

Magnetic field driven instability in the planar NJL model in the real-time formalismO. V. Gamayun,^{1,2} E. V. Gorbar,^{2,3} and V. P. Gusynin²¹*Physics Department, Lancaster University, Lancaster LA1 4YB, United Kingdom*²*Bogolyubov Institute for Theoretical Physics, 03680 Kiev, Ukraine*³*Department of Physics, Taras Shevchenko National Kiev University, 03022 Kiev, Ukraine*

(Received 13 June 2012; published 17 September 2012)

It is known that the symmetric (massless) state of the Nambu-Jona-Lasinio model in $2 + 1$ dimensions in a magnetic field B is not the ground state of the system at zero temperature due to the presence of a negative linear in $|\sigma + i\pi|$ term in the effective potential for the composite fields $\sigma \sim \bar{\psi}\psi$ and $\pi \sim \bar{\psi}i\gamma^5\psi$, while the quadratic term is always positive (a tachyon is absent). We find that finite temperature is a necessary ingredient for the tachyonic instability of the symmetric state to occur. Utilizing the Schwinger-Keldysh real-time formalism, we calculate the dispersion relations for the fluctuation modes of the composite fields σ and π . We demonstrate the presence of the tachyonic instability of the symmetric state for the coupling constant that exceeds a certain critical value which vanishes as temperature tends to zero in accordance with the phenomenon of magnetic catalysis.

DOI: [10.1103/PhysRevD.86.065021](https://doi.org/10.1103/PhysRevD.86.065021)

PACS numbers: 11.30.Qc, 11.30.Rd, 11.10.Wx

I. INTRODUCTION

For many years relativistic quantum field models in $(2 + 1)$ dimensions have attracted a significant interest due to both their sophisticated dynamics and the fact that they describe long wavelength excitations in several planar condensed matter systems [1], among them graphene [2], the d-wave state of high T_c superconductors [3], topological insulators [4], and optical lattices [5]. Recently, there has been a surge of activity in this area connected with the experimental discovery of graphene [6] whose quasiparticle excitations are described by the massless Dirac equation in $(2 + 1)$ dimensions that leads to many unusual electronic properties of this material and opens new perspectives for electronic devices (see, review papers [7]). Lattice effects necessarily produce local interactions for quasiparticles in graphene [8] and, thus, one naturally comes to the gauged Nambu-Jona-Lasinio (NJL) model in $2 + 1$ dimensions.

Historically, the Nambu-Jona-Lasinio model [9] was the first model in which the mass generation and dynamical symmetry breaking (DSB) were considered in elementary particle physics and quantum field theory. At present, NJL-type models have a significant practical value; for example, the NJL model provides a successful effective theory of low-energy quantum chromodynamics [10–12]. Dynamical symmetry breaking occurs in the NJL model only in supercritical regime when its coupling constant G exceeds a critical value G_c . This is different from the Bardeen-Cooper-Schrieffer (BCS) theory, where a gap in the quasiparticle spectrum is generated for any value of coupling constant. The physical reason for zero value of the critical coupling constant is connected with the presence of the Fermi surface in the BCS theory. According to the renormalization-group studies [13], the renormalization-group scaling takes place only in the direction perpendicular to the Fermi surface that lowers effectively the spacetime

dimension by two units to a $(1 + 1)$ -dimensional theory where, as is well known, symmetry breaking occurs for arbitrary weak attraction between fermions.

Since dynamical symmetry breaking in $(3 + 1)$ and $(2 + 1)$ -dimensional theories requires strong coupling ($g_c \gtrsim 1$), it makes the quantitative study of DSB a difficult problem [14]. Therefore, it is very interesting to consider field-theoretical models where DSB takes place in the regime of weak coupling ($g_c \approx 0$). The DSB in a magnetic field [16,17] (magnetic catalysis) gives the corresponding example (see also Refs. [18] and a short review Ref. [19]). The essence of the magnetic catalysis phenomenon is that the dynamics of the electrons in a magnetic field, B , corresponds effectively to a theory with spatial dimension reduced by two units (note a close similarity with the role of the Fermi surface in the BCS theory) if their energy is much less than the Landau gap $\sqrt{|eB|}$. The zero-energy Landau level has a finite density of states and this is a key ingredient of magnetic catalysis which plays, in fact, the role of the Fermi surface.

The magnetic catalysis is a universal phenomenon and its main features are model independent [16,20]. Other than a $(2 + 1)$ -dimensional NJL-type model, it was studied in the NJL_{3+1} model [17], quantum electrodynamics [21], and quantum chromodynamics [22]. The universality of this phenomenon is confirmed by applying holographic techniques which have proven to be a powerful analytic tool in studying the qualitative properties of strongly interacting physical systems, such as interacting quark gluon plasma, graphene, superconductivity, and superfluidity [23].

In the theory of superconductivity, the normal state of a superconductor is unstable at sufficiently low temperature with respect to the transition to a superconducting state. This instability is signaled by a pole in the scattering amplitude of the electrons with opposite momenta and is known as the Cooper instability [24]. This instability is

resolved in the superconducting state through the formation of a condensate of Cooper pairs [25] that opens a gap in the electron quasiparticle spectrum.

The instability of the normal state of a quantum statistical or field system has a precursor in the corresponding one-particle problem which is known as the fall-into-the-center phenomenon. For example, in the study of dynamical chiral symmetry breaking in strongly coupled QED [26], the corresponding one-particle problem is formulated as the Dirac equation for the electron in the field of the Coulomb center and the precursor of the normal state instability in QED corresponds to the supercritical charge problem when the lowest in energy bound state dives into the lower continuum. Then an electron-positron pair is spontaneously created from vacuum with the electron shielding the supercritical charge and positron emitted to infinity (described by a resonance state) [27,28].

It is interesting to see what is a precursor of the magnetic catalysis phenomenon in quantum field theories and what are its characteristics. Recently, the corresponding study was performed in the case of graphene in Ref. [29], where the Dirac equation for the electron in the field of the Coulomb center in a magnetic field was considered and it was shown that, as suggested by the magnetic catalysis phenomenon, indeed any charge in the gapless theory is supercritical. However, no resonance state was found that is related to the fact that charged particles cannot propagate freely to infinity in a constant magnetic field in two dimensions. Still it was found that the low-energy bound state crosses the level of filled states that suggests that the normal state of the system in a magnetic field should suffer from a tachyonic instability (i.e., an analog of the Cooper instability in the theory of superconductivity should exist).

In the present paper, we directly address the problem of instability of the symmetric state of quantum field theories with attraction between fermions and antifermions in a magnetic field in the framework of the NJL₂₊₁ model. The model is described in Sec. II. The analysis of the effective potential indicates the necessity of finite temperature for the tachyonic instability to be present. In Sec. III, using the Schwinger-Keldysh real-time formalism, we calculate the dispersion relations for composite fields in the LLL approximation and for sufficiently low temperature find a tachyonic instability. The contribution of higher Landau levels to the dispersion relations for composite fields is considered in Sec. IV. The main results are summarized in the Conclusion.

II. MODEL AND EFFECTIVE POTENTIAL

The NJL action in (2 + 1) dimensions in a magnetic field reads

$$S = \int d^3x \left[\bar{\psi} i \gamma^\mu D_\mu \psi + \frac{G_0}{2} [(\bar{\psi}(x)\psi(x))^2 + (\bar{\psi}(x)i\gamma^5\psi(x))^2] \right], \quad (1)$$

where $D_\mu = \partial_\mu + ieA_\mu$ with the vector potential $A_\mu = (0, Bx, 0)$ that describes a constant magnetic field in the Landau gauge. We use four-component spinors corresponding to a reducible representation of the Dirac algebra like in Ref. [16]. In 2 + 1 dimensions, one can also use two-dimensional representations of the Dirac algebra instead of the four-dimensional one. Using the four-dimensional representation is equivalent to doubling the fermion species under consideration [30] and allows one to have continuous chiral symmetry in 2 + 1 dimensions. In particular, the action (1) is invariant under the $U(1) \times U(1)$ symmetry with the generators I and γ_5 . According to magnetic catalysis, we expect that the symmetric state of model (1) is unstable for any $G_0 > 0$. In order to see this, we calculate the effective potential for the composite fields $\sigma \sim \bar{\psi}\psi$ and $\pi \sim \bar{\psi}i\gamma^5\psi$.

