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No-go on strictly stationary spacetimes in four/higher dimensions

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We show that strictly stationary spacetimes cannot have nontrivial configurations of form fields/ complex scalar fields. This means that the spacetime should be exactly Minkowski or anti-deSitter spacetime depending on the presence of negative cosmological constant. That is, self-gravitating complex scalar fields and form fields cannot exist.

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I. INTRODUCTION

Whether the self-gravitating and stationary/static configuration exists or not is a fundamental issue in general relativity. The famous and elegant Lichnerowicz theorem tells us that the vacuum and strictly stationary spacetimes should be static [1]. The phrase "strict stationarity" means that the existence of the timelike Killing vector field in the whole region of the spacetime is assumed (no black holes!). Since the total mass of the spacetime is zero, the positive mass theorem shows us that the spacetime should be the Minkowski spacetime [2,3]. The similar discussion has been extended into asymptotically anti-deSitter spacetimes [4]. But, we know that there is the static nontrivial solution for the Einstein-Yang-Mills systems [5] (see Ref. [6] and references therein). Holographic argument of condensed matters relied on nontrivial self-gravitating configurations in asymptotically anti-deSitter spacetimes (see Ref. [7] for a review).

Recently, there are interesting new issues on self-gravitating objects and final fate of gravitational collapse in asymptotically anti-deSitter spacetimes [8–10] (see also Refs. [11,12]). Nonstationary numerical solutions with 1-Killing vector field were found in the Einstein-complex scalar system [10]. (This is a kind of boson star. For boson stars, see Ref. [13] and reference therein). A kind of no-go theorem or Lichnerowicz-type theorem is also important. This is because they provide us some definite information about the above issue in an implicit way.

In this paper we present a no-go theorem for nontrivial self-gravitating configurations composed of p-form fields/ complex scalar fields in strictly stationary spacetimes. We will consider asymptotically flat or anti-deSitter spacetimes. This no-go does not contradict with the results of recent works [8–10] because the spacetimes considered there are not strictly stationary or there is some coupling between gauge fields and scalar fields, etc. See Ref. [14] for related issues about static configurations.

The organization of this paper is as follows. In Sec. II, we discuss the strictly stationary spacetimes in four dimensions, and show that the Maxwell field and complex scalar field cannot have nontrivial configuration. In Sec. III, we generalized this into higher dimensions with p-form fields and complex scalar fields. In Appendix A, we present the technical details. In Appendix B, we discuss an alternative argument for asymptotically anti-deSitter spacetimes.

II. FOUR DIMENSIONS

Bearing the recent work [10] in mind, we consider the following system:

$$L = R - \frac{1}{2}F^2 - 2|\partial\pi|^2 - 2\Lambda, \tag{1}$$

where F and π are the field strength of the Maxwell field and a complex scalar field, respectively. There is no source term for the Maxwell field and no potential of π. The Einstein equations are

$$R_{ab} = F_a{}^c F_{bc} - \frac{1}{4}g_{ab}F^2 + \partial_a\pi\partial_b\pi^* + \partial_a\pi^*\partial_b\pi + \Lambda g_{ab}. \tag{2}$$

Let us focus on the strictly stationary spacetimes; that is, we assume that there is a timelike Killing vector field k^a everywhere. In addition, we assume that the Maxwell field and complex scalar field are also stationary, $\mathcal{L}_k F = 0$, $k^a\partial_a\pi = 0$. Then we see $\mathcal{L}_k T_{ab} = 0$, which is consistent with the spacetime stationarity.

