $U(1)$ global invariance with associated current

$$j^0 = \bar{\psi} \gamma^0 \psi, \quad j^i = \kappa \bar{\psi} \gamma^i T \psi \quad (27)$$

so that again, the conserved $U(1)$ charge coincides with the usual one,

$$Q = \int d^d x \bar{\psi} \gamma^0 \psi = \int d^d x \psi^\dagger \psi. \quad (28)$$

B. Coupling fermions to $U(1)$ gauge fields

In order to couple massless $z = 2$ Lifshitz fermions to a $U(1)$ gauge field background A_μ , we introduce the extension of the Lifshitz-Dirac operator (19)

$$\mathcal{D} = \gamma^\mu \mathcal{D}_\mu[A] = \gamma_0 D_0[A] + \kappa \gamma_1 D_1[A] \mathcal{T}[A] \quad (29)$$

where we have defined

$$\mathcal{T}[A] = (D_j[A] D_j[A])^{1/2} \quad (30)$$

with covariant derivatives given by

$$D_0 = i\partial_0 + eA_0, \quad D_i = i\partial_i + eA_i. \quad (31)$$

Some properties of operator $\mathcal{T}[A]$ are discussed in the Appendix. In particular, $\mathcal{T}[A]$ is an Hermitian operator so that \mathcal{D} defined as in (29) is also Hermitian.

A $z = 2$ Dirac-Lifshitz action can be compactly written in terms of the Dirac-Lifshitz operator $\mathcal{D}_\mu[A]$ as

$$S_{\text{DL}}[A] = \int d^{d+1}x \bar{\psi} \gamma^\mu \mathcal{D}_\mu[A] \psi. \quad (32)$$

As in the free case, this action remains invariant under global chiral transformations (as well as under local $U(1)$ gauge transformation). So, there should be a classically conserved axial current. Concerning the quantum level, one should expect an anomaly which we shall calculate within the path-integral framework following the well-honored Fujikawa's approach [26,27]. We start from the partition function written in the form

$$Z = \int D\bar{\psi} D\psi \exp(-S[A]) \quad (33)$$

and consider the infinitesimal local path-integral change of variables given by Eq. (23). The action in the exponential now gets, as in the case of the free fermion classical action, a contribution proportional to $\partial_\mu \epsilon$; and concerning the path-integral measure, there is a Fujikawa Jacobian $J_5[\epsilon]$. So, the partition function changes to

$$Z = \int D\bar{\psi}' D\psi' J_5[\epsilon] \exp(-S[A] - \delta S[A; \partial_\mu \epsilon]). \quad (34)$$

As in the free fermion case, we shall show below that δS can be accommodated as

$$\delta S[A; \partial_\mu \epsilon] = \int d^d x (\partial_\mu \epsilon(x)) j_5^\mu \quad (35)$$

so that Z can be written as

$$Z = \int D\bar{\psi}' D\psi' J_5[\epsilon] \exp(-S[A] - \epsilon(x)(\partial_\mu j_5^\mu)). \quad (36)$$

Since the generating functional cannot depend on the parameter $\epsilon(x)$ introduced by the change of variables one has

$$\frac{1}{Z} \frac{\delta Z}{\delta \epsilon(x)} = 0 \quad (37)$$

which leads to the anomaly equation

$$\langle \partial_\mu j_5^\mu \rangle = \mathcal{A} \quad (38)$$

where

$$\mathcal{A} = \frac{\delta \log J_5}{\delta \epsilon(x)}. \quad (39)$$

Let us now prove Eq. (35). The action $S[A]$ can be written as

$$S[A] = \int d^{d+1}x \bar{\psi} (\gamma_0 D_0 + \kappa \gamma^i (D_i[A] \mathcal{T}[A])) \psi. \quad (40)$$

After the change of variables (23) one has

$$\delta S[A] = S_0[A] + S_1[A] - S[A] \quad (41)$$

with

$$S_0[A] = \int d^{d+1}x \bar{\psi}' (1 + \gamma_5 \epsilon(x)) \gamma^0 D_0 (1 + \gamma_5 \epsilon(x)) \psi$$

$$S_1[A] = \kappa \int d^{d+1}x \bar{\psi}' (1 + \gamma_5 \epsilon(x)) \gamma^i D_i \mathcal{T}[A] (1 + \gamma_5 \epsilon(x)) \psi. \quad (42)$$

The term S_0 contains the same variation arising in an ordinary Dirac theory and can then be written in the form

$$S_0 = - \int d^{d+1}x \epsilon(x) \partial_i (\bar{\psi} \gamma^0 \gamma_5 \psi) + \int d^d x \bar{\psi} i \gamma^0 D_0 \psi. \quad (43)$$

In order to handle δS_1 in a similar way, we first integrate by parts

$$\begin{aligned} \delta S_2[A] = & \kappa \int d^{d+1}x D_i^* \bar{\psi} (1 + \gamma_5 \epsilon(x)) \gamma^i \mathcal{T}[A] (1 + \gamma_5 \epsilon(x)) \psi \\ & - \kappa \int d^{d+1}x \bar{\psi} \gamma^i D_i \mathcal{T}[A] \psi \end{aligned} \quad (44)$$

finally getting

$$\begin{aligned} \delta S_2 = & -i\kappa \int d^{d+1}x (\partial_i \epsilon) \bar{\psi} \gamma^5 \gamma^i \mathcal{T}[A] \psi \\ = & i\kappa \int d^{d+1}x \epsilon(x) \partial_i (\bar{\psi} \gamma^5 \gamma^i \mathcal{T}[A] \psi). \end{aligned} \quad (45)$$

Putting all this together we have for the complete variation of the action

$$\delta S = \int d^{d+1}x \epsilon(x) \partial_\mu j_5^\mu \quad (46)$$

with

$$j_5^0 = \bar{\psi} \gamma_5 \gamma^0 \psi, \quad j_5^i = \kappa \bar{\psi} \gamma_5 \gamma^i \mathcal{T}[A] \psi \quad (47)$$

which differs from the free fermionic case because j_5^i contains the operator $\mathcal{T}[A] = (D_i D_i)^{1/2}$ instead of $T = (-\nabla^2)^{1/2}$ and hence depends on A_i .

