

**Regularization of static self-forces**

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 (Received 17 June 2012; published 18 September 2012)*

Various regularization methods have been used to compute the self-force acting on a static particle in a static, curved spacetime. Many of these are based on Hadamard's two-point function in three dimensions. On the other hand, the regularization method that enjoys the best justification is that of Detweiler and Whiting, which is based on a four-dimensional Green's function. We establish the connection between these methods and find that they are all equivalent, in the sense that they all lead to the same static self-force. For general static spacetimes, we compute local expansions of the Green's functions on which the various regularization methods are based. We find that these agree up to a certain high order, and conjecture that they might be equal to all orders. We show that this equivalence is exact in the case of ultrastatic spacetimes. Finally, our computations are exploited to provide regularization parameters for a static particle in a general static and spherically symmetric spacetime.

DOI: [10.1103/PhysRevD.86.064033](https://doi.org/10.1103/PhysRevD.86.064033)

PACS numbers: 04.20.-q, 04.40.-b, 41.20.Cv

**I. INTRODUCTION**

A test body moving freely in a curved spacetime follows a geodesic of the spacetime. When, however, the body carries a (scalar or electric) charge, the field created by the charge interacts with the spacetime curvature in such a way as to produce a deformation of the field lines from an otherwise isotropic distribution around the body. The field gives rise to a net self-force acting on the body, and the self-force prevents it from moving on a geodesic. The self-force typically contains two components, a radiation-reaction force that is accompanied by a loss of energy to radiation, and a conservative force that survives even when the body is maintained in a stationary position. A self-force can also be present in the absence of a charge, when the body's mass is too large for it to be considered a test mass; in this case the body creates a gravitational perturbation that affects its motion, which is no longer geodesic in the background spacetime. The (scalar, electromagnetic, and gravitational) self-force has been the topic of intense development in the last several years; for an extensive review see Ref. [1]. Most of this activity was focused on the gravitational case, in an effort to model the inspiral and gravitational-wave emissions of a binary system with a small mass ratio [2–4].

Self-force computations are usually attempted under the assumption that the body is a point particle, in order to avoid the largely irrelevant complications associated with internal structure. In this context, however, the very definition of the self-force requires scrutiny. Given that the field of a point particle diverges at the position occupied by the particle, it is not immediately clear how one can make

sense of its action on the particle and construct a self-force that is well defined, finite, and in agreement with the self-force acting on an extended body in the limit in which the size is taken to zero. One must find a sensible regularization procedure that not only returns a finite expression for the self-force but does so in a unique and physically well-motivated way. In this paper we examine regularization procedures that have been invoked in the computation of (scalar and electromagnetic) self-forces in the restricted context of static particles in static spacetimes. Our aim is to show that the differing procedures are equivalent and lead to the same self-force. To the best of our knowledge, this issue has not been previously addressed in the literature.

The first attempt to regularize the field of a point particle in curved spacetime to obtain its self-force was made by DeWitt and Brehme [5] in the case of an electric charge. Their formulation was later put on an axiomatic basis by Quinn and Wald [6], and the case of a scalar charge was subsequently examined by Quinn [7]. These regularization techniques were based on a decomposition of the field into “direct” and “tail” parts, with the direct part sourced by the particle at the moment it crosses the past light cone of the point at which the field is evaluated, and the tail part sourced by the particle in the interior of the light cone.

An alternative regularization scheme was later proposed by Detweiler and Whiting [8], who showed that it leads to the same standard expression for the self-force. The Detweiler-Whiting scheme involves a decomposition of the field into “singular” and “regular” pieces. The singular field is precisely identified by a local construction and is designed to provide an exact solution to the field equation

sourced by the particle, with the property that it shares the singularity structure of the particle's actual field. The regular field is the difference between the actual field and the singular field; it satisfies the source-free field equation, it is smooth at the particle's position, and it was shown by Detweiler and Whiting to be entirely responsible for the self-force. Because the singular and regular fields are both solutions to the field equations, unlike the direct and tail fields, which are not solutions, the Detweiler-Whiting scheme is more compelling than the original method of regularization, and it provides a better foundation for the self-force. Since its introduction, the Detweiler-Whiting regularization method has been thoroughly justified [9–12], and it has emerged as the method of choice in most self-force computations reported in the recent literature. It has become the standard by which other regularization methods must be compared.

Many self-force computations, however, did not make use of the Detweiler-Whiting regularization procedure nor the original method based on the direct and tail decomposition of the field, but employed instead *ad hoc* procedures that perhaps do not enjoy the same degree of justification. This is the case of all computations of self-forces acting on static particles in static spacetimes [13–27], which involved a variety of regularization methods. In the pioneering Smith-Will paper [13], for example, the field of a static electric charge in the spacetime of a Schwarzschild black hole was regularized by the Copson solution [28], which was shown to be as singular as the particle's own field but to exert no force. As other examples, self-force computations for charges in wormhole spacetimes [20–23], or for charges near global monopoles [24–27], were regularized with the help of Hadamard's two-point function, defined in each spatial section of the four-dimensional spacetime (or in a conformally related space). Because Copson's solution is known to be an exact representation of Hadamard's function in the (conformally related) spatial sections of the Schwarzschild spacetime, these regularization methods are essentially the same.

The issue that interests us in this paper is the relationship between these regularization procedures and whether they can be shown to be equivalent, so that they will lead to the same self-force. The regularization procedures mentioned previously are all based on a choice of Green's function for the (scalar or electromagnetic) field. We shall consider a number of possible choices.

The first is the four-dimensional version of the Detweiler-Whiting singular Green's function, given by

$$G_4^S(x, x') = \frac{1}{2}U(x, x')\delta(\sigma) - \frac{1}{2}V(x, x')\Theta(\sigma), \quad (1.1)$$

in which  $x$  and  $x'$  are spacetime events, assumed to be sufficiently close that they are within each other's normal convex neighborhood,  $\sigma := \sigma(x, x')$  is Synge's world function, equal to half the squared geodesic distance between  $x$  and  $x'$ ,  $\Theta$  is the Heaviside step function,  $\delta$  is the

Dirac distribution, and  $U, V$  are two-point functions that are known to be smooth when  $x \rightarrow x'$ . Because  $\sigma = 0$  when  $x$  and  $x'$  are linked by a null geodesic, we see that the Green's function is singular on the past and future light cones emerging from  $x'$  and has support outside the light cones, where  $\sigma > 0$ ; it is also symmetric in its arguments. As stated previously, the Detweiler-Whiting Green's function gives rise to a robust regularization procedure that applies to any particle moving in any spacetime.

For static particles in static spacetimes, an adequate substitute for the four-dimensional singular Green's function is its three-dimensional variant

$$G_3^S(x, x') = \int G_4^S(x, x')d\tau', \quad (1.2)$$

obtained by integrating the four-dimensional Green's function over the proper time  $\tau'$  of a static observer at the spatial position  $x'$ . For static particles in static spacetimes, this Green's function gives rise to the same regularization procedure as the four-dimensional version.

An alternative choice of Green's function, which is also appropriate in the case of static particles in static spacetimes, is the three-dimensional Hadamard function, given by

$$G_3^H(x, x') = \frac{W(x, x')}{\sqrt{2\sigma}}, \quad (1.3)$$

in which  $\sigma(x, x')$  is now half the squared geodesic distance between  $x$  and  $x'$  as measured in the purely spatial sections of the spacetime, and  $W$  is a smooth two-point function. This Green's function, also known as Hadamard's elementary solution [29], is a local construction that is known to reproduce the singular behavior of a field sourced by a point particle at  $x'$ . It is a plausible starting point for a regularization procedure, but as stated above, it does not enjoy the same level of justification as the Detweiler-Whiting Green's function.

As a final choice we shall also consider a Green's function  $\tilde{G}_3^H(x, x')$  that is related to the Hadamard function by a conformal transformation. This is to account for the fact that it is often convenient, when solving for the field in the spatial sections of the spacetime, to formulate the field equation in a conformally related space. If the metric on the original spatial sections is  $h_{ab}$ , then the metric on the conformally related space is  $\tilde{h}_{ab} = \Omega^{-2}h_{ab}$ , in which  $\Omega(x)$  is a scalar field. When  $\tilde{h}_{ab}$  is simple, the field equation simplifies in the conformally related space, and the field can then be regularized with the help of  $\tilde{G}_3^H$ , which differs from  $G_3^H$  by factors of  $\Omega$ .

Our main goal in this paper is to compare the regularization procedures that are based on  $G_4^S(x, x')$ ,  $G_3^H(x, x')$ , and  $\tilde{G}_3^H(x, x')$ . Our preferred regularization method is the one based on the Detweiler-Whiting Green's function, because the resulting singular field was proved to share the same singularity structure as the particle's actual field and to exert no force on the particle. In this case we find

that if  $\epsilon$  is a measure of distance between  $x$  and  $x'$ , then the singular Green's function admits the local expansion

$$G_3^S(x, x') = \frac{1}{\epsilon} [1 + g_1 \epsilon + g_2 \epsilon^2 + g_3 \epsilon^3 + O(\epsilon^4)], \quad (1.4)$$

with expansion coefficients  $g_1$ ,  $g_2$ , and  $g_3$  (computed below) that depend on geometrical quantities (such as the spatial Riemann tensor) evaluated at  $x'$ . Removing this from the particle's actual field returns a regularized field that is finite (indeed, once differentiable) at the position of the particle, leading to a straightforward computation of the self-force.

Our main result is the statement that the local expansions of  $G_3^H(x, x')$  and  $\tilde{G}_3^H(x, x')$  are identical to the local expansion of the Detweiler-Whiting Green's function, and that they therefore lead to equivalent regularization procedures. We view this as a significant result that clarifies and justifies the alternative regularization methods that have been employed in self-force computations; it puts these computations on a firmer footing and lends them additional credence.

We show by explicit calculation that the local expansions of all Green's functions agree through order  $\epsilon^2$ , which is more than enough to guarantee the same self-force, but we also ask whether equality could be established to all orders in  $\epsilon$ . We provide only a partial answer to this question. We collect evidence that the conjectured equality is likely to be true, sketch a proof that relies on a strong assumption about the convergence of formal power series, and present a complete proof of equality in the special case of ultrastatic spacetimes.

As an application of our results we consider the scalar and electromagnetic self-force acting on a particle held in place in a static and spherically-symmetric spacetime. The assumption of spherical symmetry implies that the self-force can be easily computed with a mode-sum method based on a spherical-harmonic decomposition of the field. The mode coefficients of the singular field then play the role of regularization parameters [30–32] that can be inserted in the mode-sum to ensure convergence. We use our local expansions to compute these regularization parameters for any static and spherically-symmetric spacetime. We expect that our explicit listing of regularization parameters will greatly facilitate future self-force computations.

Our developments below rely heavily on the general theory of bitensors and Green's functions in curved spacetime, as developed in Ref. [5] and summarized in Ref. [1], from which we import our notations. We begin in Sec. II with a description of the static spacetimes that are implicated in this work. We continue in Sec. III with a review of the scalar and electromagnetic field equations in static spacetimes, along with the associated Green's functions, and in Sec. IV we review the Hadamard construction of the three-dimensional Green's functions. The following sections contain our new work. In Secs. V and VI we compute

the local expansions of the Hadamard functions. We do the same for the Detweiler-Whiting functions in Sec. VII, and prove the equality of the local expansions in Sec. VIII. In Sec. IX we prove that in ultrastatic spacetimes, the Hadamard and Detweiler-Whiting functions are strictly equal to one another. And finally, in Sec. X we consider the scalar and electromagnetic self-forces in static, spherically-symmetric spacetimes, and involve the local expansions in a computation of regularization parameters for mode-sum computations of the self-force.

In the following we denote a spacetime event by  $x$  or  $x'$ , and a spatial position by  $\mathbf{x}$  or  $\mathbf{x}'$ , so that  $x = (t, \mathbf{x})$  and  $x' = (t', \mathbf{x}')$ . Spacetime tensors at  $x$  are denoted  $A^\alpha$ , with a Greek index  $\alpha$  that ranges over the values  $\{0, 1, 2, 3\}$ ; spacetime tensors at  $x'$  are denoted  $A^{\alpha'}$ , with a primed Greek index. Spatial tensors at  $\mathbf{x}$  are denoted  $A^a$ , with a Latin index  $a$  that ranges over  $\{1, 2, 3\}$ ; spatial tensors at  $\mathbf{x}'$  are denoted  $A^{a'}$ , with a primed Latin index. Variants of these notations will be introduced as needed.

## II. STATIC SPACETIMES

The class of spacetimes considered in this paper admits a hypersurface-orthogonal, timelike Killing vector  $t^\alpha$ , and in an adapted coordinate system the metric is expressed as

$$ds^2 = -N^2 dt^2 + h_{ab} dx^a dx^b, \quad (2.1)$$

in terms of a lapse function  $N$  and a spatial metric  $h_{ab}$  which depend on the spatial coordinates  $x^a$  only. No other assumptions are placed on the spacetime. We introduce the vector field

$$A_a := \partial_a \ln N = \frac{\partial_a N}{N}, \quad (2.2)$$

which acts as a substitute for  $\partial_a N$ ; the vector has a vanishing time component.

A straightforward computation reveals that the connection coefficients are given by

$${}^4\Gamma_{ta}^t = A_a, \quad {}^4\Gamma_{tt}^a = N^2 A^a, \quad {}^4\Gamma_{bc}^a = \Gamma_{bc}^a, \quad (2.3)$$

in which  $\Gamma_{bc}^a$  is the connection compatible with the spatial metric  $h_{ab}$ . We shall indicate covariant differentiation relative to the spacetime connection with the operator  $\nabla_\mu$  or with a semicolon (for example,  $\nabla_\mu A^\nu = A^\nu{}_{;\mu}$ ) and covariant differentiation relative to the spatial connection with the operator  $D_a$  or with a vertical stroke (for example,  $D_a A^b = A^b|_a$ ).

For future reference we examine a vector  $v^\alpha$  that is parallel-transported along a purely spatial curve described by the parametric equations  $t = \text{constant}$ ,  $x^a = z^a(s)$ , in which  $s$  is proper distance. The vector tangent to the curve is  $n^\alpha$  with components  $n^t = 0$  and  $n^a = dz^a/ds$ . The equation of parallel transport is

$$\frac{dv^\alpha}{ds} + {}^4\Gamma_{\beta\gamma}^\alpha v^\beta n^\gamma = 0, \quad (2.4)$$

and its time component reduces to  $dv^t/ds + (A_a n^a)v^t = 0$ . With  $A_a n^a = N^{-1}dN/ds$ , the solution to the differential equation is  $Nv^t = \text{constant}$ , or

$$v^t(s) = \frac{N(0)}{N(s)} v^t(0). \quad (2.5)$$

For the spatial components we find that the equation reduces to  $dv^a/ds + \Gamma_{bc}^a v^b n^c = 0$ , which states that  $v^a$  is parallel-transported as if it were a vector in a three-dimensional space with metric  $h_{ab}$ . When the spatial curve is a spacetime geodesic, we find that Eq. (2.5) produces  $n^t(s) = 0$ , while the spatial components reveal that  $n^a$  satisfies the geodesic equation in the three-dimensional space. A purely spatial geodesic in spacetime is therefore a geodesic in a three-dimensional space with metric  $h_{ab}$ .

The nonvanishing components of the Riemann tensor are

$${}^4R_{tab} = N^2(A_{a|b} + A_a A_b), \quad {}^4R_{abcd} = R_{abcd}, \quad (2.6)$$

in which  $R_{abcd}$  is the Riemann tensor associated with the spatial metric  $h_{ab}$ . The nonvanishing components of the Ricci tensor are

$${}^4R_{tt} = N^2(A^c{}_{|c} + A^c A_c), \quad (2.7a)$$

$${}^4R_{ab} = R_{ab} - (A_{a|b} + A_a A_b). \quad (2.7b)$$

The Ricci scalar is

$${}^4R = R - 2(A^c{}_{|c} + A^c A_c). \quad (2.8)$$

For future reference we also record the components of the covariant derivative of the Riemann tensor:

$${}^4R_{tab;c} = N^2(A_{a|bc} + A_{a|c}A_b + A_a A_{b|c}), \quad (2.9a)$$

$${}^4R_{tabc;t} = N^2(A_{a|b}A_c - A_{a|c}A_b - A^d R_{dabc}), \quad (2.9b)$$

$${}^4R_{abcd;e} = R_{abcd|e}. \quad (2.9c)$$

We also have

$${}^4R_{t;a} = N^2(A^c{}_{|ca} + 2A^c A_{c|a}), \quad (2.10a)$$

$${}^4R_{ta;t} = -N^2(A^c{}_{|c}A_a - A^c{}_{|a}A_c + A^c R_{ca}), \quad (2.10b)$$

$${}^4R_{ab;c} = R_{ab|c} - A_{a|bc} - A_{a|c}A_b - A_a A_{b|c} \quad (2.10c)$$

and

$${}^4R_{;a} = R_{|a} - 2(A^c{}_{|ca} + 2A^c A_{c|a}) \quad (2.11)$$

for the covariant derivative of the Ricci tensor and scalar.

Below we shall consider a scalar or electric charge at rest in the static spacetime. The charge follows an orbit of the timelike Killing vector, and the only nonvanishing component of its velocity vector is

$$u^t = \frac{1}{N}. \quad (2.12)$$

The covariant acceleration is defined by  $a^\alpha := u^\alpha{}_{;\beta}u^\beta$ , and its nonvanishing components are

$$a^a = A^a. \quad (2.13)$$

We shall also require the vectors  $\dot{a}^\alpha := a^\alpha{}_{;\beta}u^\beta$  and  $\ddot{a}^\alpha := \dot{a}^\alpha{}_{;\beta}u^\beta$ ; the nonvanishing components are

$$\dot{a}^t = \frac{1}{N}A^c A_c, \quad \ddot{a}^a = (A^c A_c)A^a. \quad (2.14)$$

### III. FIELD EQUATIONS AND GREEN'S FUNCTIONS

#### A. Scalar field

The potential  $\Phi$  generated by a scalar-charge density  $\mu$  obeys the wave equation

$$\square\Phi = -4\pi\mu \quad (3.1)$$

in any four-dimensional spacetime;  $\square := g^{\alpha\beta}\nabla_\alpha\nabla_\beta$  is the covariant wave operator. Our considerations in this paper are limited to scalar fields that are minimally coupled to the spacetime curvature; it is, however, a very straightforward exercise to extend our discussion to arbitrary couplings. When the spacetime is static, and when the potential and charge density are both time-independent, the wave equation reduces to

$$\nabla^2\Phi + A^a\partial_a\Phi = -4\pi\mu; \quad (3.2)$$

here  $\nabla^2 := h^{ab}D_aD_b$  is the covariant Laplacian operator in a three-dimensional space with metric  $h_{ab}$ .