The twist vector ω^a is defined by

$$\omega_a = \frac{1}{2}\epsilon_a{}^{bcd}k_b\nabla_c k_d. \tag{3}$$

Then, from the definition of ω_a, one can show

$$\nabla_a(\omega^a V^{-4}) = 0, \tag{4}$$

where $V^2 = -k_a k^a$. We introduce the electric and magnetic components of the Maxwell field as

$$E_a = k^b F_{ba} \tag{5}$$

and

$$B_a = -\frac{1}{2}\epsilon_{abcd}F^{bc}k^d, \tag{6}$$

respectively. Using E^a and B^a, the field strength of the Maxwell field is written as

$$V^2 F_{ab} = -2k_{[a}E_{b]} + \epsilon_{abcd}k^c B^d. \qquad (7)$$

The source-free Maxwell equation becomes

$$\nabla_{[a}E_{b]} = 0, \qquad (8)$$

$$\nabla_{[a}B_{b]} = 0, \qquad (9)$$

$$\nabla_a(E^a V^{-2}) - 2\omega_a B^a V^{-4} = 0, \qquad (10)$$

and

$$\nabla_a(B^a V^{-2}) + 2\omega_a E^a V^{-4} = 0. \qquad (11)$$

From the first two equations, we see that E^a and B^a have the potentials as

$$E_a = \nabla_a \Phi, \qquad B_a = \nabla_a \Psi. \qquad (12)$$

Here, to guarantee the existence of the potentials globally, we assumed that spacetime manifolds are contractible. Hereafter we assume this. Using the Einstein equations, we can show that

$$\nabla_{[a}\omega_{b]} = B_{[a}E_{b]} \qquad (13)$$

holds. The right-hand side is the Poynting flux. To show the above, we used the stationarity of the complex scalar field. Using the fact that the electric and magnetic fields are written in terms of the potential for each, we can rewrite the above in several different ways; for example, we have the following typical two equations

$$\nabla_{[a}(\omega_{b]} - \Psi E_{b]}) = 0 \qquad (14)$$

and

$$\nabla_{[a}(\omega_{b]} + \Phi B_{b]}) = 0. \qquad (15)$$

Therefore, the existence of scalar functions are guaranteed as

$$\omega_a - \Psi E_a = \nabla_a U_E \qquad (16)$$

and

$$\omega_a + \Phi B_a = \nabla_a U_B. \qquad (17)$$

Then, using the Maxwell equation, Eqs. (4), (16), and (17) give us

$$\nabla_a\left(U_E \frac{\omega^a}{V^4} - \frac{\Psi}{2V^2}B^a\right) = \frac{\omega_a\omega^a}{V^4} - \frac{B_a B^a}{2V^2} \qquad (18)$$

and

$$\nabla_a\left(U_B \frac{\omega^a}{V^4} - \frac{\Phi}{2V^2}E^a\right) = \frac{\omega_a\omega^a}{V^4} - \frac{E_a E^a}{2V^2}. \qquad (19)$$

They correspond to Eq. (6.35) on page 156 of Ref. [15] which has a sign error (for example, see Ref. [16]).

On the other hand, the Einstein equations give us

$$\frac{2}{V^2}R_{ab}k^a k^b = \nabla_a\left(\frac{\nabla^a V^2}{V^2}\right) + 4\frac{\omega_a\omega^a}{V^4} = -2\Lambda + \frac{E_a E^a + B_a B^a}{V^2}. \qquad (20)$$

Note that this corresponds to the Raychaudhuri equation for nongeodesics.

It is easy to see that Eqs. (18)–(20) imply

$$\nabla_a\left(\frac{\nabla^a V^2}{V^2} + W^a\right) = -2\Lambda, \qquad (21)$$

where

$$W^a = 2(U_E + U_B)\frac{\omega^a}{V^4} - \frac{\Psi B^a + \Phi E^a}{V^2}. \qquad (22)$$

Let us first consider the cases with $\Lambda = 0$. Then we see

$$\nabla_a\left(\frac{\nabla^a V^2}{V^2} + W^a\right) = 0. \qquad (23)$$

The space volume integral of the above implies the surface integral. Here note that W^a does not contribute to the surface integral because we can easily see that it behaves like $W^a = O(1/r^3)$ near the infinities ($r \to \infty$) and the contribution to the surface integral becomes $O(1/r)$. Thus, we see that it will be the total mass and then

$$M = 0 \qquad (24)$$

(see Appendix A). Here, note that the positive mass theorem holds because the dominant energy condition is satisfied. Thus, the corollary of the positive mass theorem tells us that the spacetime should be the Minkowski spacetime [2,3]. This means that the electromagnetic fields and complex scalar field vanish.