Using the Hubbard-Stratonovich method of auxiliary fields, model (1) can be equivalently rewritten as follows:

$$S_{\text{aux}} = \int d^3x \left[\bar{\psi} (i\gamma^\mu D_\mu - \sigma - i\pi\gamma^5) \psi - \frac{\sigma^2 + \pi^2}{2G_0} \right]. \quad (2)$$

Assuming that $\sigma = \text{const}$ and $\pi = \text{const}$, the effective potential for composite fields σ and π was found in the second paper in Ref. [16] (for more details of the calculation, see Ref. [31]). The following propagator for fermions with mass $m = \langle \sigma \rangle$ was used in the derivation:

$$G(x, x') = e^{i\Phi(\mathbf{x}, \mathbf{x}')} \tilde{G}(x - x'), \quad (3)$$

where the Schwinger phase [32] is separated from the translation invariant part $\tilde{G}(x - x')$. The translation invariant part of the propagator can be expanded over the Landau levels (compare with Ref. [16]) and in the mixed ω, \mathbf{r} representation it has the form

$$\begin{aligned} \tilde{G}(\omega; \mathbf{r}) = & \frac{i}{2\pi l^2} \exp\left(-\frac{\mathbf{r}^2}{4l^2}\right) \sum_{n=0}^{\infty} \frac{1}{\omega^2 - E_n^2 + i\epsilon} \\ & \times \left\{ (\gamma_0 \omega + m) \left[\mathcal{P}_{-L_n} \left(\frac{\mathbf{r}^2}{2l^2} \right) + \mathcal{P}_{+L_{n-1}} \left(\frac{\mathbf{r}^2}{2l^2} \right) \right] \right. \\ & \left. - \frac{i}{l^2} \gamma \mathbf{r} L_{n-1}^1 \left(\frac{\mathbf{r}^2}{2l^2} \right) \right\}, \quad (4) \end{aligned}$$

where $\mathcal{P}_\pm = (1 \pm i \text{sgn}(eB) \gamma_1 \gamma_2)/2$, $E_n = \sqrt{m^2 + 2|eB|n}$ are the Landau levels energies, $l = 1/\sqrt{|eB|}$ is the magnetic length, functions $L_n^\alpha(x)$ are the generalized Laguerre polynomials, and by definition, $L_n(x) = L_n^0(x)$, $L_{-1}^\alpha(x) \equiv 0$. Further, according to Refs. [16,17,21], the lowest Landau level (LLL) contribution is responsible for zero value of the critical coupling constant. Since we are interested in the instability of the normal state of the model in the weak coupling regime, it is clear that only the dynamics in the LLL can produce this instability. Equation (4) implies that the LLL fermion propagator in momentum space is given by

$$\tilde{G}_{LLL}(\omega; \mathbf{p}) = ie^{-\mathbf{p}^2 l^2} \frac{(\omega \gamma^0 + m) \mathcal{P}_-}{\omega^2 - m^2 + i\epsilon}. \quad (5)$$

The effective potential for composite fields σ and π at zero temperature and zero chemical potential in the model under consideration was calculated in Ref. [16],

$$V(\rho) = \frac{1}{\pi} \left[\frac{\Lambda}{2\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{g} - 1 \right) \rho^2 - \frac{\sqrt{2}}{l^3} \zeta \left(-\frac{1}{2}; \frac{(\rho l)^2}{2} + 1 \right) - \frac{\rho}{2l^2} \right], \quad (6)$$

where $\rho = \sqrt{\sigma^2 + \pi^2}$, $g = G_0 \Lambda / \pi$, Λ is the UV cut-off, and $\zeta(s, q)$ is the generalized Riemann zeta function. For $\rho \rightarrow 0$, at weak coupling $g \ll \sqrt{\pi}$, we have

$$V(\rho) \approx \frac{\rho^2}{2G_0} - \frac{\rho}{2\pi l^2}. \quad (7)$$

The presence of the negative linear term $\rho/(2\pi l^2)$ clearly indicates that the true minimum of the effective potential corresponds to a state with broken symmetry. However, the second derivative of the effective potential with respect to ρ is always positive, hence the tachyon is absent. This situation is rather unusual and the reason for the existence of the linear term was explained in [16]. Since the gap equation is given by $\partial V(\rho)/\partial \rho = 0$, the effective potential can be reconstructed up a constant by integrating the gap equation. It suffices to consider only the field σ (the dependence of the effective potential on π can be easily restored using the chiral symmetry). Further, the gap equation in the model under consideration equals

$$\sigma = G_0 \langle 0 | \bar{\psi} \psi | 0 \rangle. \quad (8)$$

The point crucial for the existence of the linear term in the effective potential is that the chiral condensate $\langle 0 | \bar{\psi} \psi | 0 \rangle$ does not vanish as $\sigma \rightarrow 0$ even in the free noninteracting theory. It suffices to keep only the LLL contribution. Then we have

$$\langle 0 | \bar{\psi} \psi | 0 \rangle = -i \lim_{\sigma \rightarrow 0} \frac{4\sigma}{(2\pi)^3} \int d\omega d^2 \mathbf{p} \frac{e^{-\mathbf{p}^2 l^2}}{\omega^2 - \sigma^2} = -\frac{1}{2\pi l^2}. \quad (9)$$

Integrating it, we find the linear term in the effective potential (7). Since fluctuations are described by quadratic terms in the effective action, the linear term present in the effective potential does not influence fluctuations. Consequently, a tachyonic instability is *absent* in the spectrum of the model at zero temperature.

The situation changes crucially at finite temperature. At $T \neq 0$, the effective potential was calculated in Refs. [16,31]. For small $\rho/T \ll 1$, at weak coupling it is given by

$$V_T(\rho) = \left(\frac{1}{G_0} - \frac{1}{G_c(T, B)} \right) \frac{\rho^2}{2}, \quad G_c(T, B) = 4\pi T l^2. \quad (10)$$

Clearly, for $G_0 > G_c(T, B)$, we have an instability of the conventional (tachyonic) type. The critical coupling $G_c(T, B)$

tends to zero as $T \rightarrow 0$, and the symmetry broken ground state occurs at arbitrary small attractive interaction in accordance with the phenomenon of magnetic catalysis. The absence of the linear term at finite temperature is consistent with the absence of chiral condensate in the free theory at $T \neq 0$,

$$\begin{aligned} \langle 0 | \bar{\psi} \psi | 0 \rangle_T &= -\lim_{\sigma \rightarrow 0} \frac{4\sigma T}{(2\pi)^2} \\ &\times \sum_{n=-\infty}^{+\infty} \int d^2 \mathbf{p} \frac{e^{-\mathbf{p}^2 l^2}}{(\pi T (2n+1))^2 + \sigma^2} \\ &= -\lim_{\sigma \rightarrow 0} \frac{1}{2\pi l^2} \tanh \frac{\sigma}{2T} = 0. \end{aligned} \quad (11)$$

This result suggests that in order to find a tachyonic instability, we should study quadratic fluctuations of the model at finite temperature. To do this, we will calculate in the next section the dispersion relations for composite fields σ and π at finite temperature in the LLL approximation and analyze them.

