Sigma model BPS lumps on a torus

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We study doubly periodic Bogomol'nyi-Prasad-Sommerfield lumps in supersymmetric $\mathbb{C}P^{N-1}$ nonlinear sigma models on a torus T^2. Following the philosophy of the Harrington-Shepard construction of calorons in Yang-Mills theory, we obtain the n-lump solutions on compact spaces by suitably arranging the n lumps on $\mathbb{R}^2$ at equal intervals. We examine the modular invariance of the solutions and find that there are no modular invariant solutions for $n = 1, 2$ in this construction.

I. INTRODUCTION

Instantons in Yang-Mills theories at finite temperature have been extensively investigated in the past years. Instantons at finite temperature, commonly called calorons, were first studied by Harrington and Shepard [1], who demonstrated the analytic description to the 1-instanton of $SU(2)$ gauge theory living in $\mathbb{R}^3 \times S^1$. The radius of the compact space S^1 is naturally interpreted as the inverse of the temperature T. The Harrington-Shepard caloron is constructed by locating the infinitely many BPST instantons [2] along the one direction with the equal separation T^{-1}. The authors of Ref. [1] start from the BPST instanton in the 't Hooft ansatz:

$$A_m^c = \eta_{mn}^c \partial^n \log\phi(x), \qquad \phi(x) = 1 + \frac{\rho^2}{(x_m - a_m)^2}, \quad (1)$$

where $c = 1, 2, 3$ is the index of $su(2)$, η_{mn}^c is the 't Hooft symbol and a_m, ρ are the position and the size of the instanton, respectively. It can easily be shown that the anti–self dual equation is enjoyed if the function $\phi(x)$ obeys the Laplace equation. Hence, the 1-instanton in $\mathbb{R}^3 \times S^1$ is obtained by superposing the BPST instantons placed periodically along the x^4 direction with period $1/T$. By scaling the size of each instanton $\rho \rightarrow \rho/\sqrt{2\pi T}$, one can perform the infinite sum in $\phi(x)$ as

$$\phi(\vec{x}, T) = 1 + \frac{\rho^2}{2\pi T} \sum_{k=-\infty}^{\infty} \frac{1}{\vec{x}^2 + (x_4 - kT^{-1})}$$
$$= 1 + \frac{\rho^2}{2r} \frac{\sinh(2\pi Tr)}{\cosh(2\pi Tr) - \cos(2\pi Tx_4)}, \quad (2)$$

where $\vec{x} = (x_1, x_2, x_3)$, $r = \sqrt{\vec{x}^2}$ and we have taken $a_m = 0$ for simplicity. Therefore, calorons are interpreted as the periodic instantons on $\mathbb{R}^4$ [3]. However, the Harrington-Shepard construction cannot be applied to the general solutions including all the moduli parameters. This is because the 't Hooft ansatz does not contain all the moduli. To find the most general solutions, one needs to consider

the Nahm construction [4] of calorons, which provides a strong scheme to study the structure of solutions or moduli spaces. A natural generalization of calorons are doubly periodic instantons on a torus T^2. Instantons on a torus, sometimes called torons, are studied in various contexts [5,6].

Sometimes, problems in gauge theories are simplified when one considers nonlinear sigma models that are recognized as the strong gauge coupling limit of the UV theories. Actually, explicit constructions of instantons or calorons are possible in nonlinear sigma models. For example, instantons of the sigma models in two dimensions have very simple structures. These two-dimensional instantons are called lumps. The lumps in the sigma models are studied in much detail [7,8], where the explicit construction of lumps, moduli space structure, and scattering process has been investigated. Recently, the constituent structure of the lumps in the nonlinear sigma models is studied [9–11]. In Ref. [9], it is discussed that the lumps with twisted boundary conditions in compact spaces lead to the constituent structure. This type of structure of the lumps on $\mathbb{R} \times S^1$ is quite similar to the calorons in Yang-Mills theories with nontrivial holonomies, in which there appear monopole constituents of calorons [12].

The aim of this paper is to establish the systematic construction of Bogomol'nyi-Prasad-Sommerfield (BPS) lumps in supersymmetric nonlinear sigma models on a torus T^2 for the arbitrary charge n. The sigma model lumps with $n \geq 2$ on $\mathbb{R}^2$ are obtained by multiplying the charge-1 solution by n times. We will show that the same is true even for the lumps in compact spaces. Following the Harrington-Shepard philosophy, we will collect the lumps on $\mathbb{R}^2$ aligned in two distinct directions and construct the explicit solutions on T^2. We also demonstrate that the collections of infinitely many n lumps on $\mathbb{R}^2$ with various boundary conditions result in the solutions on T^2 with twisted periodic conditions. The solutions have appropriate pole structure and correct topological charge n. We will also examine the modular invariance of the solutions.

The organization of this paper is as follows. In Sec. II, we define the model. We consider the supersymmetric $\mathbb{C}P^{N-1}$ model and the BPS equation for lumps. In

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Sec. III, focusing on the $\mathbb{C}P^1$ model, we give the constructive method to fabricate doubly periodic BPS lumps on a torus starting from the charge-n ones on $\mathbb{R}^2$. The modular invariance of the solutions will be studied. Section IV is devoted to the conclusion and discussions.

II. $\mathbb{C}P^{N-1}$ SIGMA MODEL AND BPS EQUATIONS

In this section, we start from the $\mathcal{N} = 1$ supersymmetric $\mathbb{C}P^{N-1} \sim SU(N)/[SU(N-1) \times U(1)]$ sigma model in four dimensions. Although supersymmetry is not essential for the construction of solutions, we embed the model into the superfield formalism. This is because one can easily generalize the model to the ones with other target spaces in the superspace formalism [13]. Another important point is that the supersymmetric property of solutions is necessary when one discusses the relations between the sigma model lumps and other solitonic objects in gauge theories (see footnote 1). We follow the Wess-Bagger conventions [14]. The space-time metric is given by $\eta_{mn} = \text{diag}(-1, +1, +1, +1)$. Following the quotient construction of sigma models [13], the Lagrangian in four-dimensional $\mathcal{N} = 1$ superspace is given by

$$\mathcal{L} = \int d^4\theta(\Phi_i^\dagger e^{2V}\Phi_i - cV), \qquad (i = 1, \cdots, N), \quad (3)$$

where the chiral superfields $\vec{\Phi} = \Phi_i$ are the fundamental representation (**N**) of the global $SU(N)$ symmetry, V is the $U(1)$ vector superfield and $c > 0$ is the Fayet-Iliopoulos parameter. The component expansion of the chiral superfield is given by

$$\Phi_i(y, \theta) = \phi_i(y) + \sqrt{2}\theta\psi_i(y) + \theta^2 F_i(y), \qquad (4)$$

while the vector superfield in the Wess-Zumino gauge is

$$V = -\theta\sigma^m\bar\theta A_m + i\theta\theta\bar\theta\,\bar\lambda - i\bar\theta\,\bar\theta\,\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2 D. \quad (5)$$

