Bose-Einstein condensate general relativistic stars

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We analyze the possibility that due to their superfluid properties some compact astrophysical objects may contain a significant part of their matter in the form of a Bose-Einstein condensate. To study the condensate we use the Gross-Pitaevskii equation with arbitrary nonlinearity. By introducing the Madelung representation of the wave function, we formulate the dynamics of the system in terms of the continuity and hydrodynamic Euler equations. The nonrelativistic and Newtonian Bose-Einstein gravitational condensate can be described as a gas, whose density and pressure are related by a barotropic equation of state. In the case of a condensate with quartic nonlinearity, the equation of state is polytropic with index one. In the framework of the Thomas-Fermi approximation the structure of the Newtonian gravitational condensate is described by the Lane-Emden equation, which can be exactly solved. The case of the rotating condensate is briefly discussed. General relativistic configurations with quartic nonlinearity are studied numerically with both nonrelativistic and relativistic equations of state, and the maximum mass of the stable configuration is determined. Condensates with particle masses of the order of two neutron masses (Cooper pair) and scattering length of the order of 10–20 fm have maximum masses of the order of 2 $M_\odot$, maximum central density of the order of 0.1–0.3 $\times 10^{16}$ g/cm^3 and minimum radii in the range of 10–20 km. In this way we obtain a large class of stable astrophysical objects, whose basic astrophysical parameters (mass and radius) sensitively depend on the mass of the condensed particle, and on the scattering length. We also propose that the recently observed neutron stars with masses in the range of 2–2.4 $M_\odot$ are Bose-Einstein condensate stars. We discuss the connection of our results with previous boson star models based on scalar field theory.

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I. INTRODUCTION

At very low temperatures, particles in a dilute Bose gas can occupy the same quantum ground state, forming a Bose-Einstein condensate (BEC), which appears as a sharp peak over a broader distribution in both coordinate space and momentum space. The possibility to obtain quantum degenerate gases by a combination of laser and evaporative cooling techniques has opened several new lines of research, at the border of atomic, statistical and condensed matter physics (for recent reviews see Refs. [1,2]).

To say that so many particles are in the same quantum state is equivalent to saying that these particles display the state coherence. That is, BEC is a particular case of coherence phenomena. As the gas is cooled, the condensation of a large fraction of the particles in a gas occurs via a phase transition taking place when the wavelengths of individual particles overlap and behave identically. For the transition to take place, particles have to be strongly correlated with each other [1,2].

For an ensemble of particles in thermodynamic equilibrium at temperature T, the thermal energy of a particle is given by k_BT, where k_B is Boltzmann's constant. For a particle of mass m, the thermal wavelength is $\lambda_T = \sqrt{2\pi\hbar^2/mk_BT}$. Particles become correlated with each other when their wavelengths overlap, that is, when the thermal wavelength is greater than the mean inter-particles distance l, $\lambda_T > l$. The average particle number n for N particles in a volume V, $n = N/V$, is related to the distance l through the relation $nl^3 = 1$. Hence the condition $\lambda_T > l$ can be rewritten as $n\lambda_T^3 > 1$, which yields the inequality [3]:

$$T < \frac{2\pi\hbar^2}{mk_B} n^{2/3}. \tag{1}$$

Hence a coherent state may develop if the particle density is high enough or the temperature is sufficiently low. An accurate description of the BEC for an ideal gas is based on the Bose-Einstein distribution $f(p) = \{\exp[(\varepsilon_p - \mu)/k_BT] - 1\}^{-1}$, for particles with momentum p, energy $\varepsilon_p = p^2/2m$ and chemical potential μ. In the thermodynamic limit $N \to \infty$, $V \to \infty$, $N/V \to$ constant, the fraction of particles condensing to the state with $p = 0$ below the condensation temperature T_c is $n_0 = 1 - (T/T_c)^{3/2}$, while $n_0 = 0$ above the condensation temperature. The condensation temperature is $T_c = 2\pi\hbar^2 n^{2/3}/mk_B\zeta^{2/3}$, where $\zeta = 2.612$ [3]. The dynamical process of Bose-Einstein condensation in the canonical ensemble (fixed temperature T) has been studied in Ref. [4].

A nonideal, weakly interacting, Bose gas also displays Bose-Einstein condensation though particles interactions deplete the condensate, so that at zero temperature the condensate fraction is smaller than unity, $n_0 < 1$. A system is called weakly interacting if the characteristic interaction radius r_{int} is much smaller than the mean interparticles distance l, $r_{int} \ll l$. This inequality can be rewritten equivalently as $nr_{int}^3 \ll 1$. If this condition holds, the system is called dilute [2].

Superfluid liquids, like ^{4}He, are far from being dilute. Nevertheless, one believes that the phenomenon of superfluidity is related with BEC. The experimental observations and the theoretical calculations estimate the condensate fraction for superfluid helium at $T = 0$ to be $n_0 \approx 0.10$. A strongly correlated pair of fermions can be treated approximately like a boson. This is why the arising superfluidity in ^{3}He can be interpreted as the condensation of coupled fermions. Similarly, superconductivity may be described as the condensation of the Cooper pairs that are formed by the electrons or the holes [5].

An ideal system for the experimental observation of the BEC condensation is a dilute atomic Bose gas confined in a trap and cooled to very low temperatures. BEC were first observed in 1995 in dilute alkali gases such as vapors of rubidium and sodium. In these experiments, atoms were confined in magnetic traps, evaporatively cooled down to a fraction of a microkelvin, left to expand by switching off the magnetic trap, and subsequently imaged with optical methods. A sharp peak in the velocity distribution was observed below a critical temperature, indicating that condensation has occurred with the alkali atoms condensed in the same ground state. Under the typical confining conditions of experimental settings, BECs are inhomogeneous, and hence condensates arise as a narrow peak not only in the momentum space but also in the coordinate space [6–8].

If considering only two-body mean field interactions, a dilute Bose-Einstein gas near zero temperature can be modelled using a cubic nonlinear Schrödinger equation with an external potential, which is known as the Gross-Pitaevskii equation [2].

The possibility of the Bose-Einstein condensation has also been considered in nuclear and quark matter, in the framework of the analysis of the BCS-BEC crossover. At ultra-high density, matter is expected to form a degenerate Fermi gas of quarks in which the Cooper pairs of quarks condensate near the Fermi surface (color superconductor). If the attractive interaction is strong enough, at some critical temperature the fermions may condense into the bosonic zero mode, forming a Bose-Einstein quark condensate [9]. The basic concept of the BCS-BEC crossover is as follows: As long as the attractive interaction between fermions is weak, the system exhibits the superfluidity characterized by the energy gap in the BCS mechanism. On the other hand, if the attractive interaction is strong enough, the fermions first form bound molecules (bosons). Then, they start to condense into the bosonic zero mode at some critical temperature. These two situations are smoothly connected without a phase transition [10].

One of the most striking features of the crossover is that the critical temperature in the BEC region is independent of the coupling for the attraction between fermions. This is because the increase of the coupling only affects the internal structure of the bosons while the critical temperature is determined by the boson's kinetic energy. Thus, the critical temperature reaches a ceiling for the large coupling as long as the binding effect on the boson mass can be neglected. Even in the nuclear matter where the interaction is relatively strong, the binding energy of the deuteron is much smaller than the nucleon mass. This fact allows us to work within a nonrelativistic framework to describe such a crossover [10]. However, in relativistic systems where the binding energy cannot be neglected, there could be two crossovers in the relativistic fluids: One is the ordinary BCS-BEC crossover where the critical temperature in the BEC region would not plateau because of the relativistic effect, and the second is the crossover from the BEC state to a relativistic state, the so-called relativistic BEC, where the critical temperature increases to the order of the Fermi energy [10].

In isospin symmetric nuclear matter, neutron-proton (np) pairing undergoes a smooth transition leading from an assembly of np Cooper pairs at higher densities to a gas of Bose-condensed deuterons as the nucleon density is reduced to an extremely low value. This transition may be relevant to supernova matter or for the crust of neutron stars [11]. A mixture of interacting neutral and charged Bose condensates, which is supposed to be realized in the interior of neutron stars in the form of a coexistent neutron superfluid and protonic superconductor, was considered in Ref. [12].

The possibility of the existence of some Bose condensates in neutron stars was considered for a long time (see Glendenning [13] for a detailed discussion). The possibility of a Bose condensation of dibaryons in dense nuclear matter was considered in Ref. [14]. Dibaryons do not contribute to the pressure since they have zero momentum. The effect of narrow dibaryon resonances on basic nuclear matter properties and on the structure of neutron stars was investigated in mean-field theory and in relativistic Hartree approximation in Ref. [15]. The condensation of negatively charged mesons in neutron star matter is favored because such mesons would replace electrons with very high Fermi momenta. The in-medium properties of the K^- mesons may be such that they could condense in neutron matter as well. BECs of kaons/anti-kaons in compact objects were discussed recently [16,17]. Pion as well as kaon condensates would have two important effects on neutron stars. First, condensates soften the equation of state above the critical density for onset of condensation, which reduces the maximal possible neutron star mass. At the same time, however, the central stellar density increases due to the softening. Second, meson condensates would lead to neutrino luminosities which are considerably enhanced over those of normal neutron star matter. This would speed up neutron star cooling considerably [13]. Another particle which may form a condensate is the H-dibaryon, a doubly strange six quarks composite with spin and isospin zero, and baryon number two. In neutron star matter, which may contain a significant fraction of Λ hyperons, the Λ's could combine to form H-dibaryons. H-matter condensates may thus exist at the center of neutron stars [13]. Neutrino superfluidity, as suggested by Kapusta [18], may also lead to Bose-Einstein condensation [19].

Real scalar fields considered in the framework of quantum field theory and general relativity have equilibrium configuration that were discovered by Seidel and Suen [20] and are called oscillatons. They are globally regular but are fully time dependent. As for their stability, they seem to be quite robust as far as numerical evolution is concerned [21]. The objects which can be formed by scalar fields have been investigated in detail by using mainly numerical tools [22]. Complex scalar fields can form stable equilibrium configurations, called boson stars [23–28], that are globally regular and whose energy density is time independent. The possibility that dark matter is in the form of a scalar field [29–31] or a BEC [32–35] has also been investigated extensively.