III. TACHYONIC INSTABILITY FOR COMPOSITE FIELDS IN THE REAL-TIME FORMALISM IN THE LLL APPROXIMATION

The analysis in the previous section shows that in the model under consideration, a tachyonic instability can appear only at finite temperature. Since instability is an inherently dynamical process and the Matsubara imaginary time formalism is mainly used for the study of theories at thermodynamical equilibrium, in order to analyze the tachyonic instability we will utilize the Schwinger-Keldysh real-time formalism [33,34] (for a review, see Ref. [35]). The action in the real-time formalism contains integrals over positive time branch t_+ and negative one t_- . Then action (2) transforms into

$$S = \int_p d^3 x \bar{\psi} (i\gamma^\mu D_\mu - \sigma - i\pi\gamma^5) \psi - \int_p d^3 x \frac{\sigma^2 + \pi^2}{2G_0}, \quad (12)$$

where the time integration proceeds along the closed path time contour [35]

$$\int_p dt = \int_{-\infty}^{+\infty} dt_+ + \int_{+\infty}^{-\infty} dt_- = \int_{-\infty}^{+\infty} dt_+ - \int_{-\infty}^{+\infty} dt_-. \quad (13)$$

Since the fields σ_\pm and π_\pm are defined on the positive and negative time parts of the contour, in what follows it is convenient to consider their linear combinations

$$\sigma_{c,\Delta} = \frac{\sigma_+ \pm \sigma_-}{2}, \quad \pi_{c,\Delta} = \frac{\pi_+ \pm \pi_-}{2}. \quad (14)$$

Integrating over fermions in the functional integral, we find the following effective action for the composite fields:

$$S_{\text{eff}} = - \int_p d^3x \frac{\sigma^2 + \pi^2}{2G_0} - i \text{Tr} \ln[G^{-1}], \quad (15)$$

where $G^{-1} = -i(i\gamma^\mu D_\mu - \sigma - i\pi\gamma^5)\delta_p(x-y)$ and $\delta_p(x-y)$ is a contour δ -function, the trace Tr in Eq. (15) is taken in the functional sense. The equations of motion for composite fields follow from this effective action and are given by (the physically sensible case corresponds to $\sigma_\Delta = \pi_\Delta = 0$) [35,36]

$$\begin{aligned} \frac{2\sigma_c(x)}{G_0} &= -i \frac{\delta \text{Tr} \ln[G^{-1}]}{\delta \sigma_\Delta(x)} \Big|_{\sigma_\Delta = \pi_\Delta = 0} \\ &= -i \text{Tr} \left[G \frac{\delta G^{-1}}{\delta \sigma_\Delta(x)} \right] \Big|_{\sigma_\Delta = \pi_\Delta = 0}, \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{2\pi_c(x)}{G_0} &= -i \frac{\delta \text{Tr} \ln[G^{-1}]}{\delta \pi_\Delta(x)} \Big|_{\sigma_\Delta = \pi_\Delta = 0} \\ &= -i \text{Tr} \left[G \frac{\delta G^{-1}}{\delta \pi_\Delta(x)} \right] \Big|_{\sigma_\Delta = \pi_\Delta = 0}. \end{aligned} \quad (17)$$

Here $G(x, y)$ is the two-point correlation function defined as

$$G(x, y) = \text{Tr}(T_p(\psi(x)\bar{\psi}(y))\hat{\rho}) \equiv \langle T_p(\psi(x)\bar{\psi}(y)) \rangle, \quad (18)$$

$\hat{\rho}$ is the thermal density matrix, and T_p is the time-ordering operator along a complex path p . Since x, y can take values on either positive or negative time branches, it is convenient to represent $G(x, y)$ as 2×2 matrix:

$$\begin{aligned} G(x, y) &= \begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix} \\ &= \begin{pmatrix} \langle T(\psi(x)\bar{\psi}(y)) \rangle & -\langle \bar{\psi}(y)\psi(x) \rangle \\ \langle \psi(x)\bar{\psi}(y) \rangle & \langle \bar{T}\psi(x)\bar{\psi}(y) \rangle \end{pmatrix}, \end{aligned} \quad (19)$$

where T and \bar{T} are the usual time-ordering operator and anti-time-ordering operators, respectively. Note the identity $G_{++} + G_{--} = G_{+-} + G_{-+}$ which follows from the identity for the step functions $\theta(x-y) + \theta(y-x) = 1$.

Since we are interested in physical excitations, we will consider time-dependent solutions of the above equations which deviate weakly from constant values $\sigma_c(x) = \bar{\sigma}(x) + \bar{\sigma}$ and $\pi_c(x) = \bar{\pi}(x)$, where $\bar{\sigma} = \text{const}$. Then we obtain

$$\begin{aligned} &\frac{2(\bar{\sigma}(x) + \bar{\sigma})}{G_0} \\ &= -i \text{Tr} \left[G \frac{\delta G^{-1}}{\delta \sigma_\Delta(x)} \right] \Big|_{\sigma_\Delta = \pi_\Delta = 0, \sigma_c = \bar{\sigma}} \\ &\quad + i \int_p d^3y \text{Tr} \left[G \frac{\delta G^{-1}}{\delta \sigma_\Delta(x)} G \frac{\delta G^{-1}}{\delta \sigma_c(y)} \right] \Big|_{\sigma_\Delta = \pi_\Delta = 0, \sigma_c = \bar{\sigma}} \bar{\sigma}(y), \end{aligned} \quad (20)$$

$$\begin{aligned} &\frac{2\bar{\pi}(x)}{G_0} \\ &= -i \text{Tr} \left[G \frac{\delta G^{-1}}{\delta \sigma_\Delta(x)} \right] \Big|_{\sigma_\Delta = \pi_\Delta = 0, \sigma_c = \bar{\sigma}} \\ &\quad + i \int_p d^3y \text{Tr} \left[G \frac{\delta G^{-1}}{\delta \pi_\Delta(x)} G \frac{\delta G^{-1}}{\delta \pi_c(y)} \right] \Big|_{\sigma_\Delta = \pi_\Delta = 0, \sigma_c = \bar{\sigma}} \bar{\pi}(y). \end{aligned} \quad (21)$$

In the single time representation $\delta_p(x-y) = \tau_3 \delta(x-y)$, where τ_3 is the third Pauli matrix, and the variational derivatives and functional traces are calculated according to the rules

$$\begin{aligned} \frac{\delta G^{-1}(x, y)}{\delta \sigma_\Delta(z)} &= i \frac{\delta \sigma(x)}{\delta \sigma_\Delta(z)} \delta_p(x-y) \\ &= i \delta_p(x-y) \delta_p(x-z) = i \delta(x-y) \delta(x-z), \end{aligned} \quad (22)$$

$$\begin{aligned} \text{Tr} \left[G \frac{\delta G^{-1}}{\delta \sigma_\Delta(x)} \right] &= \int_p d^3u d^3v \text{tr} \left[G(u, v) \frac{\delta G^{-1}(v, u)}{\delta \sigma_\Delta(x)} \right] \\ &= \int d^3u d^3v \text{tr} \left[\tau_3 G(u, v) \tau_3 \frac{\delta G^{-1}(v, u)}{\delta \sigma_\Delta(x)} \right] \\ &= \int d^3u d^3v \text{tr} \left[G(u, v) \frac{\delta G^{-1}(v, u)}{\delta \sigma_\Delta(x)} \right]. \end{aligned} \quad (23)$$

