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4d index to 3d index and 2d topological quantum field theory

Francesco Benini,[1] Tatsuma Nishioka,[1] and Masahito Yamazaki[2]
[1]*Department of Physics, Princeton University, Princeton, New Jersey 08544, USA*
[2]*Princeton Center for Theoretical Science, Princeton University, New Jersey 08544, USA*
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We compute the 4d superconformal index for $\mathcal{N} = 1, 2$ gauge theories on $S^1 \times L(p, 1)$, where $L(p, 1)$ is a lens space. We find that the 4d $\mathcal{N} = 1, 2$ index on $S^1 \times L(p, 1)$ reduces to a 3d $\mathcal{N} = 2, 4$ index on $S^1 \times S^2$ in the large p limit, and to a 3d partition function on a squashed $L(p, 1)$ when the size of the temporal S^1 shrinks to zero. As an application of our index, we study 4d $\mathcal{N} = 2$ superconformal field theories arising from the 6d $\mathcal{N} = (2, 0)$ A_1 theory on a punctured Riemann surface Σ, and conjecture the existence of a 2d topological quantum field theory on Σ whose correlation function coincides with the 4d $\mathcal{N} = 2$ index on $S^1 \times L(p, 1)$.

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I. INTRODUCTION

One of the beauties of supersymmetric gauge theories is that they are often amenable to exact analysis. Recent studies have uncovered powerful techniques (mostly based on localization) to extract exact results for 3d and 4d supersymmetric gauge theories, including the 4d $\mathcal{N} \geq 1$ superconformal index on $S^1 \times S^3$ [1,2], the 4d $\mathcal{N} \geq 2$ partition function on S^4 [3], the 3d $\mathcal{N} \geq 2$ partition function on S^3 [4–8] and the 3d $\mathcal{N} \geq 2$ index on $S^1 \times S^2$ [9–11].

Given the richness of the subject, a natural question is whether there are precise relations among different quantities. One such relation has been noticed by [12–14] (see also [15]), which shows that a 4d index on $S^1 \times S^3$ reduces to a 3d partition function on S^3 when the radius of the temporal S^1 goes to zero. We will present yet another connection between 4d and 3d quantities.

In this paper we study the superconformal index of 4d $\mathcal{N} = 1, 2$ superconformal field theories (SCFTs) on $S^1 \times L(p, 1)$, and obtain explicit expressions for them.[1] This is the first result of our paper, see Sec. II and, in particular, the expressions in (9)–(17) and (25)–(29) for the result and the Appendix for the derivation. Here $L(p, q)$, where p, q are coprime integers, is the lens space defined as the orbifold of S^3: $\{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}$ under the identification

$$(z_1, z_2) \sim (e^{2\pi i q/p} z_1, e^{-2\pi i/p} z_2), \tag{1}$$

where $SU(2)_1$ acts on (z_1, z_2) as a doublet (see Sec. II for our notation). Without loss of generality one can assume $0 < p$ and $0 < q \leq p - 1$. As for fermions, we choose the orbifold action such that the supercharges $\bar{\mathcal{Q}}_{I\dot{\alpha}}$ are preserved, while $\mathcal{Q}^I_\alpha$ are broken. Note that this action has no fixed points, and the manifold $L(p, q)$ is still smooth. In this paper we consider the case $q = 1$: $L(p, 1)$ is the orbifold $S^3/\mathbb{Z}_p$, where $\mathbb{Z}_p$ acts on the S^1 fiber of the Hopf fibration. Equivalently, the $\mathbb{Z}_p$ action is embedded into $U(1)_1 \subset SU(2)_1$.

Our $S^1 \times L(p, 1)$ index in itself will serve as a useful tool to quantitatively study the strongly coupled IR fixed points. For example our index could be used for checks of 4d $\mathcal{N} = 1$ Seiberg dualities. Mathematically, such a duality is expressed as an identity involving an integral of a generalization of the elliptic gamma function.[2]

The second result is about the compactification of the 4d theories. When 4d $\mathcal{N} = 1, 2$ SCFTs are compactified on S^1, they flow in the IR to 3d $\mathcal{N} = 2, 4$ SCFTs. In our setup we have two circles: one circle (denoted by S^1_T) is the temporal S^1, and another (denoted by S^1_H) is the S^1 of the Hopf fibration. Depending on the choice of S^1, we can study two limits of our index.

In the limit $S^1_H \to 0$, i.e. $p \to \infty$, the lens space $L(p, 1)$ reduces to the two-sphere and we show that the 4d index reduces to the 3d index on $S^1_T \times S^2$ (Sec. III):

$$\mathcal{I}^{\,4\text{d}}[S^1_T \times L(p, 1)] \quad \overset{p\to\infty}{\longrightarrow} \quad \mathcal{I}^{3\text{d}}[S^1_T \times S^2]. \tag{2}$$

In this limit the holonomies of the gauge field along S^1_H in the 4d theory are mapped to the monopole charges in the 3d theory. On the other hand in the limit $S^1_T \to 0$ the temporal circle shrinks to zero and the 4d index reduces to the 3d partition function on $L(p, 1)$ (Sec. IV):

$$\mathcal{I}^{\,4\text{d}}[S^1_T \times L(p, 1)] \to Z^{3\text{d}}[L(p, 1)] \quad \text{when } S^1_T \to 0. \tag{3}$$

The third result is about an application of our index (Sec. V). When the 4d theory arises from the 6d (2, 0) theory on a punctured Riemann surface Σ, we conjecture the existence of a 2d topological quantum field theory (TQFT) on Σ whose correlation function coincides with the 4d index on $S^1 \times L(p, 1)$, generalizing a similar claim of [18,19]. We summarize the relations between the 4d

[1]The index of $\mathcal{N} = 4$ super-Yang-Mills on $S^1 \times S^3/\mathbb{Z}_p$ was studied in [16,17].

[2]For the generalization of the elliptic gamma function see the infinite product form in (47) or (49), while for its hyperbolic version see (52).

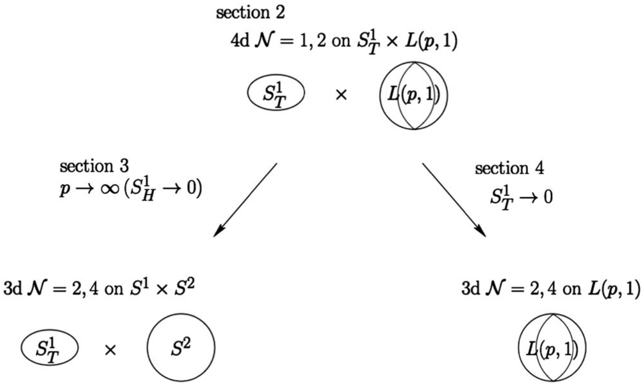

FIG. 1. A schematic summary of the relations obtained in Secs. II, III, and IV.

index and the 3d quantities we will obtain in this paper in Fig. 1.

II. 4D INDEX ON $S^1 \times L(p,1)$

In this section we will present our expression for the 4d $\mathcal{N}=1$, 2 superconformal indices[3] on $S^1 \times L(p,1)$. The derivation of these results is given in the Appendix.

Consider a 4d $\mathcal{N}=1$ ($\mathcal{N}=2$) superconformal field theory on $S^1 \times S^3$. Its superconformal algebra is given by $SU(2,2|1)$ [$SU(2,2|2)$]. We let the R-symmetry index and the supercharges be $I=1$ ($I=1,2$) and $\mathcal{Q}^I_\alpha$, $\bar{\mathcal{Q}}_{I\dot\alpha}$, $\mathcal{S}_{I\alpha}$, $\bar{\mathcal{S}}^I_{\dot\alpha}$, respectively. Here $\alpha=\pm$ ($\dot\alpha=\pm$) is the index for the $SU(2)_1$ [$SU(2)_2$] spin of the $SO(4)\simeq SU(2)_1\times SU(2)_2$ rotational symmetry of the three-sphere.