We have now to compute the Fujikawa Jacobian associated to the chiral change of variables. For simplicity we shall consider the 1+1-dimensional case in which Lagrangian in action (32) takes the simple form

$$L = \bar{\psi} \gamma^0 (D_0) \psi + \kappa \bar{\psi} (\gamma^1 D_1 \mathcal{T}[A]) \psi. \quad (48)$$

Our choice of gamma matrices is

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma_0 \gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (49)$$

Following Fujikawa's approach, we consider a change of variables as in (23) and calculate the associated Jacobian $J^5[\delta \epsilon(x)]$ which gives

$$J_5[\delta \epsilon(x)] = \exp\left(2 \int d^2x \sum_n \varphi_n^\dagger(x) \gamma_5 \varphi_n(x) \delta \epsilon(x)\right). \quad (50)$$

From the fact that the generating functional cannot depend on the parameter $\epsilon(x)$ introduced by the change of variables

$$\frac{\delta Z}{\delta \epsilon(x)} = 0 \quad (51)$$

one then gets

$$\partial_\mu j_5^\mu = \frac{\delta \log J_5}{\delta \epsilon(x)} = \mathcal{A} \quad (52)$$

with \mathcal{A} given by the ill-defined formula

$$\mathcal{A} = 2 \sum_n \varphi_n^\dagger(x) \gamma_5 \varphi_n(x) = \lim_{y \rightarrow x} 2 \text{tr} \gamma_5 \delta^{(2)}(x-y) \quad (53)$$

which thus requires an appropriate regularization

$$\mathcal{A}_{\text{reg}} = \lim_{y \rightarrow x} 2 \text{tr} \gamma_5 \delta^{(2)}(x-y)|_{\text{reg}}. \quad (54)$$

A practical regularization approach is the so-called heat-kernel method which consists of introducing an appropriate regulating operator R^2 and a cutoff mass M in the form

$$\mathcal{A}_{\text{reg}} = \lim_{y \rightarrow x} 2 \text{tr} (\gamma_5 \exp(-R^2/\Lambda[M]^2) \delta^{(2)}(x-y)) \quad (55)$$

where $\Lambda[M] = M^q$ with q chosen so that $[\Lambda] = [R] = q$. A finite result for the anomaly is obtained by interchanging trace and limit

$$\mathcal{A} = \lim_{M \rightarrow \infty} \lim_{y \rightarrow x} 2 \text{tr} (\gamma_5 \exp(-R^2/\Lambda^2) \delta^{(2)}(x-y)). \quad (56)$$

Now, the regulator should be chosen so as to preserve the basic invariances of the theory, in the present case gauge invariance and also the invariance under the anisotropic scaling (1). This makes it natural to use as a regulator the operator appearing in the Lagrangian so that $R^2 = \mathcal{D}\mathcal{D}$ and then one has to choose $\Lambda[M] = M^4$.

In two space-time dimensions the Dirac-Lifshitz operator reduces to

$$\mathcal{D} = i\gamma_0 D_0 + \kappa \gamma_1 D_1[A] \mathcal{T}[A] \quad (57)$$

and $\mathcal{T}[A]$ is simply

$$\mathcal{T}[A] = (D_1[A] D_1[A])^{1/2}. \quad (58)$$

In the Appendix we show that $\mathcal{T}[A]$ can be also written in the form

$$\mathcal{T}[A] = W^{-1}[A_1] T W[A_1] \quad (59)$$

where

$$W[A_1] = \exp(-ie \int^x A_1(y) dy). \quad (60)$$

At this point, it is important to note that one can gauge away one component of the gauge potential just by a gauge transformation. In particular, we shall consider $W[A_1]$ as a $U(1)$ -gauge group element $W[A_1]$ so that

$$(A_0, A_1) \rightarrow (W[A_1] A_0 W^{-1}[A_1], 0)$$

which is formally implemented as

$$A_\mu \rightarrow A_\mu + W[A_1] (i\partial_\mu) W^{-1}[A_1]. \quad (61)$$

This gauge will greatly simplify calculations leading to a result that can be trivially extended to a general gauge.

The regulator \mathcal{D}^2 can be now written in the form

$$\begin{aligned} \mathcal{D}^2 = & ID_0 D_0 - i\kappa \gamma_5 [D_0, W^{-1}[A_1] (i\partial_1) T W[A_1]] \\ & + \kappa^2 I W^{-1}[A_1] (-\partial_1^2) W[A_1]. \end{aligned} \quad (62)$$

Under the gauge transformation (61) D_μ changes according to

$$D_\mu \rightarrow W[A_1]D_\mu W^{-1}[A_1]$$

so that the regularizing operator becomes

$$\begin{aligned} W[A_1]\mathcal{D}^2 W^{-1}[A_1] &= IW[A_1]D_0D_0W^{-1}[A_1] + \kappa^2 I(-\partial_1^2)^2 \\ &\quad - i\kappa\gamma_5[W[A_1]D_0W^{-1}[A_1], (i\partial_1)T] \end{aligned} \quad (63)$$

with this and the $\delta^{(2)}(x-y)$ function written in term of the plane wave basis Eq. (56) becomes

$$\begin{aligned} \mathcal{A}_{\text{reg}} &= \frac{1}{(2\pi)^2} \lim_{M \rightarrow \infty} \lim_{y \rightarrow x} \\ &\quad \times \int d^2p 2\text{tr}(\gamma_5 e^{-ip_\mu y_\mu} e^{-(W[A_1]\mathcal{D}^2 W^{-1}[A_1])/M^4} e^{ip_\mu x_\mu}). \end{aligned} \quad (64)$$

Taking now the limit $y \rightarrow x$, the operator content of \mathcal{A}_{reg} becomes

$$\begin{aligned} e^{-ip_\mu x_\mu} W[A_1]\mathcal{D}^2 W^{-1}[A_1] e^{ip_\mu x_\mu} \\ = I\Phi(A, p) - i\kappa\gamma_5\Theta(A, p) + p_0^2 + \kappa^2 I p_1^4. \end{aligned}$$

Here we have used that

$$e^{-ip_1 x_1} W[A_1] = W\left[A_1 + \frac{p_1}{e}\right]$$

and introduced Φ and Θ defined as

$$\begin{aligned} \Phi(A, p) &= W\left[A_1 + \frac{p_1}{e}\right](D_0^2 - 2p_0(i\partial_0))W^{-1}\left[A_1 + \frac{p_1}{e}\right] \\ \Theta(A, p) &= \left[W\left[A_1 + \frac{p_1}{e}\right](i\partial_0)W^{-1}\left[A_1 + \frac{p_1}{e}\right], \right. \\ &\quad \left. e^{-ip_1 x_1}(i\partial_1)T e^{ip_1 x_1}\right]. \end{aligned}$$

Making the substitution $p_0 = M^2 k_0$ and $p_1 = M k_1$, the anomaly becomes

$$\begin{aligned} \mathcal{A}_{\text{reg}} &= \frac{1}{(2\pi)^2} \lim_{M \rightarrow \infty} \int d^2p 2M^3 (\exp(-(k_0^2 + \kappa^2 k_1^4)) \\ &\quad \times \text{tr}(\gamma_5 \exp(-(I\Phi(A, M^2 k_0, M k_1) \\ &\quad - i\kappa\gamma_5\Theta(A, M^2 k_0, M k_1))/M^4)). \end{aligned} \quad (65)$$