To calculate the right-hand sides of Eqs. (20) and (21), we should determine the fermion Green's function in the real-time formalism. As we discussed in the previous section, for our purposes it suffices to use the LLL approximation. The LLL Green's function in the real-time formalism equals

$$G_{LLL}(x, x') = \mathcal{P}_- K(\mathbf{x}, \mathbf{x}') \begin{pmatrix} G^{++}(t-t') & G^{+-}(t-t') \\ G^{-+}(t-t') & G^{--}(t-t') \end{pmatrix}, \quad (24)$$

where

$$K(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi l^2} \exp \left[-\frac{(\mathbf{x} - \mathbf{x}')^2}{4l^2} + i\Phi(\mathbf{x}, \mathbf{x}') \right] \quad (25)$$

is the space-dependent part of the LLL fermion propagator and

$$\begin{aligned} G^{++}(\omega) &= i \frac{\gamma_0 \omega + \bar{\sigma}}{\omega^2 - \bar{\sigma}^2 + i\epsilon} \\ &\quad - 2\pi(\gamma^0 \omega + \bar{\sigma}) n_F(\bar{\sigma}) \delta(\omega^2 - \bar{\sigma}^2), \end{aligned} \quad (26)$$

$$\begin{aligned} G^{--}(\omega) &= -i \frac{\gamma_0 \omega + \bar{\sigma}}{\omega^2 - \bar{\sigma}^2 - i\epsilon} \\ &\quad - 2\pi(\gamma^0 \omega + \bar{\sigma}) n_F(\bar{\sigma}) \delta(\omega^2 - \bar{\sigma}^2), \end{aligned} \quad (27)$$

$$G^{+-}(\omega) = -2\pi(\gamma^0 \omega + \bar{\sigma}) n_F(\omega) \text{sgn}(\omega) \delta(\omega^2 - \bar{\sigma}^2), \quad (28)$$

$$G^{-+}(\omega) = 2\pi(\gamma^0 \omega + \bar{\sigma}) n_F(-\omega) \text{sgn}(\omega) \delta(\omega^2 - \bar{\sigma}^2) \quad (29)$$

are the Fourier transforms of $G^{ij}(t)$, $i, j = \pm$. Here $n_F(\bar{\sigma}) = (\exp(\bar{\sigma}/T) + 1)^{-1}$ is the Fermi-Dirac distribution function.

Further, it is convenient to perform the unitary Keldysh transformation [34,35]

$$G \rightarrow U^\dagger G U = \begin{pmatrix} 0 & G_a \\ G_r & G_c \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (30)$$

where

$$\begin{aligned} G_a(\omega) &= \frac{1}{2}(G^{++} - G^{--} + G^{+-} - G^{-+}) \\ &= G^{++} - G^{-+} = i \frac{\gamma^0 \omega + \bar{\sigma}}{\omega^2 - \bar{\sigma}^2 - i\epsilon \text{sgn} \omega}, \end{aligned} \quad (31)$$

$$\begin{aligned} G_r(\omega) &= \frac{1}{2}(G^{++} - G^{--} - G^{+-} + G^{-+}) \\ &= G^{++} - G^{+-} = i \frac{\gamma^0 \omega + \bar{\sigma}}{\omega^2 - \bar{\sigma}^2 + i\epsilon \text{sgn} \omega}, \end{aligned} \quad (32)$$

$$\begin{aligned} G_c(\omega) &= G^{++} + G^{--} = G^{+-} + G^{-+} \\ &= 2\pi \tanh \frac{\bar{\sigma}}{2T} (\gamma^0 \omega + \bar{\sigma}) \delta(\omega^2 - \bar{\sigma}^2) \end{aligned} \quad (33)$$

are the advanced, retarded, and correlation functions. For time-dependent and spatially homogeneous modes $\tilde{\sigma}(t)$ and $\tilde{\pi}(t)$, Eqs. (20) and (21) imply the following equations:

$$\tilde{\sigma}(t) = G_0 \int dt' \Pi^\sigma(t-t') \tilde{\sigma}(t'), \quad (34)$$

$$\tilde{\pi}(t) = G_0 \int dt' \Pi^\pi(t-t') \tilde{\pi}(t'), \quad (35)$$

where

$$\begin{aligned} \Pi^\sigma(t-t') &= \frac{-i}{4\pi l^2} \text{tr}[G_r(t-t')G_c(t'-t) \\ &\quad + G_c(t-t')G_a(t'-t)] = 0, \end{aligned} \quad (36)$$

$$\begin{aligned} \Pi^\pi(t-t') &= \frac{-i}{4\pi l^2} \text{tr}[i\gamma^5 G_r(t-t')i\gamma^5 G_c(t'-t) \\ &\quad + i\gamma^5 G_c(t-t')i\gamma^5 G_a(t'-t)] \\ &= \frac{2\bar{\sigma}}{\pi l^2} \tanh \frac{\bar{\sigma}}{2T} \int \frac{d\Omega}{2\pi} \frac{e^{-i\Omega(t-t')}}{4\bar{\sigma}^2 - \Omega^2}. \end{aligned} \quad (37)$$

Hence, Eq. (34) gives $\tilde{\sigma}(t) = 0$. Note that the equality $\Pi^\sigma(t) = 0$ is due to the LLL approximation used in this section. On the other hand, $\Pi^\pi(t) \neq 0$ in the same approximation. In the next section we obtain expressions for $\Pi^{\sigma,\pi}(t)$, where all Landau levels are taken into account.

For the Fourier transform $\tilde{\pi}(\Omega)$, we find

$$\left(\frac{2\pi l^2}{G_0} - \frac{4\bar{\sigma} \tanh \frac{\bar{\sigma}}{2T}}{4\bar{\sigma}^2 - \Omega^2} \right) \tilde{\pi}(\Omega) = 0 \quad (38)$$

that implies

$$\Omega^2 = 4\bar{\sigma} \left[\bar{\sigma} - \frac{G_0}{2\pi l^2} \tanh \frac{\bar{\sigma}}{2T} \right]. \quad (39)$$

For $\bar{\sigma} \rightarrow 0$,

$$\Omega^2 = 4\bar{\sigma}^2 \left[1 - \frac{G_0}{4\pi T l^2} \right]. \quad (40)$$

Obviously, for T less than the critical value

$$T_c = \frac{G_0}{4\pi l^2}, \quad (41)$$

we have a tachyon. This result is perfectly consistent with the effective potential at finite temperature (10) whose symmetric and symmetry broken phases are separated by the curve

$$\frac{1}{G_0} - \frac{1}{4\pi l^2 T} = 0, \quad (42)$$

that leads to the critical temperature (41).

IV. EQUATIONS OF MOTION FOR COMPOSITE FIELDS IN THE REAL-TIME FORMALISM BEYOND THE LLL APPROXIMATION

In Sec. III, we calculated the correlators $\Pi^\sigma(t-t')$ and $\Pi^\pi(t-t')$ given by Eqs. (36) and (37) in the LLL approximation. In the present section, we calculate these quantities taking into account the contribution of all Landau levels. In addition, we determine the dependence of Π^σ and Π^π on spatial coordinates, i.e., calculate $\Pi^\sigma(t-t', \mathbf{x}-\mathbf{x}')$ and $\Pi^\pi(t-t', \mathbf{x}-\mathbf{x}')$ (note that Π^σ and Π^π are translation invariant in spatial coordinates because the Schwinger phases cancel out for a closed fermion loop with two vertices).

In the real time formalism, the propagator in a magnetic field and at finite temperature can be written in the form

$$\begin{aligned} G(x, y) &= K(\mathbf{x}, \mathbf{y}) \int \frac{d\omega}{2\pi} e^{-i(x^0-y^0)\omega} \sum_{n=0}^{\infty} D_n(\mathbf{x}-\mathbf{y}, \omega) \\ &\quad \times \begin{pmatrix} G_n^{++}(\omega) & G_n^{+-}(\omega) \\ G_n^{-+}(\omega) & G_n^{--}(\omega) \end{pmatrix}, \end{aligned} \quad (43)$$

where the factor $K(\mathbf{x}, \mathbf{y})$ is given by Eq. (25) and

$$\begin{aligned} D_n(\mathbf{r}, \omega) &= (\gamma^0 \omega + \bar{\sigma}) \left(\mathcal{P}_- L_n \left(\frac{\mathbf{r}^2}{2l^2} \right) \right. \\ &\quad \left. + \mathcal{P}_+ L_{n-1} \left(\frac{\mathbf{r}^2}{2l^2} \right) \right) - i \frac{\boldsymbol{\gamma} \mathbf{r}}{l^2} L_{n-1}^1 \left(\frac{\mathbf{r}^2}{2l^2} \right). \end{aligned} \quad (44)$$

Further,

$$G_n^{++}(\omega) = \frac{i}{\omega^2 - E_n^2 + i0} - 2\pi n_F(E_n) \delta(\omega^2 - E_n^2), \quad (45)$$

$$G_n^{--}(\omega) = -\frac{i}{\omega^2 - E_n^2 - i0} - 2\pi n_F(E_n)\delta(\omega^2 - E_n^2), \quad (46)$$

$$\begin{aligned} G_n^{-+}(\omega) &= 2\pi n_F(-\omega)\text{sgn}(\omega)\delta(\omega^2 - E_n^2), \\ G_n^{+-}(\omega) &= -2\pi n_F(\omega)\text{sgn}(\omega)\delta(\omega^2 - E_n^2), \end{aligned} \quad (47)$$

and $E_n = \sqrt{\bar{\sigma}^2 + 2|eB|n}$.