It is sometimes convenient to formulate Eq. (3.2) in a conformally related space with metric  $\tilde{h}_{ab}$ ; this is related to the original metric  $h_{ab}$  by the conformal transformation

$$h_{ab} = \Omega^2\tilde{h}_{ab}, \quad (3.3)$$

in which  $\Omega$  is a function of the spatial coordinates  $x^a$ . As a consequence of this transformation we find that  $h^{ab} = \Omega^{-2}\tilde{h}^{ab}$  and  $h^{1/2} = \Omega^3\tilde{h}^{1/2}$ . A simple computation reveals that in the conformal formulation, Eq. (3.2) becomes

$$\tilde{\nabla}^2\Phi + \tilde{A}^a\partial_a\Phi = -4\pi\tilde{\mu}, \quad (3.4)$$

in which  $\tilde{\nabla}^2 := \tilde{h}^{ab}\tilde{D}_a\tilde{D}_b$  is the covariant Laplacian operator associated with the metric  $\tilde{h}_{ab}$ ,

$$\tilde{A}^a := \tilde{h}^{ab}\partial_b \ln(N\Omega) = \tilde{h}^{ab}\left(\frac{\partial_b N}{N} + \frac{\partial_b \Omega}{\Omega}\right), \quad (3.5)$$

and

$$\tilde{\mu} := \Omega^2\mu. \quad (3.6)$$

The four-dimensional Green's function associated with Eq. (3.1) is  $G_4(x, x')$ , which satisfies the wave equation

$$\square G_4(x, x') = -4\pi\delta_4(x, x'), \quad (3.7)$$

in which  $\delta_4(x, x')$  is a scalarized Dirac distribution defined by

$$\delta_4(x, x') = \frac{\delta(x - x')}{\sqrt{-g'}}, \quad (3.8)$$

where  $\delta(x - x')$  is the usual product of four coordinate delta functions. A solution to Eq. (3.1) is then given by

$$\Phi(x) = \int G_4(x, x') \mu(x') \sqrt{-g'} d^4 x'. \quad (3.9)$$

The three-dimensional Green's function associated with Eq. (3.2) is  $G_3(\mathbf{x}, \mathbf{x}')$ , which satisfies the Poisson equation

$$\nabla^2 G_3(\mathbf{x}, \mathbf{x}') + A^a \partial_a G_3(\mathbf{x}, \mathbf{x}') = -4\pi \delta_3(\mathbf{x}, \mathbf{x}'), \quad (3.10)$$

in which  $\delta_3(\mathbf{x}, \mathbf{x}')$  is a scalarized Dirac distribution defined by

$$\delta_3(\mathbf{x}, \mathbf{x}') = \frac{\delta(\mathbf{x} - \mathbf{x}')}{\sqrt{h'}}, \quad (3.11)$$

where  $\delta(\mathbf{x} - \mathbf{x}')$  is the usual product of three coordinate delta functions. A solution to Eq. (3.2) is then given by

$$\Phi(\mathbf{x}) = \int G_3(\mathbf{x}, \mathbf{x}') \mu(\mathbf{x}') \sqrt{h'} d^3 x'. \quad (3.12)$$

The Green's function associated with Eq. (3.4) is  $\tilde{G}_3(\mathbf{x}, \mathbf{x}')$ , which satisfies

$$\tilde{\nabla}^2 \tilde{G}_3(\mathbf{x}, \mathbf{x}') + \tilde{A}^a \partial_a \tilde{G}_3(\mathbf{x}, \mathbf{x}') = -4\pi \tilde{\delta}_3(\mathbf{x}, \mathbf{x}'), \quad (3.13)$$

in which  $\tilde{\delta}_3(\mathbf{x}, \mathbf{x}')$  is defined by

$$\tilde{\delta}_3(\mathbf{x}, \mathbf{x}') = \frac{\delta(\mathbf{x} - \mathbf{x}')}{\sqrt{\tilde{h}'}}. \quad (3.14)$$

A solution to Eq. (3.4) is then given by

$$\Phi(\mathbf{x}) = \int \tilde{G}_3(\mathbf{x}, \mathbf{x}') \tilde{\mu}(\mathbf{x}') \sqrt{\tilde{h}'} d^3 x'. \quad (3.15)$$

The relation between  $G_4$  and  $G_3$  can be identified by performing the time integration in Eq. (3.9) and comparing with Eq. (3.12). The result is

$$G_3(\mathbf{x}, \mathbf{x}') = \int G_4(x, x') N(x') dt'. \quad (3.16)$$

Apart from the factor of  $N(x')$ , which converts from coordinate time to proper time at  $\mathbf{x}'$ , the three-dimensional Green's function is simply the time integral of the four-dimensional Green's function.

The relation between  $\tilde{G}_3$  and  $G_3$  is found by making the substitutions  $h^{1/2} = \Omega^3 \tilde{h}^{1/2}$  and  $\mu = \Omega^{-2} \tilde{\mu}$  within Eq. (3.12) and comparing with Eq. (3.15). The result is

$$\tilde{G}_3(\mathbf{x}, \mathbf{x}') = \Omega(x') G_3(\mathbf{x}, \mathbf{x}'). \quad (3.17)$$

This can be confirmed by expressing Eq. (3.10) in terms of the conformally related metric  $\tilde{h}_{ab}$ . We find that the equation becomes

$$\tilde{\nabla}^2 G_3(\mathbf{x}, \mathbf{x}') + \tilde{A}^a \partial_a G_3(\mathbf{x}, \mathbf{x}') = -4\pi \frac{\delta(\mathbf{x} - \mathbf{x}')}{\Omega' \sqrt{\tilde{h}'}}}, \quad (3.18)$$

and comparison with Eq. (3.13) allows us to make the identification of Eq. (3.17).

When the charge density  $\mu$  describes a point charge  $q$  moving on a world line  $\gamma$  described by the parametric relations  $z(\tau)$ , we have that

$$\mu(x) = q \int_{\gamma} \delta_4(x, z(\tau)) d\tau. \quad (3.19)$$

For a general world line the scalar charge produces a potential given by Eq. (3.9), which evaluates to

$$\Phi(x) = q \int_{\gamma} G_4(x, z(\tau)) d\tau. \quad (3.20)$$

For a static charge at a fixed position  $z$ , the integral of Eq. (3.19) evaluates to

$$\mu(\mathbf{x}) = q \delta_3(\mathbf{x}, z), \quad (3.21)$$

and in this case the potential, as given by Eq. (3.12), becomes

$$\Phi(\mathbf{x}) = q G_3(\mathbf{x}, z). \quad (3.22)$$

The link between Eqs. (3.20) and (3.22) can be seen directly from Eq. (3.16).

In the conformal formulation we have instead

$$\tilde{\mu}(\mathbf{x}) = q \frac{\tilde{\delta}_3(\mathbf{x}, z)}{\Omega(z)}, \quad (3.23)$$

and Eq. (3.15) produces

$$\Phi(\mathbf{x}) = q \frac{\tilde{G}_3(\mathbf{x}, z)}{\Omega(z)}. \quad (3.24)$$

This result is compatible with Eq. (3.22) by virtue of Eq. (3.17).

## B. Electromagnetic field

A current density  $j^\alpha$  creates an electromagnetic field  $F_{\alpha\beta}$  that satisfies Maxwell's equations

$$F^{\alpha\beta}{}_{;\beta} = 4\pi j^\alpha, \quad F_{\alpha\beta;\gamma} + F_{\gamma\alpha;\beta} + F_{\beta\gamma;\alpha} = 0. \quad (3.25)$$

The homogeneous equations are automatically satisfied when the field tensor is expressed in terms of a vector potential  $\Phi_\alpha$ ,

$$F_{\alpha\beta} = \nabla_\alpha \Phi_\beta - \nabla_\beta \Phi_\alpha. \quad (3.26)$$

The inhomogeneous equations then take the form of a wave equation for the vector potential,

$$\square \Phi_\alpha - R_\alpha{}^\beta \Phi_\beta = -4\pi j_\alpha, \quad (3.27)$$

provided that  $\Phi_\alpha$  is required to satisfy the Lorenz gauge condition

$$\nabla_\alpha \Phi^\alpha = 0. \quad (3.28)$$

In a static spacetime, and for a static distribution of charge, the only relevant component of Maxwell's equations is  $F^{t\beta}{}_{;\beta} = 4\pi j^t$ , and with  $F_{ta} := -\partial_a \Phi_t$  this reduces to

$$\nabla^2 \Phi_t - A^a \partial_a \Phi_t = 4\pi \mu, \quad \mu := N^2 j^t = -j_t. \quad (3.29)$$

This equation also follows from evaluating Eq. (3.27) in a static spacetime. Comparing with Eq. (3.2), we see that  $\Phi_t$  satisfies the same Poisson equation as a scalar potential  $\Phi$ , except that the sign of  $A^a$  is reversed. It is easy to see that the gauge condition of Eq. (3.28) becomes  $D_a \Phi^a + A_a \Phi^a = 0$  in a static spacetime, and that it has no impact on  $\Phi_t$ .

In a conformal formulation in which the spatial metric is expressed as  $h_{ab} = \Omega^2 \tilde{h}_{ab}$ , Eq. (3.29) becomes

$$\tilde{\nabla}^2 \Phi_t - \tilde{A}^a \partial_a \Phi_t = 4\pi \tilde{\mu}, \quad (3.30)$$

in which

$$\tilde{A}^a := \tilde{h}^{ab} \partial_b \ln\left(\frac{N}{\Omega}\right) = \tilde{h}^{ab} \left( \frac{\partial_b N}{N} - \frac{\partial_b \Omega}{\Omega} \right) \quad (3.31)$$

and

$$\tilde{\mu} = \Omega^2 \mu = N^2 \Omega^2 j^t. \quad (3.32)$$

The four-dimensional Green's function associated with Eq. (3.27) is  $G_\alpha{}^{\beta'}(x, x')$ , which satisfies the wave equation

$$\square G_\alpha{}^{\beta'}(x, x') - R_\alpha{}^\gamma G_\gamma{}^{\beta'}(x, x') = -4\pi g_\alpha{}^{\beta'}(x, x') \delta_4(x, x'), \quad (3.33)$$

in which  $g_\alpha{}^{\beta'}(x, x')$  is an operator of parallel transport, taking a vector  $A_{\beta'}$  at  $x'$  and producing a parallel-transported vector  $A_\alpha$  at  $x$ . A solution to Eq. (3.27) is then given by

$$\Phi_\alpha(x) = \int G_\alpha{}^{\beta'}(x, x') j_{\beta'}(x') \sqrt{-g'} d^4 x'. \quad (3.34)$$

The three-dimensional Green's function associated with Eq. (3.29) is  $G_3(\mathbf{x}, \mathbf{x}')$ , which satisfies the Poisson equation

$$\nabla^2 G_3(\mathbf{x}, \mathbf{x}') - A^a \partial_a G_3(\mathbf{x}, \mathbf{x}') = -4\pi \delta_3(\mathbf{x}, \mathbf{x}'). \quad (3.35)$$

A solution to Eq. (3.29) is then given by

$$\Phi_t(\mathbf{x}) = - \int G_3(\mathbf{x}, \mathbf{x}') \mu(\mathbf{x}') \sqrt{h'} d^3 x'. \quad (3.36)$$

Notice the minus sign on the right-hand side of Eq. (3.36), which can be compared with its scalar equivalent in Eq. (3.12). Notice also that while we denote both the scalar and electromagnetic Green's functions by  $G_3(\mathbf{x}, \mathbf{x}')$ , these functions are not equal to each other because they satisfy distinct differential equations.

The Green's function associated with Eq. (3.30) is  $\tilde{G}_3(\mathbf{x}, \mathbf{x}')$ , which satisfies

$$\tilde{\nabla}^2 \tilde{G}_3(\mathbf{x}, \mathbf{x}') - \tilde{A}^a \partial_a \tilde{G}_3(\mathbf{x}, \mathbf{x}') = -4\pi \tilde{\delta}_3(\mathbf{x}, \mathbf{x}'). \quad (3.37)$$

A solution to Eq. (3.30) is then given by

$$\Phi_t(\mathbf{x}) = - \int \tilde{G}_3(\mathbf{x}, \mathbf{x}') \tilde{\mu}(\mathbf{x}') \sqrt{\tilde{h}'} d^3 x', \quad (3.38)$$

which features the same minus sign as in Eq. (3.36).

The relation between  $G_\alpha{}^{\beta'}(x, x')$  and  $G_3(\mathbf{x}, \mathbf{x}')$  can be identified by performing the time integration in Eq. (3.34) and noticing that in a static situation, the integral involves  $j_{t'}$  only. Comparing with Eq. (3.36) produces

$$G_3(\mathbf{x}, \mathbf{x}') = \int G_{t'}{}^{t'}(x, x') N(x') dt', \quad (3.39)$$

essentially the same relation as in the scalar case.

The relation between  $\tilde{G}_3$  and  $G_3$  is found by making the substitutions  $h^{1/2} = \Omega^3 \tilde{h}^{1/2}$  and  $\mu = \Omega^{-2} \tilde{\mu}$  within Eq. (3.36) and comparing with Eq. (3.38). The result is

$$\tilde{G}_3(\mathbf{x}, \mathbf{x}') = \Omega(\mathbf{x}') G_3(\mathbf{x}, \mathbf{x}'), \quad (3.40)$$

the same relation as in the scalar case.

When the current density  $j^\alpha$  describes a point charge  $e$  moving on a world line  $\gamma$  described by the parametric relations  $z(\tau)$ , we have that

$$j^\alpha(x) = e \int_\gamma g^\alpha{}_\mu(x, z) u^\mu \delta_4(x, z) d\tau. \quad (3.41)$$

For a general world line, the electric charge produces a potential given by Eq. (3.34), which evaluates to

$$\Phi_\alpha(x) = e \int_\gamma G_{\alpha\mu}(x, z) u^\mu d\tau. \quad (3.42)$$

For a static charge at a fixed position  $z$ , the integral of Eq. (3.41) evaluates to  $j^t(\mathbf{x}) = e N^{-1}(z) \delta_3(\mathbf{x}, z)$ , so that

$$\mu(\mathbf{x}) = e N(z) \delta_3(\mathbf{x}, z). \quad (3.43)$$

In this case the potential, as given by Eq. (3.36), becomes

$$\Phi_t(\mathbf{x}) = -e N(z) G_3(\mathbf{x}, z). \quad (3.44)$$

Notice the extra minus sign and factor of  $N$  when comparing this with Eq. (3.22). In the conformal formulation we have instead

$$\tilde{\mu}(\mathbf{x}) = e \frac{N(z)}{\Omega(z)} \tilde{\delta}_3(\mathbf{x}, z), \quad (3.45)$$

and Eq. (3.38) produces

$$\Phi_t(\mathbf{x}) = -e \frac{N(z)}{\Omega(z)} \tilde{G}_3(\mathbf{x}, z). \quad (3.46)$$

This result is compatible with Eq. (3.44) by virtue of Eq. (3.40).

## IV. HADAMARD'S CONSTRUCTION

### A. Scalar field

We wish to find a representation for a Green's function  $G_3(\mathbf{x}, \mathbf{x}')$  that satisfies

$$\nabla^2 G_3(\mathbf{x}, \mathbf{x}') + A^a \partial_a G_3(\mathbf{x}, \mathbf{x}') = -4\pi \delta_3(\mathbf{x}, \mathbf{x}'). \quad (4.1)$$

The Hadamard's construction developed here applies to this equation—a copy of Eq. (3.10)—but it applies just as well to the conformally related formulation of Eq. (3.13).

The Hadamard construction for the Green's function is [29]

$$G_3^H(\mathbf{x}, \mathbf{x}') = \frac{W(\mathbf{x}, \mathbf{x}')}{\sqrt{2\sigma(\mathbf{x}, \mathbf{x}')}}, \quad (4.2)$$

in which the two-point function  $W(\mathbf{x}, \mathbf{x}')$  is postulated to be smooth in the coincidence limit  $\mathbf{x} \rightarrow \mathbf{x}'$ , to satisfy the differential equation

$$2\sigma^a \partial_a W + (\nabla^2 \sigma + A^a \sigma_a - 3)W - (2\sigma)(\nabla^2 W + A^a \partial_a W) = 0, \quad (4.3)$$

where  $\sigma_a := \partial \sigma / \partial x^a$  and  $\nabla^2 \sigma := h^{ab} D_a \sigma_b$ , and to satisfy the boundary condition

$$W(\mathbf{x}', \mathbf{x}') = 1. \quad (4.4)$$

Notice that with  $W(\mathbf{x}, \mathbf{x}')$  smooth at  $\mathbf{x} \rightarrow \mathbf{x}'$ , it is the factor  $(2\sigma)^{-1/2}$  that is fully responsible for the singular behavior of the Green's function at coincidence.

To construct  $W$  we express it as an expansion in powers of  $2\sigma$ ,

$$W(\mathbf{x}, \mathbf{x}') = \sum_{n=0}^{\infty} W_n(\mathbf{x}, \mathbf{x}') [2\sigma(\mathbf{x}, \mathbf{x}')]^n, \quad (4.5)$$

insert this within Eq. (4.3), and collect powers of  $2\sigma$ , making use of the identity  $\sigma^a \sigma_a = 2\sigma$ . Setting each coefficient to zero, we find that each  $W_n$  must satisfy the differential equation

$$\begin{aligned} \nabla^2 W_{n-1} + A^a \partial_a W_{n-1} \\ = 2(1 - 2n)\sigma^a \partial_a W_n + (1 - 2n)(\nabla^2 \sigma + A^a \sigma_a)W_n \\ - [3 + 4n(n - 2)]W_n. \end{aligned} \quad (4.6)$$

The burden of enforcing Eq. (4.4) is then placed solely upon  $W_0$ , which must satisfy

$$W_0(\mathbf{x}', \mathbf{x}') = 1. \quad (4.7)$$

Equation (4.6) is a recursion relation for each  $W_n$ . With  $W_{n-1}$  previously determined,  $W_n$  is obtained by selecting a base point  $\mathbf{x}'$  and integrating Eq. (4.6) along each geodesic that emanates from  $\mathbf{x}'$ . Each solution is unique.

A remarkable aspect of the expansion of Eq. (4.5) is that it was shown by Hadamard [29] to converge when  $\mathbf{x}$  is sufficiently close to  $\mathbf{x}'$ , a property that is inherited from the postulated smoothness of  $W(\mathbf{x}, \mathbf{x}')$  at coincidence. Because convergence ensures that the two-point function actually exists (in the appropriate domain), smoothness is clearly a crucial defining property of  $W(\mathbf{x}, \mathbf{x}')$ . The following point shall be of importance in later sections of the paper. Suppose that one is given a Green's function  $G_3(\mathbf{x}, \mathbf{x}')$

and proceeds to define a two-point function  $\tilde{W}(\mathbf{x}, \mathbf{x}')$  via the relation  $\tilde{W} = \sqrt{2\sigma} G_3$ . How would this  $\tilde{W}$  differ from Hadamard's function  $W$ ? Because  $G_3$  can differ from  $G_3^H$  by a function  $J(\mathbf{x}, \mathbf{x}')$  that satisfies the homogeneous version of Eq. (3.13), it follows that  $\tilde{W} = W + \sqrt{2\sigma} J$  and  $\tilde{W}$  would in general fail to be smooth at  $\mathbf{x} = \mathbf{x}'$ . Such a function may still admit an expansion in powers of  $2\sigma$ , but the expansion may not converge and may not produce a smooth function. In fact, the uniqueness of Hadamard's construction implies that when  $\tilde{W}$  is smooth, it must be equal to  $W$ .

## B. Electromagnetic field

We now wish to find the Hadamard representation for the electromagnetic Green's function  $G_3(\mathbf{x}, \mathbf{x}')$ , which satisfies

$$\nabla^2 G_3(\mathbf{x}, \mathbf{x}') - A^a \partial_a G_3(\mathbf{x}, \mathbf{x}') = -4\pi \delta_3(\mathbf{x}, \mathbf{x}'). \quad (4.8)$$

The Hadamard construction applies to this equation—a copy of Eq. (3.35)—but it applies just as well to the conformally related formulation of Eq. (3.37).