The orbifold theory has a set of degenerate vacua, labeled by a nontrivial holonomy V along the S^1_H direction, since $\pi_1(L(p,1))=\mathbb{Z}_p$. The holonomy V satisfies $V^p=1$ and can be mapped to an element of the maximal torus through conjugation by an element of the gauge group

$$V=(\omega^0 \mathbb{1}_{N_0},\cdots,\omega^{p-1}\mathbb{1}_{N_{p-1}}), \tag{4}$$

where $\omega=e^{2\pi i/p}$ and the integers N_I satisfy the relation $\sum_{I=0}^{p-1} N_I = N$ with N the rank of the gauge group. Another useful parametrization is given by $m_1,\ldots,m_N$, which is defined as

$$(m_i)=(\underbrace{0,\cdots,0}_{N_0},\cdots,\underbrace{p-1,\cdots,p-1}_{N_{p-1}}), \qquad N_I=(\# m_i=I), \tag{5}$$

where $i=1,\ldots,N$ and $I=0,\cdots,p-1$. In this notation the ith holonomy is given by ω^{m_i}. The holonomy breaks the gauge group into a product of p subgroups

$$G\to\prod_{I=0}^{p-1} G_I, \tag{6}$$

where the rank of G_I is given by N_I. For example, in case of a $U(N)$ gauge group we have $U(N)\to\prod_{I=0}^{p-1}U(N_I)$.

A. $\mathcal{N}=2$ index

Let us begin with the $\mathcal{N}=2$ index. We will comment on the $\mathcal{N}=1$ index later.

We define the index with respect to the supercharge $\mathcal{Q}\equiv\bar{\mathcal{Q}}_{2+}$ that survives the orbifold projection. This is given by [1,2]

$$\mathcal{I}=\mathrm{Tr}(-1)^{\mathcal{F}}e^{-\tilde\beta\Xi}t^{2(E+j_2)}y^{2j_1}v^{-(r+R)}z^F, \tag{7}$$

where $\mathcal{F}$ is the fermion number, the trace is taken over the states of the theory on S^3, and the quantum numbers of the R symmetries $U(1)_R\subset SU(2)_R$ and $U(1)_r$ are denoted by (R,r). The Ξ is the commutator of $\mathcal{Q}$ with its conjugate,

$$\Xi\equiv 2\{\mathcal{Q},\mathcal{Q}^\dagger\}=E-2j_2-2R+r, \tag{8}$$

and the index is independent of $\tilde\beta$. Therefore, only the states obeying $\Xi=0$ contribute to the index. The expression z^F is a shorthand for $\prod_j z_j^{F_j}$, where F_j are charges with respect to flavor symmetries (commuting with $\mathcal{Q}$, $\mathcal{Q}^\dagger$) and z_j are their chemical potentials. The operators $E+j_2$, j_1, $R+r$ and F_i appearing in (7) are the maximal set of operators commuting with $\mathcal{Q}$ and $\mathcal{Q}^\dagger$.

In the path-integral formulation, our 4d index for an $\mathcal{N}=2$ theory on $S^1\times L(p,1)$ can be written as

$$\mathcal{I}_p(t,y,v,z)=\sum_m \mathcal{I}^0_{p,m}(t,y,v,z)\int[da] \times\exp\Big[\sum_{n=1}^\infty\frac{1}{n}\hat{\mathcal{I}}_{p,m}(t^n,y^n,v^n,z^n;e^{ina})\Big]. \tag{9}$$

Here, the index consists of a sum of indices labeled by the set of holonomies $m\equiv\{m_i\}$, with $0\le m_1\le\cdots\le m_N\le p-1$. The measure $[da]$ is given by

$$[da]=\frac{1}{\prod_I|\mathcal{W}_I|}\prod_{i=1}^N\frac{da_i}{2\pi}\prod_{\substack{\alpha\in G\\ \alpha(m)=0}}2\sin\frac{\alpha(a)}{2}, \tag{10}$$

where $\mathcal{W}_I$ is the Weyl group of G_I and the last product is over the roots of the unbroken gauge group. This is the Haar measure of the unbroken gauge group $\prod_I G_I$.

The function $\hat{\mathcal{I}}_{p,m}$ is the single-letter contribution to the index, and is obtained by summing over all the fields Φ contributing to the index:

$$\hat{\mathcal{I}}_{p,m}(t,y,v,z;e^{ia})=\sum_\Phi\hat{\mathcal{I}}^\Phi_{p,m}(t,y,v,z;e^{ia}). \tag{11}$$

For a vector multiplet we find (see the Appendix)

[3]Despite the name, we can define this index for 4d $\mathcal{N}=1$, 2 theories which are nonconformal in the UV.

$$\hat{\mathcal{I}}^{\mathcal{N}=2\text{ vector}}_{p,m}(t,y,v,z;e^{ia}) = \sum_{\rho\in\text{Adj}}[(t^2v - t^4v^{-1} + t^6 - 1) \times F_p(t,y;[\![\rho(m)]\!]) + \delta_{[\![\rho(m)]\!],0}]e^{i\rho(a)}, \tag{12}$$

and for a half-hypermultiplet in a representation $\mathcal{R}$ and with flavor charges F

$$\hat{\mathcal{I}}^{\mathcal{N}=2\text{ half-hyper}}_{p,m}(t,y,v,z;e^{ia}) = \sum_{\rho\in\mathcal{R}}(t^2v^{-1/2}z^F - t^4v^{1/2}z^{-F}) \times F_p(t,y;[\![\rho(m)]\!])e^{i\rho(a)}, \tag{13}$$

where the summations are over the weights of the adjoint representation and of the representation $\mathcal{R}$ of G respectively.[4] Here the function $F_p(t,y;L)$ is defined by

$$F_p(t,y;L) = \frac{1}{1-t^6}\left(\frac{t^{3L}y^L}{1-t^{3p}y^p} + \frac{t^{3(p-L)}y^{-(p-L)}}{1-t^{3p}y^{-p}}\right), \tag{14}$$

and we use the notation

$$[\![x]\!] = \{\text{an integer } y \text{ such that } 0 \le y < p \quad \text{and} \quad y \equiv xp (\text{mod } p)\}. \tag{15}$$

Finally $\mathcal{I}^0_{p,m}(t,y,v,z)$ in front of the integral is the contribution from zero-point oscillations of the fields, which depends on the matter content of the theory. We have

$$\mathcal{I}^0_{p,m}(t,y,v,z) = \exp\Bigg[\beta\sum_{\alpha\in G}\frac{1+\mu}{2p}(p[\![\alpha(m)]\!] - [\![\alpha(m)]\!]^2) - \beta\sum_{\Phi:\text{ half-hyper}}\sum_{\rho\in\mathcal{R}_\Phi}\frac{1+\mu-2F_\Phi\nu}{4p} \times (p[\![\rho(m)]\!] - [\![\rho(m)]\!]^2)\Bigg], \tag{16}$$

where we introduced the notation

$$t = e^{-\frac{\beta}{2}}, \quad y = e^{-\beta\Omega_1}, \quad v = e^{-\beta\mu} \quad z_j = e^{-\beta\nu_j}. \tag{17}$$

Note that this vanishes when the holonomy is trivial, $m = 0$, so that in the first term we only considered the sum over the roots of G. Moreover, since $[\![-x]\!] = p - [\![x]\!]$, we have $p[\![x]\!] - [\![x]\!]^2 = p[\![-x]\!] - [\![-x]\!]^2$.