In the following, we consider the bosonic part of the Lagrangian (3). The Lagrangian in the component form is given by

$$\mathcal{L} = -(D_m\phi_i)(D^m\phi_i)^\dagger + D(\phi_i\bar\phi_i - c) + F_i\bar F_i, \quad (6)$$

where $D_m* = \partial_m* + iA_m*$ is the $U(1)$ gauge covariant derivative. From the D-term condition, we have the constraint for the scalar fields,

$$|\phi_i|^2 = c, \qquad (7)$$

while the F-term condition is trivial. Therefore, the Lagrangian is rewritten as

$$\mathcal{L} = -|D_m\phi_i|^2, \qquad |\phi_i|^2 = c. \qquad (8)$$

Since the gauge field does not have the kinetic term, it is eliminated by the equation of motion,

$$A_m = i\frac{c^{-1}}{2}(\bar\phi_i\partial_m\phi_i - \partial_m\bar\phi_i\phi_i). \qquad (9)$$

Next, we consider the BPS equation for lumps which depends on the two-dimensional directions x^a ($a = 1, 2$). The lumps are instantons in two-dimensional sigma models. In the following, we consider two-dimensional models though we started from four dimensions. The dimensional reduction from four to two dimensions is straightforward. The model becomes the two-dimensional $\mathcal{N} = (2, 2)$ $\mathbb{C}P^{N-1}$ model.[1] The energy is given by

$$E = \int d^2x\left[\frac{1}{2}|D_a\phi_i \pm i\varepsilon_{ab}D_b\phi_i|^2 \pm i\varepsilon_{ab}(D_a\phi_i)(D_b\phi_i)^\dagger\right]$$
$$\geq \pm \int d^2x\, i\varepsilon_{ab}(D_a\phi_i)(D_b\phi_i)^\dagger$$
$$= \pm 2\pi cQ, \qquad (10)$$

where $\varepsilon_{12} = -1$ is the antisymmetric epsilon symbol and the topological charge Q has been defined as

$$Q = \frac{1}{2\pi c}\int d^2z(|D_z\phi_i|^2 - |D_{\bar z}\phi_i|^2)$$
$$= -\frac{1}{4\pi}\int d^2x\,\varepsilon^{ab}F_{ab}. \qquad (11)$$

Here, the complex coordinate in two dimensions is defined as $z = \frac{1}{\sqrt{2}}(x^1 + ix^2)$. The gauge field and the covariant derivative are complexified in the same way. From the energy bound in Eq. (10), the BPS condition is given by

$$D_a\phi_i \pm i\varepsilon_{ab}D_b\phi_i = 0, \qquad (12)$$

or equivalently,

$$D_{\bar z}\phi = 0, \qquad D_z\phi = 0. \qquad (13)$$

The first and the second conditions correspond to the plus and minus signs in Eq. (12), respectively. In the following, we focus on the first condition. The solutions to the BPS equation (13) preserve a half of $\mathcal{N} = (2, 2)$ supersymmetry. Therefore, the lumps are 1/2 BPS configurations.

In order to satisfy the constraint (7), it is convenient to consider the following field decomposition:

$$\phi_i = W_i \frac{\sqrt{c}}{\sqrt{W_j^\dagger W_j}}, \qquad (14)$$

where W_i is an N-component vector. Then, one easily finds that the BPS equation becomes

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[1]This $\mathcal{N} = (2, 2)$ $\mathbb{C}P^{N-1}$ sigma model is the world-volume effective theory of a 1/2 BPS vortex in supersymmetric gauge theory in four dimensions [15]. It was discussed in Ref. [16] that the 1/2 BPS lumps in the supersymmetric sigma model are interpreted as the 1/4 BPS instantons in supersymmetric gauge theories. Clearly, this lumps/instantons correspondence is based on the supersymmetric setup.

$$D_{\bar{z}}\phi_i = \sqrt{c}P(\partial_{\bar{z}}W_i)(W^\dagger \cdot W)^{-1/2} = 0, \qquad (15)$$

where $P_{ij} \equiv 1_{ij} - W_i \frac{1}{W^\dagger \cdot W}W_j^\dagger$ is the projection operator. Therefore, solutions to the BPS equation are given by holomorphic functions $W_i = W_i(z)$ [17]. Using the gauge symmetry, we fix the gauge as

$$W_i = \begin{pmatrix} 1 \\ w_{\hat{i}} \end{pmatrix}, \qquad (\hat{i} = 2, \cdots, N). \qquad (16)$$

The topological charge for the BPS lump is, therefore, given by

$$\begin{aligned} Q &= \frac{1}{2\pi c}\int d^2z \frac{c}{W^\dagger \cdot W}\partial_{\bar{z}}W^\dagger P\partial_z W \\ &= \frac{1}{2\pi}\int d^2z \frac{1}{(1+|w_{\hat{i}}|^2)^2}[(1+|w_{\hat{i}}|^2)|\partial w_{\hat{i}}|^2 \\ &\quad - w_{\hat{i}}\bar{\partial}\bar{w}_{\hat{i}}\bar{w}_{\hat{j}}\partial w_{\hat{j}}]. \end{aligned} \qquad (17)$$

Since we have the relation $\partial\bar{\partial}\log(W^\dagger \cdot W) = \frac{1}{W^\dagger \cdot W}\bar{\partial}W_i^\dagger P_{ij}\partial W_j$, the topological charge is rewritten as

$$\begin{aligned} Q &= \frac{1}{2\pi}\int d^2z \partial\bar{\partial}\log(W^\dagger \cdot W) \\ &= \frac{1}{4\pi}\int d^2z[\partial\bar{\partial}\log(W^\dagger \cdot W) + \bar{\partial}\partial\log(W^\dagger \cdot W)] \\ &= \frac{i}{4\pi}\oint[\bar{\partial}\log(W^\dagger \cdot W)d\bar{z} - \partial\log(W^\dagger \cdot W)dz]. \end{aligned} \qquad (18)$$

Therefore, the topological charge is determined by the residue of the function $U \equiv \partial\log(W^\dagger \cdot W)$,

$$Q = \frac{1}{2}(\text{Res}_z(U) + \text{Res}_{\bar{z}}(\bar{U})). \qquad (19)$$

Since the energy (10) is invariant under the conformal transformation in the two-dimensional plane $\mathbb{R}^2$, the field is defined on the conformally compactified S^2. The lumps are, therefore, harmonic maps from S^2 to $\mathbb{C}P^{N-1}$ that are classified by integers, namely, the topological charges.