The construction is obtained directly from the scalar case by altering the sign of  $A^a$  in all equations. The Hadamard representation for the Green's function is

$$G_3^H(\mathbf{x}, \mathbf{x}') = \frac{W(\mathbf{x}, \mathbf{x}')}{\sqrt{2\sigma(\mathbf{x}, \mathbf{x}')}}, \quad (4.9)$$

in which the two-point function  $W(\mathbf{x}, \mathbf{x}')$  is smooth in the coincidence limit  $\mathbf{x} \rightarrow \mathbf{x}'$ . It satisfies the differential equation

$$\begin{aligned} 2\sigma^a \partial_a W + (\nabla^2 \sigma - A^a \sigma_a - 3)W - (2\sigma)(\nabla^2 W - A^a \partial_a W) \\ = 0 \end{aligned} \quad (4.10)$$

together with the boundary condition

$$W(\mathbf{x}', \mathbf{x}') = 1. \quad (4.11)$$

As in the scalar case we express  $W$  as an expansion in powers of  $2\sigma$ ,

$$W(\mathbf{x}, \mathbf{x}') = \sum_{n=0}^{\infty} W_n(\mathbf{x}, \mathbf{x}') [2\sigma(\mathbf{x}, \mathbf{x}')]^n, \quad (4.12)$$

in which each coefficient  $W_n$  must satisfy the differential equation

$$\begin{aligned} \nabla^2 W_{n-1} - A^a \partial_a W_{n-1} \\ = 2(1 - 2n)\sigma^a \partial_a W_n + (1 - 2n)(\nabla^2 \sigma - A^a \sigma_a)W_n \\ - [3 + 4n(n - 2)]W_n. \end{aligned} \quad (4.13)$$

## V. LOCAL EXPANSION OF HADAMARD'S FUNCTION

### A. Scalar field

We wish to express the three-dimensional Green's function  $G_3^H(\mathbf{x}, \mathbf{x}')$  as a local expansion about the base point  $\mathbf{x}'$ . We return to the Hadamard construction of Eq. (4.2) with the expansion of Eq. (4.5), and now express  $W_0$  and  $W_1$  as the local expansions

$$W_0 = 1 + W_{a'}^0 \sigma^{a'} + \frac{1}{2} W_{a'b'}^0 \sigma^{a'} \sigma^{b'} + \frac{1}{6} W_{a'b'c'}^0 \sigma^{a'} \sigma^{b'} \sigma^{c'} + O(\epsilon^4) \quad (5.1)$$

and

$$W_1 = W^1 + W_{a'}^1 \sigma^{a'} + O(\epsilon^2), \quad (5.2)$$

in which  $\sigma_{a'} := \partial \sigma / \partial x^{a'}$  and each expansion coefficient is an ordinary tensor at  $\mathbf{x}'$ . We let  $\epsilon$  be a measure of the distance between  $\mathbf{x}$  and  $\mathbf{x}'$ , and the expansions of Eqs. (5.1) and (5.2) give rise to an expression for  $W$  accurate through order  $\epsilon^3$ .

The expansion coefficients are determined by inserting Eqs. (5.1) and (5.2) within Eq. (4.6). We begin with  $W_0$ , which satisfies

$$2\sigma^a \partial_a W_0 + (\nabla^2 \sigma + A^a \sigma_a - 3)W_0 = 0. \quad (5.3)$$

We rely on the standard expansions

$$\sigma_{ab} = h^{a'}_a h^{b'}_b \left[ h_{a'b'} - \frac{1}{3} R_{a'c'b'd'} \sigma^{c'} \sigma^{d'} + \frac{1}{4} R_{a'c'b'd'|e'} \sigma^{c'} \sigma^{d'} \sigma^{e'} + O(\epsilon^4) \right] \quad (5.4)$$

and

$$A_a = h^{a'}_a \left[ A_{a'} - A_{a'|c'} \sigma^{c'} + \frac{1}{2} A_{a'|c'd'} \sigma^{c'} \sigma^{d'} + O(\epsilon^3) \right], \quad (5.5)$$

in which  $\sigma_{ab} := D_a \sigma_b$  and  $h^{a'}_a$  is the operator of parallel transport in the three-dimensional space. From the first equation we get

$$\nabla^2 \sigma = 3 - \frac{1}{3} R_{c'd'} \sigma^{c'} \sigma^{d'} + \frac{1}{4} R_{c'd'|e'} \sigma^{c'} \sigma^{d'} \sigma^{e'} + O(\epsilon^4), \quad (5.6)$$

and the second equation gives rise to

$$A_a \sigma^a = -A_{c'} \sigma^{c'} + A_{c'|d'} \sigma^{c'} \sigma^{d'} - \frac{1}{2} A_{c'|d'e'} \sigma^{c'} \sigma^{d'} \sigma^{e'} + O(\epsilon^4) \quad (5.7)$$

because  $h^{a'}_a \sigma^a = -\sigma^{a'}$ . Making use of the identity  $\sigma^{c'}_a \sigma^a = \sigma^{c'}$  we also find that

$$\sigma^a \partial_a W_0 = W_{c'd'}^0 \sigma^{c'} + W_{c'd'}^0 \sigma^{c'} \sigma^{d'} + \frac{1}{2} W_{c'd'e'}^0 \sigma^{c'} \sigma^{d'} \sigma^{e'} + O(\epsilon^4). \quad (5.8)$$

Making the substitutions within Eq. (5.3) and equating each expansion coefficient to zero, we eventually arrive at

$$W_{a'}^0 = \frac{1}{2} A_{a'}, \quad (5.9a)$$

$$W_{a'b'}^0 = -\frac{1}{2} A_{a'|b'} + \frac{1}{4} A_{a'} A_{b'} + \frac{1}{6} R_{a'b'}, \quad (5.9b)$$

$$W_{a'b'c'}^0 = \frac{1}{2} A_{(a'|b'c')} - \frac{3}{4} A_{(a'} A_{b'|c')} + \frac{1}{8} A_{a'} A_{b'} A_{c'} + \frac{1}{4} A_{(a'} R_{b'c')} - \frac{1}{4} R_{(a'b'|c')}. \quad (5.9c)$$

It should be noted that since  $A_{a'}$  is the gradient of a scalar function,  $A_{a'|b'} = A_{(a'|b')}$ .

We next turn to  $W_1$ , which satisfies the differential equation

$$2\sigma^a \partial_a W_1 + (\nabla^2 \sigma + A^a \sigma_a - 1)W_1 = -(\nabla^2 W_0 + A^a \partial_a W_0). \quad (5.10)$$

The left-hand side of the equation is computed with the same methods as for the previous computation. For the right-hand side we make use of the results  $\sigma^{a'}_{b'} = \delta^{a'}_{b'} + O(\epsilon^2)$ ,  $\sigma^{a'}_b = -h^{a'}_b + O(\epsilon^2)$ , and  $\nabla^2 \sigma^{a'} = -\frac{2}{3} R^{a'}_{c'} \sigma^{c'} + O(\epsilon^2)$  to obtain

$$\nabla^2 W_0 = h^{a'b'} W_{a'b'}^0 + \left( -\frac{2}{3} W_{a'}^0 R^{a'}_{c'} + h^{a'b'} W_{a'b'c'}^0 \right) \sigma^{c'} + O(\epsilon^2) \quad (5.11)$$

and

$$A^a \partial_a W_0 = -A^{a'} W_{a'}^0 - (A^{a'} W_{a'c'}^0 - W_{a'}^0 A^{a'}_{|c'}) \sigma^{c'} + O(\epsilon^2). \quad (5.12)$$

Making the substitutions within Eq. (5.10), equating each expansion coefficient to zero, and simplifying the results with Eq. (5.9), we eventually arrive at

$$W^1 = \frac{1}{4} A^{a'}_{|a'} + \frac{1}{8} A^{a'} A_{a'} - \frac{1}{12} R', \quad (5.13a)$$

$$W_{a'}^1 = -\frac{1}{8} A^{c'}_{|c'a'} - \frac{1}{8} A^{c'} A_{c'|a'} + \frac{1}{8} A^{c'}_{|c'} A_{a'} + \frac{1}{16} A^{c'} A_{c'} A_{a'} - \frac{1}{24} R' A_{a'} + \frac{1}{24} R'_{|a'}, \quad (5.13b)$$

in which  $R'$  stands for the Ricci scalar evaluated at  $\mathbf{x}'$ . The expression for  $W_{a'}^1$  was simplified by invoking the contracted Bianchi identity  $R^{c'}_{a'|c'} = \frac{1}{2} R'_{|a'}$  as well as Ricci's identity to write  $A_{a'}^{c'}_{|c'} + A^{c'}_{|a'c'} + A^{c'}_{|c'a'} = 3A^{c'}_{|c'a'} + 2R_{a'c'} A^{c'}$ ; recall that  $A_{a'|c'} = A_{c'|a'}$  because  $A_{a'}$  is the gradient of a scalar function.



The local expansion of the Green's function is therefore and

$$G_3^{\text{H,local}}(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{2\sigma}} \left\{ 1 + W_{a'}^0 \sigma^{a'} + \frac{1}{2} W_{a'b'}^0 \sigma^{a'} \sigma^{b'} + \frac{1}{6} W_{a'b'c'}^0 \sigma^{a'} \sigma^{b'} \sigma^{c'} + O(\epsilon^4) + 2\sigma[W^1 + W_{a'}^1 \sigma^{a'} + O(\epsilon^2)] + O(\sigma^2) \right\}, \quad (5.14)$$

with the expansion coefficients listed in Eqs. (5.9) and (5.13).

## B. Electromagnetic field

We wish to express the three-dimensional Green's function  $G_3^{\text{H}}(\mathbf{x}, \mathbf{x}')$  as a local expansion about the base point  $\mathbf{x}'$ . Once more we rely on the results from the scalar case, which we directly import after implementing the substitution  $A^a \rightarrow -A^a$ .

The local expansion of the electromagnetic Green's function is

$$G_3^{\text{H,local}}(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{2\sigma}} \left\{ 1 + W_{a'}^0 \sigma^{a'} + \frac{1}{2} W_{a'b'}^0 \sigma^{a'} \sigma^{b'} + \frac{1}{6} W_{a'b'c'}^0 \sigma^{a'} \sigma^{b'} \sigma^{c'} + O(\epsilon^4) + 2\sigma[W^1 + W_{a'}^1 \sigma^{a'} + O(\epsilon^2)] + O(\sigma^2) \right\}, \quad (5.15)$$

with

$$W_{a'}^0 = -\frac{1}{2} A_{a'}, \quad (5.16a)$$

$$W_{a'b'}^0 = \frac{1}{2} A_{a'|b'} + \frac{1}{4} A_{a'} A_{b'} + \frac{1}{6} R_{a'b'}, \quad (5.16b)$$

$$W_{a'b'c'}^0 = -\frac{1}{2} A_{(a'|b'c')} - \frac{3}{4} A_{(a'} A_{b'|c')} - \frac{1}{8} A_{a'} A_{b'} A_{c'} - \frac{1}{4} A_{(a'} R_{b'c')} - \frac{1}{4} R_{(a'b'|c')} \quad (5.16c)$$

## VI. LOCAL EXPANSION IN CONFORMAL FORMULATION

### A. Scalar field

The local expansion of Eq. (5.14) applies to the Hadamard representation of the Green's function  $G_3(\mathbf{x}, \mathbf{x}')$  defined by Eq. (3.10), but it applies just as well to the conformally related Green's function  $\tilde{G}_3(\mathbf{x}, \mathbf{x}')$  defined by Eq. (3.13); in this case one simply inserts the conformally related quantities (such as  $\tilde{A}^a$ ,  $\tilde{\sigma}^a$ , and  $\tilde{R}_{a'b'}$ ) in place of the original quantities (such as  $A^a$ ,  $\sigma^a$ , and  $R_{a'b'}$ ). As we shall now show, the expansions are then related by Eq. (3.17),

$$\tilde{G}_3^{\text{H,local}}(\mathbf{x}, \mathbf{x}') = \Omega(\mathbf{x}') G_3^{\text{H,local}}(\mathbf{x}, \mathbf{x}'), \quad (6.1)$$

a conclusion that guarantees the consistency of the two approaches to the local expansion. Thus, a local expansion formulated in the original space, and a local expansion formulated in the conformally related space, will produce the same Green's function, apart from the factor of  $\Omega(\mathbf{x}')$  that appears in the relationship between the Green's functions.

This conclusion can be verified by straightforward computation, making use of the well-known relations between conformally related quantities. These include

$$\tilde{A}_a = A_a + B_a, \quad (6.2a)$$

$$\tilde{h}_{ab} = \Omega^{-2} h_{ab}, \quad (6.2b)$$

$$\tilde{\Gamma}_{bc}^a = \Gamma_{bc}^a + \delta^a_b B_c + \delta^a_c B_b - h_{bc} B^a, \quad (6.2c)$$

$$\tilde{R}^a_{bcd} = R^a_{bcd} + \delta^a_c D_d B_b - \delta^a_d D_c B_b - h_{bc} D_d B^a + h_{bd} D_c B^a + \delta^a_c B_b B_d - \delta^a_d B_b B_c - \delta^a_c h_{bd} B_m B^m + \delta^a_d h_{bc} B_m B^m - h_{bc} B^a B_d + h_{bd} B^a B_c, \quad (6.2d)$$

$$\tilde{R}_{ab} = R_{ab} + D_a B_b + h_{ab} D_m B^m + B_a B_b - h_{ab} B_m B^m, \quad (6.2e)$$

$$\tilde{R} = \Omega^2 (R + 4D_m B^m - 2B_m B^m), \quad (6.2f)$$

$$\begin{aligned} \tilde{D}_c \tilde{R}_{ab} = & D_c R_{ab} + D_c D_a B_b + h_{ab} D_c D_m B^m + 2(B_a D_b B_c + B_c D_a B_b + B_b D_c B_a) - (h_{ac} B^m D_m B_b + h_{bc} B^m D_m B_a \\ & + 2h_{ab} B^m D_m B_c) + 2h_{ab} B_c D_m B^m + 2R_{ab} B_c + R_{ac} B_b + R_{bc} B_a - h_{ac} R_{bm} B^m - h_{bc} R_{am} B^m + 4B_a B_b B_c \\ & - (h_{ac} B_b + h_{bc} B_a + 2h_{ab} B_c) B_m B^m, \end{aligned} \quad (6.2g)$$

$$\tilde{D}_a \tilde{R} = \Omega^2 (D_a R + 4D_a D_m B^m - 4B^m D_a B_m + 8B_a D_m B^m + 2R B_a - 4B_a B_m B^m), \quad (6.2h)$$

in which  $B_a := \partial_a \ln \Omega$ , and where all indices on the right-hand side are raised with  $h^{ab}$ . They include also

$$\tilde{\sigma}^{a'} = \sigma^{a'} + \frac{1}{2} S^{a'}{}_{b'c'd'} \sigma^{a'} \sigma^{b'} \sigma^{c'} + \frac{1}{6} S^{a'}{}_{b'c'd'e'} \sigma^{a'} \sigma^{b'} \sigma^{c'} \sigma^{d'} + \frac{1}{24} S^{a'}{}_{b'c'd'e'f'} \sigma^{a'} \sigma^{b'} \sigma^{c'} \sigma^{d'} \sigma^{e'} + O(\epsilon^6), \quad (6.3)$$

an expansion of  $\tilde{\sigma}^{a'} := \tilde{h}^{a'b'} \tilde{D}_{b'} \tilde{\sigma}$  in powers of  $\sigma^{a'} := h^{a'b'} D_{b'} \sigma$ , in which  $\tilde{\sigma}$  is half the geodetic separation in the conformally related space. The expansion coefficients are given by [33]

$$S^{a'}{}_{b'c'd'} = 2\delta^{a'}{}_{b'} B_{c'd'} - h_{b'c'd'} B^{a'}, \quad (6.4a)$$

$$S^{a'}{}_{b'c'd'e'} = -2\delta^{a'}{}_{b'} D_{c'd'} B_{e'} + h_{b'c'd'e'} B^{a'} + 4\delta^{a'}{}_{b'} B_{c'd'} B_{e'} - \delta^{a'}{}_{b'} h_{c'd'e'} B_{m'} B^{m'} - 2g_{b'c'd'} B^{a'} B_{e'}, \quad (6.4b)$$

$$\begin{aligned} S^{a'}{}_{b'c'd'e'f'} &= 2\delta^{a'}{}_{b'} D_{c'd'} D_{e'f'} B_{f'} - h_{b'c'd'e'f'} B^{a'} - 12\delta^{a'}{}_{b'} B_{c'd'} D_{e'f'} B_{f'} + 4h_{b'c'd'e'f'} B^{a'} \\ &\quad + 4\delta^{a'}{}_{b'} h_{c'd'e'f'} B^{m'} D_{e'f'} B_{m'} - h_{b'c'd'e'f'} B^{m'} D_{m'} B^{a'} + 2h_{b'c'd'e'f'} B^{a'} D_{d'f'} B_{e'} + 8\delta^{a'}{}_{b'} B_{c'd'} B_{e'f'} B_{f'} \\ &\quad - 4\delta^{a'}{}_{b'} h_{c'd'e'f'} B_{e'f'} B_{m'} B^{m'} - 4h_{b'c'd'e'f'} B_{d'f'} B_{e'} B^{a'} + h_{b'c'd'e'f'} B_{m'} B^{m'} B^{a'} - h_{b'c'd'e'f'} R^a{}_{d'm'e'} B^{m'}, \end{aligned} \quad (6.4c)$$

in which the lower indices  $b'c'$ , or  $b'c'd'$ , or  $b'c'd'e'$  are understood to be fully symmetrized on the right-hand side of the equations. These relations imply

$$\tilde{\sigma}(\mathbf{x}, \mathbf{x}') = \Omega^{-2}(\mathbf{x}') \sigma(\mathbf{x}, \mathbf{x}') \left[ 1 + P_{a'} \sigma^{a'} + \frac{1}{2} P_{a'b'} \sigma^{a'} \sigma^{b'} + \frac{1}{6} P_{a'b'c'} \sigma^{a'} \sigma^{b'} \sigma^{c'} + O(\epsilon^4) \right], \quad (6.5)$$

with

$$P_{a'} = B_{a'}, \quad (6.6a)$$

$$P_{a'b'} = -\frac{2}{3} D_{a'} B_{b'} + \frac{4}{3} B_{a'} B_{b'} - \frac{1}{6} h_{a'b'} B_{m'} B^{m'}, \quad (6.6b)$$

$$P_{a'b'c'} = \frac{1}{2} D_{a'} D_{b'} B_{c'} - 3B_{a'} D_{b'} B_{c'} + \frac{1}{2} h_{a'b'} B^{m'} D_{c'} B_{m'} + 2B_{a'} B_{b'} B_{c'} - \frac{1}{2} h_{a'b'} B_{c'} B_{m'} B^{m'}, \quad (6.6c)$$

with the same understanding regarding the  $a'b'$  or  $a'b'c'$  indices on the right-hand side.

To verify that Eq. (6.1) holds, we begin with the conformal formulation of Eq. (5.14), in which we make the substitutions listed above. Simplifying, and keeping all expansions accurate through order  $\epsilon^3$ , reveals that indeed, the end result is Eq. (5.14) formulated in the original space, except for the overall factor of  $\Omega(\mathbf{x}')$  that occurs in Eq. (6.1). Consistency of the local expansions is therefore assured.