The correlators $\Pi^\sigma(x - x')$, $\Pi^\pi(x - x')$ ($x = (t, \mathbf{x})$) are defined by the expressions,

$$\begin{aligned} \Pi^\sigma(x - x') &= -i\text{Tr}[G(x, x')G(x', x)], \\ \Pi^\pi(x - x') &= -i\text{Tr}[i\gamma^5 G(x, x')i\gamma^5 G(x', x)], \end{aligned} \quad (48)$$

and the trace Tr also includes the trace (tr) over Dirac indices. Performing the Keldysh transformation, the matrix in Eq. (43) takes the form like in Eq. (30) with

$$\begin{aligned} G_{r,a}(\omega, n) &= \frac{i}{\omega^2 - E_n^2 \pm i\epsilon\text{sgn}\omega} = \frac{i}{(\omega \pm i\epsilon)^2 - E_n^2}, \\ G_c(\omega, n) &= 2\pi \tanh\frac{E_n}{2T} \delta(\omega^2 - E_n^2). \end{aligned} \quad (49)$$

Taking the Fourier transform of $\Pi^\sigma(t, \mathbf{x})$ we obtain

$$\begin{aligned} \Pi^\sigma(\Omega, \mathbf{k}) &= -i \int \frac{d\omega}{2\pi} \int d^2r e^{i\mathbf{k}\mathbf{r}} \text{tr}[G_r(\mathbf{r}, \omega)G_c(-\mathbf{r}, \omega + \Omega) \\ &\quad + G_c(\mathbf{r}, \omega)G_a(-\mathbf{r}, \omega + \Omega)] \\ &= -i \sum_{n,m=0}^{\infty} \int \frac{d\omega}{2\pi} [G_r(\omega, n)G_c(\omega + \Omega, m) \\ &\quad + G_c(\omega, n)G_a(\omega + \Omega, m)] \int \frac{d^2r}{(2\pi l^2)^2} e^{i\mathbf{k}\mathbf{r} - r^2/2l^2} \\ &\quad \times \text{tr}[D_n(\mathbf{r}, \omega)D_m(-\mathbf{r}, \omega + \Omega)]. \end{aligned} \quad (50)$$

The space integral equals

$$\begin{aligned} \int \frac{d^2r}{2\pi l^2} e^{i\mathbf{k}\mathbf{r} - r^2/2l^2} \text{tr}[D_n(\mathbf{r}, \omega)D_m(-\mathbf{r}, \omega + \Omega)] \\ = 2s_{nm}(y)(\omega(\omega + \Omega) + \bar{\sigma}^2) - 4|eB|r_{nm}(y), \end{aligned} \quad (51)$$

where

$$\begin{aligned} s_{nm}(y) &= \int \frac{d^2r}{4\pi l^2} e^{i\mathbf{k}\mathbf{r} - r^2/2l^2} \left(L_n\left(\frac{r^2}{2l^2}\right)L_m\left(\frac{r^2}{2l^2}\right) \right. \\ &\quad \left. + L_{n-1}\left(\frac{r^2}{2l^2}\right)L_{m-1}\left(\frac{r^2}{2l^2}\right) \right) \\ &= \frac{(-1)^{n+m}}{2} e^{-y} (L_m^{n-m}(y)L_n^{m-n}(y) \\ &\quad + L_{m-1}^{n-m}(y)L_{n-1}^{m-n}(y)), \\ y &= \mathbf{k}^2 l^2 / 2, \end{aligned} \quad (52)$$

$$\begin{aligned} r_{nm}(y) &= \int \frac{d^2r}{2\pi l^2} \frac{\mathbf{r}^2}{2l^2} e^{i\mathbf{k}\mathbf{r} - r^2/2l^2} L_{n-1}^1\left(\frac{r^2}{2l^2}\right)L_{m-1}^1\left(\frac{r^2}{2l^2}\right) \\ &= (-1)^{m+n} e^{-y} m L_m^{n-m}(y)L_{n-1}^{m-n}(y). \end{aligned} \quad (53)$$

[for the evaluation of the integrals $s_{nm}(y)$, $r_{nm}(y)$, see Appendix A in Ref. [37]]. Therefore, we get

$$\begin{aligned} \Pi^\sigma(\Omega, k) &= \frac{1}{\pi l^2} \int_{-\infty}^{\infty} d\omega \sum_{n,m=0}^{\infty} t_m \delta(\omega^2 - E_m^2) \left[\frac{((\omega - \Omega)\omega + \bar{\sigma}^2)s_{nm}(y) - 2|eB|r_{nm}(y)}{(\omega - \Omega + i0)^2 - E_n^2} \right. \\ &\quad \left. + \frac{((\omega + \Omega)\omega + \bar{\sigma}^2)s_{nm}(y) - 2|eB|r_{nm}(y)}{(\omega + \Omega - i0)^2 - E_n^2} \right] \\ &= \sum_{n,m=0}^{\infty} \left[\frac{t_m}{E_m} \frac{E_m^2(E_m^2 - E_n^2 - \Omega^2)s_{nm}(y) - (E_n^2 - E_m^2 - \Omega^2)(s_{nm}(y)\bar{\sigma}^2 - 2|eB|r_{nm}(y))}{\pi l^2((E_m + E_n)^2 - \Omega^2)((E_m - E_n)^2 - \Omega^2)} + (m \leftrightarrow n) \right], \end{aligned} \quad (54)$$

where $t_m = \tanh(E_m/2T)$. The calculation of the correlator Π^π results in the same expression except $\bar{\sigma}^2$ is replaced by $-\bar{\sigma}^2$. Thus, we write

$$\Pi^{\sigma,\pi}(\Omega, k) = \sum_{n,m=0}^{\infty} \left[\frac{t_m}{E_m} \frac{E_m^2(E_m^2 - E_n^2 - \Omega^2)s_{nm}(y) - (E_n^2 - E_m^2 - \Omega^2)(\pm s_{nm}(y)\bar{\sigma}^2 - 2|eB|r_{nm}(y))}{\pi l^2((E_m + E_n)^2 - \Omega^2)((E_m - E_n)^2 - \Omega^2)} + (m \leftrightarrow n) \right], \quad (55)$$

where \pm signs correspond to Π^σ and Π^π , respectively.