For concreteness, let us specialize to a $U(N)$ gauge theory with vector multiplets and trifundamental hypermultiplets. This is the theory we will discuss in Sec. V. We also set the flavor chemical potential to zero, $z = 1$. Then the measure $[da]$ becomes

$$[da] = \frac{1}{\prod_I N_I!}\prod_{i=1}^N\frac{da_i}{2\pi}\prod_{\substack{i,j\\ m_i=m_j}}2\sin\frac{a_i-a_j}{2}, \tag{18}$$

which coincides with the product of Haar measures $\prod_I[dU_I]$ with $N_I \times N_I$ unitary matrices U_I, whose eigenvalues are denoted by e^{ia_i}. We have

$$\hat{\mathcal{I}}^{\mathcal{N}=2\text{ vector}}_{p,m}(t,y,v;e^{ia}) = \sum_{i,j=1}^N f_p([\![m_i - m_j]\!])e^{i(a_i-a_j)} = \sum_{I,J=0}^{p-1} f_p([\![I-J]\!])\text{Tr}(U_I)\text{Tr}(U_J^\dagger) \tag{19}$$

for a vector multiplet, and

$$\hat{\mathcal{I}}^{\mathcal{N}=2\text{ trifund}}_{p,m}(t,y,v;e^{ia}) = \sum_{i,j,k=1}^N g_p([\![m_i+m_j+m_k]\!]) \times e^{i(a_i+a_j+a_k)} = \sum_{I,J,K=0}^{p-1} g_p([\![I+J+K]\!]) \times \text{Tr}(U_I)\text{Tr}(U_J)\text{Tr}(U_K) \tag{20}$$

for a half-hypermultiplet in the trifundamental representation. Here we defined

$$f_p(L) = (t^2v - t^4v^{-1} + t^6 - 1)F_p(t,y;L) + \delta_{L,0},$$
$$g_p(L) = (t^2v^{-\frac{1}{2}} - t^4v^{\frac{1}{2}})F_p(t,y;L). \tag{21}$$

B. $\mathcal{N} = 1$ index

Let us repeat the discussion for the 4d $\mathcal{N} = 1$ index given by

$$\mathcal{I}(t,y,z) = \text{Tr}(-1)^{\mathcal{F}}t^{2(E+j_2)}y^{2j_1}z^F, \tag{22}$$

where the trace is taken over all the fields satisfying

$$\{\mathcal{Q},\mathcal{Q}^\dagger\} = E - 2j_2 - \frac{3}{2}\tilde{r} = 0, \tag{23}$$

and $\tilde{r}$ is the R symmetry of the $\mathcal{N} = 1$ superalgebra. When we regard the $\mathcal{N} = 2$ SCFT as the $\mathcal{N} = 1$ SCFT, the R symmetries R, r of the $\mathcal{N} = 2$ supersymmetry recombine into the $\mathcal{N} = 1$ R symmetry $\tilde{r}$ and a flavor symmetry A commuting with $\mathcal{Q}$. By comparing the definitions of the indices [compare (7) and (8) with (22) and (23), and see also [20]] we obtain

$$\tilde{r} = \frac{4R-2r}{3}, \qquad A = -R - r. \tag{24}$$

The same derivation as in the Appendix works for the $\mathcal{N} = 1$ theory, but there are some important differences. First, we do not have a chemical potential v for the R

[4] For an adjoint representation, this sum is over the roots of G as well as vanishing weights.

symmetry. Second, there is a zero-point contribution $e^{iB^0_{p,m}(a)}$ to the measure of the theory. We have

$$B^0_{p,m}(a) = -\sum_{\Phi:\ \text{chiral}} \sum_{\rho\in\mathcal{R}_\Phi} \frac{\rho(a)}{2p}(p[\![\rho(m)]\!] - [\![\rho(m)]\!]^2). \tag{25}$$

This correction is absent for a vectorlike theory, including the $\mathcal{N}=2$ theories previously discussed. The index is given by

$$\mathcal{I}_p(t,y,z) = \sum_m \mathcal{I}^0_{p,m}(t,y,z)\int[da]e^{iB^0_{p,m}(a)} \times \exp\Big[\sum_{n=1}^\infty \frac{1}{n}\hat{\mathcal{I}}_{p,m}(t^n,y^n,z^n;e^{ina})\Big]. \tag{26}$$

The single-letter index $\hat{\mathcal{I}}$ for an $\mathcal{N}=1$ vector multiplet is

$$\hat{\mathcal{I}}^{\mathcal{N}=1\text{ vector}}_{p,m} = \sum_{\rho\in\text{Adj}}((t^6-1)F_p(t,y;[\![\rho(m)]\!]) + \delta_{[\![\rho(m)]\!],0})e^{i\rho(a)}, \tag{27}$$

where we used the same function (14). This is essentially the half of (12) corresponding to an $\mathcal{N}=1$ vector multiplet. For an $\mathcal{N}=1$ chiral multiplet with flavor charges F we have

$$\hat{\mathcal{I}}^{\mathcal{N}=1\text{ chiral}}_{p,m} = \sum_{\rho\in\mathcal{R}}(t^{3Q}z^F e^{i\rho(a)} - t^{6-3Q}z^{-F}e^{-i\rho(a)}) \times F_p(t,y;[\![\rho(m)]\!]). \tag{28}$$

In this expression we have included an anomalous R charge Q (cf. [21]). In many $\mathcal{N}=1$ examples, the theory in the UV is not conformal but flows to a conformal fixed point in the IR. In these situations, the IR R symmetry is a mixture of the UV R symmetry $\tilde{r}$ and flavor symmetries, and we need to discuss nontrivial anomalous dimensions. This effect can be incorporated by shifting the flavor chemical potential z^F by $t^{3(Q-2/3)} = t^{2(3Q/2-1)}$, where $3Q/2$ is the anomalous dimension and the factor 2 comes from the definition of the index [see (22)].

The total zero-point contribution $\mathcal{I}^0_{p,m}e^{iB^0_{p,m}(a)}$ from vector and chiral multiplets is

$$\mathcal{I}^0_{p,m}e^{iB^0_{p,m}(a)} = \exp\Big[\frac{3\beta}{4p}\sum_{\alpha\in G}(p[\![\alpha(m)]\!] - [\![\alpha(m)]\!]^2) - \sum_{\Phi:\ \text{chiral}}\sum_{\rho\in\mathcal{R}_\Phi}\frac{\beta(3-3Q-2F_\Phi\nu)+2i\rho(a)}{4p} \times(p[\![\rho(m)]\!] - [\![\rho(m)]\!]^2)\Big], \tag{29}$$

where the contribution of vanishing weights of the adjoint representation drops out and $\alpha\in G$ now represents the sum over roots of the gauge group.

C. Refined index

We can construct a refined 4d orbifold index which depends on holonomies for the flavor symmetries, besides the chemical potentials. To construct it, both in $\mathcal{N}=1$ and $\mathcal{N}=2$ cases, we first define the flavor chemical potentials in an alternative equivalent way (therefore setting $z=1$ in the previous expressions): we introduce external vector fields for all flavor symmetries. Let $\tilde{G} = G\times H$ be the extended symmetry group, of which G is gauged and H is external. We do not integrate over the external vector fields in (9) and (29), nor introduce a single-letter contribution for them as opposed to (12) and (27). However in the half-hyper- (13) and the chiral multiplets (28) single-letter index, as well in the zero-point energies (16) and (29), we sum over weights of the representation $\mathcal{R}$ under the full symmetry group $\tilde{G}$.

We can introduce holonomies e^{ia_α} of the external vector fields along the temporal direction S^1_T: up to conjugation, they are parametrized by parameters $\{a_\alpha\}$ in the maximal torus of H. After complexification of the cotangent bundle of the maximal torus, we can identify $e^{ia_\alpha} = z_\alpha$ with the flavor chemical potentials.

For $p>1$ we can also introduce flavor holonomies $e^{2\pi i m_\alpha/p}$, mutually commuting with the temporal holonomies, along S^1_H inside $L(p,1)$. The integer parameters $\{m_\alpha\}$, with $0\le m_\alpha < p$, provide a refined version of the index:

$$\mathcal{I}_p(t,y,v;a_\alpha,m_\alpha). \tag{30}$$

Note that the flavor holonomies break the flavor group as $H\to\prod_I H_I$, and enter both in the single-letter indices and in the zero-point energy. On the other hand as we do not integrate over temporal flavor holonomies, we do not sum over flavor holonomies.