Next we consider the cases with $\Lambda < 0$. Introducing the vector field, r^a, satisfying

$$\nabla_a r^a = -2\Lambda, \qquad (25)$$

we find

$$\nabla_a\left(\frac{\nabla^a V^2}{V^2} - r^a + W^a\right) = 0. \qquad (26)$$

The volume integral shows us

$$M = 0 \qquad (27)$$

again. Then the positive mass theorem implies that the spacetime should be the exact anti-deSitter spacetime [17,18]. In Appendix A, we discuss the existence of r^a satisfying Eq. (25). However, one may not want to introduce this r^a. This is possible for a restricted case. In Appendix B, we present an alternative proof for asymptotically anti-deSitter spacetimes. The price we have to pay is that we cannot include the Maxwell field in the argument.

One may wonder if one can extend this result to the cases with positive cosmological constant, $\Lambda > 0$. Although there are some efforts [19], we do not have the positive mass theorem which holds for general asymptotically deSitter spacetimes. Thus, we cannot have the same statement with our current result.

Note that almost of all basic equations presented here were derived in Ref. [20]. However, these equations were applied to the stationary *axisymmetric* black holes to derive the Smarr formula and so on. On the other hand, we focused on the strictly stationary spacetimes which are not restricted to be axisymmetric in general and do not contain black holes.

III. HIGHER DIMENSIONS

Let us examine the same issue in higher dimensions. The Lagrangian we consider is

$$\mathcal{L} = R - \frac{1}{p!}H_{(p)}^2 - 2|\partial\pi|^2 - 2\Lambda, \tag{28}$$

where $H_{(p)}$ is the field strength of a $(p-1)$-form field potential and π is a complex scalar field. We consider the strictly stationary spacetimes, p-form fields, and complex scalar fields $\mathcal{L}_k H_{(p)} = 0$, $\mathcal{L}_k \pi = 0$.

The Einstein equations become

$$R_{ab} = \frac{1}{p!}\left(pH_a{}^{c_1\cdots c_{p-1}}H_{bc_1\cdots c_{p-1}} - \frac{p-1}{n-2}g_{ab}H_{(p)}^2\right) + \partial_a\pi\partial_b\pi^* + \partial_a\pi^*\partial_b\pi + \frac{2}{n-2}\Lambda g_{ab}. \tag{29}$$

The field equations for the source free p-form field are

$$\nabla_a H^{a_1\cdots a_{p-1}a} = 0 \tag{30}$$

and the Bianchi identity. Let us decompose the p-form field strength into the electric ($E_{a_1\cdots a_{p-1}}$) and magnetic parts ($B_{a_1\cdots a_{n-p-1}}$) as

$$V^2 H_{a_1\cdots a_p} = -pk_{[a_1}E_{a_2\ldots a_p]} + \epsilon_{a_1\cdots a_p a_{p+1}a_{p+2}\cdots a_n}k^{a_{p+1}}B^{a_{p+2}\cdots a_n}, \tag{31}$$

where we define each component by

$$E_{a_1\ldots a_{p-1}} = H_{aa_1\cdots a_{p-1}}k^a \tag{32}$$

and

$$B_{a_1\ldots a_{n-p-1}} = \frac{1}{p!(n-p-1)!}\epsilon_{b_1\cdots b_p c a_1\cdots a_{n-p-1}}k^c H^{b_1\cdots b_{p1}}. \tag{33}$$

