Nonrelativistic conformal groups and their dynamical realizations

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Nonrelativistic conformal groups, indexed by $l = \frac{N}{2}$, are analyzed. Under the assumption that the mass parametrizing the central extension is nonvanishing, the coadjoint orbits are classified and described in terms of convenient variables. It is shown that the corresponding dynamical system describes, within Ostrogradski framework, the nonrelativistic particle obeying $(N + 1)$-th order equation of motion. As a special case, the Schrödinger group and the standard Newton equations are obtained for $N = 1$ ($l = \frac{1}{2}$).

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I. INTRODUCTION

Historically, the structure that is now called the Schrödinger group was discovered in the nineteenth century in the context of classical mechanics [1] and heat equation [2]. It was rediscovered in the twentieth century as the maximal symmetry group of free motion in quantum mechanics [3]. Much attention has been paid to the structure of the Schrödinger group and its geometrical status [4].

The Schrödinger group, when supplemented with space dilatation transformations, becomes $l = \frac{1}{2}$ member of the whole family of nonrelativistic conformal groups [5], indexed by halfinteger l. Various structural, geometric, and physical aspects of the resulting Lie algebras have been intensively studied [6]. For $l = \frac{N}{2}$, N-odd (also N-even in the case of dimension two), the nonrelativistic conformal algebra admits central extension. Then, as it has been shown in Ref. [7], it becomes the symmetry algebra of the free nonrelativistic particle obeying $(N + 1)$-th order equation of motion.

In the present paper, we use the orbit method [8] to construct the most general dynamical systems on which the nonrelativistic conformal groups act transitively as symmetries. The main advantage of the orbit method is that once the transitivity of the action of a given group G is assumed, it gives the complete classification of all Hamiltonian sytems for which G acts as the group of symmetry (canonical) transformations. Therefore, one can reconstruct the Hamiltonian systems merely from the assumptions concerning their symmetries. The necessity of imposing the transitivity condition implies that there exist systems which are not included into the classification; the orbit method yields, however, the general framework for description of all systems exhibiting a given symmetry. As a simple example let us consider the two-particle Galilei-invariant system. The Galilei group acts transitively on center-of-mass variables. The complete set of canonical variables include also the relative motion variables which Poisson-commute with those describing the center-of-mass motion. The structure of the Galilei group implies that the relative motion variables can enter the game only in a specific way. For example, the Hamiltonian must be the sum of the center-of-mass kinetic energy and the internal one, which in the transitive (i.e. single-particle) case is just a number. In the two-particle case, the "internal" degrees of freedom can enter energy only through internal energy that now becomes their function. Similar considerations concern the angular momentum: the internal variables contribute only through the spin term.

In the case of nonrelativistic conformal groups, we find that the basic variables are coordinates and momenta together with internal variables obeying $SU(2)$ commutation rules (in the sense of Poisson brackets) and underlying trivial dynamics; the remaining internal variables obey $SL(2, \mathbb{R})$ [or $SO(2, 1)$] commutation rules and the equation of motion of conformal quantum mechanics [9] in global formulation [10].

All symmetry generators split into two parts: the external one constructed out of coordinates and momenta (like orbital angular momentum) and the internal one (like spin). The symmetry transformations are implemented as canonical transformations.

The standard free dynamics is obtained by selecting the trivial orbit for $SL(2, \mathbb{R})$ variables.

The results heavily rely on the fact that the conformal algebras under consideration admit central extensions. For vanishing "mass" parameters (as well as for conformal algebras that do not admit central extension), the classification of orbits is more complicated and the physical interpretation in such cases remains slightly obscure.

Except in Sec. III B, we assume that our space is three-dimensional, $d = 3$ (i.e., the space-time is four-dimensional, $d + 1 = 4$). However, our results are generally valid for $d \geq 3$. The case $d = 2$ is considered separately in Sec. III B because in two-dimensional space the central extensions are allowed for both odd and even N. In the case of $d = 0$ (one-dimensional space-time), no rotations, translations, or boosts [and, consequently, other transformations generated by the elements outside $sl(2, \mathbb{R})$] are allowed, and we are left with the standard conformal mechanics (see below for details).

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II. THE SCHRÖDINGER SYMMETRY

We start with the $l = \frac{1}{2}$ Galilean conformal algebra (according to the terminology of Ref. [5]). It consists of rotations $\vec{J}$, translations $\vec{P}$, boosts $\vec{B}$, and time translations H that form the Galilean algebra, together with dilatations D, conformal transformations K, and, finally, space dilatations D_s. The nontrivial commutation rules read

$$\begin{aligned} &[J_i, J_k] = i\epsilon_{ikl}J_l, \quad [J_i, P_k] = i\epsilon_{ikl}P_l, \quad [J_i, B_k] = i\epsilon_{ikl}B_l,\\ &[B_i, H] = iP_i, \quad [D, H] = iH, \quad [D, K] = -iK,\\ &[K, H] = 2iD, \quad [D, P_i] = \frac{i}{2}P_i, \quad [D, B_i] = \frac{-i}{2}B_i,\\ &[K, P_i] = iB_i, \quad [D_s, P_i] = iP_i, \quad [D_s, B_i] = iB_i. \end{aligned} \tag{1}$$