To find the dispersion laws at small Ω and $|\mathbf{k}|$, it is convenient to evaluate the sum over the Landau levels. This can be done explicitly if temperature is much lower than the value of a magnetic field, $T \ll \sqrt{|eB|}$. The details

of calculations are given in the Appendix. The dispersion relations are given by the equations,

$$-\frac{1}{G_0} + \Pi^{\sigma,\pi}(\Omega, k) = 0. \quad (56)$$

For $\bar{\sigma} \neq 0$, the dispersion relations for $\tilde{\sigma}$ and $\tilde{\pi}$ modes at small $\Omega l \ll 1$ and $y = \mathbf{k}^2 l^2 / 2 \ll 1$ take the form, respectively,

$$-\frac{1}{G_0} + \frac{V^-}{\pi^{3/2} l} + \Omega^2 \frac{l Q^-}{4\pi^{3/2}} + y \left(\frac{P^-}{\pi^{3/2} l} + \frac{\bar{\sigma}(1 - \tanh(\bar{\sigma}/2T))}{\pi} \right) = 0, \quad (57)$$

$$-\frac{1}{G_0} + \frac{V^+}{\pi^{3/2} l} + \frac{\tanh \bar{\sigma}/2T - 1}{2\pi l^2 \bar{\sigma}} + \Omega^2 \left(\frac{l Q^+}{4\pi^{3/2}} + \frac{\tanh(\bar{\sigma}/2T) - 1}{8\pi l^2 \bar{\sigma}^3} \right) + y \left(\frac{P^+}{\pi^{3/2} l} + \frac{1 - \tanh(\bar{\sigma}/2T)}{2\pi l^2 \bar{\sigma}} \right) = 0, \quad (58)$$

where the quantities V^\pm , Q^\pm , P^\pm are given by Eqs. (A16)–(A19). At the minimum of the effective potential, the $\tilde{\pi}$ mode corresponds to a Nambu-Goldstone boson and $\bar{\sigma}$ satisfies the gap equation,

$$-\frac{1}{G_0} + \frac{V^+}{\pi^{3/2} l} + \frac{\tanh \frac{\bar{\sigma}}{2T} - 1}{2\pi l^2 \bar{\sigma}} = 0. \quad (59)$$

The gap equation written in the form ($\zeta(s, v)$ is the generalized zeta function),

$$-2l\delta\bar{\sigma} + \frac{1}{l} \tanh \frac{\bar{\sigma}}{2T} + \sqrt{2}\bar{\sigma}\zeta\left(\frac{1}{2}, 1 + \frac{(\bar{\sigma}l)^2}{2}\right) = 0, \quad \delta = \pi \left(\frac{1}{G_0} - \frac{1}{G_{0c}} \right), \quad G_{0c} = \frac{2\pi^{3/2}}{3\Lambda}, \quad (60)$$

is in agreement at $T = 0$ with the one obtained in Ref. [16]. Fixing the intrinsic scale δ the gap equation determines $\bar{\sigma}$ as a function of temperature T and magnetic field $eB = 1/l^2$ (we recall that in the used approximation $Tl \ll 1$). The critical line separating symmetric and symmetry broken phases is obtained from Eq. (60) when $\bar{\sigma} \rightarrow 0$:

$$v_\sigma^2 = 4 \frac{\sqrt{2}\bar{\sigma}l \tanh \frac{\bar{\sigma}}{2T} + (\bar{\sigma}l)^2 \zeta\left(\frac{1}{2}, 1 + \frac{(\bar{\sigma}l)^2}{2}\right) + \frac{(\bar{\sigma}l)^4}{2} \zeta\left(\frac{3}{2}, 1 + \frac{(\bar{\sigma}l)^2}{2}\right)}{\zeta\left(\frac{3}{2}, 1 + \frac{(\bar{\sigma}l)^2}{2}\right) - \frac{(\bar{\sigma}l)^2}{2} \zeta\left(\frac{5}{2}, 1 + \frac{(\bar{\sigma}l)^2}{2}\right)}, \quad (68)$$

for the $\tilde{\sigma}$ mode, respectively. At zero temperature Eqs. (65)–(68) coincide with those obtained in Ref. [16]. One can check that the quantities v_π^2 , v_σ^2 are positive and remain always less than 1 (we set the velocity of light $c = 1$). Their behavior for a chosen value of a temperature ($T = 10^{-5}|\delta|$) is shown in Fig. 1 where the gap $\bar{\sigma}(T, l)$ is determined from Eq. (60). The behavior of M_σ^2/δ^2 as a function of the magnetic field is shown in Fig. 2. All dimensionful quantities in Figs. 1 and 2 are measured in units of $|\delta|$. Asymptotical behavior of the quantities v_π^2 , v_σ^2 in subcritical and supercritical regions and for $|\delta l| \gg 1$ is given by the expressions (for simplicity we take $T = 0$),

$$\frac{1}{2T_c l} = 2l\delta - \sqrt{2}\zeta\left(\frac{1}{2}\right), \quad (61)$$

and in the weak coupling limit $G_0 \ll G_{0c}$, it agrees with Eq. (42). The gap equation (60) was analyzed in Ref. [16] at $T = 0$ where three regions of different behavior of $\bar{\sigma}$ as a function of a magnetic field were revealed. In the near critical region $G_0 \simeq G_{0c}$, where $|\delta l| \ll 1$, the dependence on the ultraviolet cutoff Λ disappears and we find $\bar{\sigma} l \simeq 0.45$. In the other two regions, subcritical ($G_0 < G_{0c}$) and supercritical ($G_0 > G_{0c}$), and for $|\delta l| \gg 1$, the solution of the gap equation behaves

$$\bar{\sigma} l \simeq \frac{1}{2\delta l} \ll 1, \quad \delta > 0, \quad (62)$$

$$\bar{\sigma} l \simeq |\delta l| \gg 1, \quad \delta < 0. \quad (63)$$

At finite temperature a nontrivial solution for $\bar{\sigma}$ in subcritical region ($\delta > 0$) exists for magnetic fields satisfying

$$l\delta < \frac{1}{2\sqrt{2}} \left[\zeta(1/2) + \sqrt{\zeta^2(1/2) + \frac{2\delta}{T}} \right], \quad (64)$$

i.e., for magnetic fields exceeding some critical value. The dispersion laws take the following form:

$$\Omega^2 = v_\pi^2 \mathbf{k}^2, \quad v_\pi^2 = \frac{2\sqrt{2}(\bar{\sigma}l)^2}{\sqrt{2}\tanh \frac{\bar{\sigma}}{2T} + (\bar{\sigma}l)^3 \zeta\left(\frac{3}{2}, 1 + \frac{(\bar{\sigma}l)^2}{2}\right)}, \quad (65)$$

for the $\tilde{\pi}$ mode, and

$$\Omega^2 = M_\sigma^2 + v_\sigma^2 \mathbf{k}^2, \quad (66)$$

$$M_\sigma^2 = \frac{8}{\bar{\sigma}l^3} \frac{\sqrt{2}\tanh \frac{\bar{\sigma}}{2T} + (\bar{\sigma}l)^3 \zeta\left(\frac{3}{2}, 1 + \frac{(\bar{\sigma}l)^2}{2}\right)}{2\zeta\left(\frac{3}{2}, 1 + \frac{(\bar{\sigma}l)^2}{2}\right) - (\bar{\sigma}l)^2 \zeta\left(\frac{5}{2}, 1 + \frac{(\bar{\sigma}l)^2}{2}\right)}, \quad (67)$$

$$v_\pi^2 \simeq \begin{cases} \frac{1}{2(\delta l)^2}, & \delta > 0, \\ 1 - \frac{1}{4(\delta l)^4}, & \delta < 0, \end{cases} \quad v_\sigma^2 \simeq \begin{cases} \frac{2\sqrt{2}}{\zeta(3/2)\delta l}, & \delta > 0, \\ 1 - \frac{1}{6(\delta l)^4}, & \delta < 0, \end{cases} \quad (69)$$

and for the square mass M_σ^2 ,

$$M_\sigma^2 \simeq \begin{cases} \frac{8\sqrt{2}}{\zeta(3/2)} \frac{\delta}{l}, & \delta > 0, \\ 6\delta^2 \left(1 + \frac{1}{2(\delta l)^4}\right), & \delta < 0. \end{cases} \quad (70)$$

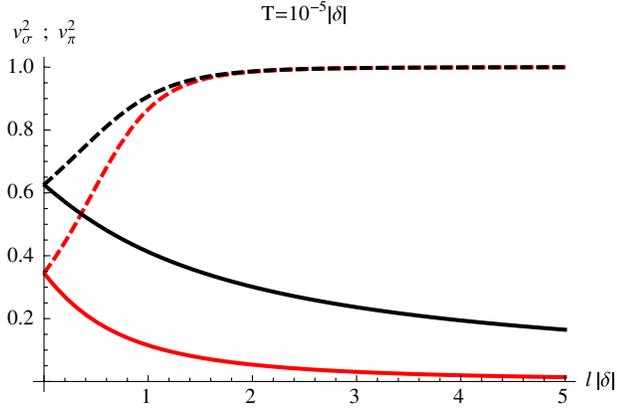


FIG. 1 (color online). Velocities v_π^2 (red lines starting from the value 0.35) and v_σ^2 (black lines starting from the value 0.63) as functions of $l|\delta|$ for temperature $T = 10^{-5}|\delta|$. The solid (dashed) line corresponds to $\delta > 0$ ($\delta < 0$).