Therefore, the physical results presented above show that the possibility of the existence of a BEC inside compact astrophysical objects or the existence of stars formed entirely from a BEC cannot be excluded *a priori*. Such a possibility has been in fact suggested recently. Wang [36] used the Gross-Pitaevskii equation, together with the associated energy functional and the Thomas-Fermi approximation, to study a cold star composed of a dilute BEC. For a static star, the exact solution for the density distribution was obtained. A number of perturbative solutions for the case of a slowly rotating star have also been derived. The effect of a scalar dark matter background on the equilibrium of degenerate stars was studied by Grifols [37], with a particular focus on white dwarfs, and the changes induced in their masses and radii.

A detailed analytical and numerical analysis of the Newtonian BEC systems was performed recently in Refs. [33,34], respectively. In Ref. [33] an approximate analytical expression of the mass-radius relation of a Newtonian self-gravitating BEC with short-range interactions, described by the Gross-Pitaevskii-Poisson system, was obtained. For repulsive short-range interactions (positive scattering lengths), configurations of arbitrary mass do exist, but their radius is always larger than a minimum value. For attractive short-range interactions (negative scattering lengths), equilibrium configurations only exist below a maximum mass. The equation of hydrostatic equilibrium describing the balance between the gravitational attraction and the pressure due to quantum effects and short-range interactions (scattering) was numerically solved in Ref. [34].

It is the purpose of the present paper to develop a general and systematic formalism for the study of gravitationally bounded BECs, in both Newtonian and general relativistic situations. Our approach is independent of the nature of the condensate. As a starting point, we consider nonrelativistic BECs and generalize the Gross-Pitaevskii equation by allowing an arbitrary form of the nonlinearity. To obtain a transparent description of the physical properties of the BECs we introduce the hydrodynamical representation of the wave function which allows the formulation of the dynamics of the condensate in terms of the continuity and hydrodynamic Euler equations. Hence, the Bose-Einstein gravitational condensate can be described as a gas whose density and pressure are related by a barotropic equation of state. In the case of a condensate with quartic nonlinearity, the equation of state of the condensate is given by a polytropic equation of state with polytropic index $n = 1$. In the framework of the Thomas-Fermi approximation, with the quantum potential neglected, the structure of the gravitational BEC is described by the Lane-Emden equation, which can be solved analytically. Hence the mass and the radius of the condensate can be easily obtained. The case of the rotating Newtonian condensate is also discussed, by using the generalized Lane-Emden equation.

By using the equation of state corresponding to the BECs with quartic nonlinearity, we consider the general relativistic properties of condensate stars by numerically integrating the structure equations [the mass continuity and the Tolman-Oppenheimer-Volkoff (TOV) equation] for a static configuration. In our general relativistic study, we consider the cases of condensates described by both nonrelativistic and relativistic equations of state, respectively. The maximum mass and the corresponding radius are obtained numerically. BEC stars with particle masses of the order of two neutron masses (Cooper pair) and scattering length of the order of 10–20 fm have maximum masses of the order of 2 $M_\odot$, maximum central density of the order of $0.1–0.3 \times 10^{16}$ g/cm^3 and minimum radii in the range of 10–20 km.

The present paper is organized as follows. The Gross-Pitaevskii equation is written down in Sec. II. The hydrodynamical representation for the study of the gravitationally bounded BECs is introduced in Sec. III. The static and slowly rotating Newtonian condensates with quartic nonlinearity are analyzed in Sec. IV. The maximum mass of relativistic BEC stars is discussed in Sec. V in a qualitative manner. The detailed properties of general relativistic BEC stars are studied in Sec. VI for both nonrelativistic and relativistic equations of state. The astrophysical implications of our results are considered in Sec. VII. We conclude by some final remarks in Sec. VIII.

Throughout the paper, we discuss the connection of our results with previous boson star models based on scalar field theory [23–28].

II. THE GROSS-PITAEVSKII EQUATION FOR THE BOSE-EINSTEIN CONDENSATE STARS

In a quantum system of N interacting condensed bosons, most of the bosons lie in the same single-particle quantum state. The many-body Hamiltonian describing the interacting bosons confined by an external potential $V_{\rm ext}$ is given, in the second quantization, by

$$\hat{H} = \int d\vec{r}\hat{\Psi}^{+}(\vec{r})\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm rot}(\vec{r}) + V_{\rm ext}(\vec{r})\right]\hat{\Psi}(\vec{r}) + \frac{1}{2}\int d\vec{r}d\vec{r}'\hat{\Psi}^{+}(\vec{r})\hat{\Psi}^{+}(\vec{r}')V(\vec{r}-\vec{r}')\hat{\Psi}(\vec{r})\hat{\Psi}(\vec{r}'), \quad (2)$$

where $\hat{\Psi}(\vec{r})$ and $\hat{\Psi}^{+}(\vec{r})$ are the boson field operators that annihilate and create a particle at the position $\vec{r}$, respectively, and $V(\vec{r}-\vec{r}')$ is the two-body interatomic potential [2]. The potential $V_{\rm rot}(\vec{r})$ associated to the rotation of the condensate is given by

$$V_{\rm rot}(\vec{r}) = -f_{\rm rot}(t)\frac{m\omega^2}{2}(x^2+y^2), \quad (3)$$

where ω is the angular velocity of the condensate in the direction z and $f_{\rm rot}(t)$ is a function which takes into account the possible time variation of the rotation potential (here, we shall take $f_{\rm rot} = 1$). For a system consisting of a large number of particles, the calculation of the ground state of the system with the direct use of Eq. (2) is impracticable, due to the high computational cost.

Therefore the use of some approximate methods can lead to a significant simplification of the formalism. One such approach is the mean field description of the condensate, which is based on the idea of separating out the condensate contribution to the bosonic field operator. For a uniform gas in a volume V, BEC occurs in the single particle state $\Psi_0 = \sqrt{N/V}$, having zero momentum. The field operator can then be decomposed in the form $\hat{\Psi}(\vec{r}) = \sqrt{N/V} + \hat{\Psi}'(\vec{r})$. By treating the operator $\hat{\Psi}'(\vec{r})$ as a small perturbation, one can develop the first order theory for the excitations of the interacting Bose gases [1,2].

In the general case of a nonuniform and time-dependent configuration, like a BEC star, the field operator in the Heisenberg representation is given by

$$\hat{\Psi}(\vec{r},t) = \psi(\vec{r},t) + \hat{\Psi}'(\vec{r},t), \quad (4)$$

where $\psi(\vec{r},t)$, called the condensate wave function, is the expectation value of the field operator, $\psi(\vec{r},t) = \langle\hat{\Psi}(\vec{r},t)\rangle$. It is a classical field and its absolute value fixes the number density of the condensate through $n(\vec{r},t) = |\psi(\vec{r},t)|^2$. The normalization condition is $N = \int n(\vec{r},t)d\vec{r}$, where N is the total number of bosons in the star.

The equation of motion for the condensate wave function is given by the Heisenberg equation corresponding to the many-body Hamiltonian given by Eq. (2):

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\vec{r},t) = [\hat{\Psi},\hat{H}] = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm rot}(\vec{r}) + V_{\rm ext}(\vec{r}) + \int d\vec{r}'\hat{\Psi}^{+}(\vec{r}',t)V(\vec{r}'-\vec{r})\hat{\Psi}(\vec{r}',t)\right]\hat{\Psi}(\vec{r},t). \quad (5)$$

Replacing $\hat{\Psi}(\vec{r},t)$ by the condensate wave function ψ gives the zeroth-order approximation to the Heisenberg equation:

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm rot}(\vec{r}) + V_{\rm ext}(\vec{r}) + \int d\vec{r}'V(\vec{r}'-\vec{r})|\psi(\vec{r}',t)|^2\right]\psi(\vec{r},t). \quad (6)$$

In the integral containing the binary potential $V(\vec{r}'-\vec{r})$, this replacement is in general a poor approximation for short distances. However, in a dilute and cold gas, only binary collisions at low energy are relevant and these collisions are characterized by a single parameter, the s-wave scattering length, independently of the details of the two-body potential. Therefore, one can replace $V(\vec{r}'-\vec{r})$ by an effective interaction $V_{\rm eff}(\vec{r}'-\vec{r}) = u_0\delta(\vec{r}'-\vec{r})$, where the coupling constant u_0 is related to the scattering length a through $u_0 = 4\pi\hbar^2 a/m$. Hence, we assume that in a medium composed of scalar particles with nonzero mass, the range of Van der Waals-type scalar mediated interactions among nucleons becomes infinite when the medium makes a transition to a Bose-Einstein condensed phase. With the use of the effective potential the integral in the bracket of Eq. (6) gives $u_0|\psi(\vec{r},t)|^2$, and the resulting equation is the Schrödinger equation for a BEC with a quartic nonlinearity (in the energy functional) [1,2]. However, in order to obtain a more general description of the BEC stars, we shall assume an arbitrary nonlinear term $g'(|\psi(\vec{r},t)|^2)$ [38].

As pointed out in Ref. [38], the Gross-Pitaevskii approximation is a long-wavelength theory widely used to describe a variety of properties of dilute Bose condensates, but for short-ranged repulsive interactions this theory fails in low dimensions, and some essential modifications of the theory are necessary. From a physical point of view, these modifications can be understood as follows [38]. The interparticle interaction can be written as $V(\vec{r}) = u_0\delta_a^d(\vec{r})$, where u_0 is the amplitude of the interparticle repulsion and $\delta_a^d(\vec{r})$ denotes any well localized d-dimensional function that transforms into the mathematical Dirac delta distribution when the range of interactions $a \to 0$. Assume that the interparticle interaction is so strong that each particle is localized within a cage formed by its neighbors. In the dilute limit $nl^d \ll 1$, the size of this cage can be estimated as $R \sim n^{-1/d}$ and the ground state energy per particle follows from the uncertainty principle as $\hbar^2/mR^2 \sim \hbar^2 n^{2/d}/m$. The ground state energy which would go into the energy functional is given by $\hbar^2 n^{(2+d)/d}/m$. The strong interaction assumption is valid if the interaction energy per particle u_0/R^d is much bigger than the ground state energy per particle, i.e., $u_0/R^d \gg \hbar^2/mR^2$. The condition for the strong coupling limit can be written as $\hbar^2 n^{(2-d)/d}/mu_0 \ll 1$. As space dimensionality decreases, it becomes increasingly harder for the repulsive particles to avoid collisions. Below the critical dimension $d_c = 2$, the quartic nonlinearity $|\psi|^4$ in the energy functional must be replaced by $|\psi|^{2(2+d)/d}$ (in $d = 2$, the quartic nonlinearity contains a logarithmic correction in $|\psi|^2$) [38]. For $d = 1$ we have a $|\psi|^6$ interaction.