Deleting D_s, one obtains twelve-dimensional Schrödinger algebra, which admits, similarly to the Galilei algebra, central extension defined by the additional nontrivial commutator

$$[B_i, P_k] = iM\delta_{ik}. \tag{2}$$

The structure of centrally extended Schrödinger algebra is well known. First, we have $su(2)$ [or $so(3)$] algebra spanned by $J_i's$; furthermore, H, D, and K span the conformal algebra which is isomorphic to $so(2,1)$ [or $sl(2,\mathbb{R})$]. To see this, one defines

$$N^0 = \frac{1}{2}(H + K), \qquad N^1 = \frac{1}{2}(K - H), \qquad N^2 = D, \tag{3}$$

which yields

$$[N^\alpha, N^\beta] = i\epsilon^{\alpha\beta}{}_\gamma N^\gamma, \qquad \alpha, \beta, \gamma = 0, 1, 2; \tag{4}$$

where $\epsilon^{012} = \epsilon_{012} = 1$ and $g_{\mu\nu} = \mathrm{diag}(+,-,-)$. Therefore $\vec{J}$, H, K, and D span direct sum $su(2) \oplus so(2,1)$. Finally, $\vec{P}$, $\vec{B}$, and M form a nilpotent algebra, which, at the same time, carries a representation of $su(2) \oplus so(2,1)$. To express this fact in a compact way, we define the spinor representation of $so(2,1)$:

$$\tilde{N}^0 = \frac{1}{2}\sigma_2, \qquad \tilde{N}^1 = \frac{i}{2}\sigma_1, \qquad \tilde{N}^2 = \frac{i}{2}\sigma_3. \tag{5}$$

Moreover, denoting $X_{1i} = P_i$, $X_{2i} = B_i$, one finds the simple form of the action of $su(2) \oplus so(2,1)$ on the space spanned by $\vec{P}$, $\vec{B}$, and M

$$[J_i, X_{ak}] = i\epsilon_{ikl}X_{al}, \qquad [N^\alpha, X_{ai}] = X_{bi}(\tilde{N}^\alpha)_{ba}, \tag{6}$$

where $a, b = 1, 2$. The commutation rule (2) takes the form

$$[X_{ai}, X_{bj}] = -iM\epsilon_{ab}\delta_{ij}. \tag{7}$$

The matrices $\tilde{N}^\alpha$ are all purely imaginary and span the defining representation of $sl(2,\mathbb{R})$. In fact, the group $SL(2,\mathbb{R})$ is nothing but the group $\mathrm{Spin}(2,1)^+$. The Schrödinger algebra can be thus integrated to the group $S = (SU(2) \times SL(2,\mathbb{R})) \ltimes R_7$, where R_7 is the seven-dimensional nilpotent group (topologically isomorphic to $\mathbb{R}^7$), and the semidirect product is defined by the $D^{(1,\frac{1}{2})} \oplus D^{(0,0)}$ representation of $SU(2) \times SL(2,\mathbb{R})$.

Let us consider the coadjoint action of the Schrödinger group S, denoting the dual-basis elements by $\tilde{\vec{J}}$, $\tilde{\vec{P}}$, $\tilde{\vec{B}}$, etc. The general element of the dual space to the Lie algebra of S is written as

$$X = \vec{j}\,\tilde{\vec{J}} + \vec{\xi}\,\tilde{\vec{P}} + \vec{\zeta}\,\tilde{\vec{B}} + h\tilde{H} + d\tilde{D} + k\tilde{K} + m\tilde{M}. \tag{8}$$

Having characterized the global structure of S, we could consider the full action of S on X. However, for our purposes, it is sufficient to compute the coadjoint action of one-parameter subgroups generated by the basic elements of the Lie algebra. The results are summarized in Table I below.

Here $(\overrightarrow{Rj})_k = R_{kl}j_l$, etc.

In order to find the structure of coadjoint orbits, note that m is invariant under the coadjoint action of S. In what follows, we assume that $m > 0$ (in fact, it is sufficient to take $m \neq 0$). Once this assumption is made, the classification of orbits becomes quite simple. Using the results collected in Table I, we conclude that each orbit contains the point corresponding to $\vec{\xi} = 0$, $\vec{\zeta} = 0$. Moreover, the stability subgroup of the submanifold $\vec{\xi} = 0$, $\vec{\zeta} = 0$ is

TABLE I. Coadjoint action of S.

Ad^*_g \ g	$e^{i\vec{a}\vec{P}}$	$e^{i\vec{v}\vec{B}}$	$e^{-i\tau H}$	$e^{i\lambda D}$	e^{iuK}	$e^{i\vec{\omega}\vec{J}}$
$\vec{j}'$	$\vec{j} - \vec{a} \times \vec{\xi}$	$\vec{j} - \vec{v} \times \vec{\zeta}$	$\vec{j}$	$\vec{j}$	$\vec{j}$	$\overrightarrow{Rj}$
$\vec{\xi}'$	$\vec{\xi}$	$\vec{\xi} + m\vec{v}$	$\vec{\xi}$	$e^{\frac{\lambda}{2}}\vec{\xi}$	$\vec{\xi} + u\vec{\zeta}$	$\overrightarrow{R\xi}$
$\vec{\zeta}'$	$\vec{\zeta} - m\vec{a}$	$\vec{\zeta}$	$\vec{\zeta} + \tau\vec{\xi}$	$e^{-\frac{\lambda}{2}}\vec{\zeta}$	$\vec{\zeta}$	$\overrightarrow{R\zeta}$
h'	h	$h + \frac{m\vec{v}^2}{2} + \vec{v}\,\vec{\xi}$	h	$e^{\lambda}h$	$h + 2ud + u^2k$	h
d'	$d - \frac{1}{2}\vec{a}\,\vec{\xi}$	$d + \frac{1}{2}\vec{v}\,\vec{\zeta}$	$d + \tau h$	d	$d + uk$	d
k'	$k - \vec{a}\,\vec{\zeta} + \frac{1}{2}m\vec{a}^2$	k	$k + 2\tau d + \tau^2 h$	$e^{-\lambda}k$	k	k
m'	m	m	m	m	m	m