It is natural to ask whether the validity of Eq. (6.1) could be established as an exact relation, instead of as an approximate local expansion pursued through order  $\epsilon^3$ . Defining a  $\tilde{W}(\mathbf{x}, \mathbf{x}')$  by the relation

$$\tilde{W}(\mathbf{x}, \mathbf{x}') := \Omega(\mathbf{x}') \sqrt{\frac{\tilde{\sigma}(\mathbf{x}, \mathbf{x}')}{\sigma(\mathbf{x}, \mathbf{x}')}} W(\mathbf{x}, \mathbf{x}'), \quad (6.7)$$

the proof would amount to a demonstration that this  $\tilde{W}$  is suitable to be implicated in a Hadamard construction of the conformal Green's function via  $\tilde{G}_3^H = \tilde{W}/\sqrt{2\tilde{\sigma}}$ .

The proof would involve three essential steps. First, the function  $\tilde{W}$ , as defined here, must be shown to satisfy the same differential equation as Eq. (4.3) expressed in its conformal formulation; this property follows directly from the fact that  $\tilde{W} = \sqrt{2\tilde{\sigma}} \tilde{G}_3$ , in which the two-point function  $\tilde{G}_3 := \Omega(\mathbf{x}') G_3^H$  is known to satisfy Eq. (3.13), the conformal formulation of Green's equation. (The issue at

stake is whether this  $\tilde{G}_3$ , which is defined as  $\tilde{W}/\sqrt{2\tilde{\sigma}}$ , is a proper Hadamard representation of the conformal Green's function.) Second,  $\tilde{W}$  must be shown to satisfy the boundary condition of Eq. (4.4); this property follows immediately from the coincidence limit of Eq. (6.5) and the fact that  $W$  itself satisfies the boundary condition. Third,  $\tilde{W}$  must be shown to be smooth at  $\mathbf{x} = \mathbf{x}'$ , by which we mean that the function must be  $C^\infty$  when viewed as a function of  $\mathbf{x}$  with  $\mathbf{x}'$  fixed; as was emphasized near the end of Sec. IVA, this property ensures that  $\tilde{W}$  admits an expansion in powers of  $\tilde{\sigma}$  as displayed in Eq. (4.5), which is known to be convergent and unique. The expansion being unique, smoothness ensures that the Hadamard construction

$$\tilde{G}_3^H(\mathbf{x}, \mathbf{x}') = \frac{\tilde{W}(\mathbf{x}, \mathbf{x}')}{\sqrt{2\tilde{\sigma}(\mathbf{x}, \mathbf{x}')}} \quad (6.8)$$

gives rise to a Green's function that satisfies Eq. (6.1) exactly.

Evidence that  $\tilde{W}$  is smooth through order  $\epsilon^3$  was presented in the context of the local expansion. Because  $W$  is known to be smooth, smoothness of  $\tilde{W}$  to all orders relies on the smoothness of  $\tilde{\sigma}/\sigma$ , which can only be assured if the series expansion of Eq. (6.5) can be proved to converge. In the absence of such a proof, we shall have to give the exact version of Eq. (6.1) the status of a plausible, but unproved, conjecture.

## B. Electromagnetic field

In Sec. VI A we were able to establish that in the case of a scalar field, a local expansion of the Hadamard Green's function formulated in the original space, and a local expansion formulated in the conformally related space, produce the same Green's function, apart from the factor of  $\Omega(x')$  that appears in Eq. (3.40). In addition, we formulated a conjecture to the effect that the two Hadamard forms may be related by

$$\tilde{G}_3^H(x, x') = \Omega(x') G_3^H(x, x') \quad (6.9)$$

as a matter of exact identity. The methods of Sec. VI A allow us to make the same statements regarding the electromagnetic Green's function. The required computations are almost identical, and all the relevant equations can be obtained from the scalar case by making the substitution  $A^a \rightarrow -A^a$ .

## VII. DETWEILER-WHITING CONSTRUCTION

### A. Scalar field

In this section we construct the three-dimensional version of the Detweiler-Whiting singular Green's function for a static scalar field in a static spacetime. By virtue of Eq. (3.16), this can be related to the four-dimensional version of Eq. (1.1) by

$$G_3^S(x, z) := \int_{\gamma} G_4^S(x, z) d\tau \quad (7.1)$$

in which  $\tau$  is proper time for an observer at rest at the spatial position  $z$ . The integral can be evaluated with the techniques described in Sec. 17.2 of Ref. [1], and we have that

$$G_3^S(x, \bar{x}) = \frac{1}{2r} U(x, x') + \frac{1}{2r_{\text{adv}}} U(x, x'') - \frac{1}{2} \int_u^v V(x, z) d\tau, \quad (7.2)$$

in which  $x' := z(u)$  is the retarded point on the (static) world line,  $x'' := z(v)$  is the advanced point,  $r := \sigma_{\alpha'} u^{\alpha'}$  is

the retarded distance,  $r_{\text{adv}} := -\sigma_{\alpha''} u^{\alpha''}$  is the advanced distance, and  $U(x, z)$  and  $V(x, z)$  are the two-point functions that appear in the construction of the four-dimensional Green's function.

To calculate  $G_3^S$  we follow the methods of Haas and Poisson (HP) [34], wherein the retarded and advanced points are related to a middle point  $\bar{x}$  on the world line. But while  $\bar{x}$  was chosen arbitrarily in HP, here we specifically choose  $\bar{x}$  to be simultaneous with  $x$ , so that  $\bar{x}$  and  $x$  have the same time coordinate. This condition implies that  $\bar{r} := \sigma_{\bar{\alpha}}(x, \bar{x}) u^{\bar{\alpha}} = 0$ .

Following HP we define the world-line functions

$$\sigma(\tau) := \sigma(x, z(\tau)), \quad (7.3a)$$

$$U(\tau) := U(x, z(\tau)), \quad (7.3b)$$

$$V(\tau) := V(x, z(\tau)), \quad (7.3c)$$

in which  $x$  is kept fixed. These functions will all be expressed as Taylor expansions about  $\tau = \bar{\tau}$ , with  $\bar{\tau}$  defined by  $\bar{x} := z(\bar{\tau})$ . We also define

$$s^2 := g^{\bar{\alpha}\bar{\beta}} \sigma_{\bar{\alpha}} \sigma_{\bar{\beta}} = 2\sigma(x, \bar{x}), \quad (7.4)$$

the squared geodesic distance between  $x$  and  $\bar{x}$ . Notice that  $r = \dot{\sigma}(u)$  and  $r_{\text{adv}} = -\dot{\sigma}(v)$ , in which an overdot indicates differentiation with respect to  $\tau$ . We define

$$\Delta_- := u - \bar{\tau}, \quad \Delta_+ := v - \bar{\tau}, \quad (7.5)$$

with  $\Delta_- < 0$  and  $\Delta_+ > 0$ ; these parameters are collectively denoted  $\Delta$ .

The  $\Delta$  parameters are determined by writing  $\sigma(u) = 0$  or  $\sigma(v) = 0$  as a Taylor expansion about  $\bar{\tau}$ :

$$0 = \sigma + \dot{\sigma}\Delta + \frac{1}{2}\ddot{\sigma}\Delta^2 + \frac{1}{6}\ddot{\sigma}\Delta^3 + \frac{1}{24}\sigma^{(4)}\Delta^4 + \frac{1}{120}\sigma^{(5)}\Delta^5 + O(\epsilon^6), \quad (7.6)$$

in which  $\sigma$  and its derivatives are evaluated at  $\tau = \bar{\tau}$ . This equation is then solved for  $\Delta$ . The derivatives of  $\sigma(\tau)$  are given by

$$\sigma = \frac{1}{2}s^2, \quad (7.7a)$$

$$\dot{\sigma} = \sigma_{\bar{\alpha}} u^{\bar{\alpha}} = 0, \quad (7.7b)$$

$$\ddot{\sigma} = \sigma_{\bar{\alpha}\bar{\beta}} u^{\bar{\alpha}} u^{\bar{\beta}} + \sigma_{\bar{\alpha}} \dot{a}^{\bar{\alpha}} = -1 - \frac{1}{3}R_{u\sigma u\sigma} + \frac{1}{12}R_{u\sigma u\sigma;\sigma} + \sigma_{\bar{\alpha}} \dot{a}^{\bar{\alpha}} + O(\epsilon^4), \quad (7.7c)$$

$$\ddot{\sigma} = \sigma_{\bar{\alpha}\bar{\beta}\bar{\gamma}} u^{\bar{\alpha}} u^{\bar{\beta}} u^{\bar{\gamma}} + 3\sigma_{\bar{\alpha}\bar{\beta}} u^{\bar{\alpha}} \dot{a}^{\bar{\beta}} + \sigma_{\bar{\alpha}} \dot{a}^{\bar{\alpha}} = -\frac{1}{4}R_{u\sigma u\sigma;u} - R_{u\sigma a\sigma} + \sigma_{\bar{\alpha}} \dot{a}^{\bar{\alpha}} + O(\epsilon^3), \quad (7.7d)$$

$$\begin{aligned} \sigma^{(4)} &= \sigma_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} u^{\bar{\alpha}} u^{\bar{\beta}} u^{\bar{\gamma}} u^{\bar{\delta}} + \sigma_{\bar{\alpha}\bar{\beta}\bar{\gamma}} (5u^{\bar{\alpha}} \dot{a}^{\bar{\beta}} u^{\bar{\gamma}} + u^{\bar{\alpha}} u^{\bar{\beta}} \dot{a}^{\bar{\gamma}}) + \sigma_{\bar{\alpha}\bar{\beta}} (3\dot{a}^{\bar{\alpha}} \dot{a}^{\bar{\beta}} + 4u^{\bar{\alpha}} \dot{a}^{\bar{\beta}}) + \sigma_{\bar{\alpha}} \ddot{a}^{\bar{\alpha}} \\ &= R_{uau\sigma} - a^2 + \sigma_{\bar{\alpha}} \ddot{a}^{\bar{\alpha}} + O(\epsilon^2), \end{aligned} \quad (7.7e)$$

$$\begin{aligned} \sigma^{(5)} &= \sigma_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}\bar{\epsilon}} u^{\bar{\alpha}} u^{\bar{\beta}} u^{\bar{\gamma}} u^{\bar{\delta}} u^{\bar{\epsilon}} + \sigma_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} (a^{\bar{\alpha}} u^{\bar{\beta}} u^{\bar{\gamma}} u^{\bar{\delta}} + 6u^{\bar{\alpha}} \dot{a}^{\bar{\beta}} u^{\bar{\gamma}} u^{\bar{\delta}} + 2u^{\bar{\alpha}} u^{\bar{\beta}} \dot{a}^{\bar{\gamma}} u^{\bar{\delta}} + u^{\bar{\alpha}} u^{\bar{\beta}} u^{\bar{\gamma}} \dot{a}^{\bar{\delta}}) + \sigma_{\bar{\alpha}\bar{\beta}\bar{\gamma}} (8\dot{a}^{\bar{\alpha}} \dot{a}^{\bar{\beta}} u^{\bar{\gamma}} \\ &\quad + 6u^{\bar{\alpha}} \dot{a}^{\bar{\beta}} \dot{a}^{\bar{\gamma}} + a^{\bar{\alpha}} u^{\bar{\beta}} \dot{a}^{\bar{\gamma}} + 9u^{\bar{\alpha}} \dot{a}^{\bar{\beta}} u^{\bar{\gamma}} + u^{\bar{\alpha}} u^{\bar{\beta}} \dot{a}^{\bar{\gamma}}) + \sigma_{\bar{\alpha}\bar{\beta}} (10\dot{a}^{\bar{\alpha}} \dot{a}^{\bar{\beta}} + 5u^{\bar{\alpha}} \ddot{a}^{\bar{\beta}}) + \sigma_{\bar{\alpha}} \ddot{a}^{\bar{\alpha}} \\ &= -5a_{\bar{\alpha}} \dot{a}^{\bar{\alpha}} + O(\epsilon). \end{aligned} \quad (7.7f)$$

These results rely on the standard expansion

$$\begin{aligned} \sigma_{\bar{\alpha}\bar{\beta}}(x, \bar{x}) &= g_{\bar{\alpha}\bar{\beta}} - \frac{1}{3}R_{\bar{\alpha}\bar{\mu}\bar{\beta}\bar{\nu}}\sigma^{\bar{\mu}}\sigma^{\bar{\nu}} \\ &+ \frac{1}{12}R_{\bar{\alpha}\bar{\mu}\bar{\beta}\bar{\nu};\bar{\lambda}}\sigma^{\bar{\mu}}\sigma^{\bar{\nu}}\sigma^{\bar{\lambda}} + O(\epsilon^4), \end{aligned} \quad (7.8)$$

which can be differentiated with respect to  $\bar{x}^\alpha$  to produce expansions for  $\sigma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}$  and so on. We use the HP notation for the components of the Riemann tensor; for example  $R_{u\sigma u\sigma} := R_{\bar{\alpha}\bar{\mu}\bar{\beta}\bar{\nu}}u^{\bar{\alpha}}\sigma^{\bar{\mu}}u^{\bar{\beta}}\sigma^{\bar{\nu}}$ ,  $R_{u\sigma a\sigma} := R_{\bar{\alpha}\bar{\mu}\bar{\beta}\bar{\nu}}u^{\bar{\alpha}}\sigma^{\bar{\mu}}a^{\bar{\beta}}\sigma^{\bar{\nu}}$ , and  $R_{u\sigma\sigma;u} := R_{\bar{\alpha}\bar{\mu}\bar{\beta}\bar{\nu};\bar{\lambda}}u^{\bar{\alpha}}\sigma^{\bar{\mu}}u^{\bar{\beta}}\sigma^{\bar{\nu}}u^{\bar{\lambda}}$ . We have defined

$$a^\mu := \frac{Du^\mu}{d\tau}, \quad \dot{a}^\mu := \frac{Da^\mu}{d\tau}, \quad \ddot{a}^\mu := \frac{D\dot{a}^\mu}{d\tau}, \quad (7.9)$$

and so on, and used the identities  $u_\mu a^\mu = 0$ ,  $u_\mu \dot{a}^\mu = -a^2 := a_\mu a^\mu$ , and  $u_\mu \ddot{a}^\mu = -3a_\mu \dot{a}^\mu$ .

Substitution of these expansions within Eq. (7.6) and solving for  $\Delta$  returns an expansion of the form

$$\Delta = \Delta_1\epsilon + \Delta_2\epsilon^2 + \Delta_3\epsilon^3 + \Delta_4\epsilon^4 + O(\epsilon^5). \quad (7.10)$$

The explicit expressions for  $\Delta_1$ ,  $\Delta_2$ ,  $\Delta_3$ , and  $\Delta_4$  are too large to be displayed here, but we may mention that  $\Delta_1^+ = s$  and  $\Delta_1^- = -s$ .

With  $\Delta$  determined,  $r$  and  $r_{\text{adv}}$  can be calculated as Taylor expansions. Since  $r = \dot{\sigma}(u)$  and  $r_{\text{adv}} = -\dot{\sigma}(v)$ , we have that

$$r = \dot{\sigma}\Delta_- + \frac{1}{2}\ddot{\sigma}\Delta_-^2 + \frac{1}{6}\sigma^{(4)}\Delta_-^3 + \frac{1}{24}\sigma^{(5)}\Delta_-^4 + O(\epsilon^5), \quad (7.11a)$$

$$r_{\text{adv}} = -\dot{\sigma}\Delta_+ - \frac{1}{2}\ddot{\sigma}\Delta_+^2 - \frac{1}{6}\sigma^{(4)}\Delta_+^3 - \frac{1}{24}\sigma^{(5)}\Delta_+^4 + O(\epsilon^5). \quad (7.11b)$$

At leading order  $r = s + O(\epsilon^2)$  and  $r_{\text{adv}} = s + O(\epsilon^2)$ , but the complete expansions for  $r^{-1}$  and  $r_{\text{adv}}^{-1}$  are too large to be displayed here.

Expressions for  $U(x, x')$  and  $U(x, x'')$  are obtained in a similar way. We write

$$U(x, x') = U + \dot{U}\Delta_- + \frac{1}{2}\ddot{U}\Delta_-^2 + \frac{1}{6}\ddot{U}\Delta_-^3 + O(\epsilon^4), \quad (7.12a)$$

$$U(x, x'') = U + \dot{U}\Delta_+ + \frac{1}{2}\ddot{U}\Delta_+^2 + \frac{1}{6}\ddot{U}\Delta_+^3 + O(\epsilon^4), \quad (7.12b)$$

in which  $U(\tau)$  and its derivatives are evaluated at  $\tau = \bar{\tau}$ . These quantities are given by

$$U = 1 + \frac{1}{12}R_{\sigma\sigma} - \frac{1}{24}R_{\sigma\sigma;\sigma} + O(\epsilon^4), \quad (7.13a)$$

$$\dot{U} = U_{;\bar{\alpha}}u^{\bar{\alpha}} = \frac{1}{6}R_{u\sigma} + \frac{1}{24}R_{\sigma\sigma;u} - \frac{1}{12}R_{u\sigma;\sigma} + O(\epsilon^3), \quad (7.13b)$$

$$\begin{aligned} \ddot{U} &= U_{;\bar{\alpha}\bar{\beta}}u^{\bar{\alpha}}u^{\bar{\beta}} + U_{;\bar{\alpha}}\dot{a}^{\bar{\alpha}} = \frac{1}{6}R_{uu} + \frac{1}{6}R_{a\sigma} + \frac{1}{6}R_{u\sigma;u} \\ &- \frac{1}{12}R_{uu;\sigma} + O(\epsilon^2), \end{aligned} \quad (7.13c)$$

$$\begin{aligned} \ddot{U} &= U_{;\bar{\alpha}\bar{\beta}\bar{\gamma}}u^{\bar{\alpha}}u^{\bar{\beta}}u^{\bar{\gamma}} + 3U_{;\bar{\alpha}\bar{\beta}}\dot{a}^{\bar{\alpha}}u^{\bar{\beta}} + U_{;\bar{\alpha}}\dot{a}^{\bar{\alpha}} \\ &= \frac{1}{2}R_{au} + \frac{1}{4}R_{uu;u} + O(\epsilon). \end{aligned} \quad (7.13d)$$

To evaluate the tail integral we expand  $V(\tau)$  as  $V(\bar{\tau}) + (\tau - \bar{\tau})\dot{V}(\bar{\tau}) + O(\epsilon^2)$  and integrate with respect to  $\tau$  between  $u = \bar{\tau} + \Delta_-$  and  $v = \bar{\tau} + \Delta_+$ . The result is

$$\int_u^v V(x, z)d\tau = V(\Delta_+ - \Delta_-) + \frac{1}{2}\dot{V}(\Delta_+^2 - \Delta_-^2) + O(\epsilon^3), \quad (7.14)$$

in which  $V := V(x, \bar{x})$  and  $\dot{V} := V_{;\bar{\alpha}}u^{\bar{\alpha}}$ . These are given by the expansions

$$V = \frac{1}{12}\bar{R} - \frac{1}{24}R_{;\bar{\alpha}}\sigma^{\bar{\alpha}} + O(\epsilon^2), \quad (7.15)$$

and

$$\dot{V} = \frac{1}{24}R_{;\bar{\alpha}}u^{\bar{\alpha}} + O(\epsilon). \quad (7.16)$$

To obtain Eq. (7.15) we rely on standard expansion techniques. The two-point function is required to satisfy the wave equation  $\square V = 0$  as well as the light-cone equation

$$V_{;\alpha}\sigma^\alpha + \frac{1}{2}(\sigma^\alpha{}_\alpha - 2)V = \frac{1}{2}\square U, \quad (7.17)$$

which is evaluated at  $\sigma(x, \bar{x}) = 0$ . The solution is expressed as an expansion

$$V(x, \bar{x}) = \sum_{n=0}^{\infty} V_n(x, \bar{x})\sigma^n, \quad (7.18)$$

and the wave equation gives rise to a sequence of equations which determine  $V_n$  from  $V_{n-1}$ ; the light-cone equation determines  $V_0$ . Because  $\sigma = O(\epsilon^2)$ ,  $V = V_0$  to order  $\epsilon$ , and this can be obtained by inserting the expansion

$$V = V^0 + V_{\bar{\alpha}}^0\sigma^{\bar{\alpha}} + O(\epsilon^2) \quad (7.19)$$

within the light-cone equation. We use the fact that  $\sigma^\alpha{}_\alpha = 4 + O(\epsilon^2)$ , and to compute  $\square U$  we start with Eq. (7.13b) and rely on the expansions

$$\sigma^{\bar{\alpha}}{}_\alpha = -g^{\bar{\alpha}}{}_\alpha + O(\epsilon^2), \quad g^{\bar{\alpha}}{}_{\alpha;\beta} = O(\epsilon); \quad (7.20)$$

we eventually arrive at

$$\square U = \frac{1}{6}\bar{R} - \frac{1}{6}R_{;\bar{\alpha}}\sigma^{\bar{\alpha}} + O(\epsilon^2). \quad (7.21)$$

The end result of the computation is Eq. (7.15).