III. BPS LUMPS

In this section, we give the constructive procedure to formulate the BPS lumps on a torus T^2 with appropriate base point conditions. Before going to the totally compactified space T^2, we establish the relations between the lump solutions in $\mathbb{R}^2$ and $\mathbb{R} \times S^1$. In the following, we consider the $N = 2$ case, namely, the $\mathbb{C}P^1$ model. In this case, only the nontrivial component in W_i is $w_2 \equiv u(z)$, and the topological charge is given by

$$Q = \frac{1}{2\pi}\int d^2z \frac{|\partial u|^2}{(1+|u|^2)^2}. \qquad (20)$$

A. Lumps on $\mathbb{R}^2$

Let us start from the 1-lump solution on $\mathbb{R}^2$ denoted as $u^{(1)}$. The solution to the BPS equation (15) should be a holomorphic function, and it is required to be settled down to the vacuum asymptotically. When we take the base point (vacuum) condition $u^{(1)}(\infty) = 0$, the 1-lump solution is given by [17]

$$u^{(1)}(z) = \frac{\lambda}{z - \check{z}_1}, \qquad \lambda \in \mathbb{R}, \qquad \check{z}_1 \in \mathbb{C}. \qquad (21)$$

The residue of the function U associated with the solution (21) is evaluated at the pole $z = \check{z}_1$, giving the expected result $Q = 1$. When one considers a different base point condition, for example $u^{(1)}(\infty) = 1$, the 1-lump solution is given by

$$u^{(1)}(z) = \frac{z - \hat{z}_1}{z - \check{z}_1}, \qquad \hat{z}_1 = \check{z}_1, \qquad \hat{z}_1, \qquad \check{z}_1 \in \mathbb{C}. \qquad (22)$$

For this solution, the topological charge density q is

$$q = \frac{1}{2\pi}\frac{\lambda^2}{(|z - z_1|^2 + \lambda^2)^2}, \qquad (23)$$

where we have defined the parameters $z_1 \equiv \frac{\hat{z}_1 - \check{z}_1}{2}$, $\lambda \equiv \frac{|\hat{z}_1 - \check{z}_1|}{2}$, interpreted as the position and the size of the lump. The profile of the energy density is found in Fig 1.
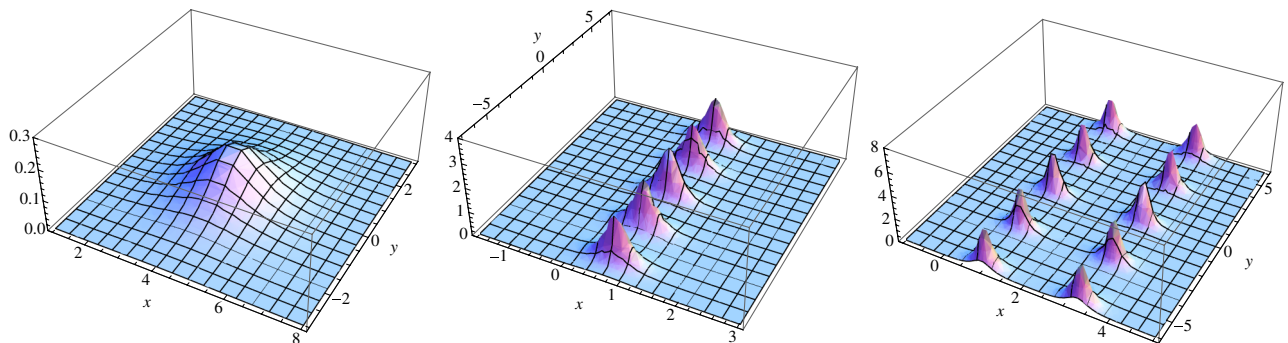

FIG. 1 (color online). Charge density for the solution (22) with $\hat{z}_1 = 4$, $\check{z}_1 = 2$ (left), for the solution (28) with $\beta = 2$, $\lambda = 0.4$, $\nu = 0.1$, (middle). Charge density for the solution (31), $n = 2$ case (right). $\beta = 2$, $\lambda_1 = 0.4$, $\lambda_2 = 1.4$, $\nu_1 = 0.1$, $\nu_2 = 1.1$.

Similarly, for the base point condition $u^{(1)}(\infty) = \infty$, we have the 1-lump solution $u^{(1)}(z) = \lambda(z - \hat{z}_1)$.

Generalizations to the multilump solutions are straightforward. The n-lump solutions $u^{(n)}$ are obtained by multiplying the 1-lump solutions n times. The result is meromorphic rational functions with degree n. For example, in the case of the base point condition $u^{(n)}(\infty) = 1$, the solutions are given by

$$u^{(n)}(z) = \prod_{k=1}^{n} \frac{z - \hat{z}_k}{z - \check{z}_k}, \qquad \hat{z}_k \neq \check{z}_j \text{(for any } j, k). \quad (24)$$

One can easily find that the residue of the function U for the solution (24) is n, which gives the desired result $Q = n$.

B. Lumps on $\mathbb{R} \times S^1$

Next, we consider the lumps on $\mathbb{R} \times S^1$ by compactifying one space-time direction. Without loss of generality, one can consider the imaginary direction in the complex plane $\mathbb{C}$ as the compact direction. We expect that solutions on $\mathbb{R} \times S^1$ are interpreted as periodically aligned lumps on $\mathbb{R}^2$. Following the Harrington-Shepard philosophy, we multiply the infinite number of the 1-lump solutions (21) located at the equal interval $\beta \in \mathbb{R}$ along the imaginary direction. Namely, we consider the following solution

$$u^{(1)}(z, \beta) = \prod_{k=-\infty}^{\infty} \frac{\lambda}{z - z_0 - i\beta k}$$
$$= \frac{\lambda}{z - z_0} \prod_{k=1}^{\infty} \frac{\lambda^2/\beta^2}{(z - z_0)^2/\beta^2 + k^2}, \quad (25)$$

where we have multiplied by the 1-lump solutions so that the solution $u^{(1)}(z, \beta)$ has poles at $z = z_0 + i\beta k$. Since the infinite product of the 1-lump solution diverges, we employ the ζ-function regularization to find the finite solution:

$$\prod_{k=1}^{\infty} \frac{\lambda^2}{\beta^2} = (\lambda/\beta)^{-1/2}. \quad (26)$$

After the regularization, we find that the solution on $\mathbb{R} \times S^1$ is obtained as

$$u^{(1)}(z, \beta) = \frac{1}{2} \frac{1}{\sinh \pi \beta^{-1}(z - z_0)}. \quad (27)$$

For this solution, the poles of the function U are at $z = z_0$, and it is easy to find that the topological charge for this solution is $Q = 1$. Since the solution (27) satisfies the antiperiodic boundary condition $u^{(1)}(z + i\beta, \beta) = -u^{(1)}(z, \beta)$, the solution is allowed only when the twisted boundary condition is imposed. This solution has been discussed in Refs. [9,10] in the context of the constituent structure of sigma model lumps on the compact space. In Refs. [9,10], the authors introduced nontrivial holonomy

parameters in the solution (27) and studied its partonic, or constituent, nature.