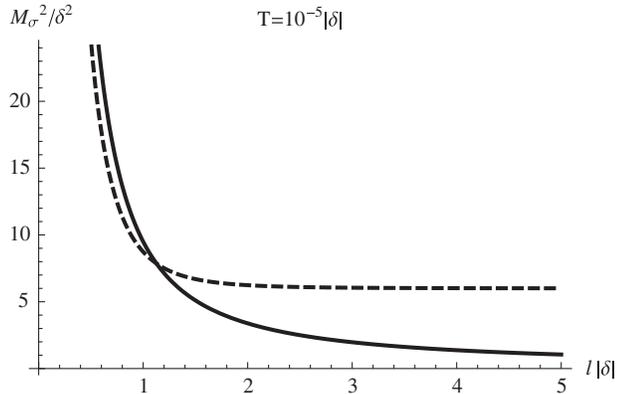


FIG. 2. M_σ^2/δ^2 dependence on $l|\delta|$ for temperature $T = 10^{-5}|\delta|$. The solid (dashed) line corresponds to $\delta > 0$ ($\delta < 0$).

These asymptotics should be compared with those obtained in Ref. [16].

V. CONCLUSION

According to the magnetic catalysis phenomenon [16], an arbitrary weak attraction between fermions and their antiparticles leads to chiral symmetry breaking and gap generation in $(3 + 1)$ and $(2 + 1)$ -dimensional theories in a magnetic field. Consequently, the normal state of these theories should be unstable in a magnetic field even in the weak coupling regime. It is worth noting that the instability of the normal state in a magnetic field is qualitatively different for theories in $(3 + 1)$ and $(2 + 1)$ dimensions. Since constant magnetic field effectively reduces [16,17] the spacetime dimension by two units for fermions in the infrared region, $(3 + 1)$ -dimensional theories are reduced to effective $(1 + 1)$ -dimensional theories, where bound states are easily formed in the weak coupling

regime and resonance states describing emitted antiparticles propagating to infinity are realized in the standard way in the corresponding quantum mechanical one-particle problems.

As noted in the Introduction, the situation is different in $(2 + 1)$ -dimensional theories. The dimensional reduction in a magnetic field means that the corresponding effective theories are $(0 + 1)$ -dimensional ones. Consequently, no emission to infinity is possible. This conclusion is explicitly confirmed by the study of the $(2 + 1)$ -dimensional Dirac equation for the electrons in the field of the Coulomb center in graphene in a magnetic field performed in Ref. [29] where no resonance state was found.

In the present paper, in order to study the normal state instability connected with the magnetic catalysis phenomenon in a $(2 + 1)$ -dimensional theory, we considered the weakly coupled NJL $_{2+1}$ model in a magnetic field at finite temperature. The choice of the model was made basically from the requirement of the simplicity of analysis. Certainly, the generalization to the case of long-range gauge models would be of significant interest.

Using the Hubbard-Stratonovich method of auxiliary fields, we looked for tachyonic excitations in the normal state of the NJL $_{2+1}$ model in a magnetic field at finite temperature. We would like to note that the consideration of the theory at finite temperature is a necessary feature of our analysis. As discussed in Sec. II, although the symmetric state of the effective potential is unstable, its quadratic form of fluctuations that follows from (7) is positive definite, hence, tachyonic excitations are absent. The situation changes at finite temperature, where the effective potential (10) has the instability typical of a second-order phase transition. Utilizing the Schwinger-Keldysh real-time formalism, the dispersion relations for the composite fields $\bar{\psi}\psi$ and $\bar{\psi}i\gamma^5\psi$ were calculated in the LLL approximation in Sec. III, and for temperature less than a critical one, a tachyonic excitation in the normal ground state was found. Thus, although there is no resonance state in the quantum mechanical one-particle problem, the corresponding quantum field-theoretic problem in a magnetic field does have a tachyonic excitation in the normal state for temperature less a critical one. The contribution of higher Landau levels into dispersion relations for the composite fields σ and π was taken into account in Sec. IV.

ACKNOWLEDGMENTS

We are grateful to V. A. Miransky and I. A. Shovkovy for useful discussions. This work is supported partially by the SCOPES Grant No. IZ73Z0_128026 of Swiss NSF, the Grant SIMTECH No. 246937 of the European FP7 program, the joint Ukrainian-Russian SFFR-RFBR Grant No. F40.2/108, and by the Program of Fundamental Research of the Physics and Astronomy Division of the NAS of Ukraine. V.P.G. acknowledges a collaborative grant from the Swedish Institute. O.V.G. is grateful to the ERC, Grant No. 279738 NEDFOQ, for financial support.

**APPENDIX: CLOSED FORM FOR THE
CORRELATORS $\Pi^{\sigma,\pi}$**

To perform the summation over the Landau levels in Eq. (55), we assume that $T \ll \sqrt{|eB|}$. Then we can set $t_m = 1$ for all $m \geq 1$ while keeping $t_0 = \tanh(\bar{\sigma}/2T)$, and expression (55) takes the form

$$\begin{aligned} \Pi^{\sigma,\pi}(\Omega, k) &= \frac{2\bar{\sigma}}{\pi l^2} \left(\tanh \frac{\bar{\sigma}}{2T} - 1 \right) \\ &\times \sum_{n=0}^{\infty} \frac{-2|eB|n(1 \pm 1) - \Omega^2(1 \mp 1)}{(E_n^2 - (\bar{\sigma} + \Omega)^2)(E_n^2 - (\bar{\sigma} - \Omega)^2)} s_{n0} \\ &+ \frac{1}{\pi l^2} \sum_{n,m=0}^{\infty} \frac{E_m + E_n}{(E_m + E_n)^2 - \Omega^2} \\ &\times \left(s_{nm}(y) - \frac{\pm s_{nm}(y)\bar{\sigma}^2 - 2|eB|r_{nm}(y)}{E_n E_m} \right). \end{aligned} \quad (\text{A1})$$

Thus, in the considered approximation, the temperature dependence is described by the terms in the first line of the above equation. To calculate the first sum over the Landau levels in Eq. (A1) we use the representation $1/a = \int_0^\infty dt e^{-at}$ valid for $\text{Re} a > 0$, and take into account that $s_{0n}(y) = s_{n0}(y) = y^n e^{-y}/2n!$. The evaluation of the second sum in Eq. (A1) is more involved. First, we use the chain of transformations