$SU(2)\times SL(2,\mathbb{R})\times\mathbb{R}$, where the last factor is the subgroup generated by M and can be neglected. The orbits of $SU(2)\times SL(2,\mathbb{R})$ are the products of orbits of both factors. For $SU(2)$, any coadjoint orbit is a 2-sphere (or a point) which can be parametrized by vector $\vec{s}$ of fixed length, $\vec{s}^2=s^2$. To describe the orbits of $SL(2,\mathbb{R})$ [which is equivalent, as far as coadjoint action is concerned, to $SO(2,1)$], we define, in analogy with Eq. (3),

$$\chi^0=\frac{1}{2}(h+k),\quad \chi^1=\frac{1}{2}(-h+k),\quad \chi^2=d. \tag{9}$$

Then, by standard arguments, the full list of orbits reads:

$$\begin{aligned}
\mathcal{H}_\sigma^+ &= \{\chi^\mu\colon g_{\mu\nu}\chi^\mu\chi^\nu=\sigma^2,\chi^0>0\},\\
\mathcal{H}_\sigma^- &= \{\chi^\mu\colon g_{\mu\nu}\chi^\mu\chi^\nu=\sigma^2,\chi^0<0\},\\
\mathcal{H}_0^+ &= \{\chi^\mu\colon g_{\mu\nu}\chi^\mu\chi^\nu=0,\chi^0>0\},\\
\mathcal{H}_0^- &= \{\chi^\mu\colon g_{\mu\nu}\chi^\mu\chi^\nu=0,\chi^0<0\},\\
\mathcal{H}_\sigma &= \{\chi^\mu\colon g_{\mu\nu}\chi^\mu\chi^\nu=-\sigma^2\},\\
\mathcal{H}_0 &= \{0\}.
\end{aligned}\tag{10}$$

Consequently, any coadjoint orbit of S (with nonvanishing m) contains the point

$$\vec{s}\tilde{\vec{J}}+(\chi^0-\chi^1)\tilde{H}+\chi^2\tilde{D}+(\chi^0+\chi^1)\tilde{K}+m\tilde{M}, \tag{11}$$

where $\vec{s}\in S^2$ and χ^μ is a point on one of the manifolds, $\mathcal{H}$, listed above. We see that any orbit is characterized by the values of m, $\vec{s}^2$, $g_{\mu\nu}\chi^\mu\chi^\nu$, and, for $g_{\mu\nu}\chi^\mu\chi^\nu\geq 0$, the sign of χ^0. Let us note that the above invariants correspond to the Casimir operators of Schrödinger algebra

$$\begin{aligned}
&C_1=M,\quad C_2=(M\vec{J}-\vec{B}\times\vec{P})^2,\\
&C_3=\Big(MH-\frac{\vec{P}^2}{2}\Big)\Big(MK-\frac{\vec{B}^2}{2}\Big)+\Big(MK-\frac{\vec{B}^2}{2}\Big)\Big(MH-\frac{\vec{P}^2}{2}\Big)+\\
&\quad -2\Big(MD-\frac{\vec{B}\vec{P}}{4}-\frac{\vec{P}\vec{B}}{4}\Big)^2.
\end{aligned}\tag{12}$$

The whole coadjoint orbit of S can be obtained by applying $g(\vec{a})$ and $g(\vec{v})$ to all points (11) with $\vec{s}$ and χ^μ varying over their orbits. Calling $\vec{a}=-\vec{x}$ and $\vec{v}=\vec{p}/m$, one finds the following parametrization of coadjoint orbits

$$\begin{aligned}
&\vec{j}=\vec{x}\times\vec{p}+\vec{s},\quad \vec{\xi}=\vec{p},\quad \vec{\zeta}=m\vec{x},\\
&h=\frac{\vec{p}^2}{2m}+\chi^0-\chi^1,\quad d=\frac{1}{2}\vec{x}\vec{p}+\chi^2,\quad k=\frac{m}{2}\vec{x}^2+\chi^0+\chi^1.
\end{aligned}\tag{13}$$