Next, we consider the 1-lump solution (22) by choosing the base point condition $u^{(1)}(\infty) = 1$. Again, we arrange the solution along $\check{z}_k = 2\nu + i\beta k$, $\beta \in \mathbb{R}$, λ, $\nu \in \mathbb{C}$, $k \in \mathbb{Z}$. We further demand that the zeros of the solution appear at $\hat{z}_k = 2\lambda + i\beta k$. By choosing these zero points, the size of the 1-lump solution (22) does not diverge at $k \to \infty$ and is fixed to be $|\lambda - \nu|$. The position of the lump in one period is $\lambda - \nu$. Then, we obtain the solution as follows:

$$u^{(1)}(z, \beta) = \prod_{k=-\infty}^{\infty} \left(\frac{z - 2\lambda - i\beta k}{z - 2\nu - i\beta k} \right)$$
$$= \frac{z - 2\lambda}{z - 2\nu} \prod_{k=1}^{\infty} \frac{k^2}{\beta^{-2}(z - 2\nu)^2 + k^2}$$
$$\times \frac{\beta^{-2}(z - 2\lambda)^2 + k^2}{k^2}$$
$$= \frac{\sinh \pi \beta^{-1}(z - 2\lambda)}{\sinh \pi \beta^{-1}(z - 2\nu)}. \quad (28)$$

Thanks to the good base point condition, we do not need any regularization. Moreover, the solution preserves the periodic boundary condition,

$$u^{(1)}(z + i\beta, \beta) = u^{(1)}(z, \beta). \quad (29)$$

This solution was found in Ref. [18] in the same way we have just shown above. The energy profile for this solution is given in Fig 1. Since the poles of the function U are at $z = 2\nu$ inside the one period, the residue is evaluated as $\text{Res}_{z=2\nu} \partial \log W^\dagger W = 1$ at the pole. Therefore, the topological charge is given by $Q = 1$. One can easily find that the decompactification limit $\beta \to \infty$ of the solution gives the correct result:

$$\lim_{\beta \to \infty} u^{(1)}(z, \beta) = \frac{z - 2\lambda}{z - 2\nu}. \quad (30)$$

The n-lump generalization is straightforward. This is obtained from the solution (24) on $\mathbb{R}^2$. The result is

$$u^{(n)}(z, \beta) = \prod_{k=1}^{n} \frac{\sinh \pi \beta^{-1}(z - 2\lambda_k)}{\sinh \pi \beta^{-1}(z - 2\nu_k)}. \quad (31)$$

The charge density profile for this solution is found in Fig 1 for the $n = 2$ case.

C. Lumps on T^2

In this subsection, we construct the multilump solutions on a torus T^2 by extending the superposition procedure established in the previous subsection. By suitably arranging the solutions on $\mathbb{R} \times S^1$, we will find the lump solutions with topological charges $Q = n \geq 1$. It is known that there is no harmonic map from the genus g Riemann surface to $\mathbb{C}P^1 \sim S^2$ when the degree n of the map is less than g [19–21]. Therefore, we expect that there is no $Q = 1$

periodic solution on a torus. Actually, as we will see, the $n = 1$ lump constructed below does not show the doubly periodic property. When $n \geq 2$, the solutions can be doubly periodic and are rewritten as elliptic functions. Even more, for the cases $n \geq 3$, the solutions show the modular invariance.

Let us begin with the solution (28) on $\mathbb{R} \times S^1$, a lump aligned in the imaginary direction. In order to find solutions on T^2, we locate the solution (28) along the real direction at the interval γ. Assuming that $\gamma/\beta > 0$, the array of $\mathbb{R} \times S^1$ lumps with interval γ is given by

$$
\begin{aligned}
u^{(1)}(z, \beta, \gamma) &= \prod_{k=-\infty}^{\infty} \frac{\sinh \pi \beta^{-1}(z - 2\lambda - \gamma k)}{\sinh \pi \beta^{-1}(z - 2\nu - \gamma k)} \\
&= \frac{\sinh \pi \beta^{-1}(z - 2\lambda)}{\sinh \pi \beta^{-1}(z - 2\nu)} \prod_{k=1}^{\infty} \frac{(1 - e^{-2\pi\beta^{-1}\gamma k}e^{2\pi\beta^{-1}(z-2\lambda)})(1 - e^{2\pi\beta^{-1}\gamma k}e^{-2\pi\beta^{-1}(z-2\lambda)})}{(1 - e^{-2\pi\beta^{-1}\gamma k}e^{2\pi\beta^{-1}(z-2\nu)})(1 - e^{2\pi\beta^{-1}\gamma k}e^{-2\pi\beta^{-1}(z-2\nu)})}. \quad (32)
\end{aligned}
$$

We can rewrite this infinite product as the pseudoperiodic θ functions by using the formula[2]

$$
\theta_1(v, \tau) = q_0 q^{\frac{1}{4}} \frac{\mathbf{z} - \mathbf{z}^{-1}}{i} \prod_{k=1}^{\infty}(1 - q^{2k}\mathbf{z}^2)(1 - q^{2k}\mathbf{z}^{-2}), \quad (33)
$$

$$
q_0 = \prod_{k=1}^{\infty}(1 - q^{2k}), \quad q = e^{i\pi\tau}, \quad \mathbf{z} = e^{i\pi v}, \quad \text{Im}\tau > 0. \quad (34)
$$