Putting all the ingredients together, we eventually arrive at the following expansion for  $G_3^S$ :

$$\begin{aligned} G_3^S(x, \bar{x}) = & \frac{1}{s} \left\{ 1 + \psi_{\bar{\alpha}}^0 \sigma^{\bar{\alpha}} + \frac{1}{2} \psi_{\bar{\alpha}\bar{\beta}}^0 \sigma^{\bar{\alpha}} \sigma^{\bar{\beta}} \right. \\ & + \frac{1}{6} \psi_{\bar{\alpha}\bar{\beta}\bar{\gamma}}^0 \sigma^{\bar{\alpha}} \sigma^{\bar{\beta}} \sigma^{\bar{\gamma}} + O(\epsilon^4) \\ & \left. + s^2 [\psi^1 + \psi_{\bar{\alpha}}^1 \sigma^{\bar{\alpha}} + O(\epsilon^2)] \right\}, \quad (7.22) \end{aligned}$$

with

$$\psi_{\bar{\alpha}}^0 = \frac{1}{2} a_{\bar{\alpha}}, \quad (7.23a)$$

$$\psi_{\bar{\alpha}\bar{\beta}}^0 = \frac{3}{4} a_{\bar{\alpha}} a_{\bar{\beta}} + \frac{1}{6} R_{\bar{\alpha}\bar{\beta}} - \frac{1}{3} u^{\bar{\mu}} u^{\bar{\nu}} R_{\bar{\mu}\bar{\alpha}\bar{\nu}\bar{\beta}}, \quad (7.23b)$$

$$\begin{aligned} \psi_{\bar{\alpha}\bar{\beta}\bar{\gamma}}^0 = & \frac{15}{8} a_{\bar{\alpha}} a_{\bar{\beta}} a_{\bar{\gamma}} - \frac{3}{2} a_{\bar{\alpha}} u^{\bar{\mu}} u^{\bar{\nu}} R_{\bar{\mu}\bar{\beta}\bar{\nu}\bar{\gamma}} + \frac{1}{4} a_{\bar{\alpha}} R_{\bar{\beta}\bar{\gamma}} \\ & + \frac{1}{4} u^{\bar{\mu}} u^{\bar{\nu}} R_{\bar{\mu}\bar{\alpha}\bar{\nu}\bar{\beta};\bar{\gamma}} - \frac{1}{4} R_{\bar{\alpha}\bar{\beta};\bar{\gamma}} \end{aligned} \quad (7.23c)$$

and

$$\psi^1 = -\frac{1}{8} a^{\bar{\mu}} a_{\bar{\mu}} + \frac{1}{12} u^{\bar{\mu}} u^{\bar{\nu}} R_{\bar{\mu}\bar{\nu}} - \frac{1}{12} \bar{R}, \quad (7.24a)$$

$$\begin{aligned} \psi_{\bar{\alpha}}^1 = & -\frac{5}{16} a^{\bar{\mu}} a_{\bar{\mu}} a_{\bar{\alpha}} + \frac{1}{8} \ddot{a}_{\bar{\alpha}} + \frac{1}{8} a_{\bar{\alpha}} u^{\bar{\mu}} u^{\bar{\nu}} R_{\bar{\mu}\bar{\nu}} - \frac{1}{24} \bar{R} a_{\bar{\alpha}} \\ & + \frac{1}{8} u^{\bar{\mu}} a^{\bar{\nu}} u^{\bar{\lambda}} R_{\bar{\mu}\bar{\nu}\bar{\lambda}\bar{\alpha}} + \frac{1}{12} a^{\bar{\mu}} R_{\bar{\mu}\bar{\alpha}} + \frac{1}{12} u^{\bar{\mu}} u^{\bar{\nu}} R_{\bar{\alpha}\bar{\mu};\bar{\nu}} \\ & - \frac{1}{24} u^{\bar{\mu}} u^{\bar{\nu}} R_{\bar{\mu}\bar{\nu};\bar{\alpha}} + \frac{1}{24} R_{;\bar{\alpha}}. \end{aligned} \quad (7.24b)$$

The actual expression for  $\psi_{\bar{\alpha}\bar{\beta}\bar{\gamma}}^0$  is obtained from what appears above by symmetrizing over all three indices; this operation was suppressed to keep the notation uncluttered.

Noting that the vector  $\sigma^{\bar{\alpha}}$  has a vanishing time component when  $x$  and  $\bar{x}$  are simultaneous events, we may re-express Eq. (7.22) as

$$\begin{aligned} G_3^S(x, \bar{x}) = & \frac{1}{s} \left\{ 1 + \psi_{\bar{a}}^0 \sigma^{\bar{a}} + \frac{1}{2} \psi_{\bar{a}\bar{b}}^0 \sigma^{\bar{a}} \sigma^{\bar{b}} + \frac{1}{6} \psi_{\bar{a}\bar{b}\bar{c}}^0 \sigma^{\bar{a}} \sigma^{\bar{b}} \sigma^{\bar{c}} \right. \\ & \left. + O(\epsilon^4) + s^2 [\psi^1 + \psi_{\bar{a}}^1 \sigma^{\bar{a}} + O(\epsilon^2)] \right\}. \quad (7.25) \end{aligned}$$

And with the results derived in Sec. II, the expansion coefficients become

$$\psi_{\bar{a}}^0 = \frac{1}{2} A_{\bar{a}}, \quad (7.26a)$$

$$\psi_{\bar{a}\bar{b}}^0 = -\frac{1}{2} A_{\bar{a}\bar{b}} + \frac{1}{4} A_{\bar{a}} A_{\bar{b}} + \frac{1}{6} R_{\bar{a}\bar{b}}, \quad (7.26b)$$

$$\begin{aligned} \psi_{\bar{a}\bar{b}\bar{c}}^0 = & \frac{1}{2} A_{(\bar{a}\bar{b}\bar{c})} - \frac{3}{4} A_{(\bar{a}} A_{\bar{b}\bar{c})} + \frac{1}{8} A_{\bar{a}} A_{\bar{b}} A_{\bar{c}} + \frac{1}{4} A_{(\bar{a}} R_{\bar{b}\bar{c})} \\ & - \frac{1}{4} R_{(\bar{a}\bar{b}\bar{c})}. \end{aligned} \quad (7.26c)$$

and

$$\psi^1 = \frac{1}{4} A^{\bar{a}}_{|\bar{a}} + \frac{1}{8} A^{\bar{a}} A_{\bar{a}} - \frac{1}{12} \bar{R}, \quad (7.27a)$$

$$\begin{aligned} \psi_{\bar{a}}^1 = & -\frac{1}{8} A^{\bar{c}}_{|\bar{c}\bar{a}} - \frac{1}{8} A^{\bar{c}} A_{\bar{c}|\bar{a}} + \frac{1}{8} A^{\bar{c}}_{|\bar{c}} A_{\bar{a}} + \frac{1}{16} A^{\bar{c}} A_{\bar{c}} A_{\bar{a}} \\ & - \frac{1}{24} \bar{R} A_{\bar{a}} + \frac{1}{24} R_{|\bar{a}}. \end{aligned} \quad (7.27b)$$

Comparing Eq. (7.26) with (5.9) and (7.27) with (5.13), we observe that the expansion coefficients of  $G_3^S$  and  $G_3^H$  are in precise agreement. This allows us to conclude that

$$G_3^S(x, \bar{x}) = G_3^H(x, \bar{x}) + O(\epsilon^3) \quad (7.28)$$

for a static spacetime.

## B. Electromagnetic field

We next turn to the three-dimensional version of the Detweiler-Whiting singular Green's function for a static electromagnetic field in a static spacetime. By virtue of Eq. (3.44), we have that the vector potential of a point charge  $e$  situated at  $z$  is given by

$$\Phi_i^S(x) = -eN(z)G_3^S(x, z), \quad (7.29)$$

with  $G_3^S(x, z)$  denoting the three-dimensional version of the Detweiler-Whiting electromagnetic Green's function. And according to Sec. 18.2 of Ref. [1], we have that the vector potential is given

$$\begin{aligned} \Phi_{\alpha}^S(x) = & \frac{e}{2r} U_{\alpha\beta'}(x, x') u^{\beta'} + \frac{e}{2r_{\text{adv}}} U_{\alpha\beta''}(x, x'') u^{\beta''} \\ & - \frac{e}{2} \int_u^v V_{\alpha\mu}(x, z) u^{\mu} d\tau, \end{aligned} \quad (7.30)$$

in which  $U_{\alpha\mu}(x, z)$ ,  $V_{\alpha\mu}(x, z)$  are the two-point functions that appear in the construction of the four-dimensional Green's function.

To calculate  $\Phi_i^S$  and obtain  $G_3^S$  we once more follow the methods of Haas and Poisson (HP) [34], as outlined in the scalar case. We thus define the world-line functions

$$\sigma(\tau) := \sigma(x, z(\tau)), \quad (7.31a)$$

$$U_{\alpha}(\tau) := U_{\alpha\mu}(x, z(\tau)) u^{\mu}(\tau), \quad (7.31b)$$

$$V_{\alpha}(\tau) := V_{\alpha\mu}(x, z(\tau)) u^{\mu}(\tau), \quad (7.31c)$$

in which  $x$  is kept fixed. These functions are all scalars with respect to their dependence upon  $z(\tau)$ . As in the scalar case they are expressed as Taylor expansions about  $\tau = \bar{\tau}$ , at which  $z = \bar{x}$ , and the results are converted into explicit expressions for  $r$ ,  $r_{\text{adv}}$ ,  $U_{\alpha\beta'} u^{\beta'}$ ,  $U_{\alpha\beta''} u^{\beta''}$ , and the tail integral. The results for  $r$  and  $r_{\text{adv}}$  appear in Eq. (7.11).

To compute  $U_{\alpha\beta'}u^{\beta'}$  and  $U_{\alpha\beta''}u^{\beta''}$  we write

$$U_{\alpha\beta'}u^{\beta'} = U_\alpha + \dot{U}_\alpha\Delta_- + \frac{1}{2}\ddot{U}_\alpha\Delta_-^2 + \frac{1}{6}\dddot{U}_\alpha\Delta_-^3 + O(\epsilon^4), \quad (7.32a)$$

$$U_{\alpha\beta''}u^{\beta''} = U_\alpha + \dot{U}_\alpha\Delta_+ + \frac{1}{2}\ddot{U}_\alpha\Delta_+^2 + \frac{1}{6}\dddot{U}_\alpha\Delta_+^3 + O(\epsilon^4), \quad (7.32b)$$

in which  $U_\alpha(\tau)$  and its derivatives are evaluated at  $\tau = \bar{\tau}$ . These quantities are given by

$$U_\alpha = U_{\alpha\bar{\alpha}}u^{\bar{\alpha}} = g_{\alpha\bar{\alpha}}u^{\bar{\alpha}}\left(1 + \frac{1}{12}R_{\sigma\sigma} - \frac{1}{24}R_{\sigma\sigma;\sigma} + O(\epsilon^4)\right), \quad (7.33a)$$

$$\dot{U}_\alpha = U_{\alpha\bar{\alpha};\bar{\beta}}u^{\bar{\alpha}}u^{\bar{\beta}} + U_{\alpha\bar{\alpha}}\dot{a}^{\bar{\alpha}} = g_{\alpha\bar{\alpha}}\left[\frac{1}{2}R^{\bar{\alpha}}{}_{uu\sigma} - \frac{1}{6}R^{\bar{\alpha}}{}_{uu\sigma;\sigma} + u^{\bar{\alpha}}\left(\frac{1}{6}R_{u\sigma} + \frac{1}{24}R_{\sigma\sigma;u} - \frac{1}{12}R_{u\sigma;\sigma}\right) + \dot{a}^{\bar{\alpha}}\left(1 + \frac{1}{12}R_{\sigma\sigma}\right) + O(\epsilon^3)\right], \quad (7.33b)$$

$$\begin{aligned} \ddot{U}_\alpha &= U_{\alpha\bar{\alpha};\bar{\beta}\bar{\gamma}}u^{\bar{\alpha}}u^{\bar{\beta}}u^{\bar{\gamma}} + U_{\alpha\bar{\alpha};\bar{\beta}}(u^{\bar{\alpha}}\dot{a}^{\bar{\beta}} + 2\dot{a}^{\bar{\alpha}}u^{\bar{\beta}}) + U_{\alpha\bar{\alpha}}\ddot{a}^{\bar{\alpha}} \\ &= g_{\alpha\bar{\alpha}}\left[\frac{1}{3}R^{\bar{\alpha}}{}_{uu\sigma;u} + \frac{1}{2}R^{\bar{\alpha}}{}_{u\sigma} + R^{\bar{\alpha}}{}_{uu\sigma} + u^{\bar{\alpha}}\left(\frac{1}{6}R_{uu} + \frac{1}{6}R_{u\sigma;u} - \frac{1}{12}R_{uu;\sigma} + \frac{1}{6}R_{a\sigma}\right) + \dot{a}^{\bar{\alpha}}\left(\frac{1}{3}R_{u\sigma}\right) + \ddot{a}^{\bar{\alpha}} + O(\epsilon^2)\right], \end{aligned} \quad (7.33c)$$

$$\begin{aligned} \dddot{U}_\alpha &= U_{\alpha\bar{\alpha};\bar{\beta}\bar{\gamma}\bar{\delta}}u^{\bar{\alpha}}u^{\bar{\beta}}u^{\bar{\gamma}}u^{\bar{\delta}} + U_{\alpha\bar{\alpha};\bar{\beta}\bar{\gamma}}(3\dot{a}^{\bar{\alpha}}u^{\bar{\beta}}u^{\bar{\gamma}} + 2u^{\bar{\alpha}}\dot{a}^{\bar{\beta}}u^{\bar{\gamma}} + u^{\bar{\alpha}}u^{\bar{\beta}}\dot{a}^{\bar{\gamma}}) + U_{\alpha\bar{\alpha};\bar{\beta}}(3\dot{a}^{\bar{\alpha}}\dot{a}^{\bar{\beta}} + 3\dot{a}^{\bar{\alpha}}u^{\bar{\beta}} + u^{\bar{\alpha}}\dot{a}^{\bar{\beta}}) + U_{\alpha\bar{\alpha}}\dddot{a}^{\bar{\alpha}} \\ &= g_{\alpha\bar{\alpha}}\left[\frac{1}{2}R^{\bar{\alpha}}{}_{uuu} + u^{\bar{\alpha}}\left(\frac{1}{2}R_{au} + \frac{1}{4}R_{uu;u}\right) + \dot{a}^{\bar{\alpha}}\left(\frac{1}{2}R_{uu}\right) + \ddot{a}^{\bar{\alpha}} + O(\epsilon)\right]. \end{aligned} \quad (7.33d)$$

The expansions involve components of the Riemann tensor such as  $R^{\bar{\alpha}}{}_{uu\sigma} := R^{\bar{\alpha}}{}_{\bar{\mu}\bar{\beta}\bar{\nu}}u^{\bar{\mu}}u^{\bar{\beta}}\sigma^{\bar{\nu}}$  and components of the Ricci tensor such as  $R_{\sigma\sigma} := R_{\bar{\alpha}\bar{\beta}}\sigma^{\bar{\alpha}}\sigma^{\bar{\beta}}$ . They involve also  $g^{\alpha\bar{\alpha}}(x, \bar{x})$ , the parallel propagator from  $\bar{x}$  to  $x$ . To arrive at these results we rely on the expansion of the two-point function  $U^\alpha{}_{\bar{\alpha}}(x, \bar{x})$  given by

$$U^\alpha{}_{\bar{\alpha}} = g^\alpha{}_{\bar{\alpha}}\left(1 + \frac{1}{12}R_{\bar{\mu}\bar{\nu}}\sigma^{\bar{\mu}}\sigma^{\bar{\nu}} - \frac{1}{24}R_{\bar{\mu}\bar{\nu};\bar{\lambda}}\sigma^{\bar{\mu}}\sigma^{\bar{\nu}}\sigma^{\bar{\lambda}} + O(\epsilon^3)\right); \quad (7.34)$$

this is derived, for example, in Appendix B of Ref. [35]. Another useful expansion is

$$g^\alpha{}_{\bar{\alpha};\bar{\beta}} = g^\alpha{}_{\bar{\gamma}}\left(\frac{1}{2}R^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}\bar{\mu}}\sigma^{\bar{\mu}} - \frac{1}{6}R^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}\bar{\mu};\bar{\nu}}\sigma^{\bar{\mu}}\sigma^{\bar{\nu}} + O(\epsilon^3)\right). \quad (7.35)$$

These are differentiated repeatedly with respect to  $\bar{x}^\alpha$ , and the results are inserted within the expressions for  $U_\alpha$  and its derivatives.