The refined index is useful if we want to compute the index of a theory obtained by gauging together two theories $\mathcal{T}_1$ and $\mathcal{T}_2$ along a common flavor symmetry factor H' (see Sec. V):

$$\mathcal{I}_p(t,y,v;a,m,c,s) = \sum_r \mathcal{I}^{0,\text{vector}H'}_{p,r}\int[db]\exp\Big[\sum_n\frac{1}{n}\hat{\mathcal{I}}^{\text{vector}H'}_{p,r}(t^n,y^n,v^n;e^{inb})\Big] \times \mathcal{I}^{\mathcal{T}_1}_p(t,y,v;a,m,b,r)\mathcal{I}^{\mathcal{T}_2}_p(t,y,v;b,r,c,s). \tag{31}$$

Here a, b, c are flavor chemical potentials, m, r, s are flavor holonomies, (b,r) refer to the common flavor symmetry H', and the inserted functions are the zero-point energy and single-letter contribution of the gauge fields along H'.

In the following sections we discuss the reduction of the 4d orbifold index to the 3d partition function and the 3d index. Correspondingly, there are refinements of the 3d index and 3d partition functions. The former is the generalized superconformal index of [22].

III. RELATION TO THE 3D INDEX ON $S^1 \times S^2$

In this section we show explicitly that the $p \to \infty$ limit of the 4d $\mathcal{N} = 1$ index on $S^1 \times L(p, 1) = S^1 \times S^3/\mathbb{Z}_p$ gives the 3d $\mathcal{N} = 2$ index on $S^1 \times S^2$. A parallel analysis shows that in the same limit the 4d $\mathcal{N} = 2$ index on $S^1 \times L(p, 1)$ reduces to the 3d $\mathcal{N} = 4$ index on $S^1 \times S^2$. This result can be regarded as yet another derivation of the 3d index, including the nontrivial monopole charges. This approach does not require more intricate information such as the choice of clever localization terms and supersymmetry transformation on curved backgrounds. Notice that the present method can be applied to 3d theories coming from dimensional reduction from the 4d parents, and cannot be used for theories with Chern-Simons terms.

Let us take the limit $p \to \infty$. The circle S^1_H shrinks to zero size in this limit and thus the chemical potential y along the direction goes to 1. The expression in the parentheses appearing in both (27) and (28) gets finite contributions in the limit from either $[\![\rho(m)]\!] \sim 0$ or $[\![\rho(m)]\!] \sim p$:

$$\hat{\mathcal{I}}^{\mathcal{N}=1\text{vector}}_{p,m} \to \sum_{\alpha\in G} (-t^{3|\alpha(m)|} + \delta_{\alpha(m),0}) e^{i\alpha(a)},$$
$$\hat{\mathcal{I}}^{\mathcal{N}=1\text{chiral}}_{p,m} \to \sum_{\rho\in\mathcal{R}} \frac{t^{3Q} z^F e^{i\rho(a)} - t^{6-3Q} z^{-F} e^{-i\rho(a)}}{1-t^6} t^{3|\rho(m)|}. \tag{32}$$

Note that the roots with $\alpha(m) = 0$ do not contribute to the vector single-letter index. In this limit, the total zero-point contribution $\mathcal{I}^0_{p,m} e^{iB^0_{p,m}(a)}$ becomes

$$\exp\left[\frac{3\beta}{4}\sum_{\alpha\in G}|\alpha(m)| - \sum_{\Phi\text{chiral}}\sum_{\rho\in\mathcal{R}_\Phi} \frac{\beta(3-3Q-2F_\Phi\nu) + 2i\rho(a)}{4}|\rho(m)|\right] = t^{3\epsilon_0} z^{q_0} e^{ib_0(a)}, \tag{33}$$

with

$$\epsilon_0 = -\frac{1}{2}\sum_{\alpha\in G}|\alpha(m)| - \frac{1}{2}\sum_{\Phi}\sum_{\rho\in\mathcal{R}_\Phi}(Q-1)|\rho(m)|,$$
$$q_{0,i} = -\frac{1}{2}\sum_{\Phi} F_i(\Phi)\sum_{\rho\in\mathcal{R}_\Phi}|\rho(m)|,$$
$$b_0(a) = -\frac{1}{2}\sum_{\Phi}\sum_{\rho\in R_\Phi}\rho(a)|\rho(m)|. \tag{34}$$

In summary, the $p \to \infty$ limit of the 4d index gives rise to

$$\mathcal{I} = \sum_m t^{3\epsilon_0} z^{q_0} \int [da] e^{ib_0(a)} \exp\left\{\sum_{n=1}^\infty \frac{1}{n}\hat{\mathcal{I}}_{p,m}(\cdot^n)\right\}, \tag{35}$$

where $\hat{\mathcal{I}}$ is given by the sum of (32). This result coincides with the formula for the 3d index given in [11],[5,6] provided that we identify $x = t^3$. Here x appears in the definition of the 3d index

$$\mathcal{I} = \mathrm{Tr}[(-1)^{\mathcal{F}} x^{(E+j)} z^F], \tag{39}$$

and the trace is taken over operators satisfying

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = E - j - \tilde{r} = 0. \tag{40}$$

Comparing (22), (23), (39), and (40), we see that $t^{2(E+j_2)} = t^{6j_2+3\tilde{r}}$ should be identified with $x^{E+j} = x^{2j+\tilde{r}}$. This explains the parameter identification $x = t^3$.

IV. RELATION TO THE 3D PARTITION FUNCTION ON $L(p, 1)$

In this section we consider the 4d $\mathcal{N} = 1,2$ index on $S^1_T \times L(p, 1)$ and show that in the limit $S^1_T \to 0$ it reproduces the 3d partition function of the dimensionally reduced 3d $\mathcal{N} = 2,4$ theory on $L(p, 1)$.[7] This is to be expected since when the circle shrinks the nontrivial modes along the circle become infinitely massive and decouple from the spectrum, leaving only the constant modes along S^1_T. Indeed, the Lagrangians of the 4d and 3d theories are the same up to terms irrelevant for the localization [14], and at the level of the one-loop determinant the $S^1_T \to 0$ limit is realized as [see (A7)]

[5]With respect to the expression in [11], we have $S^{(0)}_{\text{CS}} = 0$ because our 3d theories arise from the dimensional reduction of 4d theories and do not have a Chern-Simons term in the Lagrangian.

[6]We could make use of the identity

$$\prod_{\alpha\in G,\alpha(m)=0} 2i\sin\frac{\alpha(a)}{2} = \exp\left(\sum_{n=1}^\infty \frac{1}{n} g(e^{ina})\right)$$
$$\text{with } g(e^{ia}) = -\sum_{\alpha\in G,\alpha(m)=0} e^{i\alpha(a)} \tag{36}$$

to rewrite the Haar measure $[da]$ in terms of the flat measure $\widetilde{[da]}$:

$$\widetilde{[da]} = \frac{1}{\prod_I n_I!}\prod_{i=1}^N \frac{da_i}{2\pi}, \tag{37}$$

reabsorbing the extra factor into the vector multiplet single-letter index. In this case we have

$$\hat{\mathcal{I}}^{\text{vector,flat}}_{p,m} = -\sum_{\alpha\in G} x^{|\alpha(m)|} e^{i\alpha(a)}, \tag{38}$$

which is the expression found in [11].

[7]The same problem for $p = 1$ was analyzed in [12–14]. See also [23,24] for the 3d partition function for a pure gauge theory without matters on lens spaces and more generally on Seifert manifolds. Our 3d partition function on a lens space includes the coupling with matter.

$$\prod_E \prod_{n=-\infty}^{\infty}\left(\frac{2\pi i n}{\beta}+E\right)\to\prod_E E, \tag{41}$$

where we regularized the divergent constant $\prod_{n\neq 0}(\frac{2\pi i n}{\beta})$. The right-hand side is precisely the one-loop determinant of the 3d theory.