Then the 1-lump solution on T^2 is given in the simple closed form:

$$
u^{(1)}(z, \beta, \gamma) = \frac{\theta_1(i\beta^{-1}(z - 2\lambda), i|\beta^{-1}\gamma|)}{\theta_1(i\beta^{-1}(z - 2\nu), i|\beta^{-1}\gamma|)}, \quad (35)
$$

where $\tau = i|\beta^{-1}\gamma|$ and $\text{Im}\tau = |\beta^{-1}\gamma| > 0$. The expression is valid even for the case $\beta^{-1}\gamma < 0$. Again, we do not need any regularization for the multiplication of the solution (28). Using the property of the theta function,

$$
\theta_1(i\beta^{-1}z - 1, i\beta^{-1}\gamma) = -\theta_1(i\beta^{-1}z, i\beta^{-1}\gamma), \quad (36)
$$

$$
\begin{aligned}
\theta_1(i\beta^{-1}z &+ i\beta^{-1}\gamma, i\beta^{-1}\gamma) \\
&= -e^{2\pi\beta^{-1}z}e^{\pi\beta^{-1}\gamma}\theta_1(i\beta^{-1}z, i\beta^{-1}\gamma), \quad (37)
\end{aligned}
$$

the periodicity of the solution (35) is found to be

$$
u^{(1)}(z + i\beta, \beta, \gamma) = u^{(1)}(z, \beta, \gamma), \quad (38)
$$

$$
u^{(1)}(z + \gamma, \beta, \gamma) = e^{-4\pi\beta^{-1}(\lambda - \nu)}u^{(1)}(z, \beta, \gamma). \quad (39)
$$

Therefore, in general, the solution (35) is not periodic in the γ direction. Only the twisted boundary condition is allowed in that direction when the parameters satisfy $\text{Re}(\lambda - \nu) = 0$. For the solution (35), we find

$$
\begin{aligned}
\partial \log W^{\dagger}W = &-i\beta^{-1} \frac{\theta_1(i\beta^{-1}(z - 2\lambda))}{\theta_1(-i\beta^{-1}(z - 2\nu))} \\
&\times \frac{\theta_1'(-i\beta^{-1}(z - 2\lambda))\theta_1(-i\beta^{-1}(z - 2\nu)) - \theta_1(-i\beta^{-1}(z - 2\lambda))\theta_1'(-i\beta^{-1}(z - 2\nu))}{|\theta_1(i\beta^{-1}(z - 2\nu))|^2 + |\theta_1(i\beta^{-1}(z - 2\lambda))|^2},
\end{aligned}
$$

where the theta functions have a common modulus $\tau = |\beta^{-1}\gamma|$. Since the function $\theta_1(v, \tau)$ has a zero at $v = 0$ and no poles in the defined region (the fundamental lattice $-\frac{\sqrt{2}}{2}\gamma \leq x \leq \frac{\sqrt{2}}{2}\gamma$, $-\frac{\sqrt{2}}{2}\beta \leq y \leq \frac{\sqrt{2}}{2}\beta$), the pole of the function U is at $z = 2\nu$. The residue at the pole is evaluated as $\text{Res}_{z=2\nu}\partial \log W^{\dagger}W = 1$, which implies $Q = 1$.

Now, let us consider the decompactification limits of the solution in the real and imaginary directions. Using the expansion of the θ function,

$$
\theta_1(v, \tau) = 2q^{\frac{1}{4}}q_0 \sin \pi v \prod_{n=1}^{\infty}(1 - 2q^{2n}\cos 2\pi v + q^{4n}), \quad (40)
$$

and the fact, $q = e^{-\pi\beta^{-1}\gamma} \to 0$ in the limit $\gamma \to \infty$, we find

$$
\begin{aligned}
\lim_{\gamma \to \infty} u^{(1)}(z, \beta, \gamma) &= \frac{\sin i\pi\beta^{-1}(z - 2\lambda)}{\sin i\pi\beta^{-1}(z - 2\nu)} \\
&= \frac{\sinh \pi\beta^{-1}(z - 2\lambda)}{\sinh \pi\beta^{-1}(z - 2\nu)} = u^{(1)}(z, \beta).
\end{aligned} \quad (41)
$$

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[2]Here, $\text{Im}z > 0$ is required for $|q| = e^{-\pi\text{Im}\tau} < 1$, which is a necessary condition for the definition of θ functions.

This is just the array of the 1-lump solution on $\mathbb{R}^2$ along the imaginary direction. Next, using the Jacobi identity relation of the θ functions, the decompactification limit along the real direction is calculated as

$$
\begin{aligned}
\lim_{\beta\to\infty} u^{(1)}(z,\beta,\gamma) &= \lim_{\beta\to\infty} \frac{e^{\pi\beta^{-1}\gamma^{-1}(z-2\lambda)^2}}{e^{\pi\beta^{-1}\gamma^{-1}(z-2\nu)^2}} u^{(1)}(-i\gamma^{-1}\beta z, i\gamma^{-1}\beta) \\
&= \lim_{\beta\to\infty} \frac{e^{\pi\beta^{-1}\gamma^{-1}(z-2\lambda)^2}}{e^{\pi\beta^{-1}\gamma^{-1}(z-2\nu)^2}} \frac{2\tilde{q}^{\frac{1}{4}}\tilde{q}_0 \sin\pi\gamma^{-1}(z-2\lambda)\prod_{n=1}^{\infty}(1-2\tilde{q}^{2n}\cos2\pi\gamma^{-1}(z-2\lambda)+\tilde{q}^{4n})}{2\tilde{q}^{\frac{1}{4}}\tilde{q}_0 \sin\pi\gamma^{-1}(z-2\nu)\prod_{n=1}^{\infty}(1-2\tilde{q}^{2n}\cos2\pi\gamma^{-1}(z-2\nu)+\tilde{q}^{4n})} \\
&= \frac{\sinh\pi(i\gamma)^{-1}(z-2\lambda)}{\sinh\pi(i\gamma)^{-1}(z-2\nu)},
\end{aligned}
\tag{42}
$$

where we have defined $\tilde{q} = e^{2\pi i(-1/\tau)} = e^{-2\pi\gamma^{-1}\beta}$. The result is the 1-lump solution aligned along the real direction with interval γ as expected.

Generalization to the multilump solutions is straightforward. The n-lump solution on T^2 is given by

$$
u^{(n)}(z,\beta,\gamma) = \prod_{k=1}^{n} \frac{\theta_1(i\beta^{-1}(z-2\lambda_k), i|\beta^{-1}\gamma|)}{\theta_1(i\beta^{-1}(z-2\nu_k), i|\beta^{-1}\gamma|)}. \tag{43}
$$

Its periodicity is

$$
u^{(n)}(z+i\beta,\beta,\gamma) = u^{(n)}(z,\beta,\gamma), \tag{44}
$$

$$
u^{(n)}(z+\gamma,\beta,\gamma) = e^{-4\pi\beta^{-1}\sum_{k=1}^{n}(\lambda_k-\nu_k)} u^{(n)}(z,\beta,\gamma). \tag{45}
$$

Therefore, when the following condition,

$$
\sum_{k=1}^{n}\lambda_k = \sum_{k=1}^{n}\nu_k, \qquad \lambda_k \neq \nu_k, \tag{46}
$$

is satisfied, the solution becomes periodic. This is possible only for the $n \geq 2$ cases, thus confirming the mathematical result on harmonic maps. Note that when one relaxes the condition (46) and the parameters satisfy $\mathrm{Re}\sum_{k=1}^{n}(\lambda_k - \nu_k) = 0$, the exponential factor in (45) becomes a phase factor and the solution satisfies the twisted boundary condition.