We see that the phase-space variables are $\vec{x}$, $\vec{p}$, $\vec{s}$, and χ^μ. The Poisson brackets implied by Kirillov symplectic structure read

$$\{x_i,p_k\}=\delta_{ik},\quad \{s_i,s_k\}=\epsilon_{ikl}s_l,\quad \{\chi^\alpha,\chi^\beta\}=\epsilon^{\alpha\beta}{}_\gamma\chi^\gamma, \tag{14}$$

while the corresponding equations of motion take the form

$$\begin{aligned}
&\dot{\vec{x}}=\frac{\vec{p}}{m},\qquad \dot{\vec{p}}=0,\qquad \dot{\vec{s}}=0,\\
&\dot{\chi}^0=\chi^2,\qquad \dot{\chi}^1=\chi^2,\qquad \dot{\chi}^2=-\chi^1+\chi^0.
\end{aligned}\tag{15}$$

We can summarize our findings. The ten-dimensional orbits are parametrized by $\vec{x}$, $\vec{p}$, $\vec{s}$, and χ^μ subject to the constraints $\vec{s}^2=\text{const}$ and $g_{\mu\nu}\chi^\mu\chi^\nu=\text{const}$ and equipped with the symplectic structure defined by Eq. (14) and dynamics given by Eq. (15).

One can say that, besides the standard canonical variables $\vec{x}$ and $\vec{p}$, there are two kinds of "internal" degrees of freedom—ordinary spin variables $\vec{s}$ and $SO(2,1)$ and "pseudospin" degrees of freedom χ^μ. Note that, contrary to the true spin variables, χ^μ variables have nontrivial dynamics.

Let us consider the dynamics of internal variables in some detail. It is uniquely dictated by the symmetry group structure. The Lie algebra under consideration contains the direct sum $su(2)\oplus sl(2,\mathbb{R})$ and the Hamiltonian belongs to the $sl(2,\mathbb{R})$ part. Therefore, it commutes with angular momentum. The generators which do not belong to $su(2)\oplus sl(2,\mathbb{R})$ form a linear representation under the adjoint action of the latter, which implies that the "external" part of the angular momentum Poisson commutes with H, so the "internal" must also commute, yielding trivial dynamics for spin variables. On the other hand, the pseudospin variables related to $sl(2,\mathbb{R})$ obey, as it was noted above, the nontrivial dynamical equations. This is due to the fact that the Hamiltonian itself belongs to $sl(2,\mathbb{R})$ subalgebra and obeys nontrivial commutation rules with the remaining $sl(2,\mathbb{R})$ generators. The dynamics of "internal" and "external" contributions to $sl(2,\mathbb{R})$ generators decouple. This again follows from the property that the generators outside the $su(2)\oplus sl(2,\mathbb{R})$ subalgebra form a linear representation under the adjoint action of the latter. As it will be discussed in the last section, the dynamics of $sl(2,\mathbb{R})$ pseudospin variables is described by conformal mechanics.

Making the trivial choice $\mathcal{H}_0=\{0\}$ of the $SL(2,\mathbb{R})$ orbit, one finds the standard realization of the Schrödinger group as the symmetry of free dynamics.

From the basic functions (13), one can construct the generators (in the sense of canonical formalism) of group transformations. Due to the fact that the Hamiltonian is an element of the Lie algebra of the symmetry group, the symmetry generators depend, in general, explicitly on time. They read

$$\begin{aligned}
&j_k=j_k(t),\qquad p_k=p_k(t),\\
&x_k=x_k(t)-\frac{t}{m}p_k(t),\qquad h=h(t),\\
&k=k(t)-2td(t)+t^2h(t),\qquad d=d(t)-th(t).
\end{aligned}\tag{16}$$

III. N-GALILEAN CONFORMAL SYMMETRY

Higher-dimensional nonrelativistic conformal algebras are constructed according to the following unique scheme. One takes the direct sum $su(2) \oplus sl(2,\mathbb{R}) \oplus \mathbb{R}$, where the last term corresponds to the spatial dilatation D_s. This is supplemented by $3(N+1)$ Abelian algebra (here $l = N/2$), which carries the $D^{(1,\frac{N}{2})}$ representation of $SU(2) \otimes SL(2,\mathbb{R})$; moreover, all new generators correspond to the eigenvalue 1 of D_s. Call $\vec{C}_i = (C_i^a, a = 1,2,3)$, $i = 0,1,\ldots,N$, the new generators. The relevant commutation rules involving $\vec{C}_i$ read

$$[D_s, C_j^a] = iC_j^a, \qquad [J^a, C_j^b] = i\epsilon_{abd}C_j^d,$$
$$[H, C_j^a] = -ijC_{j-1}^a, \qquad [D, C_j^a] = i\left(\frac{N}{2} - j\right)C_j^a,$$
$$[K, C_j^a] = i(N-j)C_{j+1}^a. \tag{17}$$

As previously, we delete the space dilatation operator D_s and consider the question of the existence of the central extension of the Abelian algebra spanned by $\vec{C}$'s. To solve it, one can consider the relevant Jacobi identities or analyze the transformation properties under $SU(2) \times SL(2,\mathbb{R})$. The second order $SU(2)$ invariant tensor, i.e., Kronecker delta δ^{ab} in arbitrary dimension (and tensor ϵ^{ab} for dimension two), is symmetric (antisymmetric, respectively), so the existence of central extension is equivalent to the existence of the antisymmetric (symmetric) $SL(2,\mathbb{R})$ invariant tensor. Taking into account that $N+1$-dimensional irreducible representations of $SL(2,\mathbb{R})$ may be obtained from the symmetrized tensor product of N basic representation, one easily concludes that an invariant antisymmetric (symmetric) tensor exists only for N odd (for N even in the case dimension two) (see Ref. [11]).