By M power counting in the argument of the exponential, one concludes that only the linear term in $\Theta(A, M^2 k_0, M k_1)$ survives after taking $M \rightarrow \infty$ and the γ -matrix trace. In order to carry out the explicit calculation of this contribution, recall that before differentiating with respect to $\epsilon(x)$ the regulated Jacobian includes an integral over two-dimensional space and hence one can use the integration by parts properties of T (see the Appendix) and then differentiate, thus obtaining

$$\Theta(A, M^2 k_0, M k_1) = ieM|k_1|F_{01}.$$

The anomaly can then be finally written in a gauge invariant way as

$$\mathcal{A}_{\text{reg}} = -\frac{e}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu} \quad (66)$$

which coincides with the result for the ordinary relativistic two-dimensional

$$\partial_\mu j_5^\mu = -\frac{e}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu}. \quad (67)$$

As observed in Refs. [12,15] for $z = 3$ models in $3 + 1$ dimensions (and then generalized to arbitrary values of z in [16]), the coincidence between the anomaly of relativistic $z = 1$ and Lifshitz $z \neq 1$ theories is not surprising in retrospect since the anomaly \mathcal{A} is related to a topological density that coincides with a total divergence, so that Eq. (67) is the only possible result for the anomaly, including the universal coefficient, independently of the value of z . To confirm this in the $1 + 1$ -dimensional case, we briefly describe the $1 + 1$ anomaly calculation for the $z = 3$ model, with the action in the form usually taken in the literature,

$$S = \int d^d x dt \bar{\psi} \gamma^\mu \mathcal{D}_\mu^{(3)} \psi. \quad (68)$$

Here the Dirac-Lifshitz operator is given by

$$\begin{aligned} \mathcal{D}^{(3)} &= \gamma^\mu \mathcal{D}_\mu^{(3)} \\ &= \gamma^0 \mathcal{D}_0 + \frac{1}{2} \gamma^i (D_i (-D_k D_k) + (-D_k D_k) D_i) \end{aligned} \quad (69)$$

with the covariant derivatives defined as in (31). Again, at the classical there is a conserved Noether charge associated to a global chiral transformation. At the quantum level, one should take into account the change in the fermionic measure and one ends with the anomaly Eq. (52)

$$\mathcal{A} = \lim_{M \rightarrow \infty} \lim_{y \rightarrow x} 2\text{tr}(\gamma_5 \exp(-\mathcal{D}^{(3)} \mathcal{D}^{(3)}/M^6) \delta^{(2)}(x-y)). \quad (70)$$

As before, the regulator has been chosen so as to preserve the basic invariances of the theory (i.e. gauge invariance and also the anisotropic scaling invariance) and hence $\Lambda[M]$ in (55) should be $\Lambda[M] = M^6$.

In $1 + 1$ dimensions the Dirac-Lifshitz operator reduces to

$$\mathcal{D}^{(3)} = i\gamma_0 D_0 - \kappa\gamma_1 D_1^3$$

and $\mathcal{D}^{(3)2}$ is given by

$$\mathcal{D}^{(3)2} = ID_0^2 - i\kappa\gamma_5 [D_0, D_1^3] + \kappa^2 ID_1^6. \quad (71)$$

We shall not repeat the details of the calculation that follow closely those that leading to (67) in the $z = 2$ case. We just quote the only nonvanishing $M \rightarrow \infty$ term in \mathcal{A} which instead of integral (67) is now given by

$$\mathcal{A} = -3\epsilon\kappa \frac{2}{(2\pi)^2} F_{01} \left(\int d^2k e^{-(k_0^2 + \kappa^2 k_1^2)} k_1^2 \right)$$

which can be easily integrated to give

$$\begin{aligned} \mathcal{A} &= -3\epsilon\kappa \frac{2}{(2\pi)^2} \sqrt{\pi} F_{01} \left(\int k_1^2 dk_1 e^{-\kappa^2 k_1^2} \right) \\ &= -\frac{e}{2\pi} F_{01} = -\frac{e}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu} \end{aligned}$$

which again coincides with the ordinary result for the two-dimensional anomaly. Let us note that the above-mentioned problem regarding the smoothness of operator T at $k = 0$ has not played any role in the derivation of this result since at each stage of the calculation the integrands are infrared safe.

IV. LIFSHITZ FERMIONS AND STOCHASTIC QUANTIZATION

It has been recently observed that field theory models with the anisotropic scaling (1) can be thought of as stemming from a stochastic quantization process [6,7,28,29]. In this way, a $d + 1$ -dimensional bosonic Lifshitz theory like that defined by action (4), with $z = 2$ can be thought of as descending from a free bosonic action in d Euclidean dimensions,

$$S^{(d)} = \int d^d x \partial_i \phi \partial_i \phi, \quad i = 1, 2, \dots, d. \quad (72)$$

The dynamics in extra time variable τ is ruled by a Langevin equation, namely, a first-order differential equation with a dissipative (drift) term and driven by a source of stochastic noise. As far as the quantization of a d -dimensional theory is concerned (the asymptotic equilibrium state), there exists some arbitrariness in the election of this stochastic process. One may take some profit from this freedom to build up theories in $d + 1$ dimensions with different behaviors under “time” and space scaling.

In the case of the bosonic field (4), to get a $z = 2$ Lifshitz theory one introduce an extra dependence of the fields on this extra time variable, $\phi(x) \rightarrow \phi(x, \tau)$, and compose the Langevin equation with $\delta S^{(d)}/\delta \phi$ playing the role of a drift force, and a white noise source as the stochastic force [30]. The case of fermionic fields can be treated analogously [31,32], with ψ and $\bar{\psi}$ taken as independent fields, each one with its own Langevin equation

$$\begin{aligned} \frac{\partial \psi(x, \tau)}{\partial \tau} &= -\frac{\delta S[\bar{\psi}, \psi]}{\delta \bar{\psi}} + \eta(x, \tau) \\ \frac{\partial \bar{\psi}(x, \tau)}{\partial \tau} &= \frac{\delta S[\psi, \bar{\psi}]}{\delta \psi} + \bar{\eta}(x, \tau) \end{aligned} \quad (73)$$

where $\eta, \bar{\eta}$ are anticommuting fermionic Gaussian noises satisfying

$$\langle \eta_\alpha(x, \tau) \rangle = \langle \bar{\eta}_\alpha(x, \tau) \rangle = 0 \quad (74)$$

$$\langle \eta_\alpha(x_1, \tau_1) \bar{\eta}_\beta(x_2, \tau_2) \rangle = 2\delta(t_1 - t_2) \delta^{(d)}(x_1 - x_2) \delta_{\alpha\beta} \quad (75)$$

with α, β the spinor indices. Such normalization arises from the following definition of the (Fokker-Planck) generating functional

$$\begin{aligned} Z[J, \bar{J}] &= \int D\bar{\eta} D\eta \exp\left(-\frac{1}{2} \int dx \int_0^\tau d\tau' \bar{\eta} \eta\right) \\ &\quad \times \exp\left(\int dx \int_0^\tau d\tau' (\bar{\eta} J + \bar{J} \eta)\right) \\ &= \exp\left(2 \int dx \int_0^\tau d\tau' \bar{J} J\right) \end{aligned} \quad (76)$$

where spinor indexes have been omitted and J, \bar{J} are fermionic external sources.