Here we define the twist tensor $\omega_{a_1\cdots a_{n-3}}$ as

$$\omega_{a_1\cdots a_{n-3}} = \alpha\epsilon_{a_1\cdots a_{n-3}bcd}k^b\nabla^c k^d, \tag{34}$$

where α is a constant. From the definition of the twist, it is easy to check that

$$\nabla_{a_{n-3}}\left(\frac{\omega^{a_1\cdots a_{n-3}}}{V^4}\right) = 0 \tag{35}$$

holds. From the field equations, we have

$$\nabla_{[a_1}E_{a_2\cdots a_p]} = 0 \tag{36}$$

and

$$\nabla_{[a_1}B_{a_2\cdots a_{n-p}]} = 0. \tag{37}$$

Then there are the potentials, that is,

$$E_{a_1\cdots a_{p-1}} = \nabla_{[a_1}\Phi_{a_2\cdots a_{p-1}]} \tag{38}$$

and

$$B_{a_1\cdots a_{n-p-1}} = \nabla_{[a_1}\Psi_{a_2\cdots a_{n-p-1}]}. \tag{39}$$

It is seen from the definition of $E_{a_1\cdots a_{p-1}}$ and $B_{a_1\cdots a_{n-p-1}}$ that $k^{a_1}\Phi_{a_1\ldots a_{p-2}} = 0$ and $k^{a_1}\Psi_{a_1\ldots a_{n-p-2}} = 0$ hold. The other field equations give us

$$\Phi_{a_1\cdots a_{p-2}}\nabla_{a_{p-1}}\left(\frac{E^{a_1\ldots a_{p-2}a_{p-1}}}{V^2}\right) = \alpha^{-1}(-1)^n\omega^{a_1\cdots a_{p-2}b_1\cdots b_{n-p-1}}\Phi_{a_1\cdots a_{p-2}}\frac{B_{b_1\cdots b_{n-p-1}}}{V^4} \tag{40}$$

and

$$\Psi_{a_1\cdots a_{n-p-2}}\nabla_a\left(\frac{B^{a_1\ldots a_{n-p-2}a}}{V^2}\right) = -\alpha^{-1}\frac{(-1)^p}{(n-p-1)!(p-1)!} \times \omega^{b_1\cdots b_{p-1}a_1\cdots a_{n-p-2}}\Psi_{a_1\cdots a_{n-p-2}}\frac{E_{b_1\cdots b_{p-1}}}{V^4}. \tag{41}$$

Using the Einstein equations, we can show

$$\alpha^{-1}\epsilon^{abcd_1\cdots d_{n-3}}\nabla_c\omega_{d_1\cdots d_{n-3}} = 2(n-3)!(-1)^n(k^aR^b{}_c - k^bR^a{}_c)k^c = -\frac{2(-1)^{n+p}(n-3)!}{(p-1)!}\epsilon^{abcd_1\cdots d_{n-3}}E_{cd_1\cdots d_{p-2}}B_{d_{p-1}\cdots d_{n-3}}. \tag{42}$$

Note that we used the stationarity of the complex scalar field. Then we see that there are the $(n-4)$ forms U^E and U^B satisfying

$$\nabla_{[a_1}U^E_{a_2\cdots a_{n-3}]} = \omega_{a_1\cdots a_{n-3}} - \frac{(-1)^n 2\alpha(n-3)!}{(p-1)!}E_{[a_1\cdots a_{p-1}}\Psi_{a_p\cdots a_{n-3}]} \tag{43}$$

and

$$\nabla_{[a_1} U^B_{a_2\cdots a_{n-3}]} = \omega_{a_1\cdots a_{n-3}} + \frac{(-1)^{n+p} 2\alpha(n-3)!}{(p-1)!}\Phi_{[a_1\cdots a_{p-2}} B_{a_{p-1}\cdots a_{n-3}]}, \tag{44}$$

respectively. Using $U^{E,B}$ and Eq. (35), we can have the following equations:

$$\nabla_a\left(U^E_{a_1\cdots a_{n-4}}\frac{\omega^{a_1\cdots a_{n-4}a}}{V^4} - (-1)^p\alpha^2\beta\frac{\Psi_{a_1\cdots a_{n-p-2}}B^{a_1\cdots a_{n-p-2}a}}{V^2}\right) = (-1)^n\left(\frac{\omega^2}{V^4} - \alpha^2\beta\frac{B^2}{V^2}\right) \tag{45}$$