$$\begin{aligned} &\frac{E_m + E_n}{(E_n + E_m)^2 - \Omega^2} \left(1, \frac{1}{E_n E_m} \right) \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{(\omega(\omega + i\Omega), 1)}{((\omega + i\Omega)^2 + E_n^2)(\omega^2 + E_m^2)} \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dt_1 dt_2}{\sqrt{t_1 + t_2}} e^{\frac{\Omega^2 t_1 t_2}{t_1 + t_2} - t_1 E_n^2 - t_2 E_m^2} \left(\frac{t_1 + t_2 + 2t_1 t_2 \Omega^2}{2(t_1 + t_2)^2}, 1 \right), \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \Pi^{\sigma,\pi}(\Omega, k) &= \frac{2\bar{\sigma}}{\pi l^2} \left(\tanh \frac{\bar{\sigma}}{2T} - 1 \right) e^{-y} \left[\frac{1}{4\bar{\sigma}^2 - \Omega^2} \frac{1 \mp 1}{2} - \int_0^\infty dt_1 dt_2 e^{(t_1 + t_2)\Omega^2 + 2\bar{\sigma}\Omega(t_1 - t_2)} \right. \\ &\times \left. \left(2|eB|y e^{-2|eB|(t_1 + t_2)} e^{y e^{-2|eB|(t_1 + t_2)}} \frac{1 \pm 1}{2} + \Omega^2 (e^{y e^{-2|eB|(t_1 + t_2)}} - 1) \frac{1 \mp 1}{2} \right) \right] \\ &+ \frac{1}{\pi \sqrt{\pi} l^2} \int_0^\infty \frac{dt_1 dt_2}{\sqrt{t_1 + t_2}} e^{\frac{\Omega^2 t_1 t_2}{t_1 + t_2}} \left(\frac{t_1 + t_2 + 2t_1 t_2 \Omega^2 \mp 2\bar{\sigma}^2(t_1 + t_2)^2}{2(t_1 + t_2)^2} S(t_1, t_2) + 2|eB|R(t_1, t_2) \right), \end{aligned} \quad (\text{A7})$$

which is convenient for expansions in k^2 and Ω^2 . It is also very useful for obtaining the zero field limit, for which we get

$$\begin{aligned} \Pi^{\sigma,\pi}(\Omega, k) &= \frac{1}{2\pi^{3/2}} \int_{1/\Lambda^2}^\infty \frac{d\rho e^{-\bar{\sigma}^2 \rho}}{\rho^{3/2}} \int_0^1 dx e^{-(k^2 - \Omega^2)\rho x(1-x)} \\ &\times \left[\frac{3}{2} + \rho((\Omega^2 - k^2)x(1-x) \mp \bar{\sigma}^2) \right], \end{aligned} \quad (\text{A8})$$

valid for $\Omega^2 < E_0^2$. Then the sum

$$S(t_1, t_2) = \sum_{n,m=0}^{\infty} s_{nm}(y) e^{-t_1 E_n^2 - t_2 E_m^2} \quad (\text{A3})$$

is evaluated using the integral representation (52) and the summation formula,

$$\sum_{n=0}^{\infty} L_n^\alpha(z) x^n = (1-x)^{-(\alpha+1)} \exp\left(\frac{xz}{x-1}\right), \quad |x| < 1. \quad (\text{A4})$$

Finally, the space integral over r in Eq. (52) gives

$$\begin{aligned} S(t_1, t_2) &= \frac{1}{2} \coth(|eB|(t_1 + t_2)) \\ &\times \exp\left(-\bar{\sigma}^2(t_1 + t_2) - \frac{2y \sinh|eB|t_1 \sinh|eB|t_2}{\sinh|eB|(t_1 + t_2)}\right). \end{aligned} \quad (\text{A5})$$

Similarly, for another sum we obtain ($r_{n0}(y) = r_{0m}(y) = 0$),

$$\begin{aligned} R(t_1, t_2) &\equiv \sum_{n,m=1}^{\infty} r_{nm}(y) e^{-t_1 E_n^2 - t_2 E_m^2} \\ &= \frac{e^{-\bar{\sigma}^2(t_1 + t_2)}}{4\sinh^2|eB|(t_1 + t_2)} \\ &\times \exp\left(-\frac{2y \sinh|eB|t_1 \sinh|eB|t_2}{\sinh|eB|(t_1 + t_2)}\right) \\ &\times \left(1 - \frac{2y \sinh|eB|t_1 \sinh|eB|t_2}{\sinh|eB|(t_1 + t_2)} \right). \end{aligned} \quad (\text{A6})$$

Thus, we get the following representation for the correlators:

where an ultraviolet cutoff Λ is introduced at the lower limit of the integral.

It is obvious that the contribution of the first term in square brackets in Eq. (A7) is given by

$$\Pi_1^\sigma(\Omega, k) = 0, \quad \Pi_1^\pi(\Omega, k) = \frac{2}{\pi l^2} \frac{\bar{\sigma}(\tanh \frac{\bar{\sigma}}{2T} - 1)}{4\bar{\sigma}^2 - \Omega^2} e^{-y}. \quad (\text{A9})$$

The contribution of other terms can be expanded in y and $\Omega^2/|eB|$, and keeping only the first order terms, we get

$$\begin{aligned} \Pi^\sigma(\Omega, k) &= \Pi_1^\sigma(\Omega, k) + \frac{y}{\pi} \bar{\sigma} \left(1 - \tanh \frac{\bar{\sigma}}{2T}\right) \\ &\quad + \frac{1}{\pi^{3/2} l} \Pi^-(\Omega, k), \end{aligned} \quad (\text{A10})$$

$$\Pi^\pi(\Omega, k) = \Pi_1^\pi(\Omega, k) + \frac{1}{\pi^{3/2} l} \Pi^+(\Omega, k), \quad (\text{A11})$$

where

$$\Pi^\pm = V^\pm + \frac{\Omega^2 l^2}{4} Q^\pm + y P^\pm, \quad (\text{A12})$$

and

$$\begin{aligned} V^\pm &= \int_\epsilon^\infty \frac{d\rho}{4\sqrt{\rho}} e^{-m^2 \rho} \left[(1 \pm 2m^2 \rho) \coth \rho + \frac{2\rho}{\sinh^2 \rho} \right], \\ m^2 &= \bar{\sigma}^2 l^2, \end{aligned} \quad (\text{A13})$$

$$Q^\pm = \frac{1}{2} \int_0^\infty d\rho \sqrt{\rho} e^{-m^2 \rho} \left[\left(1 \pm \frac{2m^2 \rho}{3}\right) \coth \rho + \frac{2\rho}{3 \sinh^2 \rho} \right], \quad (\text{A14})$$

$$\begin{aligned} P^\pm &= \int_0^\infty \frac{d\rho}{\sqrt{\rho}} e^{-m^2 \rho} \left[\frac{1 \pm 2m^2 \rho}{4\rho} \coth \rho + \frac{1}{\sinh^2 \rho} \right] \\ &\quad \times (1 - \rho \coth \rho). \end{aligned} \quad (\text{A15})$$

The integral in the expression for V^\pm is divergent, and we regularized it by introducing a lower limit cutoff $\epsilon = 1/\Lambda^2 l^2$. Finally, we get

$$\begin{aligned} V^\pm &= \frac{3}{2\sqrt{\epsilon}} + \sqrt{\frac{\pi}{2}} \left[\zeta\left(\frac{1}{2}, 1 + \frac{m^2}{2}\right) \right. \\ &\quad \left. - m^2 \zeta\left(\frac{3}{2}, 1 + \frac{m^2}{2}\right) \frac{1 \mp 1}{4} + \frac{1 \pm 1}{2\sqrt{2}m} \right], \end{aligned} \quad (\text{A16})$$

$$\begin{aligned} Q^\pm &= \frac{1}{2} \sqrt{\frac{\pi}{2}} \left[\zeta\left(\frac{3}{2}, 1 + \frac{m^2}{2}\right) \right. \\ &\quad \left. - m^2 \zeta\left(\frac{5}{2}, 1 + \frac{m^2}{2}\right) \frac{1 \mp 1}{4} + \frac{1 \pm 1}{m^3 \sqrt{2}} \right], \end{aligned} \quad (\text{A17})$$

$$P^+ = -\sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{2}m}, \quad (\text{A18})$$

$$P^- = -\sqrt{\frac{\pi}{2}} \left\{ m^2 \zeta\left(\frac{1}{2}, 1 + \frac{m^2}{2}\right) + \frac{m^4}{2} \zeta\left(\frac{3}{2}, 1 + \frac{m^2}{2}\right) + \sqrt{2}m \right\}, \quad (\text{A19})$$

where $\zeta(s, v)$ is the generalized zeta function.

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