Now, as already noted in [31,32], when S is taken as the Dirac action one should in general include an appropriate kernel $K(x, y)$ to ensure that the relaxation process is such that the system tends to the equilibrium when $\tau \rightarrow \infty$. Different kernels produce stochastic process which can differ in the behavior under scaling. In any case, Langevin Eqs. (73) are then replaced by

$$\begin{aligned} \frac{\partial \psi(x, \tau)}{\partial \tau} &= -\int dy K(x, y) \frac{\delta S[\bar{\psi}, \psi]}{\delta \bar{\psi}(y, \tau)} + \eta(x, \tau) \\ \frac{\partial \bar{\psi}(x, \tau)}{\partial \tau} &= \int dy \frac{\delta S[\psi, \bar{\psi}]}{\delta \psi(y, \tau)} K(y, x) + \bar{\eta}(x, \tau) \end{aligned} \quad (77)$$

and Eq. (75) by

$$\langle \eta_\alpha(x_1, \tau_1) \bar{\eta}_\beta(x_2, \tau_2) \rangle = 2\delta(t_1 - t_2) K(x_1 - x_2) \delta_{\alpha\beta}. \quad (78)$$

The usual choice for K

$$K(x, y) = (i\not{\partial}_x + m)\delta(x - y) \quad (79)$$

in the case S is the Dirac action for a massive fermion,

$$S = \int d^d x \bar{\psi}(x) (i\not{\partial} - m) \psi(x) \quad (80)$$

gives rise to the Langevin equations

$$\begin{aligned} \frac{\partial \psi(x, \tau)}{\partial \tau} &= (\nabla^2 - m^2) \psi + \eta(x, \tau) \\ \frac{\partial \bar{\psi}(x, \tau)}{\partial \tau} &= (\nabla^2 - m^2) \bar{\psi} + \bar{\eta}(x, \tau) \end{aligned} \quad (81)$$

where a $z = 2$ anisotropic scaling turns evident.

Let us now come back to the general fermionic stochastic process defined by the Langevin Eqs. (78). In the presence of the kernel K the partition function associated to (76) takes the form

$$Z = \int D\bar{\eta} D\eta \exp\left(-\frac{1}{2} \int d^d x d^d y d\tau \bar{\eta}(x, \tau) K^{-1}(x, y) \eta(y, \tau)\right). \quad (82)$$

We now proceed to change variables from $\bar{\eta}, \eta$ to $\bar{\psi}, \psi$

$$Z = \int D\bar{\psi} D\psi \det^{-1} \frac{\delta \bar{\eta}}{\delta \psi} \det^{-1} \frac{\delta \eta}{\delta \psi} \times \exp\left(-\frac{1}{2} \int d^d x d^d y d\tau \bar{\eta}(x, \tau) K^{-1}(x, y) \eta(y, \tau)\right). \quad (83)$$

Using Langevin Eqs. (81) written in the form

$$K^{-1} \frac{\partial \psi}{\partial \tau} = -\frac{\delta S}{\delta \bar{\psi}} + K^{-1} \eta \quad \frac{\partial \bar{\psi}}{\partial \tau} K^{-1} = \frac{\delta S}{\delta \psi} + \bar{\eta} K^{-1} \quad (84)$$

we end with

$$Z = \int D\bar{\psi} D\psi \det^{-1} \left(K^{-1} \frac{\partial}{\partial \tau} - \frac{\delta^2 S[\bar{\psi}, \psi]}{\delta \bar{\psi} \delta \psi} \right) \times \det^{-1} \left(K^{-1} \frac{\partial}{\partial \tau} + \frac{\delta^2 S[\bar{\psi}, \psi]}{\delta \psi \delta \bar{\psi}} \right) \times \exp\left(-\frac{1}{2} \int d^d x d\tau \left(\frac{\partial \bar{\psi}}{\partial \tau} K^{-1} - \frac{\delta S[\bar{\psi}, \psi]}{\delta \psi} \right) \times \left(\frac{\partial \psi(x, \tau)}{\partial \tau} + K \frac{\delta S[\bar{\psi}, \psi]}{\delta \bar{\psi}} \right)\right) \quad (85)$$

where we have used the identity

$$\frac{\delta}{\delta \psi_\beta} K_{\alpha\gamma}^{-1} \partial_\tau \psi_\gamma = K_{\alpha,\beta}^{-1} \partial_\tau. \quad (86)$$

Determinants in (85) can be defined as the (regularized) product of eigenvalues of the equation

$$\left((K^{-1})^{\alpha\beta} \frac{\partial}{\partial \tau} - \frac{\delta^2 S}{\delta \bar{\psi}^\alpha \delta \psi^\beta} \right) \phi_i^\beta = \lambda \phi_i^\alpha \quad (87)$$

and a similar one for the other factor. These determinants can then be represented as quadratic path-integrals over two complex commuting fields $\phi_i^\alpha (i = 1, 2)$ carrying a spinorial index which can be identified with the spinorial ghosts for ghosts [33]. With this, the partition function Z can be written as

$$Z = \int D\bar{\psi} D\psi D\bar{\phi}_1 D\bar{\phi}_2 D\phi_1 D\phi_2 \times \exp\left(-\int dx d\tau L_{\text{eff}}[\bar{\psi}, \psi, \phi_1, \phi_2]\right) \quad (88)$$

where

$$L_{\text{eff}} = \frac{1}{2} \left(\frac{\partial \bar{\psi}(x, \tau)}{\partial \tau} K^{-1} - \frac{\delta S[\bar{\psi}, \psi]}{\delta \psi} \right) \times \left(\frac{\partial \psi(x, \tau)}{\partial \tau} + K \frac{\delta S[\bar{\psi}, \psi]}{\delta \bar{\psi}} \right) + \bar{\phi}_1 \left(K^{-1} \frac{\partial}{\partial \tau} - \frac{\delta^2 S[\bar{\psi}, \psi]}{\delta \bar{\psi} \delta \psi} \right) \phi_1 + \bar{\phi}_2 \left(K^{-1} \frac{\partial}{\partial \tau} + \frac{\delta^2 S[\bar{\psi}, \psi]}{\delta \psi \delta \bar{\psi}} \right) \phi_2 \quad (89)$$

or

$$L_{\text{eff}} = \frac{1}{2} \frac{\partial \bar{\psi}(x, \tau)}{\partial \tau} K^{-1} \frac{\partial \psi(x, \tau)}{\partial \tau} - \frac{1}{2} \frac{\delta S[\bar{\psi}, \psi]}{\delta \psi} K \frac{\delta S[\bar{\psi}, \psi]}{\delta \bar{\psi}} + \bar{\phi}_1 \left(K^{-1} \frac{\partial}{\partial \tau} - \frac{\delta^2 S[\bar{\psi}, \psi]}{\delta \bar{\psi} \delta \psi} \right) \phi_1 + \bar{\phi}_2 \left(K^{-1} \frac{\partial}{\partial \tau} + \frac{\delta^2 S[\bar{\psi}, \psi]}{\delta \psi \delta \bar{\psi}} \right) \phi_2. \quad (90)$$