and

$$\nabla_a\left(U^B_{a_1\cdots a_{n-4}}\frac{\omega^{a_1\cdots a_{n-4}a}}{V^4} - (-1)^{n+p}\alpha^2\gamma\frac{\Phi_{a_1\cdots a_{p-2}}E^{a_1\cdots a_{p-2}a}}{V^2}\right) = (-1)^n\left(\frac{\omega^2}{V^4} - \alpha^2\gamma\frac{E^2}{V^2}\right), \tag{46}$$

where $\beta = 2(n-3)!(n-p-1)!$ and $\gamma = 2(n-3)!/(p-1)!$. On the other hand, the Einstein equations give us

$$\frac{2}{V^2}R_{ab}k^a k^b = \nabla_a\left(\frac{\nabla^a V^2}{V^2}\right) + \frac{1}{\alpha^2(n-3)!}\frac{\omega^2}{V^4} = \frac{2(n-p-1)}{(p-1)!(n-2)}\frac{E^2}{V^2} + \frac{2(p-1)(n-p-1)!}{n-2}\frac{B^2}{V^2} - \frac{4}{n-2}\Lambda. \tag{47}$$

Then, together with Eqs. (45) and (46), this implies

$$\nabla_a\left(\frac{\nabla^a V^2}{V^2} + X^a\right) = -\frac{4}{n-2}\Lambda, \tag{48}$$

where X^a is defined by

$$X^a = \frac{(-1)^n}{\alpha^2(n-2)!}((p-1)U^E_{a_1\cdots a_{n-4}} + (n-p-1)U^B_{a_1\cdots a_{n-4}})\frac{\omega^{a_1\cdots a_{n-4}a}}{V^4} - (-1)^{n+p} \times \frac{2(p-1)(n-p-1)!}{n-2}\frac{\Psi_{a_1\cdots a_{n-p-2}}B^{a_1\cdots a_{n-p-2}a}}{V^2} - (-1)^p\frac{2(n-p-1)}{(p-1)!(n-2)}\frac{\Phi_{a_1\cdots a_{p-2}}E^{a_1\cdots a_{p-2}a}}{V^2}. \tag{49}$$

In a similar argument with the four-dimensional case, we can show that the mass vanishes. Then, using the positive mass theorem in higher dimensions,[1] we can see that the spacetime is exactly Minkowski/anti-deSitter spacetime depending on the presence of the negative cosmological constant. In any cases, the p-form fields and complex scalar fields vanish. For the asymptotically anti-deSitter case, we had to introduce the vector field r^a satisfying

$$\nabla_a r^a = -\frac{4}{n-2}\Lambda. \tag{50}$$

The existence of this r^a is discussed in Appendix A. As in the four dimensions, the argument without introducing r^a is given in Appendix B.

IV. SUMMARY AND DISCUSSION

In this paper, we showed that strictly stationary spacetimes with p-form and complex scalar fields should be Minkowski or anti-deSitter spacetime depending on the presence of the negative cosmological constant.

From our result, there is no room to have the self-gravitating solution composed of complex scalar fields in strictly stationary spacetimes. Therefore, if one wishes to explore a new solution, one has to think of a setup that breaks some of the assumptions imposed here. For example, we can find a new configuration which is nonstationary, but has a nonstationary 1-Killing vector field [10].

For asymptotically anti-deSitter spacetimes, we had to introduce a vector r^a to show the no-go. As shown in Appendix B, there is a way to avoid this additional treatment for the Einstein-complex scalar system. However, it is quite hard to extend this into the cases with p-form fields. This is left for future work.

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APPENDIX A: EVALUATION OF BOUNDARY TERM

In this Appendix, we present details of the calculation of the integral of Eqs. (26) and (48). Since the argument for

[1] If one considers spin manifolds, the positive mass theorem is easily proven as in the four-dimensional Witten's version [3]

asymptotically flat spacetimes is included in asymptotically anti-deSitter cases, we will focus on the latter.