Finally, to describe a BEC star, we shall treat the gravitational interaction within a mean field approximation, introducing the self-consistent gravitational potential $V_{\rm grav}(\vec{r}, t) = \int d\vec{r}' u(\vec{r}' - \vec{r}) |\psi(\vec{r}', t)|^2$ in Eq. (6) where $u(\vec{r}' - \vec{r}) = -Gm^2/|\vec{r}' - \vec{r}|$ is the usual gravitational potential of interaction. Therefore, the generalized Gross-Pitaevskii equation describing a self-gravitating BEC with short-range interactions is given by

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{r}, t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm rot}(\vec{r}) + V_{\rm ext}(\vec{r}) + V_{\rm grav}(\vec{r}, t) + g'(|\psi(\vec{r}, t)|^2)\right]\psi(\vec{r}, t), \tag{7}$$

where we denoted $g' = dg/dn$ and where the gravitational potential $V_{\rm grav}(\vec{r}, t)$ is determined self-consistently by the Poisson equation

$$\nabla^2 V_{\rm grav} = 4\pi G n m^2, \tag{8}$$

where

$$n(\vec{r}, t) = |\psi(\vec{r}, t)|^2, \tag{9}$$

is the mass density inside the BEC star. Finally, we shall take $V_{\rm ext}(\vec{r}) = 0$ since the system is confined by its self-gravity.

III. THE HYDRODYNAMICAL REPRESENTATION OF THE GRAVITATIONAL BOSE-EINSTEIN CONDENSATE

The physical properties of a BEC described by the generalized Gross-Pitaevskii equation given by Eq. (7) can be understood much easily by using the so-called Madelung representation of the wave function [1,2], which consists in writing ψ in the form

$$\psi(\vec{r}, t) = \sqrt{n(\vec{r}, t)}\exp\left[\frac{i}{\hbar}S(\vec{r}, t)\right], \tag{10}$$

where the function $S(\vec{r}, t)$ has the dimension of an action. By substituting the above expression of the wave function in Eq. (7), it decouples into a system of two differential equations for the real functions n and $\vec{v}$, given by

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) = 0, \tag{11}$$

$$m\frac{\partial \vec{v}}{\partial t} + \nabla\left(\frac{m\vec{v}^2}{2} + V_Q + V_{\rm rot} + V_{\rm grav} + g'\right) = 0, \tag{12}$$

where we have introduced the quantum potential

$$V_Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}, \tag{13}$$

and the velocity of the quantum fluid

$$\vec{v} = \frac{\nabla S}{m}, \tag{14}$$

respectively. From its definition, it follows that the velocity field is irrotational, satisfying the condition $\nabla \times \vec{v} = \vec{0}$. The quantum potential V_Q has the property [1]:

$$n\nabla_i V_Q = \nabla_j\left(-\frac{\hbar^2}{4m}n\nabla_i\nabla_j \ln n\right) = \nabla_j \sigma_{ij}^Q, \tag{15}$$

where $\sigma_{ij}^Q = -(\hbar^2 n/4m)\nabla_i\nabla_j \ln n$ is the quantum stress tensor which has the dimension of a pressure and is an intrinsically anisotropic quantum contribution to the equations of motion.

By taking into account that the flow is irrotational, the equations of motion of the gravitational ideal BEC take the form of the equation of continuity and of the hydrodynamic Euler equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) = 0, \tag{16}$$

$$nm\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla P(n) - n\nabla V_{\rm rot} - n\nabla V_{\rm grav} - \nabla\cdot\sigma^Q, \tag{17}$$

where we have denoted

$$P(n) = g'(n)n - g(n). \tag{18}$$

Therefore the Bose-Einstein gravitational condensate can be described as a gas whose density and pressure are related by a barotropic equation of state [1]. The explicit form of this equation depends on the form of the nonlinearity term g.

For a static ideal condensate, $\vec{v} \equiv \vec{0}$. In this case, from Eq. (12), we obtain

$$V_Q + V_{\rm rot} + V_{\rm grav} + g' = {\rm const}. \tag{19}$$

Applying the operator ∇^2 to both sides of Eq. (19) gives

$$\nabla^2(V_Q + V_{\rm rot} + g') + \nabla^2 V_{\rm grav} = 0. \tag{20}$$

In the case of a condensate with a nonlinearity of the form $g(n) = u_0 n^2/2$, where u_0 is a constant, it follows that the generalized potential $V_{\rm gen} = -V_Q - V_{\rm rot} - u_0 n$ satisfies the Poisson equation

$$\nabla^2 V_{\rm gen} = 4\pi G n m^2. \tag{21}$$

If the quantum potential can be neglected, then from Eq. (21), by using the relation $\nabla^2 V_{\rm rot} = -2m\omega^2$, it follows that the mass density of the condensate is described by a Helmholtz type equation given by

$$\nabla^2 n + \frac{4\pi G}{u_0} n m^2 - \frac{2m\omega^2}{u_0} = 0. \tag{22}$$

IV. STATIC AND SLOWLY ROTATING NEWTONIAN BOSE-EINSTEIN CONDENSATE STARS

When the number of particles in the gravitationally bounded BEC becomes large enough, the quantum pressure term makes a significant contribution only near the boundary of the condensate. Hence it is much smaller than the nonlinear interaction term. Thus, the quantum stress term in the equation of motion of the condensate can be neglected. This is the Thomas-Fermi approximation which has been extensively used for the study of the BECs [1]. As the number of particles in the condensate becomes infinite, the Thomas-Fermi approximation becomes exact [33,34,36]. This approximation also corresponds to the classical limit of the theory (it corresponds to neglecting all terms with powers of $\hbar$) or as the regime of strong repulsive interactions among particles. From a mathematical point of view the Thomas-Fermi approximation corresponds to neglecting all terms containing ∇n and ∇S in the equation of motion.

A. Static Bose-Einstein condensate stars

In the case of a static BEC, all physical quantities are independent of time. Moreover, in the first approximation we also neglect the rotation of the star, taking $V_{\rm rot} = 0$. Introducing the mass density $\rho = nm$ and writing the gravitational potential as $V_{\rm grav} = m\Phi$, the equations describing the static self-gravitating BEC take the form

$$\nabla P = -\rho \nabla \Phi, \tag{23}$$

$$\nabla^2 \Phi = 4\pi G \rho, \tag{24}$$

where Eq. (23) is just the usual equation of hydrostatic equilibrium. These equations must be integrated together with the equation of state $P = P(\rho)$, which follows from Eq. (18), and some appropriately chosen boundary conditions. By assuming that the nonlinearity in the Gross-Pitaevskii equation is of the form

$$g(n) = \alpha n^\gamma, \tag{25}$$

where α and γ are positive constants, it follows that the equation of state of the gravitational BEC is the polytropic equation of state

$$P(\rho) = K\rho^\gamma, \tag{26}$$

where we denoted $K = \alpha(\gamma - 1)/m^\gamma$.

By representing γ in the form $\gamma = 1 + 1/n$, where n is the polytropic index, it follows that the structure of the static BEC star is described by the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n, \tag{27}$$

where θ is a dimensionless variable defined via $\rho = \rho_c \theta^n$, ξ is a dimensionless coordinate introduced via the transformation $r = [(n+1)K\rho_c^{1/n-1}/4\pi G]^{1/2}\xi$ and ρ_c is the central density of the condensate [39].

Hence, the radius and the mass of the condensate are given by

$$R = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/2n} \xi_1, \tag{28}$$

and

$$M = 4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \xi_1^2 |\theta'(\xi_1)|, \tag{29}$$

respectively, where ξ_1 defines the zero-pressure and zero-density surface of the condensate: $\theta(\xi_1) = 0$ [39].

In the standard approach to the BECs, the nonlinearity term g is given by

$$g(n) = \frac{u_0}{2}|\psi|^4 = \frac{u_0}{2} n^2, \tag{30}$$

where $u_0 = 4\pi\hbar^2 a/m$ [1]. The corresponding equation of state of the condensate is

$$P(\rho) = K\rho^2, \tag{31}$$

with

$$K = \frac{2\pi\hbar^2 a}{m^3}. \tag{32}$$

Therefore, the equation of state of the BEC is a polytrope with index $n = 1$. In this case the solution of the Lane-Emden equation can be obtained in an analytical form, and the solution satisfying the boundary condition $\theta(0) = 1$ is [39]:

$$\theta(\xi) = \frac{\sin\xi}{\xi}. \tag{33}$$

The radius of the star is defined by the condition $\theta(\xi_1) = 0$, giving $\xi_1 = \pi$. Therefore the radius R of the BEC is given by

$$R = \pi\sqrt{\frac{\hbar^2 a}{Gm^3}}. \tag{34}$$

The radius of the gravitationally bounded BEC is independent on the central density and on the mass of the star, and depends only on the physical characteristics of the condensate.

The mass of a BEC star with quartic nonlinearity is given as a function of the central density and of the coherent scattering length a by

$$M = 4\pi^2\left(\frac{\hbar^2 a}{Gm^3}\right)^{3/2}\rho_c, \tag{35}$$

where we have used $|\theta'(\xi_1)| = 1/\pi$. Using Eq. (34), it can be expressed in terms of the radius and central density by

$$M = \frac{4}{\pi}\rho_c R^3, \tag{36}$$

which shows that the mean density of the star $\bar{\rho} = 3M/4\pi R^3$ can be obtained from the central density of the condensate by the relation $\bar{\rho} = 3\rho_c/\pi^2$.

With respect to a scaling of the parameters m, a and ρ_c of the form $m \to \alpha_1 m$, $a \to \alpha_2 a$, $\rho_c \to \alpha_3\rho_c$, the radius and the mass of the condensate have the following scaling properties:

$$R \to \alpha_1^{-3/2}\alpha_2^{1/2}R, \qquad M \to \alpha_1^{-9/2}\alpha_2^{3/2}\alpha_3 M. \tag{37}$$

B. Slowly rotating Bose-Einstein condensate stars

The case of slowly rotating BECs can also be straightforwardly analyzed by taking into account the fact that the condensate obeys a polytropic equation of state. The study of the slowly rotating polytropes is performed in detail in Ref. [40].