We can see now how the choice of the stochastic process, defined by a kernel K , gives rise to different scaling behaviors. Indeed, starting with a d -dimensional free massive fermion Dirac action and using a trivial kernel

$$K(x, y) = \delta(x - y) \quad (91)$$

(this being consistent for $m \neq 0$ the fermionic part of the effective fermion Lagrangian (90) will have second derivatives both in time and space variables

$$L_{\text{fer}} = \frac{1}{2} \frac{\partial \bar{\psi}(x, \tau)}{\partial \tau} \frac{\partial \psi(x, \tau)}{\partial \tau} + \bar{\psi}(\nabla^2 + m^2)\psi \quad (92)$$

which corresponds to a Lifshitz theory with $z = 1$, i.e., a theory with isotropic scaling. If one instead chooses as d -dimensional action a Klein-Gordon fermionic action

$$S = \int d^d x \bar{\psi}(x) (\nabla^2 + m^2) \psi(x) \quad (93)$$

one ends with a $z = 2$ Lifshitz theory with Lagrangian

$$L_{\text{fer}}^{KG2} = \frac{1}{2} \frac{\partial \bar{\psi}(x, \tau)}{\partial \tau} \frac{\partial \psi(x, \tau)}{\partial \tau} + \bar{\psi}(\nabla^2 + m^2)^2 \psi. \quad (94)$$

We shall end this section by making use of another remarkable attribute of the stochastic quantization scheme, namely, the inherent supersymmetry of the partition function, in order to supersymmetrize the Lifshitz fermionic models constructed above [34]. Alternatively, and following the ideas introduced in the Sec. II, we may consider the d -dimensional action for fermions

$$S_{LF} = \int d^d x \bar{\psi} (i\kappa(-\nabla^2)^{1/2} \gamma_i \partial_i - m) \psi$$

which gives rise through the stochastic quantization approach to the fermionic Lifshitz Lagrangian

$$L_{\text{eff}} = \frac{1}{2} \frac{\partial \bar{\psi}(x, \tau)}{\partial \tau} \frac{\partial \psi(x, \tau)}{\partial \tau} - \frac{1}{2} \bar{\psi} (-\kappa^2 (-\nabla^2)^{1/2} \times \gamma_i \partial_i (-\nabla^2)^{1/2} \gamma_j \partial_j + m^2) \psi = \frac{1}{2} \frac{\partial \bar{\psi}(x, \tau)}{\partial \tau} \frac{\partial \psi(x, \tau)}{\partial \tau} + \frac{1}{2} \bar{\psi} (\kappa^2 (\nabla^2)^2 - m^2) \psi \quad (95)$$

which also exhibits a $z = 2$ anisotropic behavior under scaling.

Starting from partition function Z as given in (88) with Lagrangian (90), we introduce fermionic auxiliary fields \bar{F} , F so that Z becomes [35]

$$Z = \int D\bar{\psi} D\psi D\phi_1 D\phi_2 D\bar{\phi}_1 D\bar{\phi}_2 D\bar{F} DF \times \exp\left(-\int d^d x d\tau L_{\text{eff}}[\bar{\psi}, \psi, \phi_1, \phi_2, \bar{F}, F]\right) \quad (96)$$

$$L_{\text{eff}} = 2\bar{F}K^{-1}F + i\left(\frac{\partial\bar{\psi}(x, \tau)}{\partial\tau}K^{-1} - \frac{\delta S_f^E[\bar{\psi}, \psi]}{\delta\bar{\psi}}\right)F + i\bar{F}\left(\frac{\partial\psi(x, \tau)}{\partial\tau} + K\frac{\delta S_f^E[\bar{\psi}, \psi]}{\delta\bar{\psi}}\right) + \bar{\phi}_1\left(K^{-1}\frac{\partial}{\partial\tau} - \frac{\delta^2 S_f^E[\bar{\psi}, \psi]}{\delta\bar{\psi}\delta\psi}\right)\phi_1 + \bar{\phi}_2\left(K^{-1}\frac{\partial}{\partial\tau} + \frac{\delta^2 S_f^E[\bar{\psi}, \psi]}{\delta\bar{\psi}\delta\psi}\right)\phi_2. \quad (97)$$

We now define fermion superfields in the form

$$\begin{aligned} \Psi &= \psi + \bar{\theta}\phi_1 + \bar{\phi}_2\theta + i\bar{\theta}\theta F \\ \bar{\Psi} &= \bar{\psi} + \bar{\theta}\phi_2 + \bar{\phi}_1\theta + i\bar{\theta}\theta\bar{F} \end{aligned} \quad (98)$$

and supercovariant derivatives D and \bar{D} acting on superfields according to

$$D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\tau}, \quad \bar{D} = \frac{\partial}{\partial\theta} \quad (99)$$

$$D^2 = \bar{D}^2 = 0, \quad \{D, \bar{D}\} = -\frac{\partial}{\partial\tau} = -H. \quad (100)$$

Supersymmetry generators Q and \bar{Q} take the form

$$Q = \frac{\partial}{\partial\theta}, \quad \bar{Q} = \frac{\partial}{\partial\theta} + \bar{\theta}\frac{\partial}{\partial\tau} \quad (101)$$

so that

$$\{Q, \bar{Q}\} = \frac{\partial}{\partial\tau} = H. \quad (102)$$

A supersymmetry transformation with parameters $(\epsilon, \bar{\epsilon})$ on superfields reads

$$Q\Psi = \bar{\epsilon}Q + \bar{Q}\epsilon. \quad (103)$$

In order to write the supersymmetry transformations in component fields we consider the action of Q and \bar{Q} on superfields

$$\begin{aligned} Q\Psi &= \frac{\partial}{\partial\theta}(\psi + \bar{\theta}\phi_1 + \bar{\phi}_2\theta + i\bar{\theta}\theta F) = \phi_1 + i\theta F \\ \bar{Q}\Psi &= \left(\frac{\partial}{\partial\theta} + \bar{\theta}\frac{\partial}{\partial\tau}\right)(\psi + \bar{\theta}\phi_1 + \bar{\phi}_2\theta + i\bar{\theta}\theta F) \\ &= \bar{\phi}_2 - \bar{\theta}F + \bar{\theta}\dot{\psi} + \bar{\theta}\theta\dot{\bar{\phi}}_2 \end{aligned} \quad (104)$$

so that

$$Q\Psi = \bar{\epsilon}(\phi_1 + i\theta F) + (\bar{\phi}_2 - i\bar{\theta}F + \bar{\theta}\dot{\psi} + \bar{\theta}\theta\dot{\bar{\phi}}_2)\epsilon \quad (105)$$

and then

$$\begin{aligned} \delta\psi &= \bar{\epsilon}\phi_1 + \bar{\phi}_2\epsilon & \delta\phi_1 &= \bar{\epsilon}(\psi - iF) \\ \delta\phi_2 &= i\bar{\epsilon}F & \delta F &= -i\dot{\bar{\phi}}_2\epsilon. \end{aligned} \quad (106)$$