In asymptotically anti-deSitter spacetimes, the leading behavior of the metric is

$$ds^2 = -V^2dt^2 + V^{-2}dr^2 + r^2d\Omega_{n-2}^2 + \cdots, \tag{A1}$$

where

$$V^2 = 1 - \frac{2M}{r^{n-3}} - \frac{2}{(n-2)(n-1)}\Lambda r^2. \tag{A2}$$

Let us consider the vector r^a satisfying

$$\nabla_a r^a = -\frac{4}{n-2}\Lambda. \tag{A3}$$

Near the infinity, r^a will be given by

$$r^a \simeq -\frac{4}{(n-2)(n-1)}\Lambda r(\partial_r)^a + \cdots. \tag{A4}$$

The global existence of r^a is guaranteed as follows. Without a loss of generality, one can suppose the form of $r^a = \nabla^a\varphi$. Then the above equation becomes $\nabla^2\varphi = -2\Lambda$. We can redefine φ so that -2Λ is subtracted and then $\nabla^2\tilde{\varphi} = S$, where S is a nonsingular source term. The existence of the solution to this is a well-known fact in regular Riemannian manifolds. Therefore, we can always introduce that r^a in general.

For the vectors V^a satisfying $V^a k_a = 0$, the spacetime divergence is written as

$$\nabla_a V^a = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}V^\mu) = \frac{1}{V\sqrt{q}}\partial_i(V\sqrt{q}V^i), \tag{A5}$$

where q is the determinant of the spacial metric and the index i stands for the space component. Therefore, the volume integral of the left-hand side of Eqs. (26) or (48) becomes the surface integral and then it is evaluated as

$$\begin{aligned}\int_{S_\infty}&(\partial_i V^2 - Vr_i)dS^i \\ &= \omega_{n-2}r^{n-2}\left(\partial_r V^2 + \frac{4}{(n-2)(n-1)}\Lambda r\right) \\ &= 2(n-3)\omega_{n-2}M,\end{aligned} \tag{A6}$$

where ω_{n-2} is the volume of the unit $(n-2)$-dimensional sphere.

For asymptotically flat spacetimes, we do not introduce r^a and the evaluation at the boundary is the same as the above in the limit of $\Lambda = 0$.

APPENDIX B: ALTERNATIVE PROOF

For asymptotically anti-deSitter spacetimes, one may want to avoid introducing the vector r^a satisfying Eq. (50). This is because its introduction is artificial. Therefore, we try to present an alternative proof. Here we focus on the Einstein gravity with complex scalar fields. In the following proof, unfortunately, we cannot include the p-form fields. The argument here basically follows Ref. [21] which was devoted to the vacuum case (see also Ref. [22]).

In the absence of the p-form fields, the volume integrals of Eqs. (18) and (45) give us

$$\omega_{a_1\cdots a_{n-2}} = 0 \tag{B1}$$

and then the spacetimes must be static. Here we assumed that the scalar fields are also stationary, $\mathcal{L}_k\pi = 0$. Then we can employ the following metric form:

$$ds^2 = -V^2(x^i)dt^2 + g_{ij}(x^k)dx^i dx^j. \tag{B2}$$

The Ricci tensor becomes

$$R_{00} = VD^2V = \frac{n-1}{\ell^2}V^2 + S_{00,} \tag{B3}$$

$$R_{ij} = {}^{(n-1)}R_{ij} - \frac{1}{V}D_iD_jV = -\frac{n-1}{\ell^2}g_{ij} + S_{ij}, \tag{B4}$$

where $\ell^{-2} = -2\Lambda/(n-1)(n-2)$ and

$$S_{ab} = T_{ab} - \frac{1}{n-2}g_{ab}T = \partial_a\pi\partial_b\pi^* + \partial_a\pi^*\partial_b\pi. \tag{B5}$$

For the current case, $S_{00} = 0$ and $S_{ij} = D_i\pi D_j\pi^* + D_i\pi^* D_j\pi$.