The Lane-Emden equation for a rotating BEC is

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right) + \frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right] = -\Theta^n + \Omega, \tag{38}$$

where $\mu = \cos\theta$ and $\Omega = \omega^2/2\pi G\rho_c$. The volume V_ω and the mass M_ω of the condensate in slow rotation are given in the first order in Ω by

$$V_\omega = V_0\left[1 + \frac{3\psi_0(\xi_1)}{\xi_1|\theta'(\xi_1)|}\Omega\right], \tag{39}$$

and

$$M_\omega = M_0\left[1 + \frac{\xi_1/3 - \psi_0'(\xi_1)}{|\theta'(\xi_1)|}\Omega\right], \tag{40}$$

respectively. In these expressions, M_0 is given by Eq. (29) and $V_0 = (4/3)\pi R_0^3$ where R_0 is given by Eq. (28). The values of the function ψ_0 are tabulated in Ref. [40]. Equations (39) and (40) represent the mass and volume relations for two stars with equal central density, one rotating with an angular velocity ω and the other nonrotating.

In the case of BECs with quartic nonlinearity, corresponding to a polytropic index $n = 1$, the Lane-Emden equation can be integrated exactly (yielding $\theta(\xi) = \sin\xi/\xi$ and $\psi_0(\xi) = 1 - \sin\xi/\xi$), giving for the volume V_ω and mass M_ω of the rotating condensate the following simple relations

$$V_\omega = V_0(1 + 3\Omega), \tag{41}$$

$$M_\omega = M_0\left[1 + \left(\frac{\pi^2}{3} - 1\right)\Omega\right]. \tag{42}$$

Remark: since the velocity field of a BEC is irrotational, a BEC cannot have a solid-body rotation $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$ (as we have assumed), unless it nucleates a lattice of uniformly spaced singular vortices in an attempt to mimic solid-body rotation [41]. However, it is known that vortices appear only for sufficiently large values of angular velocity [42]. For $\Omega < \Omega_c$, a BEC cannot have a solid-body rotation but it can have a differential rotation. For example, Rindler-Daller and Shapiro [35] propose to model self-gravitating rotating BECs by irrotational Riemann-S ellipsoids. Therefore, the BEC is rotating but its motion is irrotational in agreement with the condition (14). In view of this discussion, our model of self-gravitating BECs in solid-body rotation is highly idealized and, at most, approximate. This is just a very first step in the modeling of rotating self-gravitating BECs.

V. MAXIMUM MASS OF THE STATIC RELATIVISTIC BOSE-EINSTEIN CONDENSATE STARS: QUALITATIVE TREATMENT

The numerical values of the basic parameters (mass and radius) of the condensed object sensitively depend on the mass m of the particles and on the scattering length a. Of course, in general, the values of the mass and radius of the gravitational condensate also depend on the adopted model for the nonlinearity. However, a quartic nonlinearity is a good working hypothesis as a starting point.

The scattering length a is defined as the zero-energy limit of the scattering amplitude f, and it can be related to the particle scattering cross section σ by the relation $\sigma = 4\pi a^2$ [1]. On the other hand, the notion that particles like, for example, the quarks, retain their usual properties and interactions at the very high densities in the neutron stars may not be viable [13,43]. In our calculations, we use a "hard" core approximation of the potential. Therefore, we accept that, at high densities, the "hard" core potential is in the QCD range of 1 fm and the allowed values of the scattering length a may generally be in the interval $0.5\ \text{fm} \leq a < 1-2$ fm, corresponding to a scattering cross section of about 1 mb.

The transition temperature to a BEC of dense matter can be written as

$$T_c = \frac{2\pi\hbar^2}{\zeta^{2/3}k_B}\frac{\rho^{2/3}}{m^{5/3}} = 1.650 \times 10^{12} \times \left(\frac{\rho}{10^{16}\text{g/cm}^3}\right)^{2/3}\left(\frac{m}{2m_n}\right)^{-5/3} \text{K}, \tag{43}$$

where $m_n = 1.6749 \times 10^{-24}$ g is the mass of the neutron. Neutron stars are born with interior temperatures of the order of 2–5×10^{11} K, but they rapidly cool down via neutrino emission to temperatures of less than 10^{10} K within minutes. Also strange matter, pion condensates, λ hyperons, δ isobars, or free quark matter might form under the initial thermal conditions prevailing in the very young neutron star. Thus, a condensation process can take place in the very early stages of stellar evolution. If the core is composed of only "ordinary" matter (neutrons, protons, and electrons), then when the temperature drops below about 10^9 K all particles are degenerate. We expect that after a hundred years or so the core will become superfluid [13], and this may also favor the possibility of a Bose-Einstein condensation through the BCS-BEC crossover.[1]

By introducing the dimensionless parameter κ defined as

$$\kappa = \left(\frac{a}{1\text{fm}}\right)^{1/2}\left(\frac{m}{2m_n}\right)^{-3/2}, \tag{44}$$

the radius of the BEC star given by Eq. (34) can be written

$$R = \pi\sqrt{\frac{\hbar^2 a}{Gm^3}} = 6.61\kappa \text{ km}. \tag{45}$$

For $m = 2m_n$ and $a = 1$ fm, we find $R \simeq 7$ km, similar to the size of neutron stars.[2] For a correct determination of the maximum mass and maximum density of BEC stars, we cannot ignore the effects induced by the space-time curvature, and a relativistic treatment is necessary (see Sec. VI). Before that, we shall present simple heuristic arguments to take relativistic effects into account.

Restriction on the maximum central density and maximum mass of the BEC stars with quartic nonlinearity can be obtained from the study of the speed of sound, defined as $c_s^2 = dP/d\rho$. With the use of Eq. (31), we obtain $c_s^2 = 2K\rho$. The causality condition implies $c_s \leq c$, where c is the speed of light. This leads to the following upper bound for the central density of the condensate:

$$\rho_c \leq \frac{m^3c^2}{4\pi a\hbar^2} = 2.42 \times 10^{16}\kappa^{-2} \text{ g/cm}^3. \tag{46}$$

With the use of Eqs. (46) and (35), we obtain the following restriction on the maximum mass of the BEC stars with quartic nonlinearity:

$$M \leq \pi\frac{\hbar c^2\sqrt{a}}{(Gm)^{3/2}} = 4.46\kappa\ M_\odot. \tag{47}$$

For $m = 2m_n$, we obtain the condition $\rho_c \leq [2.42/a(\text{fm})] \times 10^{16}$ g/cm^3. By taking into account that for this range of high densities a physically reasonable value for the scattering length is $a \approx 1$ fm, we obtain the restriction on the maximum mass of the BEC star from the causality condition as $M \leq 4.46\ M_\odot$.

A stronger bound on the central density can be derived from the condition that the radius of the star R must be greater than the Schwarzschild radius $R_S = 2GM/c^2$, $R \geq R_S$. For a BEC star, R_S can be expressed as a function of the central density and of the radius as $R_S = 8G\rho_c R^3/\pi c^2$ [see Eq. (36)]. Then, using Eq. (34), the condition of stability against gravitational collapse gives

$$\rho_c \leq \frac{m^3c^2}{8\pi a\hbar^2} = 1.21 \times 10^{16}\kappa^{-2} \text{ g/cm}^3, \tag{48}$$

a relation which, for the condensate star with $m = 2m_n$ and $a = 1$ fm, leads to the constraint $\rho_c \leq 1.21 \times 10^{16}$ g/cm^3. The constraint on the maximum mass for the stellar type BEC can be formulated as [see Eq. (35)]:

$$M \leq \frac{\pi}{2}\frac{\hbar c^2\sqrt{a}}{(Gm)^{3/2}} = 2.23\kappa\ M_\odot. \tag{49}$$

This inequality can also be directly obtained by substituting Eq. (34) in the condition $R \geq R_S = 2GM/c^2$. With $a = 1$ fm and $m = 2m_n$ we obtain for the maximum mass of the BEC star the restriction $M \leq 2.23\ M_\odot$.

For the $n = 1$ polytrope, the radius of the star is independent on the central density. Generally, one may consider a as a free parameter, which must be constrained by the physics of the nuclear interactions taking place in the system. However, due to the possible dependence of the free scattering length a on the mass density, in the case of BECs there may be (indirect) dependence of the radius on the central density of the star.

Finally, we would like to emphasize that the estimates on the maximum mass obtained in the present section are qualitative with respect to the numerical factors, and more precise values of the maximum mass of the BEC stars will be obtained in the next section by using a fully general relativistic approach.

VI. GENERAL RELATIVISTIC BOSE-EINSTEIN CONDENSATE STARS

In Sec. IV, we have considered the gravitationally bounded BEC stars in the framework of Newtonian gravity.

[1] In adopting the scaling of the mass in Eq. (43), we have in mind the possibility that neutrons in the core of neutron stars form the equivalent of Cooper pairs and behave as bosons of mass $2m_n$. This means that we treat the core of neutron stars as a superfluid (see Sec. VIII for additional comments). However, our study may be valid in other circumstances so that we leave the mass m unspecified.

[2] As will become clear below, the expression (45) of the radius is valid both in the Newtonian and relativistic regimes.

As discussed in Sec. V, general relativistic effects may change the physical properties of compact objects in both a qualitative and quantitative way. For example, general relativity imposes a strict limit on the maximum mass of a stable compact astrophysical object, a feature that is missing for classical Newtonian stars. Therefore, the study of the general relativistic Bose-Einstein condensates offers a better understanding of their physical properties. In the present section, we study the properties of static general relativistic Bose-Einstein condensate stars.