Let us consider the action

$$S_1 = \int d\bar{\theta}d\theta\bar{D}\bar{\Psi}D\Psi \quad (107)$$

that is written in terms of component fields given by

$$\begin{aligned} S_1 &= \int d\bar{\theta}d\theta\frac{\partial}{\partial\theta}(\bar{\psi} + \bar{\theta}\phi_2 + \bar{\phi}_1\theta + i\bar{\theta}\theta\bar{F}) \\ &\quad \times \left(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\tau}\right)(\psi + \bar{\theta}\phi_1 + \bar{\phi}_2\theta + i\bar{\theta}\theta F) \\ &= \int d\bar{\theta}d\theta(\bar{\phi}_1 - i\bar{\theta}\bar{F})(\phi_1 + i\theta F - \theta\dot{\psi} + \bar{\theta}\theta\dot{\phi}_1) \\ &= \bar{\phi}_1\dot{\phi}_1 + \bar{F}F + i\bar{F}\dot{\psi}. \end{aligned} \quad (108)$$

Define also

$$S_2 = \int d\bar{\theta}d\theta\bar{D}\Psi D\bar{\Psi} \quad (109)$$

or

$$\begin{aligned} S_2 &= \int d\bar{\theta}d\theta\frac{\partial}{\partial\theta}(\psi + \bar{\theta}\phi_1 + \bar{\phi}_2\theta + i\bar{\theta}\theta F) \\ &\quad \times \left(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\tau}\right)(\bar{\psi} + \bar{\theta}\phi_2 + \bar{\phi}_1\theta + i\bar{\theta}\theta\bar{F}) \\ &= \int d\bar{\theta}d\theta(\bar{\phi}_2 - i\bar{\theta}F)(\phi_2 + i\theta\bar{F} - \theta\dot{\bar{\psi}} + \bar{\theta}\theta\dot{\phi}_2) \\ &= \bar{\phi}_2\dot{\phi}_2 + F\bar{F} + iF\dot{\bar{\psi}} \end{aligned} \quad (110)$$

so that

$$\begin{aligned} S_1 + S_2 &= \int d\bar{\theta}d\theta(\bar{D}\bar{\Psi}D\Psi + \bar{D}\Psi D\bar{\Psi}) \\ &= \bar{\phi}_1\dot{\phi}_1 + \bar{\phi}_2\dot{\phi}_2 + 2\bar{F}F + i(\bar{F}\dot{\psi} + \dot{\bar{\psi}}F). \end{aligned} \quad (111)$$

For the case $K = K^{-1} = I$ the terms in the right-hand side exhaust all terms in (97) containing temporal derivatives. For nontrivial K one has just to make the following change of variables:

$$\begin{aligned} \bar{F}K^{-1} &\rightarrow \bar{F}, & F &\rightarrow F & \bar{\psi}K^{-1} &\rightarrow \bar{\psi}, & \psi &\rightarrow \psi \\ \bar{\phi}_1K^{-1} &\rightarrow \bar{\phi}_1, & \phi_1 &\rightarrow \phi_1 & \bar{\phi}_2K^{-1} &\rightarrow \bar{\phi}_2, & \phi_2 &\rightarrow \phi_2 \end{aligned} \quad (112)$$

and the calculation reduces to the $K = K^{-1}$ one.

As for the terms in (97) depending on action S , one has just to add to Lagrangian (111) the following term:

$$S_S = \int d\bar{\theta}d\theta S[\bar{\Psi}, \Psi]. \quad (113)$$

Expanding the action in powers of the superfield the only terms that contribute to the $\bar{\theta}, \theta$ integral are

$$\begin{aligned}
S_S &= \int d\bar{\theta}d\theta S[\bar{\Psi}, \Psi] \\
&= \bar{F} \frac{\delta S}{\delta \bar{\psi}} + \frac{\delta S}{\delta \psi} F + \bar{\phi}_1 \frac{\delta^2 S}{\delta \psi \delta \bar{\psi}} \phi_1 + \bar{\phi}_2 \frac{\delta^2 S}{\delta \psi \delta \bar{\psi}} \phi_2
\end{aligned}
\tag{114}$$

so that the fermionic sector of superaction

$$S = \int d\bar{\theta}d\theta d^d x d\tau (\bar{D}\bar{\Psi}D\Psi + \bar{D}\Psi D\bar{\Psi} + S_f^E[\bar{\Psi}, \Psi])
\tag{115}$$

coincides with the action defined from Lagrangian (90).

V. DISCUSSION

The source of the anisotropic scaling in the Lagrangian formulation of Lifshitz theories is the choice of spatial and time derivatives of different orders. This is generally achieved by replacing first-order spatial derivatives in relativistic Lagrangians by the Laplacian operator (or integer powers of it) while leaving unchanged the term with time derivatives. Concerning bosonic theories, whose standard relativistic Lagrangian is quadratic in spatial derivatives, such replacement gives rise to a $z = 2$ anisotropy (or $z > 2$ for higher Laplacian powers [8]). For fermionic theories, application of the same strategy to the Dirac Lagrangian leads to an anisotropy degree $z = 3$ (or $z > 3$ [16]). Now, Lifshitz theories have recently received much attention in connection with condensed matter systems for which the $z = 2$ case is of particular interest. In particular, in [3,4] the issue of a $z = 2$ fermionic systems in $2 + 1$ dimension has been related to a bosonic $z = 2$ Hamiltonian, invoking a bosonization approach.

We have proposed in this work another route to study Lifshitz fermion actions with $z = 2$ in $d + 1$ space-time dimensions, by introducing the square root of the Laplacian, and its natural extension when fermions are coupled to gauge fields. Roughly, it consists of replacing spatial derivatives ∂_i by the operator $(-\nabla^2)^{1/2} \partial_i$, which naturally leads both to bosonic and fermionic $z = 2$ Lifshitz theories. We have stressed (and briefly explained in the Appendix) that although $(-\nabla^2)^{1/2}$ is a nonlocal operator, it can be handled as a local one in the context of the harmonic extension to an additional dimension. Exploiting this fact, one can prove useful identities that allow to discuss relevant physical properties of the systems with dynamics governed by Lagrangians containing such operator.