When the periodicity condition (46) is satisfied, we expect that the solutions can be rewritten as elliptic functions. For example, when we choose $\nu_1 = \nu_2 \equiv \nu$ and $\lambda_1 = \nu - \frac{i\gamma}{4}$, $\lambda_2 = \nu + \frac{i\gamma}{4}$ for $n = 2$ case, the solution (43) is rewritten as

$$
\begin{aligned}
u^{(2)}(z,\beta,\gamma) &= -4\beta^2\left(\frac{\theta_4^0}{\theta_1^{0\prime}}\right)^2\{\wp(2(z-2\nu)) - e_1\}, \\
e_1 &= \left(\frac{\pi}{2\beta}\right)^2\frac{1}{3}((\theta_2^0)^2 + (\theta_3^0)^2), \qquad \theta_l^0 \equiv \theta_l(0,\tau), \\
&\quad (l = 1,2,3), \qquad \tau = |\beta^{-1}\gamma|,
\end{aligned}
\tag{47}
$$

where $\wp$ is the Weierstrass $\wp$ function, the degree-2 elliptic function. The moduli space of this 2-lump solution on T^2 was studied in Ref. [22].

The profile of the energy density for the $n = 3$ solution is found in Fig. 2. There are "interference fringes" among the three peaks since the lumps are trapped on the

finite-size lattice and the notion of the "well separated" is essentially lost in fully compact spaces.

Next, we study the modular invariance of the multilump solutions on a torus. Let us consider a torus endowed with a generic modulus τ. A torus T^2_τ with modulus $\tau \in \mathbb{C}$ is defined by the equivalence class $z \sim z - \beta(m + n\tau)$, $\beta \in \mathbb{R}$, m, $n \in \mathbb{Z}$. The torus is invariant under the $PSL(2,\mathbb{Z})$ modular transformation,

$$
\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1, \qquad a,b,c,d \in \mathbb{Z}. \tag{48}
$$

Following the same procedure as before, the n-lump solution in the torus T^2_τ is found to be

$$
u^{(n)}(z,\tau) = \prod_{k=1}^{n} \frac{\theta_1(\beta^{-1}(z-2\lambda_k),\tau)}{\theta_1(\beta^{-1}(z-2\nu_k),\tau)}. \tag{49}
$$

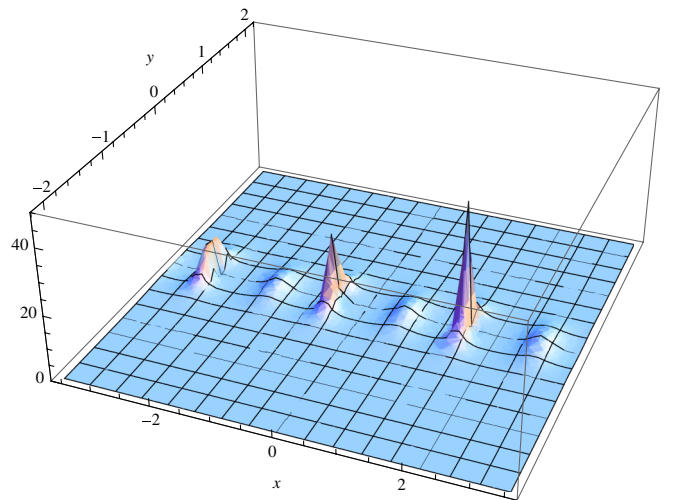The modular transformation (48) is generated by the following fundamental transformations,

FIG. 2 (color online). Energy density for the solution (43) in the one fundamental lattice $-\frac{\sqrt{2}}{2}\gamma \leq x \leq \frac{\sqrt{2}}{2}\gamma$, $-\frac{\sqrt{2}}{2}\beta \leq y \leq \frac{\sqrt{2}}{2}\beta$. $\beta = 3$, $\gamma = 5$. $\lambda_1 = -0.253$, $\lambda_2 = -1.19$, $\lambda_3 = -0.680$, $\nu_1 = -0.918$, $\nu_2 = 0.629$, $\nu_3 = -0.680$. The parameters λ_i, ν_i are a numerical solution to the modular invariance constraints (54).

$$\tau \rightarrow \tau + 1, \qquad \tau \rightarrow -\frac{1}{\tau} \quad \text{with } z \rightarrow \tau z. \qquad (50)$$

Under the first transformation in the above, the θ function changes as

$$\theta_1(v, \tau + 1) = e^{\pi i/4} \theta_1(v, \tau). \qquad (51)$$

The solution (49) is therefore invariant under the transformation (51), cancelling the phase factor $e^{\frac{\pi i}{4}}$. Next, using the relation,

$$\theta_1(v, -1/\tau) = e^{i\pi v^2 \tau} e^{-3\pi i/4} \tau^{1/2} \theta_1(\tau v, \tau), \qquad (52)$$

the solution (49) transforms as

$$u^{(n)}(z, \tau) \rightarrow u^{(n)}(z, -1/\tau)$$
$$= \exp\left[4\pi i \tau \beta^{-2}\left\{\left(\sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n} \nu_n\right)z\right.\right.$$
$$\left.\left. - \left(\sum_{i=1}^{n} \lambda_i^2 - \sum_{i=1}^{n} \nu_i^2\right)\right\}\right] u^{(n)}(\tau z, \tau). \qquad (53)$$

Therefore, the solution is modular invariant if the following conditions are satisfied:

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \nu_i, \qquad \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} \nu_i^2,$$
$$\lambda_i \neq \nu_j \quad \text{for all } i, j. \qquad (54)$$

Again, $n = 1$ is the special case. It is easy to find that the condition (54) cannot be satisfied for the $n = 1$ case. When $n = 2$, we find that the first two conditions in (54) imply $\lambda_1 = \nu_2$, $\lambda_2 = \nu_1$, which contradicts the third condition. Therefore, the modular invariance is generically lost. On the other hand, there are infinitely many solutions to the conditions (54) for $n \geq 3$. It is apparent that the modular invariance conditions (54) contain the periodicity condition (46). Hence, the modular invariance is sufficient for the periodicity of the solutions.