A. 3d partition function on $L(p, 1)$

Let us first present the 3d partition function on the lens space for general 3d $\mathcal{N}=2$ theories, in the absence of Chern-Simons terms. This in itself is a useful result regardless of the reduction from the 4d index. The answer can be obtained by generalizing the localization procedure of [4,7,8]. The reduction from the 4d index provides another derivation of this result.

The partition function takes the following form:

$$Z_{\rm 3d}[L(p,1)]=\sum_m\int[da]_{\rm 3d}Z^{\rm vector}_{\rm 1\text{-}loop}[a,m]Z^{\rm chiral}_{\rm 1\text{-}loop}[a,m], \tag{42}$$

where $[da]_{\rm 3d}$ is a Vandermonde measure of the residual gauge symmetry

$$[da]_{\rm 3d}=\frac{1}{\prod_I N_I!}\prod_{i=1}^N da_i\prod_{\alpha\in G\,\alpha(m)=0}\alpha(a), \tag{43}$$

and $Z^{\rm vector}_{\rm 1\text{-}loop}$ and $Z^{\rm chiral}_{\rm 1\text{-}loop}$ are the one-loop determinants of the gauge and matter sectors. For a vector multiplet,

$$Z^{\rm vector}_{\rm 1\text{-}loop}[a,m]=\prod_{\alpha>0}\frac{\sinh[\frac{\pi}{p}(\alpha(a)+i\alpha(m))]\sinh[\frac{\pi}{p}(\alpha(a)-i\alpha(m))]}{(\alpha(a))^{2\delta_{\alpha(m),0}}}, \tag{44}$$

whose denominator cancels the Vandermonde measure (43).

For a chiral multiplet with an anomalous R charge Q, the one-loop determinant becomes

$$Z^{\rm chiral}_{\rm 1\text{-}loop}[a,m]=\prod_{\rho\in\mathcal{R}}\prod_{l=0}^{\infty}\left(\frac{l+2-Q+i\rho(a)}{l+Q-i\rho(a)}\right)^{N_\rho(l)}, \tag{45}$$

where $N_\rho(l)$ is defined to be the number of half-integers $m_1\in\{-\frac{l}{2},-\frac{l}{2}+1,\ldots,\frac{l}{2}-1,\frac{l}{2}\}$ satisfying

$$2m_1=\rho(m)\pmod p. \tag{46}$$

The one-loop determinant (45) becomes trivial for the $\mathcal{N}=2$ chiral multiplet (which has $Q=1$) inside the $\mathcal{N}=4$ vector multiplet.

B. From the 4d index to the 3d partition function

Consider the reduction from the 4d $\mathcal{N}=1$ index to the 3d $\mathcal{N}=2$ partition function.

Let us start with the chiral multiplet. The orbifold index given in (A20) can be written, with the use of (A14) and (A9), as

$$I^{\mathcal{N}=1\,\rm chiral}_{p,m}=\prod_{\rho\in\mathcal{R}}\prod_{n_1,n_2\geq 0}{}^{'}\frac{1-t^{3(n_1+n_2)+6-3Q}y^{n_1-n_2}z^{-F}e^{-i\gamma\rho(a)}}{1-t^{3(n_1+n_2)+3Q}y^{n_1-n_2}z^{F}e^{i\gamma\rho(a)}}, \tag{47}$$

where we rescaled a by a factor γ for later purpose, and prime means that the product is over the non-negative integers n_1, n_2 satisfying the orbifold condition $n_1-n_2=\rho(m)p \pmod p$ in (A17).[8] The formula above provides a generalization of the elliptic gamma function. Let us define an integer l and a half-integer m_1 as

$$l=n_1+n_2,\qquad m_1=(n_1-n_2)/2; \tag{48}$$

then the orbifold condition (A17) agrees with that of the 3d partition function (46) and we can rewrite

$$I^{\mathcal{N}=1\,\rm chiral}_{p,m}=\prod_{\rho\in\mathcal{R}}\prod_{l\geq 0\,m_1\in\{-\frac{l}{2},-\frac{l}{2}+1,\ldots,\frac{l}{2}\}}{}^{'}\frac{1-t^{3l+6-3Q}y^{2m_1}z^{-F}e^{-i\gamma\rho(a)}}{1-t^{3l+3Q}y^{2m_1}z^{F}e^{i\gamma\rho(a)}}. \tag{49}$$

Now we set $t=e^{-\gamma/3}$, $y=z=1$ and take $\gamma\to 0$ limit [13].[9] The index then reduces exactly to the 3d partition function of the chiral multiplet (45).

In the same way we can write the 4d index of the vector multiplet (A19) as

$$I^{\mathcal{N}=1\,\rm vector}_{p,m}=\prod_{\alpha\in G}\left[\frac{1}{(1-e^{i\gamma\alpha(a)})^{\delta_{\alpha(m),0}}}\times\prod_{l,m_1}{}^{'}\frac{1-t^{3l}y^{2m_1}e^{-i\gamma\alpha(a)}}{1-t^{3l+6}y^{2m_1}e^{i\gamma\alpha(a)}}\right]. \tag{50}$$

[8] This is the infinite product in (A8), and includes the zero-point contribution.

[9] The γ here is different from β in (17) by a factor 3/2, and is the same as the β in [13].

As before, setting $y = 1$ and taking the $\gamma \to 0$ limit we obtain (up to overall constants independent of the holonomies)

$$\begin{aligned}\prod_{\alpha\in G}\Big[\frac{1}{(\alpha(a))^{\delta_{\alpha(m),0}}}\prod_{l=0}^{\infty}\Big(\frac{l+i\alpha(a)}{l+2+i\alpha(a)}\Big)^{N_\alpha(l)}\Big] &= \prod_{\alpha\in G}\Big[\frac{1}{(\alpha(a))^{\delta_{\alpha(m),0}}}\prod_{l=0}^{\infty}(l+i\alpha(a))^{N_\alpha(l)-N_\alpha(l-2)}\Big] \\ &= \prod_{\alpha\in G}\Big[\frac{1}{(\alpha(a))^{\delta_{\alpha(m),0}}}\prod_{l\geq 0 - l-\alpha(m)\in p\mathbb{Z}}(l+i\alpha(a))\prod_{l\geq 0 l-\alpha(m)\in p\mathbb{Z}}(l+i\alpha(a))\Big] \\ &= \prod_{\alpha>0}\frac{\sinh[\frac{\pi}{p}(\alpha(a)+i\alpha(m))]\sinh[\frac{\pi}{p}(\alpha(a)-i\alpha(m))]}{(\alpha(a))^{2\delta_{\alpha(m),0}}}, \end{aligned} \tag{51}$$

where we defined $N_\alpha(l) = 0$ when $l < 0$. This result coincides with the 3d partition function of the vector multiplet (44). The measure term in the 4d index (18) becomes that of the 3d partition function (43) after rescaling a by a factor of γ, and this verifies our claim of the relation between the 4d index and the 3d partition function on the lens spaces.