A $z = 2$ Lifshitz action for massive free fermions was presented in Eq. (18) and for massless fermions coupled to a $U(1)$ gauge field in Eq. (32). It should be pointed that ‘‘squaring’’ the resulting Dirac-Lifshitz equation (17) one gets the prototypical equation for Lifshitz bosons (Eq. (2) with $z = 2$) in the same way as the Klein-Gordon equation can be obtained by squaring the Dirac one.

The $z = 2$ Dirac-Lifshitz action that we have proposed for massless fermions is invariant under chiral transformations

and we have constructed the associated Noether current which leads to classically conserved charges. The j_5^0 component coincides with the usual chiral charge density arising in the relativistic case but the spatial components j_5^i differ. They have however a very simple form given by Eq. (27) for free fermions and Eq. (47) for fermions coupled to an Abelian gauge field. This should be contrasted with the $z = 3$ case in which j_5^i turns out to be much more complicated [15]. In any case the coupling of gauge fields to Lifshitz fermions makes the resulting chiral current depend on the gauge field even in the $U(1)$ case, in contrast with what happens with the relativistic theory. It should be noticed that the same phenomenon takes place for the case of $z = 3$ theories constructed from local operators [15,16].

As it was to be expected, there is an anomaly at the quantum level which we have calculated in the two-dimensional case, showing that it takes the same form as in the ordinary Dirac theory. The fact that the anomaly is the same for relativistic and Lifshitz fermions is in agreement with the results in Refs. [13,15,16] for the $z = 3$ case in $3 + 1$ dimensions, and should be related to the connection between the obstruction to the chiral current conservation law and the topological charge of the gauge field configuration.

In view of the connection between the anomaly in two-dimensional space-time and bosonization (see [36] and references therein) one could expect to find the rules to connect the $z = 2$ fermionic theory we propose with a $z = 2$ two-dimensional bosonic theory, possibly with action (2).

Finally, we discussed in Sec. IV fermionic Lifshitz theories with $z = 2$ within the stochastic quantization framework, a natural one recalling that in Parisi and Wu approach [30] one passes from a d -dimensional Euclidean action to a $d + 1$ effective actions which in general exhibit anisotropic scaling. Since one can introduce in the Langevin equation different kernels acting on the drift force, one can find different effective $d + 1$ actions which nevertheless correspond to the same d -dimensional quantum theory. Exploiting this fact we were able to construct several $z = 2$ fermionic Lifshitz theories. Moreover, through the well-known connection between the Fokker-Plank partition function and supersymmetry, we were able to find the supersymmetric extensions of the resulting fermionic $z = 2$ theories.

Let us end this section by discussing possible applications of our proposal on Lifshitz fermionic actions governing the dynamics of condensed matter systems. As stated in the introduction, Lifshitz theories were introduced in the context of condensed matter phenomena in Refs. [1–4], as a way to study models describing strongly correlated materials. In particular, Ardone *et al.* discussed in Ref. [3] a two-dimensional quantum systems that exhibits a time-independent form of conformal invariance related to the Lifshitz points of three-dimensional quantum dimer models of relevance, for example, in the context of superconductivity. Lines of fixed points as those resulting from

theories with action (18) arise at finite temperature multi-critical points in the phase diagrams of known materials (as, for example, layered uranium compounds, see [2] and references therein for a large number of systems in which Lifshitz points may occur).

A bosonic Hamiltonian which belongs to the same universality class as the square-lattice quantum dimer model was analyzed in Ref. [3] in terms of a $2 + 1$ -dimensional Euclidean action of the form (2). The resulting zero-energy ground state wave functional $\Psi_0[\phi]$ of such system takes the form

$$\Psi_0[\phi] = \frac{1}{\sqrt{Z}} \exp\left(-\frac{\kappa}{2} \int d^2x (\nabla\phi)^2\right) \quad (116)$$

where Z is the partition function for a (Lorentz invariant) two-dimensional free boson theory.

Fermionic counterparts of ground state functionals for bosonic theories like that in Eq. (116) are relevant in quantum dimer models which interpolate between the square and triangular lattices. In order to treat fermionic models, a bosonized version of the fermionic theory was proposed in [3]. Now, bosonization in $2 + 1$ dimensions has only been formulated in Lorentz invariant theories [37] and this to lowest order in the inverse of the fermion mass. The fermionic action presented in Eq. (18), for $d = 2$ spatial dimensions and with $m = 0$, provides a natural candidate to parallel Ardone *et al.* bosonic studies [3] to obtain ground state functionals and correlation functions in the case of fermionic models. Such investigation would help to understand, in particular, $z = 2$ Lifshitz models originally introduced to describe paramagnetic ground states of quantum antiferromagnets, exhibiting topological phases. We hope to develop this approach and discuss possible applications in condensed matter problems in a forthcoming work.

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APPENDIX: SQUARE ROOT OPERATORS

In order to handle the operator $(-\nabla^2)^{1/2}$ on \mathbb{R}^n we follow Ref. [18] where it is defined in terms of the harmonic extension problem to the upper half space $\mathbb{R}^n \times (0, \infty)$ of functions on \mathbb{R}^n . That is, given a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, one looks for an harmonic function $u: \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ such that its restriction to \mathbb{R}^n coincides with f . Thus, the problem reduces to solve the Dirichlet problem

$$u(x, 0) = f(x) \quad \nabla_{n+1}^2 u(x, y) = 0 \quad (A1)$$

where ∇_{n+1}^2 is the Laplacian operator in \mathbb{R}^{n+1} , $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. This is a well studied problem: for a smooth

function on $C_0^\infty(\mathbb{R}^n)$, there is a unique harmonic extension $u \in C^\infty(\mathbb{R}^n \times (0, \infty))$.