The $(n-1)$-dimensional Ricci scalar becomes

$$\begin{aligned}{}^{(n-1)}R &= -\frac{(n-1)(n-2)}{\ell^2} + \frac{1}{V^2}S_{00} + S_i^i \\ &= -\frac{(n-1)(n-2)}{\ell^2} + 2|D\pi|^2.\end{aligned} \tag{B6}$$

Using the Einstein equations, we have the equation

$$\begin{aligned}D^2\psi &- V^{-1}D_iVD^i\psi \\ &= 2\left(D_iD_jV - \frac{1}{\ell^2}g_{ij}V\right)^2 + 2S^{ij}D_iVD_jV \\ &\quad + \frac{2D^iV}{V}D_iS_{00} - \frac{2}{V^2}S_{00}(DV)^2 + \frac{2}{\ell^2}S_{00} \\ &= 2\left(D_iD_jV - \frac{1}{\ell^2}g_{ij}V\right)^2 + 4|D^iVD_i\pi|^2 \geq 0,\end{aligned} \tag{B7}$$

where $\psi = (DV)^2 + \ell^{-2}(1-V^2)$. Since $\psi \to 0$ as $r \to \infty$, the maximum principle implies $\psi \leq 0$. Thus, we have the following inequality:

$$(DV)^2 \leq (V^2 - 1)\ell^{-2}. \tag{B8}$$

We will use this soon.

Let us perform the conformal transformation defined by

$$\bar{g}_{ij} = (1+V)^{-2}g_{ij}. \tag{B9}$$

Then we see that the Ricci scalar of $\bar{g}_{ij}$ is non-negative,

$$\begin{aligned}
&{}^{(n-1)}\bar{R}(1+V)^{-2}\\
&= {}^{(n-1)}R + 2(n-2)\frac{D^2V}{V+1} - (n-1)(n-2)\frac{(DV)^2}{(1+V)^2}\\
&= \left(\frac{1}{V^2} + \frac{2(n-2)}{V(V+1)}\right)S_{00} + S_i^i\\
&\quad + \frac{(n-1)(n-2)}{\ell^2}\left(-\frac{1-V}{1+V} - \ell^2\frac{(DV)^2}{(1+V)^2}\right)\\
&= -\frac{(n-1)(n-2)}{(1+V)^2}\psi + 2|D\pi|^2 \geq 0. \qquad \text{(B10)}
\end{aligned}$$

The $r = \infty$ boundary is the unit sphere in the conformally transformed space and the trace of the extrinsic curvature is $\bar{k}|_{r=\infty} = n-2$. Thus, the conformally transformed space is a compact Riemannian manifold, $\bar{M}$, with the boundary of the unit sphere, $\partial\bar{M} =: S^{n-2}$. Pasting the flat space removing the unit ball with $\bar{M}$ along S^{n-2}, we can construct the manifold with the zero mass. Since the Ricci scalar is non-negative there, we can apply the positive mass theorem [2,23] and then see that the space should be flat. This means $\bar{R} = 0$ and then Eq. (B10) implies

$$D_i\pi = 0 \qquad \text{(B11)}$$

and

$$\psi = 0. \qquad \text{(B12)}$$

Then Eq. (B7) implies

$$D_iD_jV = \frac{1}{\ell^2}g_{ij}V. \qquad \text{(B13)}$$

Therefore, ${}^{(n-1)}R_{ij} = -(n-2)\ell^{-2}g_{ij}$. Due to the conformal flatness, the Weyl tensor with respect to g_{ij} is zero and then the Riemann tensor becomes ${}^{(n-1)}R_{ijkl} = -\frac{1}{\ell^2}(g_{ik}g_{jl} - g_{il}g_{jk})$. Using the Einstein equations, we can compute the Riemann tensor of spacetime as follows:

$$R_{0i0j} = VD_iD_jV = \frac{V^2}{\ell^2}g_{ij} = -\frac{1}{\ell^2}g_{00}g_{ij}, \qquad \text{(B14)}$$

$$R_{ijkl} = {}^{(n-1)}R_{ijkl} = -\frac{1}{\ell^2}(g_{ik}g_{jl} - g_{il}g_{jk}). \qquad \text{(B15)}$$

Therefore, $R_{abcd} = -\ell^{-2}(g_{ac}g_{bd} - g_{ad}g_{bc})$ holds and then the spacetime is exactly anti-deSitter spacetime.

Once the p-form fields are turned on, we cannot show the inequality as Eqs. (B7) and (B10). So the current proof does not work for such cases.

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