Finally, let us take a more general limit $t = e^{-\gamma/3}$, $y = e^{-\gamma\eta}$, $z = e^{-\gamma\nu}$ with $\gamma \to 0$ while keeping η, ν finite. The one-loop determinants then become

$$\begin{aligned} Z^{\text{vector},\eta}_{\text{1-loop}}[a,m] &= \prod_{\alpha\in G}\prod_{l,m_1}^{l}\frac{l+2m_1\eta+i\alpha(a)}{l+2+2m_1\eta-i\alpha(a)} = \prod_{\alpha\in G}\prod_{n_1,n_2\geq 0}^{l}\frac{n_1(1+\eta)+n_2(1-\eta)+i\alpha(a)}{n_1(1+\eta)+n_2(1-\eta)+2-i\alpha(a)}, \\ Z^{\text{chiral},\eta}_{\text{1-loop}}[a,m] &= \prod_{\rho\in\mathcal{R}}\prod_{l,m_1}^{l}\frac{l+2-Q-\nu F+2m_1\eta+i\rho(a)}{l+Q+\nu F+2m_1\eta-i\rho(a)} = \prod_{\rho\in\mathcal{R}}\prod_{n_1,n_2\geq 0}^{l}\frac{n_1(1+\eta)+n_2(1-\eta)+2-Q-\nu F+i\rho(a)}{n_1(1+\eta)+n_2(1-\eta)+Q+\nu F-i\rho(a)}, \end{aligned} \tag{52}$$

where we used (48). Notice that these provide generalizations of the hyperbolic hypergeometric gamma function. Moreover we see that $\nu \neq 0$ has the effect of changing the anomalous R charge Q. The remaining question is the effect of the parameter η: when $\nu = 0$, $\eta \neq 0$, the answer coincides with the 3d partition function on a squashed lens space (generalizing the result of [25]) with the squashing parameter $b = \sqrt{\frac{1+\eta}{1-\eta}}$.

V. RELATION TO 2D TQFT

Let us apply our formalism to the class of 4d $\mathcal{N} = 2$ SCFTs discovered by Gaiotto [26] (see also [27–29]): these are obtained by compactifying the 6d (2,0) A_{N-1} theory on punctured Riemann surfaces Σ. In this paper we specialize to the case $N = 2$.

To obtain a Lagrangian description, we fix a pants decomposition of the surface. This is specified by a graph, whose set of internal edges (trivalent vertices) we denote by $\mathcal{G}$ ($\mathcal{V}$). An internal edge $l \in \mathcal{G}$ corresponds to an $SU(2)$ gauge group, and a trivalent vertex $(l, m, n) \in \mathcal{V}$ corresponds to a trifundamental hypermultiplet. Since the total gauge group is $SU(2)^{|\mathcal{G}|}$, the holonomy is determined by a set of integers m^I_l.

The 4d orbifold index of this theory can be computed to be [recall (19) and (20)]

$$\mathcal{I}_p = \sum_{m_l^I}\mathcal{I}^0_{p,m}\int\prod_{l\in\mathcal{G}}\prod_I[dU^I_l]\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}\Big[\sum_{l\in\mathcal{G}}f_p(\cdot^n)\mathrm{Tr}_{\mathrm{Adj}}(U^n_l)+\sum_{(l,m,n)\in\mathcal{V}}g_p(\cdot^n)\mathrm{Tr}_{\mathrm{trifund}}(U^n_l,U^n_m,U^n_n)\Big]\Big), \tag{53}$$

where f_p and g_p are defined in (21). Let us define

$$C_{\alpha_i,\alpha_j,\alpha_k} \equiv \exp\Big(\sum_n\frac{1}{n}g_p(\cdot^n)\mathrm{Tr}_{\mathrm{trifund}}(n\alpha_l,n\alpha_m,n\alpha_n)\Big), \qquad \eta^{\alpha_k,\alpha_l} \equiv \exp\Big(\sum_{n=1}^{\infty}f_p(\cdot^n)\mathrm{Tr}_{\mathrm{Adj}}(n\alpha_i)\Big)\Delta(\alpha)^{-1}, \tag{54}$$

where $\Delta(\alpha)$ is the square root of the measure given in (10). As in [18], this trivially satisfies the axioms of TQFT, except for the associativity. We have checked the associativity by series expansion. The associativity holds not for a fixed holonomy, but after summing over the holonomies. We conjecture that the associativity holds in general; this can be regarded as a nontrivial test of the S duality of 4d $\mathcal{N} = 2$ SCFTs. It is desirable to give an analytic proof of the associativity. If this is the case, we have a 2d TQFT whose correlation function on the Riemann surface coincides with the orbifold index for the 4d $\mathcal{N} = 2$ theory characterized by the same Riemann surface. When the lens space is a three-sphere, the 2d theory is proposed to be the q-deformed Yang-Mills theory [19]. To identify the 2d counterpart of the orbifold index, it would be important to understand its relation to the Alday-Gaiotto-Tachikawa correspondence [30] between 4d $\mathcal{N} = 2$

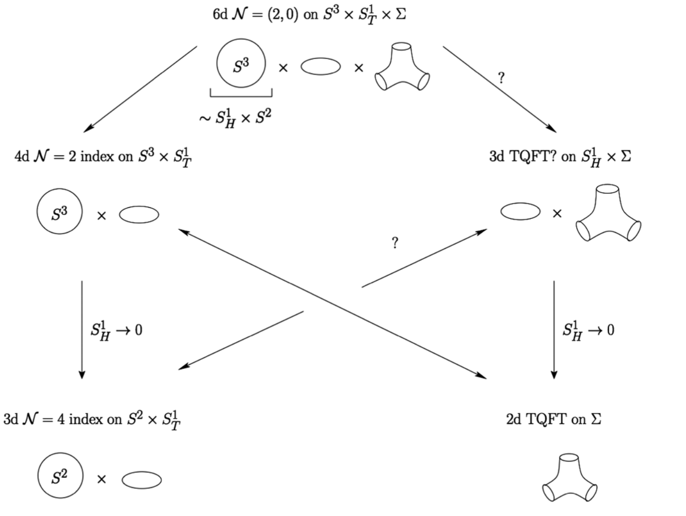

FIG. 2. The dimensional reduction from the 4d $\mathcal{N}=2$ index to the 3d $\mathcal{N}=4$ index, as discussed in this paper, should correspond to a dimensional oxidation from the 2d TQFT to a one higher dimensional theory.

partition functions on asymptotically locally Euclidean spaces and 2d Para-Liouville/Toda theories [31–35] (see also [36–38]). In a similar way, one could consider the 6d $\mathcal{N}=(2,0)$ A_1 theory compactified on a Riemann surface with $\mathcal{N}=1$ twist, giving rise to 4d $\mathcal{N}=1$ theories [39]. We expect their superconformal index and orbifold index to describe some 2d topological theory on the Riemann surface.

In the limit $p\to\infty$ the 4d theory reduces to a 3d theory (which by mirror symmetry is dual to a conventional quiver theory [40]), the 4d index reduces to the 3d index (Sec. III) and it is expected that the 2d TQFT lifts to a 3d TQFT, perhaps along the lines of [41]. See Fig. 2 for the schematic relations. Similarly, in the limit $S^1_T\to 0$ ($\gamma\to 0$ in the notation of Sec. IV), we expect to recover a Chern-Simons theory with a noncompact gauge group [42–47]. It would be interesting to study these points further.

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APPENDIX: DERIVATION OF THE 4D INDEX ON $S^1\times L(p,1)$

In this Appendix we derive our orbifold index in the path-integral formulation (cf. [48]). Let us deal with the case of the 4d $\mathcal{N}=2$ SCFT first, and comment on the $\mathcal{N}=1$ case later.

The index (7) is written as

$$\mathcal{I}=\mathrm{Tr}(-1)^{\mathcal{F}}e^{-\beta(E+j_2+2\Omega_1 j_1-\mu(r+R)+\nu F)},\qquad (A1)$$

where we used the notation (17), and $\mathcal{H}$ will denote the set of all fields contributing to the index. The expression (A1) is equivalent to the path integral over all the fields Φ in $\mathcal{H}$,

$$\mathcal{I}=\int_{\mathcal{H}}\mathcal{D}\Phi e^{-S[\Phi]},\qquad (A2)$$

where we impose a periodic boundary condition along S^1 for fermions, and the chemical potentials modify the covariant derivative with respect to the Euclidean time, acting on the field Φ in a representation $\mathcal{R}_\Phi$, to be

$$D_0=\partial_0-i\rho(a)-j_2-2\Omega_1 j_1+\mu(r+R)-\nu F,\qquad (A3)$$

where $\rho\in\mathcal{R}_\Phi$ stands for the weight of the representation. Since the index does not depend on gauge couplings, one can perform a path integral exactly in the free field limit. The one-loop contribution from a field Φ is

$$Z_\Phi=\prod_{\rho\in\mathcal{R}_\Phi}\mathrm{Det}^{(-1)^{\mathcal{F}+1}}(-D_0^2+\Delta_\Phi),\qquad (A4)$$

where Δ_Φ is an operator whose eigenvalues are denoted by E^2_Φ. For example, we have $\Delta_\Phi=-\Delta_{S^3}+1$ for a scalar, where the term 1 comes from the conformal coupling of the scalar field.