A. N-odd

In this case the relevant central extension reads [7]

$$[C_j^a, C_k^b] = i\delta^{ab}\delta^{N,j+k}(-1)^{\frac{k-j+1}{2}}k!j!M, \tag{18}$$

for $j,k = 0,1,\ldots,N$ and $a,b = 1,2,3$. In order to classify the coadjoint orbits we put, in analogy to Eq. (8),

$$X = \vec{j}\tilde{\vec{J}} + \vec{c}_i\tilde{\vec{C}}_i + h\tilde{H} + d\tilde{D} + k\tilde{K} + m\tilde{M}. \tag{19}$$

Again, m is invariant under the coadjoint action; we assume that $m > 0$.

Consider the coadjoint action of $\exp(ix_k^aC_k^a)$. It reads

$$\begin{aligned}
m' &= m,\\
j'^b &= j^b - \epsilon_{bad}\sum_{j=0}^{N}x_j^ac_j^d - \frac{m}{2}\sum_{j=0}^{N}(-1)^{j-\frac{N+1}{2}}\epsilon_{bca}x_j^ax_{N-j}^c j!(N-j)!,\\
c_j'^b &= c_j^b + (-1)^{j-\frac{N-1}{2}}mj!(N-j)!x_{N-j}^b,\\
h' &= h + \sum_{j=0}^{N-1}(j+1)x_{j+1}^bc_j^b + \frac{m}{2}\sum_{j=1}^{N}(-1)^{j-\frac{N+1}{2}}j!(N-j+1)!x_j^ax_{N-j+1}^a,\\
d' &= d - \sum_{j=0}^{N}\left(\frac{N}{2}-j\right)x_j^bc_j^b + \frac{m}{2}\sum_{j=0}^{N}\left(\frac{N}{2}-j\right)(-1)^{j-\frac{N+1}{2}}j!(N-j)!x_j^ax_{N-j}^a,\\
k' &= k - \sum_{j=1}^{N}(N-j+1)x_{j-1}^bc_j^b + \frac{m}{2}\sum_{j=0}^{N-1}(-1)^{j-\frac{N-1}{2}}(j+1)!(N-j)!x_j^ax_{N-j-1}^a.
\end{aligned} \tag{20}$$

We see that, as in the case of the Schrödinger group, any orbit contains the points

$$\vec{s}\tilde{\vec{J}} + (\chi^0 - \chi^1)\tilde{H} + \chi^2\tilde{D} + (\chi^0 + \chi^1)\tilde{K} + m\tilde{M}, \tag{21}$$

where, again, $\vec{s} \in S^2$ and χ^μ belong to one of the orbits (10). The whole orbit is produced by acting with $\exp(ix_k^aC_k^a)$ on the above points. As a result we arrive at the following parametrization

$$j^b = s^b - \frac{m}{2}\sum_{j=0}^{N}(-1)^{j-\frac{N+1}{2}}\epsilon_{bca}x_j^ax_{N-j}^cj!(N-j)!, \qquad c_j^b = (-1)^{j-\frac{N-1}{2}}mj!(N-j)!x_{N-j}^b,$$
$$h = \chi^0 - \chi^1 + \frac{m}{2}\sum_{j=1}^{N}(-1)^{j-\frac{N+1}{2}}j!(N-j+1)!x_j^ax_{N-j+1}^a, \qquad d = \chi^2 + \frac{m}{2}\sum_{j=0}^{N}\left(\frac{N}{2}-j\right)(-1)^{j-\frac{N+1}{2}}j!(N-j)!x_j^ax_{N-j}^a,$$
$$k = \chi^0 + \chi^1 + \frac{m}{2}\sum_{j=0}^{N-1}(-1)^{j-\frac{N-1}{2}}(j+1)!(N-j)!x_j^ax_{N-j-1}^a. \tag{22}$$

The invariants $\vec{s}^2$ and $g_{\mu\nu}\chi^\mu\chi^\nu$, which characterize the orbits, correspond to the Casimir operators

$$
\begin{aligned}
C_1 &= M,\\
C_2 &= \left(M\vec{J} - \frac{1}{2}\sum_{j=0}^{N}\frac{(-1)^{j-\frac{N+1}{2}}}{j!(N-j)!}\vec{C}_j \times \vec{C}_{N-j}\right)^2,\\
C_3 &= (MH - A)(MK - B) + (MK - B)(MH - A)\\
&\quad - 2(MD - C)^2,
\end{aligned} \tag{23}
$$

where

$$
\begin{aligned}
A &= \frac{1}{2}\sum_{j=1}^{N}\frac{(-1)^{j-\frac{N+1}{2}}}{(j-1)!(N-j)!}\vec{C}_{j-1}\vec{C}_{N-j},\\
B &= -\frac{1}{2}\sum_{j=0}^{N-1}\frac{(-1)^{j-\frac{N+1}{2}}}{j!(N-j-1)!}\vec{C}_{j+1}\vec{C}_{N-j},\\
C &= \frac{1}{2}\sum_{j=0}^{N}\frac{(-1)^{j-\frac{N+1}{2}}}{j!(N-j)!}\left(j - \frac{N}{2}\right)\vec{C}_j\vec{C}_{N-j}.
\end{aligned} \tag{24}
$$