Once the n-lump solutions satisfy the modular invariance conditions (54), the solutions are generically rewritten as elliptic functions. To show this fact, let us consider the following relations between the θ function and the Weierstrass σ function:

$$\sigma(2\omega_1 z) = 2\omega_1 e^{2\eta_1 \omega_1 z^2} \theta_1(z, \tau)/\theta_1'^0, \qquad \tau = \omega_3/\omega_1,$$
$$\eta_1 = \zeta(\omega_1) = \frac{\pi^2}{\omega_1}\left(\frac{1}{12} - 2\sum_{k=1}^{\infty} \frac{kq^{2k}}{1 - q^{2k}}\right), \qquad q = e^{\pi i \tau},$$
$$(55)$$

where $2\omega_1$, $2\omega_3$ are two distinct periods of doubly periodic functions. Then the n-lump solution (43) can be rewritten as

$$u^{(n)}(z, \tau) = e^{A(z)} \prod_{k=1}^{n} \frac{\sigma(2\omega_1 \beta^{-1}(z - 2\lambda_k))}{\sigma(2\omega_1 \beta^{-1}(z - 2\nu_k))}, \qquad (56)$$

where the exponential factor is evaluated as

$$A(z) = 8\eta_1 \omega_1 \beta^{-2}\left[\left(\sum_{k=1}^{n} \lambda_k - \sum_{k=1}^{n} \nu_k\right)z\right.$$
$$\left. + \left(\sum_{k=1}^{n} \lambda_k^2 - \sum_{k=1}^{n} \nu_k^2\right)\right]. \qquad (57)$$

Applying the modular invariance conditions (54), this exponential factor vanishes and the solutions are totally expressed by the elliptic functions. The expression (56) is nothing but the solution discussed in Ref. [23]. The contributions of these solutions to the partition function of the nonlinear sigma models on a torus are discussed in Ref. [24]. The Nahm transformation and moduli spaces of $\mathbb{C}P^{N-1}$ models on a torus were discussed in Ref. [25]. However, our solution (43) is more generic and constructive, allowing the clear decompactification limits and the modular invariance.

So far we have focused on the base point condition $u^{(n)}(\infty) = 1$ on $\mathbb{R}^2$. When we switch to the other base point conditions, for example $u^{(1)}(\infty) = 0$ on $\mathbb{R}^2$, the solution on T^2 becomes

$$u^{(1)}(z, \beta, \gamma) = i\eta(\tau)\theta_1^{-1}(i\beta^{-1}z, \tau), \qquad (58)$$

where we have again employed the ζ-function regularization. The function η is the Dedekind η function defined by

$$\eta(\tau) = q^{1/12} \prod_{k=1}^{\infty} (1 - q^{2k}), \qquad q = e^{i\pi\tau}. \qquad (59)$$

The periodicity of this solution is found to be

$$u^{(1)}(z + i\beta, \beta, \gamma) = -u^{(1)}(z, \beta, \gamma), \qquad (60)$$

TABLE I. Solutions associated with each base point condition. The θ functions have common modulus τ.

Base point cond.	Solution	Regularization	Modular inv.
$u^{(n)}(\infty) = \infty$	$\prod_{k=1}^{n}(i\eta(\tau))^{-1}\theta_1(\beta^{-1}(z - 2\lambda_k))$	needed	lost
$u^{(n)}(\infty) = 1$	$\prod_{k=1}^{n} \theta_1(\beta^{-1}(z - 2\lambda_k))/\theta_1(\beta^{-1}(z - 2\nu_k))$	no	exist
$u^{(n)}(\infty) = 0$	$\prod_{k=1}^{n} i\eta(\tau)/\theta_1(\beta^{-1}(z - 2\nu_k))$	needed	lost

$$u^{(1)}(z + \gamma, \beta, \gamma) = -e^{-2\pi\beta^{-1}z}e^{-\pi\beta^{-1}}u^{(1)}(z, \beta, \gamma). \quad (61)$$

This solution does not show any modular invariance even for the $n \geq 3$ case. One can also find that the solution (58) cannot be periodic even when the multilump generalization of the solution (58) is considered. The properties of the solutions for different base point conditions are summarized in Table I.

Let us comment on the generalization of our construction to the $\mathbb{C}P^{N-1}$ models for $N \geq 3$ cases. One can easily find that this is straightforward. The vector W_i has $N - 1$ independent components $w_{\hat{i}}$. Each component are holomorphic functions and we can construct solutions on T^2 by the same way shown in the $N = 2$ case. The topological charges are determined by the highest degree of the holomorphic functions $w_{\hat{i}}(z)$.

Finally, let us see the topological charge of the BPS lumps on a torus. Without loss of generality, one can consider a rectangle torus defined by $z \sim z + (i\beta + \gamma)$. The topological charge of lumps is given by the first Chern number

$$Q = -\frac{1}{4\pi} \int d^2x \varepsilon^{ab} F_{ab}. \quad (62)$$

We demand that the $U(1)$ gauge field and, hence, the scalar field are periodic up to the gauge transformation:

$$A_1(x_1, x_2 = \beta) = A_1(x_1, x_2 = 0) - \partial_1 \lambda^{(2)}(x_1),$$
$$A_2(x_1 = \gamma, x_2) = A_2(x_1 = 0, x_2) - \partial_2 \lambda^{(1)}(x_2). \quad (63)$$

Note that the gauge parameters $\lambda^{(m)}(x_n)$ depend only on $x_n(n \neq m)$. Then, the topological charge is given by

$$Q = \frac{1}{2\pi}[\lambda^{(1)}(\beta) - \lambda^{(1)}(0) + \lambda^{(2)}(0) - \lambda^{(2)}(\gamma)]. \quad (64)$$

This is the gauge transformation parameter along the closed path depicted in Fig 3. On the other hand, once one goes around the closed path, the scalar field acquires the phase $\lambda^{(1)}(\beta) - \lambda^{(1)}(0) + \lambda^{(2)}(0) - \lambda^{(2)}(\gamma)$. The single-valuedness requires that this phase factor
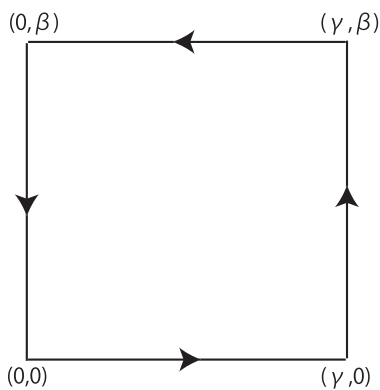

FIG. 3. The closed path on the torus.

must be an integer multiple of 2π. Therefore, the topological charge on T^2 must be integer,

$$Q = n, \qquad n \in \mathbb{Z}. \quad (65)$$

Configurations with nonzero topological number Q are caused by the large gauge transformation $\lambda^{(m)}(x_n)$ that is defined modulo 2π.