A. Static general relativistic BEC stars

For a static spherically symmetric star, the interior line element is given by

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\mu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (50)$$

The structure equations describing a general relativistic compact star are the mass continuity equation and the TOV equation of standard general relativity. They are given by [13]:

$$\frac{dM}{dr} = 4\pi\rho r^2, \quad (51)$$

$$\frac{dP(r)}{dr} = -\frac{G(\rho + P/c^2)(4\pi P r^3/c^2 + M(r))}{r^2[1 - 2GM(r)/c^2 r]}. \quad (52)$$

These equations extend the classical condition of hydrostatic equilibrium for a self-gravitating gas to the context of general relativity. We have written the energy density as $\epsilon = \rho c^2$. The system of equations (51) and (52) must be closed by choosing the equation of state for the thermodynamic pressure of the matter inside the star

$$P = P(\rho). \quad (53)$$

At the center of the star, the mass must satisfy the boundary condition

$$M(0) = 0. \quad (54)$$

For the thermodynamic pressure P, we assume that it vanishes on the surface: $P(R) = 0$.

The exterior of the BEC star is characterized by the Schwarzschild metric, describing the vacuum outside the star, and given by [13]:

$$(e^{\nu})^{\rm ext} = (e^{-\mu})^{\rm ext} = 1 - \frac{2GM}{c^2 r}, \qquad r \geq R. \quad (55)$$

The interior solution must match with the exterior solution on the vacuum boundary of the star.

B. Maximum mass of relativistic BECs with short-range interactions: Semirelativistic treatment

We assume that, in general relativity, the BEC can still be described by the nonrelativistic equation of state[3]

$$P = K\rho^2, \qquad \text{with} \quad K = \frac{2\pi\hbar^2 a}{m^3}, \quad (56)$$

corresponding to a polytropic equation of state with index $n = 1$. The theory of polytropic fluid spheres in general relativity has been developed by Tooper [44], and we shall use his formalism and notations. Therefore, we set

$$\rho = \rho_c\theta, \qquad P = K\rho_c^2\theta^2, \qquad \sigma = \frac{K\rho_c}{c^2}, \quad (57)$$

$$r = \frac{\xi}{A}, \qquad M(r) = \frac{4\pi\rho_c}{A^3}v(\xi), \qquad A = \left(\frac{2\pi G}{K}\right)^{1/2}, \quad (58)$$

where ρ_c is the central density. In terms of these variables, the TOV equation and the mass continuity equation become

$$\frac{d\theta}{d\xi} = -\frac{(1 + \sigma\theta)(v + \sigma\xi^3\theta^2)}{\xi^2(1 - 4\sigma v/\xi)}, \quad (59)$$

$$\frac{dv}{d\xi} = \theta\xi^2. \quad (60)$$

For a given value of the relativity parameter σ, they have to be solved with the initial condition $\theta(0) = 1$ and $v(0) = 0$. Since $v \sim \xi^3$ as $\xi \to 0$, it is clear that $\theta'(0) = 0$. On the other hand, the density vanishes at the first zero ξ_1 of θ:

$$\theta(\xi_1) = 0. \quad (61)$$

This determines the boundary of the sphere. In the nonrelativistic limit $\sigma \to 0$, the system (59) and (60) reduces to the Lane-Emden equation (27) with $n = 1$.

[3]While we use the equation of state (56) derived from the nonrelativistic Gross-Pitaevskii equation (7), we treat gravity in the framework of general relativity using the TOV equations (51) and (52). The fully-relativistic problem is treated in Sec. VI C. Although there are qualitative differences between the two treatments, the "semirelativistic" treatment developed here already provides the correct order of magnitude of the maximum mass of a relativistic BEC (see below). Furthermore, this semirelativistic treatment may be relevant for pulsars. Indeed, pulsars as neutron stars obey a nonrelativistic equation of state since neutrons are too massive to be relativistic even at extremely high densities, but gravitation must be treated with general relativity (this is the opposite of white dwarf stars in which the electrons are relativistic but gravitation can be treated in the Newtonian framework). So, if a Bose-Einstein condensation takes place through Cooper pairing in a neutron star (pulsar), the neutrons in the star will be nonrelativistic and the equation of state $P = K\rho^2$ (that can be rigorously derived from the GP equation) can be applied. Therefore, pulsars could be BEC stars with a nonrelativistic equation of state ($n = 1$ polytrope).

From the foregoing relations, we find that the radius, the mass and the central density of the configuration are given by

$$R = \xi_1 R_*, \qquad M = 2\sigma v(\xi_1) M_*, \qquad \rho_c = \sigma \rho_*, \quad (62)$$

where the scaling parameters R_*, M_* and ρ_* can be expressed in terms of the fundamental constants and the parameter κ as

$$R_* = \left(\frac{a\hbar^2}{Gm^3}\right)^{1/2} = 2.106\kappa \text{ km}, \quad (63)$$

$$M_* = \frac{\hbar c^2 \sqrt{a}}{(Gm)^{3/2}} = 1.420\kappa\, M_\odot, \quad (64)$$

$$\rho_* = \frac{m^3 c^2}{2\pi a \hbar^2} = 4.846 \times 10^{16} \kappa^{-2} \text{ g/cm}^3. \quad (65)$$

We note that the expression of the scaled radius R_* is the same as in the Newtonian regime (in particular it is independent on c), while the scaling of the mass and of the density are due to relativistic effects. By varying σ from 0 to $+\infty$, we obtain the series of equilibria in the form $M(\rho_c)$, $R(\rho_c)$ and $M(R)$.

The velocity of sound is $c_s^2 = P'(\rho) = 2K\rho$. The condition that the velocity of sound at the center of the configuration (where it achieves its largest value) is smaller than the velocity of light can be expressed as $2K\rho_c \le c^2$, or equivalently as $\sigma \le \sigma_s$ with

$$\sigma_s = \frac{1}{2}. \quad (66)$$

The values of ξ_1 and $v(\xi_1)$ at this point have been tabulated by Tooper (and confirmed by our numerical study):

$$\xi_1 = 1.801, \qquad v(\xi_1) = 0.4981. \quad (67)$$

The corresponding values of radius, mass, and central density are

$$R_s = 1.801\left(\frac{a\hbar^2}{Gm^3}\right)^{1/2} = 3.790\kappa \text{ km}, \quad (68)$$

$$M_s = 0.498 \frac{\hbar c^2 \sqrt{a}}{(Gm)^{3/2}} = 0.707\kappa\, M_\odot, \quad (69)$$

$$(\rho_c)_s = \frac{m^3 c^2}{4\pi a \hbar^2} = 2.423 \times 10^{16} \kappa^{-2} \text{ g/cm}^3. \quad (70)$$

However, it is not granted that the criterion $\sigma > \sigma_s$ is equivalent to the condition of dynamical instability. The principle of causality is a necessary, but not a sufficient, condition of stability.[4] The condition of dynamical instability corresponds to the turning point of mass ($dM = 0$) and there is no reason why this should be equivalent to $c_s = c$. In fact, our numerical study demonstrates that this is not the case. We find that the maximum mass does not exactly correspond to the point where the velocity of sound becomes equal to the velocity of light. In the series of equilibria (parametrized by the central density $\rho_c = \sigma\rho_*$), the instability occurs *sooner* than predicted by the criterion (66). We find indeed that instability (corresponding to the mass peak) occurs for $\sigma \ge \sigma_c$ with

$$\sigma_c = 0.42. \quad (71)$$

The values of ξ_1 and $v(\xi_1)$ at this point are

$$\xi_1 = 1.888, \qquad v(\xi_1) = 0.5954. \quad (72)$$

The corresponding values of radius, mass and central density are

$$R_{\min} = 1.888\left(\frac{a\hbar^2}{Gm^3}\right)^{1/2} = 3.974\kappa \text{ km}, \quad (73)$$

$$M_{\max} = 0.5001 \frac{\hbar c^2 \sqrt{a}}{(Gm)^{3/2}} = 0.710\kappa\, M_\odot, \quad (74)$$

$$(\rho_c)_{\max} = 0.42 \frac{m^3 c^2}{2\pi a \hbar^2} = 2.035 \times 10^{16} \kappa^{-2} \text{ g/cm}^3, \quad (75)$$

respectively. We also note that the radius of a BEC star is necessarily smaller than

$$R_{\max} = \pi\sqrt{\frac{\hbar^2 a}{Gm^3}} = 6.61\kappa \text{ km}, \quad (76)$$

corresponding to the Newtonian limit ($\sigma \to 0$). Therefore, its value $3.974\kappa \le R(\text{km}) \le 6.61\kappa$ is very much constrained.

The dimensionless curves giving the mass-central density, radius-central density, mass-radius relations and some density profiles are plotted in Figs. 1–5. In Fig. 6 we present the mass-radius relation for $a = 1$ fm and different values of m.

C. Maximum mass of relativistic BECs with short-range interactions: Fully-relativistic treatment

The previous treatment is approximate because we use the equation of state (56) obtained in the nonrelativistic regime (i.e., from the Gross-Pitaevskii equation) but solve the TOV equations (51) and (52) expressing the condition of hydrostatic equilibrium in general relativity. A fully relativistic approach based on the Klein-Gordon-Einstein system has been developed by several authors [23–28] in the context of boson stars described by a scalar field. Self-interacting boson stars were first considered by Mielke and Scherzer [24] and further studied by Colpi *et al.* [25] who showed that the self-interaction can increase their mass up to the Chandrasekhar mass (i.e., like for fermion stars).

[4]Similarly, in statistical mechanics, the condition that the specific heat is positive in the canonical ensemble is a necessary, but not a sufficient, condition of canonical stability [45].

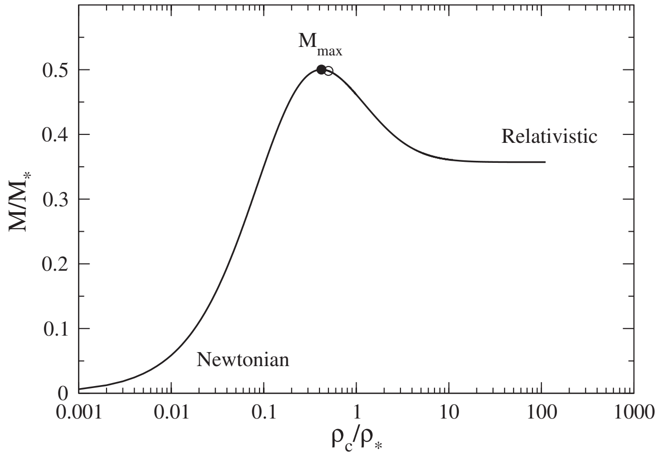

FIG. 1. Dimensionless mass-central density relation of a relativistic BEC with short-range interactions modeled by a $n = 1$ polytrope. There exist a maximum mass $M_{max}/M_* = 0.5001$ (black bullet) at which the system becomes dynamically unstable. This mass does *not* coincide with the point (white bullet) at which the velocity of sound becomes equal to the velocity of light (see Fig. 2). In particular, gravitational instability occurs slightly sooner than what is predicted from the criterion based on the velocity of sound (principle of causality).