Let us expand the determinant in terms of the eigenvalues of ∂_0:

$$\mathrm{Det}(-D_0^2+E_\Phi^2)=\prod_{n=-\infty}^{\infty}\left(\frac{2\pi in}{\beta}+E_{\Phi,+}\right)\left(-\frac{2\pi in}{\beta}+E_{\Phi,-}\right),\qquad (A5)$$

where

$$E_{\Phi,\pm}=E_\Phi\pm(-i\rho(a)-j_2-2\Omega_1 j_1+\mu(r+R)-\nu F).\qquad (A6)$$

Consequently,

$$Z_\Phi=Z_{\Phi,+}Z_{\Phi,-},\quad Z_{\Phi,\pm}=\prod_{E_\Phi}\prod_{n=-\infty}^{\infty}\left(\frac{2\pi in}{\beta}+E_{\Phi,+}\right).\qquad (A7)$$

Since $E_{\Phi,+}$ and $E_{\Phi,-}$ share the same energy but opposite charges, we recognize the two terms as the contributions from a particle and its antiparticle. We therefore have

$$\mathcal{I}=\sum_m\int[da]\prod_{\mathcal{H}}Z_{\Phi,\pm}^{(-1)^{\mathcal{F}+1}},\qquad (A8)$$

where the measure $[da]$ comes from the gauge fixing (see [17], Sec. 2.2), and we summed over the holonomies $m=\{m_i\}$. The product is over all states contributing to the index.

We can rewrite (A8) with the help of the formula

$$\prod_{n=-\infty}^{\infty}(2\pi i n + x) = 2\sinh\frac{x}{2} = e^{\frac{x}{2}}(1-e^{-x}) = e^{\frac{x}{2}}\exp\Bigg[-\sum_{m=1}^{\infty}\frac{1}{m}e^{-mx}\Bigg], \tag{A9}$$

which results in (neglecting overall constants)

$$\mathcal{I} = \sum_m \int [da] e^{-\beta\sum_{\mathcal{H}}(-1)^{\mathcal{F}}\frac{E_{\Phi,\pm}}{2}} \times \exp\Bigg[\sum_{n=1}^{\infty}\frac{1}{n}\hat{\mathcal{I}}(t^n, y^n, v^n, z^n; e^{in\beta a})\Bigg], \tag{A10}$$

where we defined the single-letter index $\hat{\mathcal{I}}$ by

$$\hat{\mathcal{I}}(t, y, v, z; e^{i\beta a}) = \sum_{\mathcal{H}}(-1)^{\mathcal{F}} e^{-\beta E_{\Phi,\pm}}. \tag{A11}$$

The exponential factor $e^{-\beta\sum_{\mathcal{H}}(-1)^{\mathcal{F}}\frac{E_{\Phi,\pm}}{2}}$ is the zero-point (Casimir) contribution to the energy and chemical potentials. The Casimir part is easily obtained by differentiating the single-letter index with respect to β:

$$\sum_{\mathcal{H}}(-1)^{\mathcal{F}}E = -\text{Finite}_{\beta\to 0}\Bigg[\frac{\partial\hat{\mathcal{I}}}{\partial\beta}\Bigg], \tag{A12}$$

where we remove the divergent part in the $\beta \to 0$ limit [10,11]. We also neglect the holonomy-independent part of the zero-point contribution, since this is merely an overall shift of the index and does not affect the sum over the holonomies.

The remaining task is to explicitly evaluate the one-loop determinant, or equivalently $\hat{\mathcal{I}}$. This can be carried out by using the expressions for Δ_Φ and E_Φ. Alternatively, we can count the operators contributing to the index (see Table I).

For a half-hypermultiplet (q, ψ) with a weight ρ, the nontrivial contributions come from $\partial_{++}^{n_1}\partial_{-+}^{n_2}q$ and $\partial_{++}^{n_1}\partial_{-+}^{n_2}\bar{\psi}_+$, where n_1, n_2 are non-negative integers. We therefore have

$$\hat{\mathcal{I}}^{\mathcal{N}=2\text{half-hyper}} = \sum_{\rho\in\mathcal{R}}(t^2 v^{-\frac{1}{2}} z^F e^{i\rho(a)} - t^4 v^{\frac{1}{2}} z^{-F} e^{-i\rho(a)}) \times \tilde{F}_p(t, y; \rho(m)), \tag{A13}$$

where we defined

$$\tilde{F}_p(t, y; \rho(m)) = \sum_{n_1,n_2\geq 0}{}' (t^{3(n_1+n_2)} y^{n_1-n_2}), \tag{A14}$$

and the prime in the sum means that we sum over non-negative integers n_1, n_2 satisfying the orbifold projection condition (A17), to be given shortly.[10] Since the dependence

[10]To compute $\hat{\mathcal{I}}_{\text{orbifold}}$ we can first compute $\hat{\mathcal{I}}$ for the unorbifolded theory and then impose the orbifold projection afterwards. This is because the supercharge $\mathcal{Q}$ used in the definition of the index commutes with the orbifold action.

TABLE I. The operators contributing to the single-letter 4d $\mathcal{N}=2$ index $\hat{\mathcal{I}}$. In Euclidean signature, the vector multiplet is given by $(\phi, \bar{\phi}, \lambda_\alpha^I, \bar{\lambda}_{I\dot{\alpha}}, F_{\alpha\beta}, \tilde{F}_{\dot{\alpha}\dot{\beta}})$ while the half-hypermultiplet by $(q, \bar{q}, \psi_\alpha, \bar{\psi}_{\dot{\alpha}})$. We also included a constraint from the equation of motion for λ^1.

Operators	E	j_1	j_2	R	r	Contribution to $\hat{\mathcal{I}}$
ϕ	1	0	0	0	-1	$t^2 v$
$\lambda_\pm^1$	$\frac{3}{2}$	$\pm\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-t^3 y,\ -t^3 y^{-1}$
$\bar{\lambda}_{2+}$	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-t^4 v^{-1}$
$\tilde{F}_{++}$	2	0	1	0	0	t^6
$\partial_{-+}\lambda_+^1 + \partial_{++}\lambda_-^1 = 0$	$\frac{5}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	t^6
q	1	0	0	$\frac{1}{2}$	0	$t^2 v^{-1/2} z^F$
$\bar{\psi}_+$	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-t^4 v^{1/2} z^{-F}$
$\partial_{\pm+}$	1	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	0	$t^3 y,\ t^3 y^{-1}$

of $\tilde{F}_p$ on $\rho(m)$ is through (A17), it clearly only depends on $\rho(m)$ mod p. Since $\mathcal{R}$ is a pseudoreal representation, we have $\sum_{\rho\in\mathcal{R}} = \sum_{-\rho\in\mathcal{R}}$ and we can write

$$\hat{\mathcal{I}}^{\mathcal{N}=2\text{half-hyper}} = \sum_{\rho\in\mathcal{R}}(t^2 v^{-\frac{1}{2}} z^F - t^4 v^{\frac{1}{2}} z^{-F}) \times \tilde{F}_p(t, y; \rho(m)) e^{i\rho(a)}. \tag{A15}$$

The computation for a vector multiplet is similar, except that the fields $\lambda_\pm^1$ in Table I require special attention since they have nonzero $2j_1$ charge and come with the constraint of the equation of motion. The answer is given by[11]

$$\hat{\mathcal{I}}^{\mathcal{N}=2\text{vector}} = \sum_{\rho\in\text{Adj}}[(t^2 v - t^4 v^{-1} + t^6 - 1) \times \tilde{F}_p(t, y; \rho(m)) + \delta_{[\![\rho(m)]\!],0}] e^{i\rho(a)}, \tag{A16}$$

where we used $\sum_{\rho\in\text{Adj}} = \sum_{-\rho\in\text{Adj}}$ and the function $[\![x]\!]$ is defined in (15).