The basic dynamical variables are χ^μ, s^a and x_j. The Poisson bracket resulting from Kirillov symplectic structure reads

$$
\{c_j^a, c_k^b\} = \delta^{ab}\delta^{N,j+k}(-1)^{\frac{k-j+1}{2}}k!j!m, \tag{25}
$$

and implies

$$
\{x_k^a, x_{N-k}^b\} = \frac{\delta^{ab}(-1)^{k-\frac{N+1}{2}}}{mk!(N-k)!}, \qquad k = 0, 1, \ldots, N. \tag{26}
$$

It is easy to define Darboux coordinates for "external" variables. They read

$$
x_k^a = \frac{(-1)^{k-\frac{N+1}{2}}}{k!}q_k^a, \qquad x_{N-k}^a = \frac{1}{m(N-k)!}p_k^a, \tag{27}
$$

for $k = 0, \ldots, \frac{N-1}{2}$, yielding the standard form of Poisson brackets

$$
\{q_k^a, p_l^b\} = \delta^{ab}\delta_{kl}. \tag{28}
$$

In terms of new variables the remaining one read

$$
\begin{aligned}
h &= \chi^0 - \chi^1 + \frac{1}{2m}\vec{p}_{(N-1)/2}\vec{p}_{(N-1)/2} + \sum_{k=1}^{\frac{N-1}{2}}\vec{q}_k\vec{p}_{k-1},\\
d &= \chi^2 + \sum_{k=0}^{\frac{N-1}{2}}\left(\frac{N}{2} - k\right)\vec{q}_k\vec{p}_k,\\
k &= \chi^0 + \chi^1 + \frac{m}{2}\left(\frac{N+1}{2}\right)^2\vec{q}_{(N-1)/2}\vec{q}_{(N-1)/2}\\
&\quad - \sum_{k=0}^{\frac{N-3}{2}}(N-k)(k+1)\vec{q}_k\vec{p}_{k+1},\\
\vec{j} &= \vec{s} + \sum_{k=0}^{\frac{N-1}{2}}\vec{q}_k \times \vec{p}_k.
\end{aligned} \tag{29}
$$

The above findings can be compared with those of Ref. [7]. In particular, the Hamiltonian h is the sum of two terms depending on "internal" ($sl(2,\mathbb{R})$) and "external" variables. The external part coincides with the Ostrogradski Hamiltonian [12] corresponding to the Lagrangian $L = \frac{m}{2}(\frac{d^{(N+1)/2}\vec{q}}{dt^{(N+1)/2}})^2$. This can be easily seen by writing out the canonical equations of motion

$$
\begin{aligned}
&\dot{\vec{q}}_k = \vec{q}_{k+1}, \qquad k = 0, \ldots, \frac{N-3}{2}; \qquad \dot{\vec{q}}_{\frac{N-1}{2}} = \frac{1}{m}\vec{p}_{\frac{N-1}{2}}\\
&\dot{\vec{p}}_k = -\vec{p}_{k-1}, \qquad k = 1, \ldots, \frac{N-1}{2}; \qquad \dot{\vec{p}}_0 = 0,
\end{aligned} \tag{30}
$$

which, for the basic variable $\vec{q} = \vec{q}_0$, imply $\vec{q}^{(N+1)} = 0$.

B. N-even

As we have mentioned, in the case of dimension 2, also for even N there exists the central extension of the Abelian algebra spanned by $\vec{C}$'s. The relevant commutators read:

$$
[C_j^a, C_k^b] = -i\epsilon^{ab}\delta^{N,j+k}(-1)^{\frac{j-k}{2}}k!j!m, \tag{31}
$$

where $a, b = 1, 2$, $j, k = 0, 1, \ldots, N$. Let us take an arbitrary element X of dual space to the Lie algebra

$$
X = j\tilde{J} + \vec{c}_i\tilde{\vec{C}}_i + h\tilde{H} + d\tilde{D} + k\tilde{K} + m\tilde{M}. \tag{32}
$$

As previously, m is invariant under the coadjoint action; we can assume that $m > 0$. Consider the coadjoint action of $\exp(ix_k^aC_k^a)$. It reads

$$
\begin{aligned}
m' &= m,\\
j' &= j - \epsilon^{ba}\sum_{j=0}^{N}x_j^b c_j^a + \frac{m}{2}\sum_{j=0}^{N}(-1)^{\frac{2j-N}{2}}\\
&\quad \times \epsilon^{ad}\epsilon^{bd}x_j^b x_{N-j}^a j!(N-j)!,\\
c_j'^b &= c_j^b - (-1)^{\frac{N-2j}{2}}mj!(N-j)!\epsilon^{ab}x_{N-j}^a,\\
h' &= h + \sum_{j=0}^{N-1}(j+1)x_{j+1}^b c_j^b\\
&\quad + \frac{m}{2}\sum_{j=1}^{N}(-1)^{\frac{2j-N}{2}}j!(N-j+1)!\epsilon^{ab}x_j^b x_{N-j+1}^a,\\
d' &= d - \sum_{j=0}^{N}\left(\frac{N}{2} - j\right)x_j^b c_j^b\\
&\quad - \frac{m}{2}\sum_{j=0}^{N}\left(-\frac{N}{2} + j\right)(-1)^{\frac{2j-N}{2}}j!(N-j)!\epsilon^{ab}x_j^b x_{N-j}^a,\\
k' &= k - \sum_{j=0}^{N-1}(N-j)x_j^b c_{j+1}^b\\
&\quad - \frac{m}{2}\sum_{j=0}^{N-1}(-1)^{\frac{2j-N}{2}}(j+1)!(N-j)!\epsilon^{ba}x_j^b x_{N-j-1}^a.
\end{aligned} \tag{33}
$$