In order to make the correspondence between BECs with short-range interactions described by the Gross-Pitaevskii equation and scalar fields with a $\frac{1}{4}\lambda|\phi|^4$ interaction described by the Klein-Gordon equation, we set [33]:

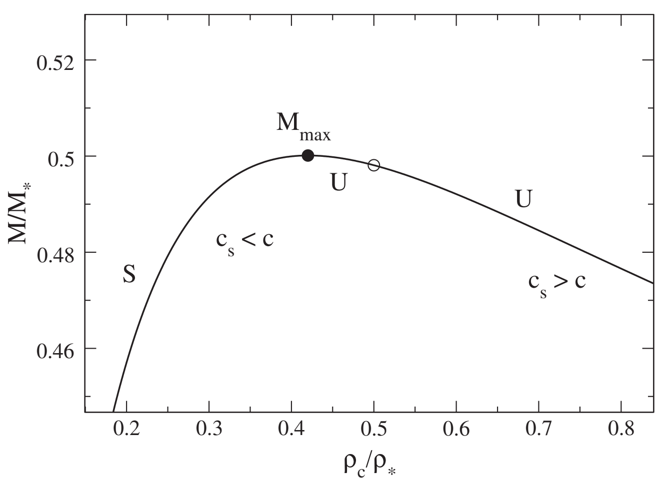

FIG. 2. Zoom of Fig. 1 showing that the maximum mass does not coincide with the point at which the velocity of sound becomes equal to the velocity of light. The system is stable (S) before the turning point of mass ($\sigma < \sigma_c$) and unstable (U) after the turning point of mass ($\sigma > \sigma_c$). The velocity of sound is smaller than the velocity of light before the white bullet ($\sigma < \sigma_s$) and larger after that point ($\sigma > \sigma_s$). Therefore, there exists a small region ($\sigma_c < \sigma < \sigma_s$) where the system is unstable although the velocity of sound is smaller than the velocity of light.

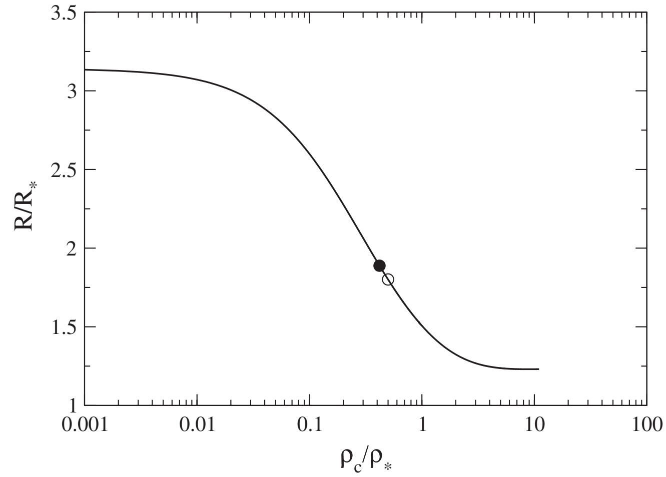

FIG. 3. Dimensionless radius-central density relation of a relativistic BEC with short-range interactions modeled by a $n = 1$ polytrope.

$$\frac{\lambda}{8\pi} \equiv \frac{a}{\lambda_c} = \frac{amc}{\hbar}, \tag{77}$$

where $\lambda_c = \hbar/mc$ is the Compton wavelength of the bosons. The equation of state (56) can be written

$$P = K\rho^2, \qquad \text{with} \quad K = \frac{\lambda\hbar^3}{4m^4c}. \tag{78}$$

This returns the equation of state obtained by Arbey *et al.* [31] for a self-interacting scalar field in the nonrelativistic

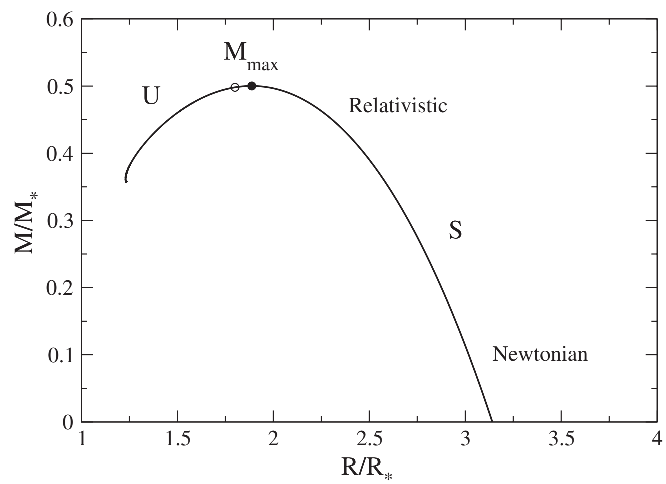

FIG. 4. Dimensionless mass-radius relation of a relativistic BEC with short-range interactions modeled by a $n = 1$ polytrope. The series of equilibria is parametrized by the relativity parameter σ. There exist a maximum mass $M_{max}/M_* = 0.5001$ and a minimum radius $R_{min}/R_* = 1.888$ corresponding to a maximum central density $(\rho_c)_{max} = 0.42\rho_*$. There also exist a maximum radius $R_{max}/R_* = \pi$ corresponding to the Newtonian limit $\sigma \to 0$.

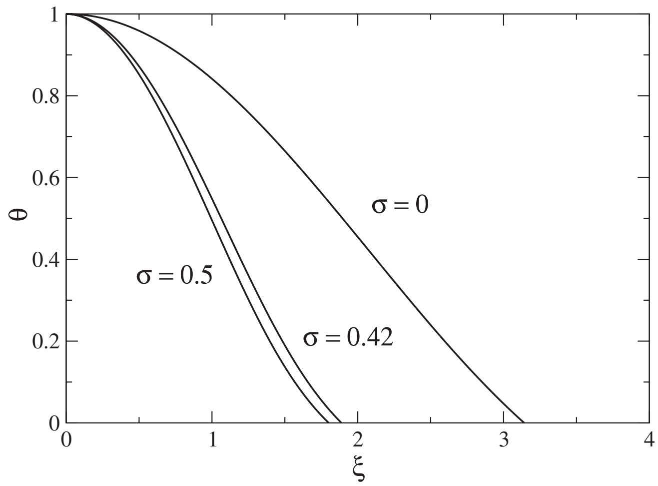

FIG. 5. Dimensionless density profiles corresponding to $\sigma = 0$ (Newtonian), $\sigma = \sigma_c = 0.42$ (maximum mass) and $\sigma = \sigma_s = 1/2$ (where $c_s = c$).

regime showing that the relation between a and λ given by Eq. (77) is correct. We note that the parameter λ can be expressed as

$$\frac{\lambda}{8\pi} = 9.523 \frac{a}{1\ \mathrm{fm}} \frac{m}{2m_n}. \tag{79}$$

We can then use (λ, m) instead of (a, m) as independent physical variables. Finally, the dimensionless parameter κ can be written

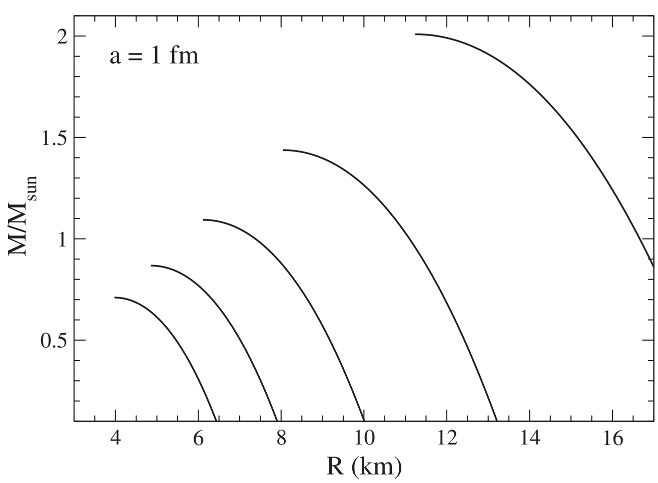

FIG. 6. Mass-radius relation for relativistic Bose-Einstein condensates with quartic nonlinearity for $a = 1$ fm and different values of the mass m (only stable configurations are shown in the framework of the semi-relativistic treatment). From top to bottom: $m = m_n$, $m = 1.25m_n$, $m = 1.5m_n$, $m = 1.75m_n$ and $m = 2m_n$. For all configurations $\rho_c \geq \rho_n$, where $\rho_n = 2.026 \times 10^{14}$ g/cm^3 is the nuclear density, and the causality condition $c_s \leq c$ is satisfied.

$$\kappa = 0.324\left(\frac{\lambda}{8\pi}\right)^{1/2}\left(\frac{2m_n}{m}\right)^2. \tag{80}$$

We can now express the results in terms of λ. The scaling of the maximum mass is given by

$$M_* = \frac{\hbar c^2\sqrt{a}}{(Gm)^{3/2}} = \sqrt{\frac{\lambda}{8\pi}\frac{1}{m^2}}\left(\frac{\hbar c}{G}\right)^{3/2} = \sqrt{\frac{\lambda}{8\pi}\frac{M_P^3}{m^2}}, \tag{81}$$

where $M_P = (\hbar c/G)^{1/2}$ is the Planck mass. This is the scaling of the maximum mass obtained by Colpi *et al.* [25] for a self-interacting scalar field in the Thomas-Fermi limit.[5] For $\lambda \sim 1$, it is of the order of the Chandrasekhar mass $\sim M_P^3/m^2$ of relativistic fermion stars while the maximum mass of noninteracting boson stars is $M_{Kaup} = 0.633\ M_P^2/m$ [23]. On the other hand, the scaling of the minimum radius is given by

$$R_* = \left(\frac{a\hbar^2}{Gm^3}\right)^{1/2} = \left(\frac{\lambda\hbar^3}{8\pi Gc}\right)^{1/2}\frac{1}{m^2} = \sqrt{\frac{\lambda}{8\pi}}\frac{M_P}{m}\lambda_c. \tag{82}$$

This is the scaling of the minimum radius given by Arbey *et al.* [31] for a self-interacting scalar field in the Thomas-Fermi limit. Finally, the scaling of the maximum density is given by

$$\rho_* = \frac{m^3c^2}{2\pi a\hbar^2} = \frac{4m^4c^3}{\lambda\hbar^3}. \tag{83}$$

Now, the maximum mass obtained by Colpi *et al.* [25] in the fully relativistic regime is

$$M_{\max} = 0.22\sqrt{\frac{\lambda}{4\pi}\frac{M_P^3}{m^2}} = 0.22\sqrt{2}M_* = 0.31M_*, \tag{84}$$

which is smaller than our previous estimate $M_{\max} = 0.5001M_*$ based on a nonrelativistic equation of state. Therefore, relativistic effects in the equation of state tend to reduce the maximum mass.