To evaluate the tail integral we expand  $V_\alpha(\tau)$  as  $V_\alpha(\bar{\tau}) + (\tau - \bar{\tau})\dot{V}_\alpha(\bar{\tau}) + O(\epsilon^2)$  and integrate with respect to  $\tau$  between  $u = \bar{\tau} + \Delta_-$  and  $v = \bar{\tau} + \Delta_+$ . The result is

$$\int_u^v V_{\alpha\mu}u^\mu d\tau = V_\alpha(\Delta_+ - \Delta_-) + \frac{1}{2}\dot{V}_\alpha(\Delta_+^2 - \Delta_-^2) + O(\epsilon^3), \quad (7.36)$$

in which  $V_\alpha = V_{\alpha\bar{\alpha}}u^{\bar{\alpha}}$  and  $\dot{V}_\alpha = V_{\alpha\bar{\alpha};\bar{\beta}}u^{\bar{\alpha}}u^{\bar{\beta}} + V_{\alpha\bar{\alpha}}\dot{a}^{\bar{\alpha}}$ . To compute these quantities we rely on the expansion

$$V^\alpha{}_{\bar{\alpha}} = g^\alpha{}_{\bar{\gamma}}\left[-\frac{1}{2}\left(R^{\bar{\gamma}}{}_{\bar{\alpha}} - \frac{1}{6}\delta^{\bar{\gamma}}{}_{\bar{\alpha}}\bar{R}\right) + \frac{1}{12}R^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\mu}\bar{\beta}}{}^{;\bar{\mu}}\sigma^{\bar{\beta}} + \frac{1}{4}\left(R^{\bar{\gamma}}{}_{\bar{\alpha};\bar{\beta}} - \frac{1}{6}\delta^{\bar{\gamma}}{}_{\bar{\alpha}}\bar{R}_{;\bar{\beta}}\right)\sigma^{\bar{\beta}} + O(\epsilon^2)\right], \quad (7.37)$$

which leads to

$$V_\alpha = g_{\alpha\bar{\alpha}}\left[-\frac{1}{2}R^{\bar{\alpha}}{}_u + \frac{1}{4}R^{\bar{\alpha}}{}_{u;\sigma} + \frac{1}{12}R^{\bar{\alpha}}{}_{u\bar{\mu}\sigma}{}^{;\bar{\mu}} + u^{\bar{\alpha}}\left(\frac{1}{12}\bar{R} - \frac{1}{24}\bar{R}_{;\sigma}\right) + O(\epsilon^2)\right], \quad (7.38a)$$

$$\dot{V}_\alpha = g_{\alpha\bar{\alpha}}\left[-\frac{1}{4}R^{\bar{\alpha}}{}_{u;u} + \frac{1}{12}R^{\bar{\alpha};\bar{\mu}}{}_{u\bar{\mu}u} - \frac{1}{2}R^{\bar{\alpha}}{}_a + \frac{1}{24}u^{\bar{\alpha}}\bar{R}_{;u} + \frac{1}{12}\dot{a}^{\bar{\alpha}}\bar{R} + O(\epsilon)\right]. \quad (7.38b)$$

Here we make use of the notation  $R^{\bar{\alpha}}{}_a := R^{\bar{\alpha}}{}_{\bar{\beta}}a^{\bar{\beta}}$ ,  $R^{\bar{\alpha}}{}_{u;\sigma} := R^{\bar{\alpha}}{}_{\bar{\beta};\bar{\gamma}}u^{\bar{\beta}}\sigma^{\bar{\gamma}}$ , and  $R^{\bar{\alpha}}{}_{u\bar{\mu}u}{}^{;\bar{\mu}} := R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\mu}\bar{\gamma}}{}^{;\bar{\mu}}u^{\bar{\beta}}u^{\bar{\gamma}}$ . In addition,  $\bar{R}$  is the Ricci scalar evaluated at  $\bar{x}$ , and  $\bar{R}_{;u} := \bar{R}_{;\bar{\beta}}u^{\bar{\beta}}$ .

To obtain Eq. (7.37) we rely on standard expansion techniques. The two-point function is required to satisfy the wave equation

$$\square V^\alpha{}_{\bar{\alpha}} - R^\alpha{}_\beta V^\beta{}_{\bar{\alpha}} = 0 \quad (7.39)$$

as well as the light-cone equation

$$V^{\alpha}_{\bar{\alpha};\beta}\sigma^{\beta} + \frac{1}{2}(\sigma^{\beta}_{\beta} - 2)V^{\alpha}_{\bar{\alpha}} = \frac{1}{2}(\square U^{\alpha}_{\bar{\alpha}} - R^{\alpha}_{\beta}U^{\beta}_{\bar{\alpha}}), \quad (7.40)$$

which is evaluated at  $\sigma(x, \bar{x}) = 0$ . The solution is expressed as the expansion

$$V^{\alpha}_{\bar{\alpha}}(x, \bar{x}) = \sum_{n=0} V^{\alpha}_{n\bar{\alpha}}(x, \bar{x})\sigma^n, \quad (7.41)$$

and the wave equation gives rise to a sequence of equations which determine  $V^{\alpha}_{n\bar{\alpha}}(x, \bar{x})$  from  $V^{\alpha}_{n-1\bar{\alpha}}$ ; the light-cone equation determines  $V^{\alpha}_{0\bar{\alpha}}$ . Because  $\sigma = O(\epsilon^2)$ ,  $V^{\alpha}_{\bar{\alpha}} = V^{\alpha}_{0\bar{\alpha}}$  to order  $\epsilon$ , and this can be obtained by inserting the expansion

$$V^{\alpha}_{0\bar{\alpha}} = g^{\alpha}_{\bar{\gamma}}[A^{\bar{\gamma}}_{\bar{\alpha}} + A^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}}\sigma^{\bar{\beta}} + O(\epsilon^2)] \quad (7.42)$$

within the light-cone equation. We use the fact that  $\sigma^{\beta}_{\beta} = 4 + O(\epsilon^2)$  and to compute  $\square U^{\alpha}_{\bar{\alpha}}$  we rely on the expansion

$$g^{\alpha}_{\bar{\alpha};\beta} = g^{\alpha}_{\bar{\gamma}}g^{\bar{\beta}}_{\beta}\left[\frac{1}{2}R^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}\bar{\mu}}\sigma^{\bar{\mu}} - \frac{1}{3}R^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}\bar{\mu};\bar{\nu}}\sigma^{\bar{\mu}}\sigma^{\bar{\nu}} + O(\epsilon^3)\right]; \quad (7.43)$$

we eventually arrive at

$$\square U^{\alpha}_{\bar{\alpha}} = g^{\alpha}_{\bar{\gamma}}\left[\frac{1}{6}\delta^{\bar{\gamma}}_{\bar{\alpha}}\bar{R} - \frac{1}{6}\delta^{\bar{\gamma}}_{\bar{\alpha}}\bar{R}_{;\bar{\nu}}\sigma^{\bar{\nu}} + \frac{1}{3}R^{\bar{\gamma}}_{\bar{\alpha}\bar{\mu}\bar{\nu};\bar{\mu}}\sigma^{\bar{\nu}} + O(\epsilon^2)\right]. \quad (7.44)$$

The end result is Eq. (7.37).

$$\phi^1_{\bar{\lambda}} = u_{\bar{\lambda}}\left(-\frac{1}{8}a^{\bar{\mu}}a_{\bar{\mu}} + \frac{1}{12}u^{\bar{\mu}}u^{\bar{\nu}}R_{\bar{\mu}\bar{\nu}} - \frac{1}{12}\bar{R}\right) + \frac{1}{2}\dot{a}_{\bar{\lambda}} + \frac{1}{2}u^{\bar{\mu}}R_{\bar{\lambda}\bar{\mu}}, \quad (7.47a)$$

$$\begin{aligned} \phi^1_{\bar{\lambda}\bar{\alpha}} = u_{\bar{\lambda}}\left(-\frac{5}{16}a^{\bar{\mu}}a_{\bar{\mu}}a_{\bar{\alpha}} + \frac{1}{8}\ddot{a}_{\bar{\alpha}} + \frac{1}{8}a_{\bar{\alpha}}u^{\bar{\mu}}u^{\bar{\nu}}R_{\bar{\mu}\bar{\nu}} - \frac{1}{24}\bar{R}a_{\bar{\alpha}} + \frac{1}{8}u^{\bar{\mu}}a^{\bar{\nu}}u^{\bar{\rho}}R_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\alpha}} + \frac{1}{12}a^{\bar{\mu}}R_{\bar{\mu}\bar{\alpha}} + \frac{1}{12}u^{\bar{\mu}}u^{\bar{\nu}}R_{\bar{\alpha}\bar{\mu};\bar{\nu}} \right. \\ \left. - \frac{1}{24}u^{\bar{\mu}}u^{\bar{\nu}}R_{\bar{\mu}\bar{\nu};\bar{\alpha}} + \frac{1}{24}R_{;\bar{\alpha}}\right) + a_{\bar{\lambda}}\left(\frac{1}{2}\dot{a}_{\bar{\alpha}} + \frac{1}{6}u^{\bar{\mu}}R_{\bar{\alpha}\bar{\mu}}\right) + \frac{3}{4}\dot{a}_{\bar{\lambda}}a_{\bar{\alpha}} + \frac{1}{4}u^{\bar{\mu}}a^{\bar{\nu}}R_{\bar{\lambda}\bar{\mu}\bar{\nu}\bar{\alpha}} + \frac{1}{2}u^{\bar{\nu}}a^{\bar{\mu}}R_{\bar{\lambda}\bar{\mu}\bar{\nu}\bar{\alpha}} \\ \left. + \frac{1}{6}u^{\bar{\mu}}u^{\bar{\nu}}u^{\bar{\rho}}R_{\bar{\lambda}\bar{\mu}\bar{\nu}\bar{\alpha};\bar{\rho}} - \frac{1}{12}u^{\bar{\mu}}\nabla^{\bar{\rho}}R_{\bar{\lambda}\bar{\mu}\bar{\rho}\bar{\alpha}} + \frac{1}{4}u^{\bar{\mu}}R_{\bar{\lambda}\bar{\mu}}a_{\bar{\alpha}} - \frac{1}{4}u^{\bar{\mu}}R_{\bar{\lambda}\bar{\mu};\bar{\alpha}}\right). \end{aligned} \quad (7.47b)$$

The actual expression for  $\phi^0_{\bar{\lambda}\bar{\alpha}\bar{\beta}\bar{\gamma}}$  is obtained from what appears above by symmetrizing over the last three indices; this operation was suppressed to keep the notation uncluttered.

From Eq. (7.45) we wish to obtain a more explicit expression for  $\Phi_t^S$ , and this requires a computation of the operator of parallel transport. Our considerations near Eq. (2.5) imply that its components are given by

$$g^t_{\bar{t}} = \frac{N(\bar{x})}{N(x)}, \quad g^a_{\bar{b}} = h^a_{\bar{b}}, \quad (7.48)$$

in which  $h^a_{\bar{b}}$  is the operator of parallel transport in the three-dimensional space; the mixed components  $g^t_{\bar{a}}$  and

Putting all the ingredients together, we eventually arrive at the following expansion for  $\Phi_t^S(x)$ :

$$\begin{aligned} \Phi_t^S(x) = \frac{e}{s}g_{\bar{\alpha}}^{\bar{\lambda}}(x, \bar{x})\left\{\phi_{\bar{\lambda}}^0 + \phi_{\bar{\lambda}\bar{\alpha}}^0\sigma^{\bar{\alpha}} + \frac{1}{2}\phi_{\bar{\lambda}\bar{\alpha}\bar{\beta}}^0\sigma^{\bar{\alpha}}\sigma^{\bar{\beta}} \right. \\ \left. + \frac{1}{6}\phi_{\bar{\lambda}\bar{\alpha}\bar{\beta}\bar{\gamma}}^0\sigma^{\bar{\alpha}}\sigma^{\bar{\beta}}\sigma^{\bar{\gamma}} + O(\epsilon^4) \right. \\ \left. + s^2\left[\phi_{\bar{\lambda}}^1 + \phi_{\bar{\lambda}\bar{\alpha}}^1\sigma^{\bar{\alpha}} + O(\epsilon^2)\right]\right\}, \end{aligned} \quad (7.45)$$

with

$$\phi_{\bar{\lambda}}^0 = u_{\bar{\lambda}}, \quad (7.46a)$$

$$\phi_{\bar{\lambda}\bar{\alpha}}^0 = \frac{1}{2}u_{\bar{\lambda}}a_{\bar{\alpha}}, \quad (7.46b)$$

$$\phi_{\bar{\lambda}\bar{\alpha}\bar{\beta}}^0 = u_{\bar{\lambda}}\left(\frac{3}{4}a_{\bar{\alpha}}a_{\bar{\beta}} + \frac{1}{6}R_{\bar{\alpha}\bar{\beta}} - \frac{1}{3}u^{\bar{\mu}}u^{\bar{\nu}}R_{\bar{\mu}\bar{\alpha}\bar{\nu}\bar{\beta}}\right), \quad (7.46c)$$

$$\begin{aligned} \phi_{\bar{\lambda}\bar{\alpha}\bar{\beta}\bar{\gamma}}^0 = u_{\bar{\lambda}}\left(\frac{15}{8}a_{\bar{\alpha}}a_{\bar{\beta}}a_{\bar{\gamma}} - \frac{3}{2}a_{\bar{\alpha}}u^{\bar{\mu}}u^{\bar{\nu}}R_{\bar{\mu}\bar{\beta}\bar{\nu}\bar{\gamma}} \right. \\ \left. + \frac{1}{4}u^{\bar{\mu}}u^{\bar{\nu}}R_{\bar{\mu}\bar{\alpha}\bar{\nu}\bar{\beta};\bar{\gamma}} + \frac{1}{4}a_{\bar{\alpha}}R_{\bar{\beta}\bar{\gamma}} - \frac{1}{4}R_{\bar{\alpha}\bar{\beta};\bar{\gamma}}\right) \end{aligned} \quad (7.46d)$$

and

$g^a_{\bar{t}}$  vanish. Noting that the vector  $\sigma^{\bar{\alpha}}$  has a vanishing time component when  $x$  and  $\bar{x}$  are simultaneous events, and making use of the results derived in Sec. II, we may re-express Eq. (7.45) as

$$\begin{aligned} \Phi_t^S(x) = -\frac{e}{s}N(x)\left\{1 + \phi_{\bar{a}}^0\sigma^{\bar{a}} + \frac{1}{2}\phi_{\bar{a}\bar{b}}^0\sigma^{\bar{a}}\sigma^{\bar{b}} \right. \\ \left. + \frac{1}{6}\phi_{\bar{a}\bar{b}\bar{c}}^0\sigma^{\bar{a}}\sigma^{\bar{b}}\sigma^{\bar{c}} + O(\epsilon^4) \right. \\ \left. + s^2[\phi^1 + \phi_{\bar{a}}^1\sigma^{\bar{a}} + O(\epsilon^2)]\right\}, \end{aligned} \quad (7.49)$$

with

$$\phi_{\bar{a}}^0 = \frac{1}{2}A_{\bar{a}}, \quad (7.50a)$$

$$\phi_{\bar{a}\bar{b}}^0 = -\frac{1}{2}A_{\bar{a}|\bar{b}} + \frac{1}{4}A_{\bar{a}}A_{\bar{b}} + \frac{1}{6}R_{\bar{a}\bar{b}}, \quad (7.50b)$$

$$\begin{aligned} \phi_{\bar{a}\bar{b}\bar{c}}^0 &= \frac{1}{2}A_{(\bar{a}|\bar{b}\bar{c})} - \frac{3}{4}A_{(\bar{a}A_{\bar{b}|\bar{c})}} + \frac{1}{8}A_{\bar{a}}A_{\bar{b}}A_{\bar{c}} \\ &\quad + \frac{1}{4}A_{(\bar{a}}R_{\bar{b}\bar{c})} - \frac{1}{4}R_{(\bar{a}\bar{b}|\bar{c})}. \end{aligned} \quad (7.50c)$$

and

$$\phi^1 = -\frac{1}{4}A_{\bar{a}}^{\bar{a}}|_{\bar{a}} + \frac{1}{8}A_{\bar{a}}^{\bar{a}}A_{\bar{a}} - \frac{1}{12}\bar{R}, \quad (7.51a)$$

$$\begin{aligned} \phi_{\bar{a}}^1 &= \frac{1}{8}A_{|\bar{c}\bar{a}}^{\bar{c}} - \frac{1}{8}A_{\bar{c}|\bar{a}}^{\bar{c}} - \frac{1}{8}A_{|\bar{c}}^{\bar{c}}A_{\bar{a}} + \frac{1}{16}A_{\bar{c}}^{\bar{c}}A_{\bar{c}}A_{\bar{a}} \\ &\quad - \frac{1}{24}\bar{R}A_{\bar{a}} + \frac{1}{24}R_{|\bar{a}}. \end{aligned} \quad (7.51b)$$

From Eq. (7.29) and (7.49) we see that the three-dimensional Green's function involves the ratio  $N(\mathbf{x})/N(\bar{\mathbf{x}})$ . This can be expressed as an expansion about  $\mathbf{x} = \bar{\mathbf{x}}$  by making use of the generalized Taylor series

$$\begin{aligned} N(\mathbf{x}) &= N(\bar{\mathbf{x}}) - N_{|\bar{a}}\sigma^{\bar{a}} + \frac{1}{2}N_{|\bar{a}\bar{b}}\sigma^{\bar{a}}\sigma^{\bar{b}} - \frac{1}{6}N_{|\bar{a}\bar{b}\bar{c}}\sigma^{\bar{a}}\sigma^{\bar{b}}\sigma^{\bar{c}} \\ &\quad + O(\epsilon^4), \end{aligned} \quad (7.52)$$

which leads to

$$\begin{aligned} \frac{N(\mathbf{x})}{N(\bar{\mathbf{x}})} &= 1 - A_{\bar{a}}\sigma^{\bar{a}} + \frac{1}{2}(A_{\bar{a}|\bar{b}} + A_{\bar{a}}A_{\bar{b}})\sigma^{\bar{a}}\sigma^{\bar{b}} \\ &\quad - \frac{1}{6}(A_{\bar{a}|\bar{b}\bar{c}} + 3A_{\bar{a}}A_{\bar{b}|\bar{c}} + A_{\bar{a}}A_{\bar{b}}A_{\bar{c}})\sigma^{\bar{a}}\sigma^{\bar{b}}\sigma^{\bar{c}} + O(\epsilon^4). \end{aligned} \quad (7.53)$$

With this we finally arrive at

$$\begin{aligned} G_3^S(\mathbf{x}, \bar{\mathbf{x}}) &= \frac{1}{s} \left\{ 1 + \psi_{\bar{a}}^0\sigma^{\bar{a}} + \frac{1}{2}\psi_{\bar{a}\bar{b}}^0\sigma^{\bar{a}}\sigma^{\bar{b}} + \frac{1}{6}\psi_{\bar{a}\bar{b}\bar{c}}^0\sigma^{\bar{a}}\sigma^{\bar{b}}\sigma^{\bar{c}} \right. \\ &\quad \left. + O(\epsilon^4) + s^2[\psi^1 + \psi_{\bar{a}}^1\sigma^{\bar{a}} + O(\epsilon^2)] \right\}, \end{aligned} \quad (7.54)$$

with

$$\psi_{\bar{a}}^0 = -\frac{1}{2}A_{\bar{a}}, \quad (7.55a)$$

$$\psi_{\bar{a}\bar{b}}^0 = \frac{1}{2}A_{\bar{a}|\bar{b}} + \frac{1}{4}A_{\bar{a}}A_{\bar{b}} + \frac{1}{6}R_{\bar{a}\bar{b}}, \quad (7.55b)$$

$$\begin{aligned} \psi_{\bar{a}\bar{b}\bar{c}}^0 &= -\frac{1}{2}A_{(\bar{a}|\bar{b}\bar{c})} - \frac{3}{4}A_{(\bar{a}A_{\bar{b}|\bar{c})}} - \frac{1}{8}A_{\bar{a}}A_{\bar{b}}A_{\bar{c}} - \frac{1}{4}A_{(\bar{a}}R_{\bar{b}\bar{c})} \\ &\quad - \frac{1}{4}R_{(\bar{a}\bar{b}|\bar{c})}. \end{aligned} \quad (7.55c)$$

and

$$\psi^1 = -\frac{1}{4}A_{\bar{a}}^{\bar{a}}|_{\bar{a}} + \frac{1}{8}A_{\bar{a}}^{\bar{a}}A_{\bar{a}} - \frac{1}{12}\bar{R}, \quad (7.56a)$$

$$\begin{aligned} \psi_{\bar{a}}^1 &= \frac{1}{8}A_{|\bar{c}\bar{a}}^{\bar{c}} - \frac{1}{8}A_{\bar{c}|\bar{a}}^{\bar{c}} + \frac{1}{8}A_{|\bar{c}}^{\bar{c}}A_{\bar{a}} - \frac{1}{16}A_{\bar{c}}^{\bar{c}}A_{\bar{c}}A_{\bar{a}} + \frac{1}{24}\bar{R}A_{\bar{a}} \\ &\quad + \frac{1}{24}R_{|\bar{a}}. \end{aligned} \quad (7.56b)$$

We notice that the expansion coefficients can be obtained from Eqs. (7.26) and (7.27) by making the replacement  $A_a \rightarrow -A_a$ ; this was expected since Eq. (3.35) for the electromagnetic Green's function differs from Eq. (3.10) for the scalar Green's function by the sign of  $A^a$ .