IV. CONCLUSION AND DISCUSSION

In this paper, we have studied the topological BPS lumps in supersymmetric $\mathbb{C}P^{N-1}$ nonlinear sigma models on a torus T^2. Following the philosophy of Harrington-Shepard, we have established the constructive procedure to give the BPS lump solutions for arbitrary topological number $Q = n$ by collecting the "fundamental lumps" aligned periodically. The charge-n BPS lump solutions on T^2 are obtained by arranging the charge-n lumps on $\mathbb{R}^2$ at equal intervals along two distinct directions. The function form of the solutions depends on the choice of the base point condition of the fundamental lumps on $\mathbb{R}^2$. Choosing the base point condition $u(\infty) = 0$ or $u(\infty) = \infty$ requires the regularization of the infinite products of rational maps. We have employed the ζ-function regularization and found the explicit solutions that exhibit suitable pole structures.

On the other hand, for the base point condition $u(\infty) = 1$, we do not need any regularization scheme. For the $n = 1$ case, we have found that there is no solution that satisfies the periodic boundary condition and the solution is not modular invariant anymore. This is consistent with the statement that there is no degree 1 elliptic functions on a torus. However, if the twisted boundary conditions are allowed, the solution turns out to be acceptable provided that the parameters of the solutions are chosen appropriately. For $n = 2$, there are no parameters λ_i, ν_i that satisfy the modular invariance conditions. In the cases of $n \geq 3$, however, we find that there are infinitely many parameters that satisfy the modular invariance conditions.

Although the lumps on a torus were discussed in several contexts in the past [19], our construction is quite simple and constructive, and the solutions have the definite decompactification limit by construction. Since our construction of the solutions is so simple, we can obtain solutions on T^2, even for sigma models with other target spaces, in the same way. Moreover, utilizing our construction, we expect that we can find solutions with nontrivial holonomy parameters on compact spaces. Such solutions on $\mathbb{R} \times S^1$ have been investigated in Ref. [9]. When a solution has nontrivial holonomy along the compact spaces, one expects that it has fractional topological charges. This fact can be seen also in the gauge theory instantons in four dimensions. It was discussed that gauge theories in a box (hypertorus) admit instantons with a fractional Pontryagin number when the twisted boundary conditions are imposed [5,26]. These instantons have constituents in their inner parts. For example, the constituents of doubly periodic

instantons in $SU(2)$ Yang-Mills theories are discussed in Ref. [27], and instantons with fractional charges are studied in Ref. [28].

Finally, let us comment on the applications of our construction in the other contexts. The two-dimensional supersymmetric sigma models are considered as the effective action of a vortex in supersymmetric gauge theories. Therefore, the lumps on the compact spaces are interpreted as four-dimensional gauge theory instantons inside the vortex wrapping the compact spaces. We will explore this possibility in the future works. Time evolutions of

the solutions on T^2 would be also interesting topics. Although the generalization of our construction to $\mathbb{C}P^{N-1}$ with $N \geq 3$ is straightforward, its dynamics would be different compared with the $N = 2$ case as in the case of $\mathbb{R}^2$ [17]. The other time-dependent solutions, for example the Q-lumps [29] on the torus, can be constructed in the same way. Lumps with fractional topological charges and their constituents in gauged sigma models [30] and the other context [31] have been studied. It would be interesting to investigate these kinds of fractional lumps in the sigma models on a torus with twisted boundary conditions.

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[1] B. J. Harrington and H. K. Shepard, Phys. Rev. D **17**, 2122 (1978).

[2] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Y. S. Tyupkin, Phys. Lett. **59B**, 85 (1975).

[3] E. Witten, Phys. Rev. Lett. **38**, 121 (1977).

[4] W. Nahm, Lect. Notes Phys. **201**, 189 (1984).

[5] P. van Baal, Commun. Math. Phys. **85**, 529 (1982).

[6] M. Jardim, Commun. Math. Phys. **216**, 1 (2001).

[7] R. Leese, Nucl. Phys. **B344**, 33 (1990).

[8] R. S. Ward, Phys. Lett. **158B**, 424 (1985).

[9] F. Bruckmann, Phys. Rev. Lett. **100**, 051602 (2008).

[10] W. Brendel, F. Bruckmann, L. Janssen, A. Wipf, and C. Wozar, Phys. Lett. B **676**, 116 (2009).

[11] B. Collie and D. Tong, J. High Energy Phys. 08 (2009) 006.

[12] K.-M. Lee and C. Lu, Phys. Rev. D **57**, 5260 (1998); T. C. Kraan and P. van Baal, Phys. Lett. B **435**, 389 (1998); Nucl. Phys. **B533**, 627 (1998); Phys. Lett. B **428**, 268 (1998); F. Bruckmann, D. Nogradi, and P. van Baal, Nucl. Phys. **B666**, 197 (2003); **B698**, 233 (2004); D. Harland, J. Math. Phys. (N.Y.) **48**, 082905 (2007); A. Nakamula and J. Sakaguchi, J. Math. Phys. (N.Y.) **51**, 043503 (2010).

[13] K. Higashijima and M. Nitta, Prog. Theor. Phys. **103**, 635 (2000).

[14] J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton University, Princeton, NJ, 1992).

[15] A. Hanany and D. Tong, J. High Energy Phys. 07 (2003) 037.

[16] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Phys. Rev. D **72**, 025011 (2005).

[17] N. S. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, England, 2004).

[18] E. Mottola and A. Wipf, Phys. Rev. D **39**, 588 (1989).

[19] P. M. Sutcliffe, Nonlinearity **8**, 411 (1995).

[20] K. Knopp, *Theory of Functions* (Dover, New York, 1947).

[21] J. Eells and J. C. Wood, Topology **15**, 263 (1976).

[22] J. M. Speight, Commun. Math. Phys. **194**, 513 (1998).

[23] R. J. Cova and W. J. Zakrzewski, Nonlinearity **10**, 1305 (1997); R. J. Cova, Eur. Phys. J. B **15**, 673 (2000).

[24] J.-L. Richard and A. Rouet, Nucl. Phys. **B211**, 447 (1983).

[25] M. Aguado, M. Asorey, and A. Wipf, Ann. Phys. (Leipzig) **298**, 2 (2002).

[26] G. 't Hooft, Commun. Math. Phys. **81**, 267 (1981).

[27] C. Ford and J. M. Pawlowski, Phys. Lett. B **540**, 153 (2002); Phys. Rev. D **69**, 065006 (2004).

[28] A. Montero, J. High Energy Phys. 05 (2000) 022.

[29] R. Leese, Nucl. Phys. **B366**, 283 (1991).

[30] M. Nitta and W. Vinci, J. Phys. A **45**, 175401 (2012).

[31] M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, T. Nagashima, M. Nitta, K. Ohashi, and W. Vinci, Phys. Rev. D **80**, 045018 (2009).