The projection condition is given by

$$n_1 - n_2 = \rho(m) \pmod{p}. \tag{A17}$$

To see this, recall that the effect of the holonomy can be locally removed by a gauge transformation; however, this modifies the global boundary condition, and we have a twisted boundary condition. The integers n_1 and n_2 are the spins under the phase rotation of z_1, and z_2 in (1) and (A17) ensures the single-valuedness of the wave function.[12] Note that the conditions are the same for bosons and fermions.

[11]Before taking the orbifold, we have a function

$$\frac{t^2 v - t^4 v - t^3(y + y^{-1}) + 2t^6}{(1-t^3 y)(1-t^3 y^{-1})} = \frac{t^2 v - t^4 v + t^6 - 1}{(1-t^3 y)(1-t^3 y^{-1})} + 1.$$

The expression inside the square bracket in (A16) is the orbifold of this expression.

[12]The spins (n_1, n_2) are related to the spins $(m_1, \frac{l}{2})$ of $U(1)_1 \times U(1)_2 \in SU(2)_1 \times SU(2)_2$ by a linear combination [see (48)].

We can use (A17) to evaluate the sum in (A14). Let us write $n_1 - n_2 = L + kp$, with $L = [\![\rho(m)]\!]$ so that $0 \leq L < p$, and $k \in \mathbb{Z}$. We divide the sum into $k \geq 0$ and $k < 0$, i.e. (a) $k \in \mathbb{Z}_{\geq 0}$ and (b) $\tilde{k} \equiv -k - 1 \in \mathbb{Z}_{\geq 0}$. Summing over n_2, k in (a) and $n_1, \tilde{k}$ in (b) we obtain

$$\tilde{F}_p(t, y; \rho(m)) = F_p(t, y; [\![\rho(m)]\!]), \tag{A18}$$

where F_p is the function given in (14).

It is straightforward to repeat the analysis for $\mathcal{N} = 1$ theories. For an $\mathcal{N} = 1$ vector multiplet

$$\hat{\mathcal{I}}^{\mathcal{N}=1\text{vector}} = \sum_{\rho \in \text{Adj}} [(t^6 - 1)\tilde{F}_p(t, y; \rho(m)) + \delta_{[\![\rho(m)]\!],0}] e^{i\rho(a)}, \tag{A19}$$

and for a chiral multiplet with flavor charge F and anomalous R charge Q,

$$\hat{\mathcal{I}}^{\mathcal{N}=1\text{chiral}} = \sum_{\rho \in \mathcal{R}} (t^{3Q} z^F e^{i\rho(a)} - t^{6-3Q} z^{-F} e^{-i\rho(a)}) \times \tilde{F}_p(t, y; \rho(m)). \tag{A20}$$

[1] C. Romelsberger, Nucl. Phys. **B747**, 329 (2006).
[2] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, Commun. Math. Phys. **275**, 209 (2007).
[3] V. Pestun, Commun. Math. Phys. **313**, 71 (2012).
[4] A. Kapustin, B. Willett, and I. Yaakov, J. High Energy Phys. 03 (2010) 089.
[5] N. Drukker, M. Marino, and P. Putrov, Commun. Math. Phys. **306**, 511 (2011).
[6] C. P. Herzog, I. R. Klebanov, S. S. Pufu, and T. Tesileanu, Phys. Rev. D **83**, 046001 (2011).
[7] D. L. Jafferis, J. High Energy Phys. 05 (2012) 159.
[8] N. Hama, K. Hosomichi, and S. Lee, J. High Energy Phys. 03 (2011) 127.
[9] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and S. Raju, J. High Energy Phys. 02 (2008) 064.
[10] S. Kim, Nucl. Phys. **B821**, 241 (2009).
[11] Y. Imamura and S. Yokoyama, J. High Energy Phys. 04 (2011) 007.
[12] F. A. H. Dolan, V. P. Spiridonov, and G. S. Vartanov, Phys. Lett. B **704**, 234 (2011).
[13] A. Gadde and W. Yan, arXiv:1104.2592.
[14] Y. Imamura, J. High Energy Phys. 09 (2011) 133.
[15] T. Nishioka, Y. Tachikawa, and M. Yamazaki, J. High Energy Phys. 08 (2011) 003.
[16] H. Lin and J. M. Maldacena, Phys. Rev. D **74**, 084014 (2006).
[17] Y. Hikida, J. High Energy Phys. 12 (2006) 042.
[18] A. Gadde, E. Pomoni, L. Rastelli, and S. S. Razamat, J. High Energy Phys. 03 (2010) 032.
[19] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, Phys. Rev. Lett. **106**, 241602 (2011).
[20] A. D. Shapere and Y. Tachikawa, J. High Energy Phys. 09 (2008) 109.
[21] C. Romelsberger, arXiv:0707.3702.
[22] A. Kapustin and B. Willett, arXiv:1106.2484.
[23] D. Gang, arXiv:0912.4664.
[24] J. Kallen, J. High Energy Phys. 08 (2011) 008.
[25] N. Hama, K. Hosomichi, and S. Lee, J. High Energy Phys. 05 (2011) 014.
[26] D. Gaiotto, arXiv:0904.2715.
[27] F. Benini, S. Benvenuti, and Y. Tachikawa, J. High Energy Phys. 09 (2009) 052.
[28] D. Gaiotto, G. W. Moore, and A. Neitzke, arXiv:0907.3987.
[29] O. Chacaltana and J. Distler, J. High Energy Phys. 11 (2010) 099.
[30] L. F. Alday, D. Gaiotto, and Y. Tachikawa, Lett. Math. Phys. **91**, 167 (2010).
[31] V. Belavin and B. Feigin, J. High Energy Phys. 07 (2011) 079.
[32] T. Nishioka and Y. Tachikawa, Phys. Rev. D **84**, 046009 (2011).
[33] G. Bonelli, K. Maruyoshi, and A. Tanzini, J. High Energy Phys. 08 (2011) 056.
[34] A. Belavin, V. Belavin, and M. Bershtein, J. High Energy Phys. 09 (2011) 117.
[35] G. Bonelli, K. Maruyoshi, and A. Tanzini, Lett. Math. Phys. **101**, 103 (2012).
[36] L. F. Alday, F. Benini, and Y. Tachikawa, Phys. Rev. Lett. **105**, 141601 (2010).
[37] T. Kimura, J. High Energy Phys. 09 (2011) 015.
[38] S. Matsuura and K. Ohta, Phys. Rev. D **73**, 046006 (2006).
[39] F. Benini, Y. Tachikawa, and B. Wecht, J. High Energy Phys. 01 (2010) 088.
[40] F. Benini, Y. Tachikawa, and D. Xie, J. High Energy Phys. 09 (2010) 063.
[41] N. Arkani-Hamed, A. G. Cohen, and H. Georgi, Phys. Rev. Lett. **86**, 4757 (2001).
[42] T. Dimofte, S. Gukov, and L. Hollands, Lett. Math. Phys. **98**, 225 (2011).
[43] Y. Terashima and M. Yamazaki, J. High Energy Phys. 08 (2011) 135.
[44] Y. Terashima and M. Yamazaki, arXiv:1106.3066.
[45] T. Dimofte and S. Gukov, arXiv:1106.4550.
[46] V. Spiridonov and G. Vartanov, arXiv:1107.5788.
[47] T. Dimofte, D. Gaiotto, and S. Gukov, arXiv:1108.4389.
[48] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, Adv. Theor. Math. Phys. **8**, 603 (2004).