One then introduces an operator T defined on functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ having an harmonic extension $u: \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$, such that

$$(Tf)(x) = -\left. \frac{\partial u(x, y)}{\partial y} \right|_{y=0}. \quad (A2)$$

Thus, since $(Tf)(x)$ has also an harmonic extension to $\mathbb{R}^n \times (0, \infty)$, namely $u_y(x, y)$ provided $u(x, y)$ is the harmonic extension of f , the successive applications of T give

$$(T(Tf))(x) = \left. \frac{\partial^2 u(x, y)}{\partial y^2} \right|_{y=0} = -\nabla_n^2 u(x, y)|_{y=0} = -\nabla_n^2 f(x).$$

It is equivalent to write

$$T^2 f(x) = (-\nabla_n^2) f(x).$$

Then one can identify

$$T = (-\nabla_n^2)^{1/2}. \quad (A3)$$

Let us consider the case $n = 1$, so $f: \mathbb{R} \rightarrow \mathbb{R}$ and we look for harmonic extensions to $\mathbb{R} \times (0, \infty)$. For instance, taking $f(x) = \cos kx$, it is easy to see that $u(x, y) = e^{-|k|y} \cos kx$, $y \in (0, \infty)$, is a bounded harmonic extension of $f(x)$,

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x, y) &= -k^2 e^{-|k|y} \cos kx + |k|^2 e^{-|k|y} \cos kx \\ &= 0. \end{aligned}$$

Hence, we may evaluate the action of the operator $(-\nabla_1^2)^{1/2} = (-\partial_1^2)^{1/2}$ on f

$$(-\partial_1^2)^{1/2} \cos kx = T(\cos kx) = -u_y(x, y)|_{y=0} = |k| \cos kx.$$

The same arguments hold if one takes $f = \sin kx$ since the function $v(x, y) = e^{-|k|y} \sin kx$ is a bounded harmonic extension on $\mathbb{R} \times (0, \infty)$ of $\sin kx$

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v(x, y) &= -k^2 e^{-|k|y} \sin kx + |k|^2 e^{-|k|y} \sin kx \\ &= 0 \end{aligned}$$

and

$$(-\partial_1^2)^{1/2} \sin kx = T(\sin kx) = -v_y(x, y)|_{y=0} = |k| \sin kx.$$

We can then write the action of $(-\partial_1^2)^{1/2}$ on e^{ikx} as

$$(-\partial_1^2)^{1/2} e^{ikx} = (-\partial_1^2)^{1/2} \cos kx + i(-\partial_1^2)^{1/2} \sin kx$$

or

$$(-\partial_1^2)^{1/2} e^{ikx} = |k| e^{ikx} \quad (A4)$$

thus proving Eq. (16).

Remark 1: Suppose that the function $f(x)$ admits an harmonic extension $u(x, y)$, so $(Tf)(x) = -u_y(x, 0)$.

Then, functions $\partial_i f(x)$, $i = 1, \dots, n$, have harmonic extensions $\partial_i u(x, y)$ and

$$\begin{aligned} (T(\partial_i f))(x) &= -\partial_y(\partial_i u(x, y))|_{y=0} = -\partial_i u_y(x, y)|_{y=0} \\ &= \partial_i(Tf)(x) \end{aligned}$$

provided that $u \in C^2(\mathbb{R} \times (0, \infty))$. From this, we conclude that

$$(-\nabla_n^2)^{1/2} \partial_i = \partial_i (-\nabla_n^2)^{1/2}. \quad (\text{A5})$$

Remark 2: We now address the self-adjointness of the operator T , following Ref. [24]. Let $u(x, y)$ and $v(x, y)$ be harmonic extensions to $\mathcal{C} = \mathbb{R}^n \times (0, \infty)$ of $g(x)$ and $f(x)$, respectively. Also, assume that both the harmonic extensions vanish for $|x|, |y| \rightarrow \infty$, then

$$\begin{aligned} &\int_{\mathcal{C}} \nabla_{n+1} u(x, y) \cdot \nabla_{n+1} v(x, y) dx dy \\ &= \int_{\mathcal{C}} \nabla_{n+1} \cdot (u(x, y) \nabla_{n+1} v(x, y)) dx dy \\ &= \int_{\partial \mathcal{C}} u(x, y) \nabla_{n+1} v(x, y) dx dy \\ &= - \int_{\mathbb{R}^n} u(x, y) \frac{\partial}{\partial y} v(x, y)|_{y=0} dx. \end{aligned} \quad (\text{A6})$$

Here, $\partial \mathcal{C}$ indicates the border of \mathcal{C} . To obtain this identity we have written the integral of the $(n+1)$ -dimensional divergence (A6) as a surface integral over the component of the vector field normal to the surface $\partial \mathcal{C}$ which is taken as a finite $(n+1)$ -dimensional cube with a face at $y=0$. Then we have made the $2n+1$ faces other than $y=0$ to go to infinity, where functions u and v vanish.

Hence, for the harmonic extensions u, v to $\mathcal{C} = \mathbb{R}^n \times (0, \infty)$ of g, f , respectively, we have

$$\int_{\mathcal{C}} u(x, y) \nabla_{n+1}^2 v(x, y) dx dy - \int_{\mathcal{C}} v(x, y) \nabla_{n+1}^2 u(x, y) dx dy = 0$$

or, using (A8),

$$\int_{\mathbb{R}^n} (u(x, y) \frac{\partial}{\partial y} v(x, y) - v(x, y) \frac{\partial}{\partial y} u(x, y))|_{y=0} dx = 0.$$

In terms of $g(x)$ and $f(x)$ this identity reads

$$\int_{\mathbb{R}^n} \left(g(x) \frac{\partial f(x)}{\partial y} - \frac{\partial g(x)}{\partial y} f(x) \right) dx = 0$$

or, equivalently

$$\int_{\mathbb{R}^n} g(x) T f(x) dx = \int_{\mathbb{R}^n} f(x) T g(x) dx. \quad (\text{A7})$$

Concerning the case in which fermions are coupled to the $U(1)$ gauge field, we introduce the operator $\mathcal{T}[A]$ defined as

$$\mathcal{T}[A] = (-D_i[A] D_i[A])^{1/2} \quad (\text{A8})$$

with the covariant derivative given by

$$D_i = i\partial_i + eA_i. \quad (\text{A9})$$

In the case of $d=1$ space dimensions this simplifies to

$$\mathcal{T}[A] = (-D_1[A] D_1[A])^{1/2}. \quad (\text{A10})$$

The covariant derivative $D_1[A]$ can be written as

$$D_1[A] = \exp\left(ie \int^x A_1 dy\right) \left(i \frac{d}{dx}\right) \exp\left(-ie \int^x A_1 dy\right) \quad (\text{A11})$$

which in terms of a Wilson line

$$W[A_1] = \exp\left(-ie \int^x A_1 dy\right) \quad (\text{A12})$$

takes the compact form

$$D_1[A] = W^{-1}[A_1] i \frac{d}{dx} W[A_1]. \quad (\text{A13})$$

One then has

$$(D_1[A])^2 = W^{-1}[A_1] \left(i \frac{d}{dx}\right)^2 W[A_1] = W^{-1}[A_1] T^2 W[A_1] \quad (\text{A14})$$

or

$$(D_1[A])^2 = W^{-1}[A_1] T W[A_1] W^{-1}[A_1] T W[A_1] \quad (\text{A15})$$

so that

$$\mathcal{T}[A_1] = W^{-1}[A_1] T W[A_1] \quad (\text{A16})$$

showing that $\mathcal{T}[A_1]$ is Hermitian.

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