Colpi *et al.* [25] showed that, in the Thomas-Fermi limit, the scalar field becomes equivalent to a fluid with an equation of state

[5]The Thomas-Fermi (or strong coupling) approximation is valid for "large" λ. Actually, the condition of validity is $\lambda \gg (m/M_P)^2 \sim 10^{-38}$ [25,33], so that this condition is almost always satisfied as soon as the system has a self-interaction. The condition of strong coupling is usually written $\Lambda = \lambda(M_P/m)^2 \gg 1$.

$$P = \frac{c^4}{36K}\left[\left(1 + \frac{12K}{c^2}\rho\right)^{1/2} - 1\right]^2, \tag{85}$$

where K is given by Eq. (78). We shall adopt this equation of state for the description of our relativistic BEC star.[6] For $\rho \to 0$ (low or moderate densities), we recover the polytropic equation of state $P = K\rho^2$ corresponding to a nonrelativistic BEC with short-range interactions. For $\rho \to +\infty$ (extremely high densities), we obtain the ultra-relativistic equation of state $P = \rho c^2/3$, similar to the one describing the core of neutron stars modeled by the ideal Fermi gas [47–49]. We know that a linear equation of state $P = q\rho c^2$ yields damped oscillations of the mass-central density relation, and a spiral structure of the mass-radius relation [49], similarly to the isothermal equation of state in Newtonian gravity [50]. Therefore, our BEC model will exhibit this behavior, just like standard neutron stars. However, our BEC model differs from standard neutron star models in that, at low or moderate densities, $P = K\rho^2$ with $K = 2\pi a\hbar^2/m^3$ instead of $P = K'\rho^{5/3}$ with $K' = (1/5)(3/8\pi)^{2/3}h/m_n^{8/3}$ [47–49]. This implies, in particular, the existence of a maximum radius given by Eq. (76), corresponding to the Newtonian limit, which has no counterpart for standard neutron stars. Furthermore, the maximum mass of the BEC star depends, through the constant K, on the ratio $\kappa^2 \sim a/m^3$ which is not well known while the maximum mass of ordinary neutron stars is fixed, through the constant K', by the neutron mass m_n.

Substituting the equation of state (85) in the TOV equations (51) and (52), using

$$P'(\rho) = \frac{1}{3}c^2\left[1 - \frac{1}{\sqrt{1 + 12K\rho/c^2}}\right], \tag{86}$$

and introducing the same notations as in the preceding section, we obtain

$$\frac{d\theta}{d\xi} = -\frac{6\left[\frac{1}{36}(\sqrt{1+12\sigma\theta} - 1)^2 + \sigma\theta\right]\left[v + \frac{\xi^3}{36\sigma}(\sqrt{1+12\sigma\theta} - 1)^2\right]}{\xi^2(1 - 4\sigma v/\xi)(1 - 1/\sqrt{1+12\sigma\theta})}, \tag{87}$$

$$\frac{dv}{d\xi} = \theta\xi^2, \tag{88}$$

instead of Eqs. (59) and (60). If we expand the square roots for $\sigma \ll 1$, we recover Eqs. (59) and (60). However, this is not a uniform expansion and the two equations (87) and (59) are in fact different even for small values of σ (of course, they both reduce to the Lane-Emden equation (27) for $\sigma = 0$).

The velocity of sound at the center of the configuration is

$$(c_s^2)_0 = \frac{1}{3}c^2\left(1 - \frac{1}{\sqrt{1 + 12\sigma}}\right), \tag{89}$$

and we always have $(c_s)_0 < c$. The series of equilibria becomes unstable after the first mass peak. We find that instability occurs for $\sigma \geq \sigma_c'$ with

$$\sigma_c' = 0.398. \tag{90}$$

The values of ξ_1 and $v(\xi_1)$ at this point are

$$\xi_1' = 1.923, \qquad v(\xi_1') = 0.3865. \tag{91}$$

The corresponding values of the radius, mass and central density are

$$R_{\rm min}' = 1.923\left(\frac{a\hbar^2}{Gm^3}\right)^{1/2} = 4.047\kappa \text{ km}, \tag{92}$$

$$M_{\rm max}' = 0.307\frac{\hbar c^2\sqrt{a}}{(Gm)^{3/2}} = 0.436\kappa\, M_\odot, \tag{93}$$

$$(\rho_c)_{\rm max}' = 0.398\frac{m^3c^2}{2\pi a\hbar^2} = 1.929\times 10^{16}\kappa^{-2} \text{ g/cm}^3, \tag{94}$$

respectively. The radius of the BEC star is now constrained to the range $4.047\kappa \leq R(\text{km}) \leq 6.61\kappa$. The maximum mass $M_{\rm max}' = 0.307M_*$ is very close to the one [see Eq. (84)] found by Colpi *et al.* [25] by solving the Klein-Gordon-Einstein equations. This shows the accuracy of the hydrodynamical approach in the Thomas-Fermi limit. The dimensionless curves giving the mass-central density, radius-central density, mass-radius relations and some density profiles are plotted in Figs. 7–10.

Since the boson star model of Colpi *et al.* [25] has a fluid limit in the strong coupling regime $\Lambda \gg 1$, corresponding to the equation of state (85), it is not surprising to obtain the same results. In particular, Fig. 7 is the same as Fig. 4 of [25]. Nevertheless, it is interesting (and maybe enlightening) to rederive these results directly from a hydrodynamical formalism, by solving the TOV equations with the equation of state (85) instead of the Klein-Gordon-Einstein system. This hydrodynamic model may be less

[6]At present, there is no successful relativistic BEC theory, based on the relativistic extension of the Gross-Pitaevskii equation. The main reason is the presence of anti-bosons, and how to handle them [46]. Taking into account the present status of the research in this field, we are tentatively proposing an effective description of relativistic BECs in terms of a scalar field. Since the relativistic BECs are also described by the Klein-Gordon equation, such a description seems reasonable. We do not adopt the model of Colpi *et al.* [25] as such, but we suggest that a similar model, with a different physical interpretation of the parameters, could describe relativistic BECs (for a comparison between BEC models and scalar field models, see the conclusion).

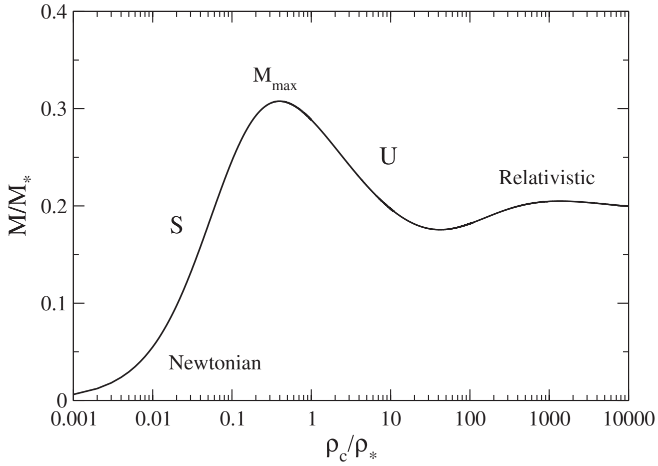

FIG. 7. Dimensionless mass-central density relation of a relativistic BEC with short-range interactions modeled by the equation of state (85). There exist a maximum mass $M'_{\rm max}/M_* = 0.307$ at which the series of equilibria becomes dynamically unstable. The velocity of sound is always smaller than the velocity of light. We note that the mass-central density relation presents damped oscillations at high densities similarly to standard neutron stars [48,49].

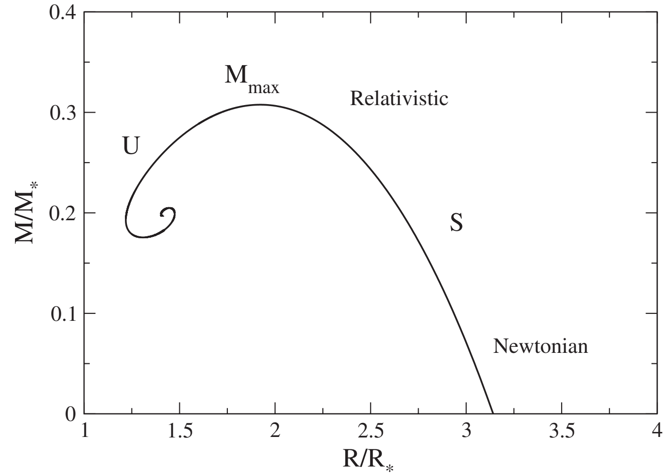

FIG. 9. Dimensionless mass-radius relation of a relativistic BEC with short-range interactions modeled by the equation of state (85). The mass-radius relation presents a snail-like structure (spiral) at high densities similarly to standard neutron stars [49,69]. There exist a maximum mass $M_{\rm max}/M_* = 0.307$ and a minimum radius $R_{\rm min}/R_* = 1.923$ corresponding to a maximum central density $(\rho_c)_{\rm max} = 0.398\rho_*$. There also exist a maximum radius $R_{\rm max}/R_* = \pi$ corresponding to the Newtonian limit $\sigma \to 0$.

abstract than scalar field theory and may clarify the physics of relativistic BEC stars.