We see that, similarly to the case of N-odd, any orbit contains the points

$$s\tilde{J} + (\chi^0 - \chi^1)\tilde{H} + \chi^2\tilde{D} + (\chi^0 + \chi^1)\tilde{K} + m\tilde{M}, \quad (34)$$

where $s \in \mathbb{R}$ and χ^μ belong to one of the orbits (10). Moreover, the whole orbit is produced by acting with $\exp(ix_k^a C_k^a)$ on the above points. Consequently, we have the following parametrization:

$$\begin{aligned}
j &= s + \frac{m}{2}\sum_{j=0}^{N}(-1)^{\frac{2j-N}{2}}\epsilon^{ad}\epsilon^{bd}x_j^b x_{N-j}^a j!(N-j)!,\\
c_j^b &= (-1)^{\frac{N-2j}{2}} m j!(N-j)!\epsilon^{ba}x_{N-j}^a,\\
h &= \chi^0 - \chi^1 + \frac{m}{2}\sum_{j=1}^{N}(-1)^{\frac{2j-N}{2}}\\
&\quad \times j!(N-j+1)!\epsilon^{ab}x_j^b x_{N-j+1}^a,\\
d &= \chi^2 - \frac{m}{2}\sum_{j=0}^{N}\left(-\frac{N}{2}+j\right)(-1)^{\frac{2j-N}{2}} j!(N-j)!\epsilon^{ab}x_j^b x_{N-j}^a,\\
k &= \chi^0 + \chi^1 - \frac{m}{2}\sum_{j=0}^{N-1}(-1)^{\frac{2j-N}{2}}\\
&\quad \times (j+1)!(N-j)!\epsilon^{ab}x_j^b x_{N-j-1}^a. \quad (35)
\end{aligned}$$

By direct, but rather tedious, computations, we check that the corresponding Casimir operators are of the form

$$\begin{aligned}
C_1 &= M, \qquad C_2 = MJ - \frac{1}{2}\sum_{j=0}^{N}\frac{(-1)^{\frac{2j-N}{2}}}{j!(N-j)!}C_{N-j}^a C_j^a,\\
C_3 &= (MH-A)(MK-B) + (MK-B)(MH-A)\\
&\quad - 2(MD-C)^2,
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{1}{2}\sum_{j=1}^{N}\frac{(-1)^{\frac{2j-N}{2}}}{(j-1)!(N-j)!}\epsilon^{ab}C_{j-1}^b C_{N-j}^a,\\
B &= -\frac{1}{2}\sum_{j=0}^{N-1}\frac{(-1)^{\frac{2j-N}{2}}}{j!(N-j-1)!}\epsilon^{ab}C_{j+1}^b C_{N-j}^a, \quad (36)\\
C &= \frac{1}{2}\sum_{j=0}^{N}\frac{(-1)^{\frac{2j-N}{2}}}{j!(N-j)!}\left(j-\frac{N}{2}\right)\epsilon^{ab}C_j^b C_{N-j}^a.
\end{aligned}$$

The induced Poisson brackets of $\vec{C}$'s take the form

$$\{c_j^a, c_k^b\} = -\epsilon^{ab}\delta^{N,j+k}(-1)^{\frac{k-j}{2}}k!j!m, \quad (37)$$

[for χ^μ see Eq. (14)]. Now let us define new coordinates as follows

$$\begin{aligned}
x_j^a &= \frac{(-1)^{\frac{N-2j}{2}}}{j!}q_j^a, \qquad j = 0,\ldots,\frac{N}{2}, \qquad a,b = 1,2;\\
x_{N-j}^a &= \frac{1}{m(N-j)!}p_j^a, \qquad j = 0,\ldots,\frac{N}{2}-1, \qquad a,b = 1,2.
\end{aligned} \quad (38)$$

Then, the nonvanishing Poisson brackets read

$$\begin{aligned}
\{q_j^a, p_k^b\} &= \delta^{ab}\delta_{jk}, \qquad j,k = 0,\ldots,\frac{N}{2}-1, \qquad a,b = 1,2;\\
\{q_{\frac{N}{2}}^a, q_{\frac{N}{2}}^b\} &= \frac{1}{m}\epsilon^{ba}, \qquad a,b = 1,2. \quad (39)
\end{aligned}$$