A comparison between Eq. (7.54) and (5.15) allows us to conclude that

$$G_3^S(\mathbf{x}, \bar{\mathbf{x}}) = G_3^H(\mathbf{x}, \bar{\mathbf{x}}) + O(\epsilon^3) \quad (7.57)$$

for a static charge in a static spacetime.

## VIII. EQUALITY OF $G_3^S$ AND $G_3^H$ : A CONJECTURE

### A. Scalar field

The result of Eq. (7.28) suggests that the equality between the Hadamard and singular Green's functions might be exact, holding to all orders in  $\epsilon$ . We re-express Eq. (7.2) as

$$G_3^S(\mathbf{x}, \bar{\mathbf{x}}) = \frac{1}{s}W^S(\mathbf{x}, \bar{\mathbf{x}}) \quad (8.1)$$

with

$$W^S(\mathbf{x}, \bar{\mathbf{x}}) := \frac{1}{2} \left[ \frac{s}{r}U(x, x') + \frac{s}{r}U(x, x'') - s \int_u^v V(x, z(\tau))d\tau \right], \quad (8.2)$$

and conjecture that  $W^S = W^H$ , where  $W^H$  is the two-point function introduced in Eq. (4.2). We recall that  $s^2 := 2\sigma(x, \bar{x})$  is the squared geodesic distance between  $x$  and the simultaneous event  $\bar{x}$ ,  $x' := z(u)$  is the retarded point on the (static) world line,  $x'' := z(v)$  is the advanced point,  $r := \sigma_{\alpha'}u^{\alpha'}$  is the retarded distance,  $r_{\text{adv}} := -\sigma_{\alpha''}u^{\alpha''}$  is the advanced distance, and  $U(x, z)$ ,  $V(x, z)$  are the two-point functions that appear in the construction of the four-dimensional Green's function.

As in Sec. VI A above, a proof of equality would involve three essential steps. First, the function  $W^S$  must be shown to satisfy the same differential equation as  $W^H$ , as displayed in Eq. (4.3); this property follows immediately from the fact that  $G_3^S$  is known to satisfy Eq. (3.10), just like  $G_3^H$ . Second,  $W^S$  must be shown to satisfy the boundary condition of Eq. (4.4); this property was established previously and can be seen directly from Eq. (7.25). Third, the function  $W^S$  must be shown to be smooth at  $\mathbf{x} = \bar{\mathbf{x}}$ , by which we mean that the function must be  $C^\infty$  when viewed as a function of  $\mathbf{x}$  with  $\bar{\mathbf{x}}$  fixed; this property ensures that  $W^S$  admits an expansion in powers of  $\sigma$  as displayed in Eq. (4.5), which is known to be convergent and unique.



The expansion being unique, smoothness therefore ensures that  $W^S = W^H$ . Evidence that  $W^S$  is smooth to order  $\epsilon^4$  is provided by Eq. (7.25).

Of the ingredients involved in the makeup of  $W^S$ , the two-point functions  $U(x, z)$  and  $V(x, z)$  are known to be smooth, but  $s, r, r_{\text{adv}}, u,$  and  $v$  are not. Nevertheless, we conjecture that the combinations

$$s/r, \quad s/r_{\text{adv}}, \quad s(v-u) \quad (8.3)$$

are in fact smooth at  $x = \bar{x}$ . The first two are directly involved in Eq. (8.2), and the third one also is involved by

$$s = \sqrt{\delta_{ab}x^ax^b}, \quad (8.5a)$$

$$r = s \left[ 1 + \frac{1}{2}a_ax^a - \frac{1}{8}(a_ax^a)^2 - \frac{1}{8}\dot{a}_t s^2 + \frac{1}{6}R_{\text{t}ab}x^ax^b + O(s^3) \right], \quad (8.5b)$$

$$r_{\text{adv}} = r + O(s^4), \quad (8.5c)$$

$$u = t - s \left[ 1 - \frac{1}{2}a_ax^a + \frac{3}{8}(a_ax^a)^2 + \frac{1}{24}\dot{a}_t s^2 - \frac{1}{6}R_{\text{t}ab}x^ax^b + O(s^3) \right], \quad (8.5d)$$

$$v = t + s \left[ 1 - \frac{1}{2}a_ax^a + \frac{3}{8}(a_ax^a)^2 + \frac{1}{24}\dot{a}_t s^2 - \frac{1}{6}R_{\text{t}ab}x^ax^b + O(s^3) \right], \quad (8.5e)$$

in which  $a_a, \dot{a}_t,$  and  $R_{\text{t}ab}$ , respectively, represent components of the acceleration vector, its covariant derivative, and the Riemann tensor evaluated on the static world line, at which  $x^a = 0$ ; terms involving  $\dot{a}_a$  were discarded because these components vanish for a static world line in a static spacetime. These results reveal that  $s, r, r_{\text{adv}}, u,$  and  $v$  are indeed not smooth at  $x^a = 0$ . But they do show that  $r/s, r_{\text{adv}}/s,$  and

$$s(v-u) = 2s^2 \left[ 1 - \frac{1}{2}a_ax^a + \frac{3}{8}(a_ax^a)^2 + \frac{1}{24}\dot{a}_t s^2 - \frac{1}{6}R_{\text{t}ab}x^ax^b + O(s^3) \right] \quad (8.6)$$

are smooth to leading orders in an expansion in powers of  $x^a$ .

We now proceed with a sketch of what might constitute a general proof. The method of proof relies on formal power series, which are all assumed to converge in a sufficiently small domain. This rather strong assumption is the main limitation of our argument, and the reason why we present it as a conjecture and not a proof. It would be desirable to either establish the convergence property, or to devise an alternative method of proof. This shall be left for future work.

We return to Eq. (7.6) and observe that the odd terms in the expansion vanish by time-reversal invariance:  $\dot{\sigma}, \ddot{\sigma},$  and all other odd derivatives of  $\sigma(\tau)$  must vanish on a static world line in a static spacetime. We recall that  $\sigma(\tau) := \sigma(x, z(\tau))$  with  $x$  fixed, and state that each derivative of  $\sigma(\tau)$  is smooth at  $x = \bar{x}$ . Equation (7.6) can therefore be written as

virtue of the mean-value theorem, which allows us to write the integral as

$$s \int_u^v V(x, z) d\tau = V(x, x^*)s(v-u), \quad (8.4)$$

with  $x^* := z(\tau^*)$  ( $u < \tau^* < v$ ) representing a middle point on the world line. Establishing that  $s/r, s/r_{\text{adv}},$  and  $s(v-u)$  are smooth is sufficient to prove that  $W^S$  itself is smooth.

Some insight can be gained by examining these quantities in Fermi normal coordinates  $(t, x^a)$  attached to the static world line. With results collected from Sec. 11 of Ref. [1], we have that

$$s^2 = \Delta^2 \sum_{n=0}^{\infty} p_n (\Delta^2)^n, \quad (8.7a)$$

$$p_n := \frac{2}{(2n+2)!} (-\sigma^{(2n+2)}), \quad (8.7b)$$

in which a bracketed number attached to  $\sigma$  indicates the number of differentiations with respect to  $\tau$ ; each expansion coefficient  $p_n$  is smooth at  $x = \bar{x}$ . Time-reversal invariance implies that  $\Delta_{\pm} = \pm\sqrt{\Delta^2} := \pm\Delta$ , and the expansions of Eq. (7.11) can be expressed as

$$r = r_{\text{adv}} = \Delta \sum_{n=0}^{\infty} q_n (\Delta^2)^n, \quad (8.8a)$$

$$q_n := \frac{1}{(2n+1)!} (-\sigma^{(2n+2)}), \quad (8.8b)$$

with  $q_n$  smooth at  $x = \bar{x}$ . Combining these results, we have that

$$\frac{r}{s} = \frac{r_{\text{adv}}}{s} = \frac{\sum_{n=0}^{\infty} q_n (\Delta^2)^n}{\sqrt{\sum_{n=0}^{\infty} p_n (\Delta^2)^n}}. \quad (8.9)$$

As stated previously, each sum in this expression is assumed to converge for  $\Delta^2$  sufficiently small.

We now wish to reverse the expansion of Eq. (8.7). According to Sec. 3.6.25 of Ref. [36], if  $y = ax + bx^2 + cx^3 + \dots$ , then  $x = Ay + By^2 + Cy^3 + \dots$  with  $aA = 1, a^3B = -b, a^5C = 2b^2 - ac$ , and an algorithm is known to generate all remaining expansion coefficients. The power series can thus be reversed when  $a \neq 0$ . In our

case  $a = p_0 = -\dot{\sigma}$  is indeed nonzero, and  $a^{-1}$  is smooth at  $\mathbf{x} = \bar{\mathbf{x}}$ . The reversed series can then be written as

$$\Delta^2 = s^2 \sum_{n=0}^{\infty} a_n (s^2)^n, \quad (8.10)$$

for some coefficients  $a_n$  that are known to be smooth at  $\mathbf{x} = \bar{\mathbf{x}}$ . Because  $s^2$  is itself smooth, the assumed convergence of the sum for sufficiently small  $s^2$  ensures that  $\Delta^2$  is smooth at  $\mathbf{x} = \bar{\mathbf{x}}$ . Making the substitution in Eq. (8.9), we find that  $r/s$  and  $r_{\text{adv}}/s$  can be expressed as

$$\frac{r}{s} = \frac{r_{\text{adv}}}{s} = \frac{\sum_{n=0}^{\infty} b_n (s^2)^n}{\sqrt{\sum_{n=0}^{\infty} c_n (s^2)^n}} \quad (8.11)$$

for some coefficients  $b_n$  and  $c_n$  that are smooth at  $\mathbf{x} = \bar{\mathbf{x}}$ . This reveals that  $r/s$  and  $r_{\text{adv}}/s$  are smooth at  $\mathbf{x} = \bar{\mathbf{x}}$ . We next turn to  $s(v - u) = 2s\Delta$ , which is given by

$$s(v - u) = s^2 \sqrt{\sum_{n=0}^{\infty} a_n (s^2)^n} \quad (8.12)$$

and is also seen to be smooth at  $\mathbf{x} = \bar{\mathbf{x}}$ .

With the stated assumption on the convergence of formal power series, we have shown that  $r/s$ ,  $r_{\text{adv}}/s$ , and  $s(v - u)$  are all smooth at  $\mathbf{x} = \bar{\mathbf{x}}$ . This implies that  $W^{\text{S}}$  is smooth and establishes the statement that  $G_3^{\text{S}}$  and  $G_3^{\text{H}}$  are strictly equal.

## B. Electromagnetic field

The result of Eq. (7.57) suggests that the equality between the Hadamard and singular Green's functions might also be exact in the case of the electromagnetic field. A proof of this statement would involve the same steps as in the scalar case, and the modifications required for the electromagnetic Green's functions are too modest to merit a separate discussion. As in the scalar case, the essential element is the proof  $s/r$ ,  $s/r_{\text{adv}}$ , and  $s(v - u)$  are all smooth at  $\mathbf{x} = \bar{\mathbf{x}}$ . If this can be established, then we can claim immediately that  $G_3^{\text{S}}$  and  $G_3^{\text{H}}$  are indeed equal to all orders.

## IX. EQUALITY OF $G_3^{\text{S}}$ AND $G_3^{\text{H}}$ FOR ULTRASTATIC SPACETIMES

### A. Scalar field

In this section we return to the theme explored in Sec. VIII and provide a complete proof of equality between the Hadamard construction  $G_3^{\text{H}}$  and the three-dimensional version of the Detweiler-Whiting Green's function  $G_3^{\text{S}}$  in the case of ultrastatic spacetimes. These spacetimes have the property that  $N = 1$ , so that their metric is

$$ds^2 = -dt^2 + h_{ab} dx^a dx^b, \quad (9.1)$$

a special case of Eq. (2.1). The geometrical quantities associated with ultrastatic spacetimes can be obtained

from the equations displayed in Sec. II by setting  $A_a := \partial_a \ln N = 0$ .

The geodesics of ultrastatic spacetimes are described by the equation  $t(\lambda) = t(0) + i(0)\lambda$ , in which  $\lambda$  is an affine parameter and an overdot indicates differentiation with respect to  $\lambda$ , as well as the statement that  $x^a(\lambda)$  describes geodesics of the spatial metric  $h_{ab}$ . This implies that the world function is necessarily given by

$$\sigma(x, x') = -\frac{1}{2}(t - t')^2 + \sigma_3(\mathbf{x}, \mathbf{x}'), \quad (9.2)$$

in which  $\sigma_3(\mathbf{x}, \mathbf{x}')$  is the three-dimensional version of the world function, defined with respect to the spatial metric.

The simplicity extends to the two-point function  $U(x, x')$  that enters the Detweiler-Whiting construction. We may show, in particular, that  $U$  has no dependence on the time coordinates, so that

$$U = U(\mathbf{x}, \mathbf{x}'). \quad (9.3)$$

This statement is a consequence of the defining properties of the two-point function (see Sec. 14.2 of Ref. [1]), that it must satisfy the differential equation

$$2\sigma^\alpha \partial_\alpha U + (\sigma^\alpha{}_\alpha - 4)U = 0 \quad (9.4)$$

in the ultrastatic spacetime, together with the coincidence limit  $U(x', x') = 1$ . With the stated properties of the world function, this becomes

$$(t - t')\partial_t U + \sigma_3^a \partial_a U + \frac{1}{2}(\sigma_3^a{}_a - 3)U = 0. \quad (9.5)$$

The differential equation can be integrated along any spacetime geodesic that originates at  $x'$ . We may, in particular, choose a time-directed geodesic with no spatial displacement, such that  $t(\lambda) = t' + \lambda$  and  $x^a(\lambda) = x'^a$ . For such a geodesic we have that  $\sigma_3^a = 0$  and  $\sigma_3^a{}_a = \sigma_3^a{}_a(x', x') = 3$ , and the differential equation reduces to  $(t - t')\partial_t U = 0$ . This implies that the two-point function cannot depend on  $t$ , and since its dependence on  $t'$  can only be through the combination  $t - t'$ , it cannot depend on  $t'$ . We have, therefore, established the stated property.

The absence of a dependence upon  $t$  implies that the two-point function satisfies the purely spatial differential equation

$$\sigma_3^a \partial_a U + \frac{1}{2}(\sigma_3^a{}_a - 3)U = 0 \quad (9.6)$$

together with the boundary condition  $U(x', x') = 1$ . These are precisely the defining relations for the Hadamard function  $W_0(\mathbf{x}, \mathbf{x}')$ , as stated in Eqs. (4.6) and (4.7). We conclude, therefore, that

$$U(\mathbf{x}, \mathbf{x}') = W_0(\mathbf{x}, \mathbf{x}') \quad (9.7)$$

for ultrastatic spacetimes.

Next we turn our attention to the two-point function  $V(x, x')$  and prove that it admits the expansion

$$V(x, x') = \sum_{n=0}^{\infty} V_n(x, x') \sigma^n, \quad (9.8)$$

in which the coefficients  $V_n$  are smooth and time-independent; the expansion involves the four-dimensional world function, and it is known to converge within a sufficiently small neighborhood of  $x'$ . The proof of the statement relies on the recurrence relations satisfied by the expansion coefficients [5],

$$\sigma^\alpha \partial_\alpha V_0 + \frac{1}{2}(\sigma^\alpha{}_\alpha - 2)V_0 = \frac{1}{2} \square U|_{\sigma=0}, \quad (9.9)$$

when  $n = 0$ , and

$$\sigma^\alpha \partial_\alpha V_n + \frac{1}{2}(\sigma^\alpha{}_\alpha + 2n - 2)V_n = -\frac{1}{2n} \square V_{n-1}, \quad (9.10)$$

when  $n > 0$ .

We begin with an examination of  $V_0$ . In ultrastatic spacetimes its differential equation becomes

$$(t - t') \partial_t V_0 + \sigma_3^a \partial_a V_0 + \frac{1}{2}(\sigma_3^a{}_a - 1)V_0 = \frac{1}{2} \nabla^2 U. \quad (9.11)$$

Once more this equation can be integrated along any space-time geodesic that originates at  $x'$ , and once more we choose a time-directed geodesic. In this case we have

$$(t - t') \partial_t V_0 + V_0 = \frac{1}{2} \nabla^2 U|_{x=x'} = \frac{1}{12} R(x'), \quad (9.12)$$

in which  $R(x')$  is the spatial Ricci scalar evaluated at  $x'$ , obtained from the known expression for  $\nabla^2 U$  evaluated in the coincidence limit (see Sec. 14.2 of Ref. [1]). The general solution to this equation is  $V_0 = \frac{1}{12} R(x') + c(t - t')^{-1}$  where  $c$  is a constant, and we see that  $V_0$  fails to be smooth at  $x = x'$  unless  $c = 0$ . We conclude that  $V_0$  cannot depend on time.

Turning next to  $V_n$ , we proceed by induction. We assume that  $V_{n-1}$  is known to be time-independent, and prove that  $V_n$  must in turn be time-independent. We begin with the differential equation

$$(t - t') \partial_t V_n + \sigma_3^a \partial_a V_n + \frac{1}{2}(\sigma_3^a{}_a + 2n - 1)V_n = -\frac{1}{2n} \nabla^2 V_{n-1}, \quad (9.13)$$

which we integrate along a time-directed geodesic. The equation reduces to

$$(t - t') \partial_t V_n + (n + 1)V_n = -\frac{1}{2n} \nabla^2 V_{n-1}|_{x=x'}, \quad (9.14)$$

and we find that the general solution contains a term  $c(t - t')^{-(n+1)}$  that fails to be smooth at  $x = x'$  unless  $c = 0$ . This allows us to conclude that  $V_n$  cannot depend on time, and we have established Eq. (9.8).