Let us introduce auxiliary notation (see Eq. (32) in Ref. [7]) $p_{\frac{N}{2}}^a = \frac{m}{2}\epsilon^{ba}q_{\frac{N}{2}}^b$. Then, the remaining dynamical variables take the form

$$\begin{aligned}
h &= \chi^0 - \chi^1 + \sum_{k=0}^{\frac{N}{2}-1}\vec{p}_k\vec{q}_{k+1},\\
d &= \chi^2 + \sum_{k=0}^{\frac{N}{2}-1}\left(\frac{N}{2}-k\right)\vec{p}_k\vec{q}_k,\\
k &= \chi^0 + \chi^1 - \sum_{k=1}^{\frac{N}{2}-1}(N-k+1)k\vec{p}_k\vec{q}_{k-1} \quad (40)\\
&\quad - N\left(\frac{N}{2}+1\right)\vec{q}_{\frac{N}{2}-1}\vec{p}_{\frac{N}{2}},\\
j &= s + \sum_{k=0}^{\frac{N}{2}}\vec{q}_k \times \vec{p}_k.
\end{aligned}$$

These results, in the case of trivial orbit $\mathcal{H}_0$, agree with the ones obtained in Ref. [7].

IV. CONCLUSIONS

We have shown that the most general dynamical system admitting the N-Galilean conformal group, with N-odd in the three-dimensional case and arbitrary N in the two-dimensional one, as the symmetry group acting transitively is described by the "external" variables obeying the equations of motion generated by the free higher-derivative Lagrangian and two kinds of "internal" ones: spin variables undergoing trivial dynamics and pseudospin ones obeying $SL(2,\mathbb{R})$-invariant equations of motion. The form of dynamical equations is uniquely determined by the symmetry group under consideration.

Let us discuss in some detail the dynamics of $SL(2,\mathbb{R})$ variables. It is, in fact, the standard conformal mechanics [9] in disguise. This has been shown in detail in Ref. [10]; here, we present only a brief discussion. First, let us note that the complete list of phase manifolds on which $SL(2,\mathbb{R})$ acts transitively as a group of canonical transformations is provided by Eq. (10). Let us take as an example $\mathcal{H}_\sigma^+$. It is two-sheeted hyperboloid equipped with the Poisson structure defined by Eq. (14). Consider the upper sheet (the lower can be described in a similar way [10]). Due to its trivial topological structure, it is not difficult to find the single global smooth map covering the whole sheet; it reads [10]

$$\chi^0 = \frac{P^2}{4\mu} + \frac{\sigma^2}{\mu X^2} + \frac{\mu X^2}{4},$$
$$\chi^1 = -\frac{P^2}{4\mu} - \frac{\sigma^2}{\mu X^2} + \frac{\mu X^2}{4}, \qquad \chi^2 = -\frac{XP}{2}, \tag{41}$$

where $0 < X < \infty$, $-\infty < P < \infty$, and μ is an arbitrary positive constant. Moreover, X and P are global Darboux coordinates, $\{X, P\} = 1$. It follows from (29) that the pseudospin contribution to the total Hamiltonian h, when expressed in terms of new coordinates, reads:

$$h_\chi = \frac{P^2}{2\mu} + \frac{2\sigma^2}{\mu X^2}, \tag{42}$$

i.e., we arrive at the standard form of conformal mechanics with positive coupling constant [9].

The case of one-sheeted hyperboloid is slightly more involved. The phase space cannot be covered by one Darboux map; however, the dynamics is completely regular, as it can be seen from Eq. (15). Locally, the Darboux coordinates can be introduced by making the replacement $\sigma^2 \mapsto -\sigma^2$ in Eq. (41). They yield, again, the standard form of conformal mechanics with negative coupling constant. It is interesting to note that, in this context, the "falling on the center" phenomenon that appears in the attractive case is an artifact produced by the nontrivial topology of phase manifold which does not admit global coordinates.

Let us also note that the geometrical structure of conformal mechanics may be expressed [13] in terms of nonlinear realizations [14] of $SL(2,\mathbb{R})$ group. This is a particular example of the general fact that the orbit method can be described using the ideas and methods of nonlinear realizations theory [15].

Let us come back to the simple case of pseudospin living on $\mathcal{H}_\sigma^+$. Then we can summarize our findings as follows (we restrict ourselves to the case $d = 3$, i.e., N-odd): the basic dynamical variables are the coordinate $\vec{q}$, spin $\vec{s}$, and the additional coordinate X obeying $0 < X < \infty$. The dynamics is given by the higher-derivative Lagrangian

$$L = \frac{m}{2}\left(\frac{d^{(N+1)/2}\vec{q}}{dt^{(N+1)/2}}\right)^2 + \frac{\mu}{2}\left(\frac{dX}{dt}\right)^2 - \frac{2\sigma^2}{\mu X^2}. \tag{43}$$

For the remaining cases [i.e. remaining orbits (10)], similar description can be given except that the internal coordinate is not global.

As in the case of Schrödinger algebra [cf. Eq. (16)], it is easy to construct the explicitly time-dependent integrals of motion. They are generators (in the sense of canonical formalism) of the relevant symmetry transformations.

We conclude that the general dynamical system admitting N-Galilean conformal symmetry with N-odd (N-even in dimension two) as the symmetry group acting transitively is described by the "external" variables corresponding to higher derivative Lagrangian and two kinds of "internal" ones: spin variables $\vec{s}$ (s, respectively) with trivial dynamics and $SL(2,\mathbb{R})$ pseudospin variables χ^μ with the nontrivial conformal invariant one.

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