We may now demonstrate the equality of the Green's functions. The four-dimensional version of the Detweiler-Whiting singular Green's function is

$$G_4^S(x, x') = \frac{1}{2} U(x, x') \delta(\sigma) - \frac{1}{2} V(x, x') \Theta(\sigma), \quad (9.15)$$

in which  $\Theta$  is the Heaviside step function and  $\delta$  the Dirac distribution. According to Eq. (3.16), the three-dimensional version is

$$G_3^S(x, x') = \int G_4^S(x, x') dt', \quad (9.16)$$

when  $N(x') = 1$ . With  $U$  independent of time and  $\sigma$  factorized as

$$\sigma = -\frac{1}{2}(\Delta t - \sqrt{2\sigma_3})(\Delta t + \sqrt{2\sigma_3}) \quad (9.17)$$

with  $\Delta t = t - t'$ , we find that the integral becomes

$$G_3^S(x, x') = \frac{U(x, x')}{\sqrt{2\sigma_3}} - \frac{1}{2} \int_{-\sqrt{\sigma_3}}^{\sqrt{\sigma_3}} V(x, x') d\Delta t. \quad (9.18)$$

In this we insert Eq. (9.8), integrate term by term using

$$\begin{aligned} \int_{-\sqrt{\sigma_3}}^{\sqrt{\sigma_3}} \sigma^n d\Delta t &= \left(-\frac{1}{2}\right)^n \int_{-\sqrt{\sigma_3}}^{\sqrt{\sigma_3}} (\Delta t^2 - 2\sigma_3)^n d\Delta t \\ &= \frac{\sqrt{\pi} \Gamma(n+1)}{2^n \Gamma(n + \frac{3}{2})} (2\sigma_3)^{n+\frac{1}{2}}, \end{aligned} \quad (9.19)$$

and simplify. Our final expression for the singular Green's function is

$$G_3^S(x, x') = \frac{W^S(x, x')}{\sqrt{2\sigma_3}} \quad (9.20)$$

with

$$W^S(x, x') = U(x, x') - \sum_{n=1}^{\infty} \frac{(n-1)!}{(2n-1)!!} V_{n-1}(x, x') (2\sigma_3)^n. \quad (9.21)$$

These equations reveal that  $G_3^S$  does admit a three-dimensional Hadamard form and that we may make the identifications

$$W_0^S(x, x') := U(x, x') \quad (9.22)$$

as in Eq. (9.7), and

$$W_n^S(x, x') := -\frac{(n-1)!}{(2n-1)!!} V_{n-1}(x, x'). \quad (9.23)$$

These coefficients satisfy the recursion relation of Eq. (4.6), as can be seen by invoking Eqs. (9.9) and (9.10), and are therefore the same coefficients that appear in Eq. (4.5). The proof of equality between  $G_3^S(x, x')$  and  $G_3^H(x, x')$  in ultrastatic spacetimes is complete, and the calculations have revealed the relationship between  $U$  and  $W_0$ , and between  $V_n$  and  $W_n$ .

## B. Electromagnetic field

The proof of equality between the Hadamard construction  $G_3^H$  and the three-dimensional version of the Detweiler-Whiting Green's function  $G_3^S$  for ultrastatic spacetimes proceeds along the same lines as in the scalar case. In fact, the calculational details are strictly identical, because the two-point functions  $U_i^{t'}$  and  $V_i^{t'}$  that are involved in the relevant component of the electromagnetic Green's function,

$$G^{S t'}_i(x, x') = \frac{1}{2} U_i^{t'}(x, x') \delta(\sigma) - \frac{1}{2} V_i^{t'}(x, x') \Theta(\sigma), \quad (9.24)$$

are strictly identical to their scalar counterparts:  $U_i^{t'} = U$  and  $V_i^{t'} = V$ . The first equality follows from the general relation  $U_\alpha^{\beta'} = g_\alpha^{\beta'} U$ —see Eqs. (14.8) and (15.9) in Ref. [1]—together with the property that  $g_i^{t'} = 1$  for ultrastatic spacetimes. The second equality follows from the fact that if  $V_\alpha^{\beta'}$  is expanded as

$$V_\alpha^{\beta'} = \sum_{n=0} V_{n\alpha}^{\beta'} \sigma^n, \quad (9.25)$$

then the recursion relations satisfied by  $V_{ni}^{t'}$  are strictly identical to those satisfied by  $V_n$ . The results of Sec. IX A, therefore, allow us to state that for ultrastatic spacetimes,  $G_3^H = G_3^S$  in the electromagnetic case also.

## X. SELF-FORCE IN SPHERICAL SPACETIMES

### A. Scalar field

We consider the self-force acting on a static scalar charge  $q$  in a static and spherically symmetric spacetime. The metric is written as

$$ds^2 = -e^{2\psi} dt^2 + f^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (10.1)$$

in which  $\psi$  and  $f$  are functions of  $r$ . In this notation  $N = e^\psi$  and  $A_r = \psi'$  is the only nonvanishing component of the vector  $A_a$ . The potential  $\Phi$  generated by the point scalar charge is a solution to

$$\nabla^2 \Phi + A^a \partial_a \Phi = -4\pi q \delta_3(\mathbf{x}, \mathbf{z}), \quad (10.2)$$

which is obtained from Eqs. (3.2) and (3.21).

In order to integrate the field equation, we decompose the potential and source in spherical harmonics:

$$\Phi(t, \theta, \phi) = \sum_{\ell m} \Phi_{\ell m}(r) Y_{\ell m}(\theta, \phi) \quad (10.3)$$

and

$$\delta_3(\mathbf{x}, \mathbf{z}) = \frac{f_0^{1/2}}{r_0^2} \delta(r - r_0) \sum_{\ell m} Y_{\ell m}^*(\theta_0, \phi_0) Y_{\ell m}(\theta, \phi), \quad (10.4)$$

in which  $(r_0, \theta_0, \phi_0)$  represent the spherical coordinates of the particle's position  $\mathbf{z}$ , and  $f_0 := f(r_0)$ . Without

loss of generality we may place the particle along the polar axis ( $\theta_0 = 0$ ) and exploit the property  $Y_{\ell m}(0, \phi) = \sqrt{(2\ell + 1)/(4\pi)} \delta_{m,0}$  of spherical-harmonic functions. Substitution within the field equation then produces

$$\begin{aligned} r^2 \Phi''_{\ell 0} + \left(2 + \frac{rf'}{2f} + r\psi'\right) r \Phi'_{\ell 0} - \frac{\ell(\ell + 1)}{f} \Phi_{\ell 0} \\ = -4\pi q \sqrt{\frac{2\ell + 1}{4\pi}} f_0^{-1/2} \delta(r - r_0), \end{aligned} \quad (10.5)$$

in which a prime indicates differentiation with respect to  $r$ . The modes with  $m \neq 0$  necessarily vanish.

The self-force acting on the scalar charge is given by  $F^\alpha = q(g^{\alpha\beta} + u^\alpha u^\beta) \nabla_\beta \Phi^R$ , in which  $\Phi^R := \Phi - \Phi^S$  is the difference between the actual potential  $\Phi$  and the Detweiler-Whiting singular field  $\Phi^S$ ; the regular potential is known to be smooth at  $\mathbf{x} = \mathbf{z}$ . In a static situation the self-force has a vanishing time component, and its spatial components are given by  $F^a = q h^{ab} \partial_b \Phi^R$ . In a spherically symmetric spacetime the angular components vanish, and the radial component is

$$F^r = q f_0 \partial_r \Phi^R(r_0, \theta_0, \phi_0). \quad (10.6)$$

Recalling the spherical-harmonic decomposition of the potential, we may express this as

$$F^r = q f_0 \lim_{\mathbf{x} \rightarrow \mathbf{z}} \sum_{\ell} [(\partial_r \Phi)_\ell - (\partial_r \Phi^S)_\ell], \quad (10.7)$$

in which

$$(\partial_r \Phi)_\ell := \sum_{m=-\ell}^{\ell} \Phi'_{\ell m}(r) Y_{\ell m}(\theta, \phi) \quad (10.8)$$

are the multipole coefficients of  $\partial_r \Phi$ , while  $(\partial_r \Phi^S)_\ell$  are those of the singular potential  $\Phi^S$ . Recalling the relation of Eq. (3.22) between the potential and the scalar Green's function, we may write this in the form

$$F^r = q^2 f_0 \lim_{\mathbf{x} \rightarrow \mathbf{z}} \sum_{\ell} [q^{-1} (\partial_r \Phi)_\ell - (\partial_r G_3^S)_\ell], \quad (10.9)$$

in which  $G_3^S(\mathbf{x}, \mathbf{z})$  is the three-dimensional version of the Detweiler-Whiting singular Green's function introduced in Sec. VII A.

The limit in Eq. (10.9) can be taken by setting  $r = r_0 + \Delta$ ,  $\theta = \theta_0$ ,  $\phi = \phi_0$ , and letting  $\Delta \rightarrow 0$  (from either direction). With this choice, we shall show below that

$$\begin{aligned} (\partial_r G_3^S)_\ell = A \left( \ell + \frac{1}{2} \right) + B + \frac{C}{(\ell + \frac{1}{2})} + \frac{D}{(\ell - \frac{1}{2})(\ell + \frac{3}{2})} \\ + O(\ell^{-3}), \end{aligned} \quad (10.10)$$

in which the regularization parameters  $A$ ,  $B$ ,  $C$ , and  $D$  depend on  $r_0$  but are independent of  $\ell$ ; explicit expressions will be presented below. The  $A$ ,  $B$ , and  $C$  terms originate from terms in  $\partial_r G_3^S$  that diverge or stay bounded in the

limit  $\mathbf{x} \rightarrow \mathbf{z}$ , and correspondingly, their sums over  $\ell$  clearly diverge. On the other hand, the  $D$  and subsequent terms originate from terms in  $\partial_r G_3^S$  that vanish in the limit, and correspondingly, their sums over  $\ell$  vanish (as can be seen explicitly for the  $D$  term). The neglected terms are therefore known to sum to zero, and inserting Eq. (10.10) within Eq. (10.9) provides a practical method of computing the self-force by means of a regularized mode sum that converges to the correct answer. With the particle placed on the polar axis ( $\theta_0 = 0$ ), the multipole coefficients reduce to

$$(\partial_r \Phi)_\ell = \sqrt{\frac{2\ell + 1}{4\pi}} \Phi'_{\ell 0}(r_0 + \Delta). \quad (10.11)$$

To establish the relation of Eq. (10.10) and calculate the regularization parameters, we follow the method described in Sec. V of Haas and Poisson (HP) [34], which we adapt to the situation at hand. In HP the motion of the particle was geodesic and the spacetime was that of a Schwarzschild black hole; here the particle is kept in place in any static, spherically symmetric spacetime. In HP the motion was taking place in the equatorial plane, and a transformation of the angular coordinates was implemented to put the particle momentarily on the polar axis; here the particle is kept on the axis at all times, and the transformation is not required. Following Sec. III of HP, the singular Green's function of Eq. (7.25) is expressed as an expansion in powers of the coordinate displacements  $w^a := x^a - \bar{x}^a$ , in which  $\bar{\mathbf{x}} := \mathbf{z}$  denotes the particle's position. As in Sec. V of HP we express the angular separations  $w^\theta$  and  $w^\phi$  in terms of functions  $Q := \sqrt{1 - \cos\theta}$ ,  $\sin\phi$ , and  $\cos\phi$  that are globally well defined on the sphere. In this case of static motion, the squared-distance function introduced in Eq. (5.22) of HP reduces to

$$\rho^2 = f_0^{-1} \Delta^2 + 2r_0^2 Q = 2r_0^2 (\delta^2 + 1 - \cos\theta), \quad (10.12)$$

where

$$\delta^2 = \frac{\Delta^2}{2r_0^2 f_0} \quad (10.13)$$

with  $\Delta := w^r = r - r_0$ . Following the steps outlined in Sec. VE of HP, we obtain an expansion for  $\partial_r G_3^S$  that takes the schematic form of

$$\begin{aligned} \partial_r G_3^S &= (\partial_r G_3^S)_{-2} + (\partial_r G_3^S)_{-1} + (\partial_r G_3^S)_0 \\ &\quad + (\partial_r G_3^S)_1 + O(\epsilon^2), \end{aligned} \quad (10.14)$$

in which a subscript attached to enclosing brackets indicates the scaling with powers of  $\epsilon$ . The various terms are schematically given by

$$(\partial_r G_3^S)_{-2} = M_{-2} (\Delta/\rho^3), \quad (10.15a)$$

$$(\partial_r G_3^S)_{-1} = M_{-1} (1/\rho) + O(\Delta^2/\rho^3) + O(\Delta^4/\rho^5), \quad (10.15b)$$

$$\begin{aligned} (\partial_r G_3^S)_0 &= O(\Delta/\rho) + O(\Delta^3/\rho^3) + O(\Delta^5/\rho^5) \\ &\quad + O(\Delta^7/\rho^7), \end{aligned} \quad (10.15c)$$

$$\begin{aligned} (\partial_r G_3^S)_1 &= M_1 \rho + O(\Delta^2/\rho) + O(\Delta^4/\rho^3) + O(\Delta^6/\rho^5) \\ &\quad + O(\Delta^8/\rho^7) + O(\Delta^{10}/\rho^9). \end{aligned} \quad (10.15d)$$

The terms involving the coefficients  $M_{-2}$ ,  $M_{-1}$ , and  $M_1$  are those giving rise to the regularization parameters; all other terms are unimportant.

The multipole decomposition of  $\partial_r G_3^S$  is next carried out with the help of Eq. (A19) of Haas and Poisson; because the expressions are all  $\phi$ -independent (by virtue of the axial symmetry of the problem), there is no need to perform the  $\phi$  average described by Eq. (A13). We make use of the relations

$$(\Delta/\rho^3)_\ell = \left( \ell + \frac{1}{2} \right) \frac{f_0^{1/2}}{r_0^2} \text{sign}(\Delta) + O(\Delta), \quad (10.16a)$$

$$(1/\rho)_\ell = \frac{1}{r_0} + O(\Delta), \quad (10.16b)$$

$$(\rho)_\ell = -\frac{r_0}{(\ell - \frac{1}{2})(\ell + \frac{3}{2})} + O(\Delta) \quad (10.16c)$$

and arrive at the expression of Eq. (10.10) with  $A = M_{-2} f_0^{1/2} r_0^{-2} \text{sign}(\Delta)$ ,  $B = M_{-1}/r_0$ ,  $C = 0$ , and  $D = -M_1 r_0$ . The detailed computation reveals that

$$A = -\frac{1}{r^2} f^{-1/2} \text{sign}(\Delta), \quad (10.17a)$$

$$B = -\frac{1}{2r^2} (1 + r\psi'), \quad (10.17b)$$

$$C = 0, \quad (10.17c)$$

$$\begin{aligned} D &= -\frac{1}{16r^2} [(1 + r\psi') - (1 + r\psi' + 3r^2\psi'^2 + r^3\psi'^3 \\ &\quad - 6r^2\psi'' - 2r^3\psi''')f + (1 + 4r\psi' + 3r^2\psi'')rf' \\ &\quad + (1 + r\psi')r^2f''], \end{aligned} \quad (10.17d)$$

in which all functions are to be evaluated at  $r = r_0$ . These are the regularization parameters for a static scalar charge in any static, spherically symmetric spacetime.

## B. Electromagnetic field

We next consider the self-force acting on a static electric charge  $e$  in a static and spherically-symmetric spacetime with the metric of Eq. (10.1). The vector potential  $\Phi_t$  generated by the point charge is a solution to

$$\nabla^2 \Phi_t - A^a \partial_a \Phi_t = 4\pi e N(\mathbf{z}) \delta_3(\mathbf{x}, \mathbf{z}), \quad (10.18)$$

which is obtained from Eqs. (3.29) and (3.43).

As in the scalar case we decompose the potential and source in spherical harmonics and place the particle along

the polar axis ( $\theta_0 = 0$ ). Substitution within the field equation then produces

$$\begin{aligned} r^2 \Phi''_{\ell 0} + \left(2 + \frac{rf'}{2f} - r\psi'\right) r \Phi'_{\ell 0} - \frac{\ell(\ell+1)}{f} \Phi_{\ell 0} \\ = 4\pi e \sqrt{\frac{2\ell+1}{4\pi}} e^{\psi_0} f_0^{-1/2} \delta(r-r_0), \end{aligned} \quad (10.19)$$

in which  $f_0 := f(r_0)$ ,  $\psi_0 := \psi(r_0)$ , and a prime indicates differentiation with respect to  $r$ . The modes with  $m \neq 0$  necessarily vanish.

The self-force acting on the scalar charge is given by  $F^\alpha = e F_{\text{R}}^\alpha{}_\beta u^\beta$ , in which  $F_{\text{R}}^\alpha{}_\beta := F^\alpha{}_\beta - F_{\text{S}}^\alpha{}_\beta$  is the difference between the actual electromagnetic field and the Detweiler-Whiting singular field; the regular field is known to be smooth at  $\mathbf{x} = \mathbf{z}$ . In a static situation the self-force has a vanishing time component, and in spherical symmetry its radial component is

$$F^r = e e^{-\psi_0} f_0 \partial_r \Phi_r^{\text{R}}(r_0, \theta_0, \phi_0). \quad (10.20)$$

We express this as

$$F^r = e e^{-\psi_0} f_0 \lim_{\mathbf{x} \rightarrow \mathbf{z}} \sum_{\ell} [(\partial_r \Phi_t)_{\ell} - (\partial_r \Phi_t^{\text{S}})_{\ell}], \quad (10.21)$$

in which

$$\begin{aligned} (\partial_r \Phi_t)_{\ell} &:= \sum_{m=-\ell}^{\ell} \Phi'_{\ell m}(r) Y_{\ell m}(\theta, \phi) \\ &= \sqrt{\frac{2\ell+1}{4\pi}} \Phi'_{\ell 0}(r_0 + \Delta) \end{aligned} \quad (10.22)$$

are the multipole coefficients of  $\partial_r \Phi_t$ , while  $(\partial_r \Phi_t^{\text{S}})_{\ell}$  are those of the singular potential. Recalling the relation of Eq. (3.44) between the potential and the scalar Green's function, we may write this in the form

$$F^r = e^2 f_0 \lim_{\mathbf{x} \rightarrow \mathbf{z}} \sum_{\ell} [e^{-1} e^{-\psi_0} (\partial_r \Phi_t)_{\ell} + (\partial_r G_3^{\text{S}})_{\ell}], \quad (10.23)$$

in which  $G_3^{\text{S}}(\mathbf{x}, \mathbf{z})$  is the three-dimensional version of the Detweiler-Whiting singular Green's function introduced in Sec. VII B.

The limit in Eq. (10.9) is taken by setting  $r = r_0 + \Delta$ ,  $\theta = \theta_0 = 0$ , and  $\phi = \phi_0 = 0$  and letting  $\Delta \rightarrow 0$  (from either direction). With this choice, the multipole coefficients of the singular Green's function take the same form as in Eq. (10.10). In this case, however, because of the different sign in front of  $A^a$  in the Poisson equation for  $\Phi_t$ , the regularization parameters are given by

$$A = -\frac{1}{r^2} f^{-1/2} \text{sign}(\Delta), \quad (10.24a)$$

$$B = -\frac{1}{2r^2} (1 - r\psi'), \quad (10.24b)$$

$$C = 0, \quad (10.24c)$$

$$\begin{aligned} D = -\frac{1}{16r^2} [(1 - r\psi') - (1 - r\psi' + 3r^2\psi'^2 - r^3\psi'^3 \\ + 6r^2\psi'' + 2r^3\psi''')f + (1 - 4r\psi' - 3r^2\psi'')rf' \\ + (1 - r\psi')r^2f''], \end{aligned} \quad (10.24d)$$

in which all functions are to be evaluated at  $r = r_0$ . These are the regularization parameters for a static electric charge in any static, spherically symmetric spacetime. The computations that lead to Eq. (10.24) involve the same steps as those described in Sec. X A.

## ACKNOWLEDGMENTS

This work was supported by the Natural Sciences and Engineering Research Council of Canada. M. C. acknowledges funding support from a IRCSET-Marie Curie International Mobility Fellowship in Science, Engineering and Technology.

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