Remark: At $T = 0$, the first law of thermodynamics takes the form

$$d\rho = \frac{P/c^2 + \rho}{n} dn, \tag{95}$$

where nm is the rest-mass density. Integrating this relation with the equation of state (85), we can obtain the relation $n(\rho)$. For $\rho \to 0$ (nonrelativistic limit), we get $\rho = nm$, leading to $P \sim K\rho^2 = K(nm)^2$ corresponding to a polytrope $n = 1$. For $\rho \to +\infty$ (ultra-relativistic limit), we get $\rho \propto (nm)^{4/3}$, leading to $P \sim \rho c^2/3 \propto (nm)^{4/3}$ corresponding to a polytrope $n = 3$ like for an ultra-relativistic Fermi gas at $T = 0$ (standard neutron star). These results are consistent with those obtained by Goodman [30]. The proper number of particles is

$$N = \int_0^R n(r)\left[1 - \frac{2GM(r)}{c^2 r}\right]^{-1/2} 4\pi r^2 dr. \tag{96}$$

It can be shown that a general relativistic, spherically symmetric, gaseous star at $T = 0$ is dynamically stable with respect to the Einstein equations if, and only if, it is

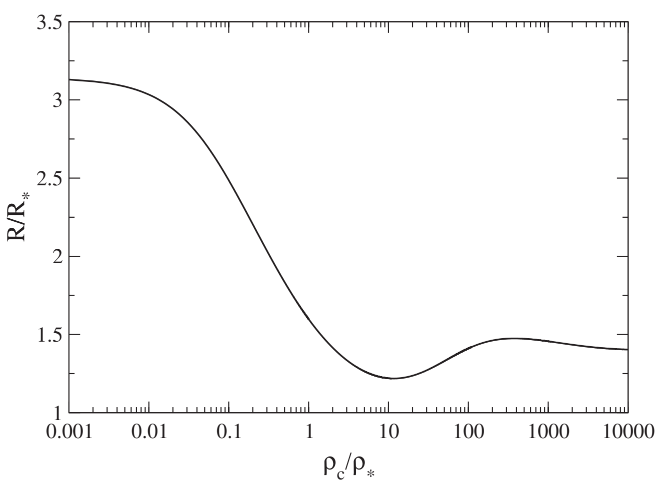

FIG. 8. Dimensionless radius-central density relation of a relativistic BEC with short-range interactions modeled by the equation of state (85).

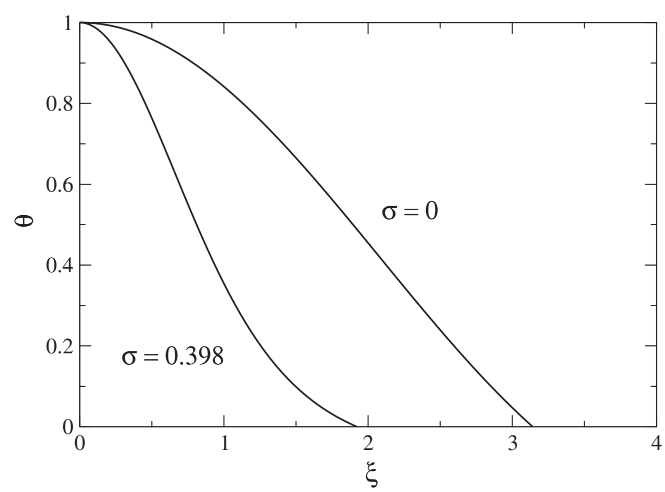

FIG. 10. Dimensionless density profiles corresponding to $\sigma = 0$ (Newtonian) and $\sigma = \sigma'_c = 0.398$ (maximum mass).

a maximum of $N[\rho]$ at fixed mass $M[\rho] = M$ (see, e.g., Refs. [51,52]). The first order variations $\delta N - \alpha\delta M = 0$, where α is a Lagrange multiplier, yield the TOV equations determining steady state solutions. The ensemble of these solutions forms the series of equilibria. Then, using the Poincaré theorem [45], one can conclude that the series of equilibria becomes unstable at the first mass peak and that a new mode of instability appears at each turning point of mass in the series of equilibria (see Ref. [53] for an alternative derivation of these results). At these points, we have $\delta N = \delta M = 0$ so that the curve $N(M)$ presents *cusps* at each point where $M(\rho_c)$ reaches an extremal value. An illustration of this behavior is given in Fig. 5 of Ref. [52]. These results of dynamical stability for general relativistic stars are similar to results of dynamical and thermodynamical stability for Newtonian self-gravitating systems [45,50,54].

VII. ASTROPHYSICAL IMPLICATIONS

One of the most important results in general relativistic astrophysics is the existence of a maximum mass of the neutron stars [47]. Ultra-dense compact objects may have a stable equilibrium configuration until their mass M is equal to the maximum mass $M_{\max}$. By integrating the mass continuity equation and the hydrostatic equilibrium equation for a star made of free, noninteracting, neutron gas (fermion star), Oppenheimer and Volkoff [47] have shown that the maximum equilibrium mass is $M_{\rm OV} = 0.7\ M_\odot$, with a corresponding radius of the order of $R_{\rm OV} = 9.6$ km, and a central density of the order of $\rho_c = 5 \times 10^{15}$ g/cm^3. Using a variational method in which the equation of state was constrained to have subluminal sound velocity and to be stable against microscopic collapse, Rhoades and Ruffini [55] proved that, in the regions where it is uncertain, the equation of state that produces the maximum neutron star mass is the one for which the sound speed is equal to the speed of light, i.e., $P = \rho c^2$. As a result, they found a maximum neutron star mass $M_{\max} \approx 3.2\ M_\odot$, assuming uncertainty in the equation of state above a fiducial density $\rho_0 \approx 4.6 \times 10^{14}$ g/cm^3. More realistic models that take into account the composition of the star and the interaction between neutrons lead to values of the maximum mass of the neutron stars in the range 1.5–3.2 $M_\odot$ [56]. The main reason for the lack of a better theoretical value of the maximum mass of the neutron stars is the poor knowledge of the equation of state of hadronic matter at high densities.

With the use of Eqs. (63) and (65) it follows that the scaled mass, radius, and central density satisfy the relations

$$\frac{R_*}{M_*} = \frac{G}{c^2}, \qquad \rho_* M_*^2 = \frac{c^6}{2\pi G^3}, \qquad \rho_* R_*^2 = \frac{c^2}{2\pi G}. \tag{97}$$

With the use of Eqs. (62), (71), (72), and (97), we obtain the following radius-mass, central density-mass, and central density-radius relations for the maximally stable BEC configuration in the semirelativistic treatment:

$$R_{\min} = \frac{\xi_1}{2\sigma_c v(\xi_1)} \frac{GM_{\max}}{c^2} = 5.599 \frac{M_{\max}}{M_\odot}\ \text{km}, \tag{98}$$

$$(\rho_c)_{\max} = 4\sigma_c^3 v^2(\xi_1) \frac{c^6}{2\pi G^3 M_{\max}^2} = 1.026 \times 10^{16} \left(\frac{M_\odot}{M_{\max}}\right)^2 \frac{\text{g}}{\text{cm}^3}, \tag{99}$$

$$(\rho_c)_{\max} = \sigma_c \xi_1^2 \frac{c^2}{2\pi G R_{\min}^2} = 3.215 \times 10^{15} \left(\frac{10\ \text{km}}{R_{\min}}\right)^2 \frac{\text{g}}{\text{cm}^3}. \tag{100}$$

In the fully-relativistic treatment, using Eqs. (90) and (91), we find

$$R_{\min} = 9.271 \frac{M_{\max}}{M_\odot}\ \text{km}, \tag{101}$$

$$(\rho_c)_{\max} = 3.682 \times 10^{15} \left(\frac{M_\odot}{M_{\max}}\right)^2 \frac{\text{g}}{\text{cm}^3}, \tag{102}$$

$$(\rho_c)_{\max} = 3.160 \times 10^{15} \left(\frac{10\ \text{km}}{R_{\min}}\right)^2 \frac{\text{g}}{\text{cm}^3}. \tag{103}$$

In the first case, the mass-radius ratio of the star can be expressed as

$$\frac{2GM_{\max}}{c^2 R_{\min}} = \frac{4\sigma_c v(\xi_1)}{\xi_1} = 0.529, \tag{104}$$

while, in the second case, we obtain

$$\frac{2GM_{\max}}{c^2 R_{\min}} = 0.319. \tag{105}$$

A classical result by Buchdahl [57] shows that for static solutions of the spherically symmetric Einstein-matter systems, the total mass M and the area radius R of the boundary of the body obey the relation $2GM/c^2R \leq 8/9 = 0.888$, the equality sign corresponding to constant density stars. For BEC stars, Eqs. (104) and (105) obviously satisfy the Buchdahl inequality for the mass-radius ratio.

We emphasize that the radius $R_{\min}$ and the central density $(\rho_c)_{\max}$ given by Eqs. (98)–(103) only depend on the mass $M_{\max}$. In particular, they do not explicitly depend on the two physical parameters of the model, the scattering length a, and the particle mass m. On the other hand, the maximum mass $M_{\max}$ depends on these two parameters only through their ratio a/m^3, or equivalently, through the parameter κ, and it can be obtained from the relations

$$M_{\max} = 0.71\kappa\ M_\odot, \qquad M_{\max} = 0.4368\kappa\ M_\odot, \tag{106}$$

in the semirelativistic and fully-relativistic treatments, respectively. Hence, all physical parameters of the model are determined by the mass $M_{\max}$ of the star, which can be obtained from observations. Therefore, in the present model, we have only *one* free parameter $\kappa = (a/\mathrm{fm})^{1/2} \times (2m_n/m)^{3/2}$. With respect to a scaling of the scattering length and of the particle mass of the form

$$a/\mathrm{fm} \to \beta_1(a/\mathrm{fm}), \qquad m/2m_n \to \beta_2(m/2m_n), \quad (107)$$

where β_1, β_2 are constants, the parameter κ scales as

$$\kappa \to \beta_1^{1/2}\beta_2^{-3/2}\kappa. \quad (108)$$

Since $\kappa = 0.324(\lambda/8\pi)^{1/2}(2m_n/m)^2$, we equivalently conclude that the maximum mass $M_{\max}$ depends on the two parameters (λ, m) only through their ratio λ/m^4.

By assuming that the mass of the star is $M_{\max} = 2\ M_\odot$, in the semirelativistic treatment we obtain for the parameters of the star $(\rho_c)_{\max} = 0.256 \times 10^{16}\ \mathrm{g/cm^3}$ and $R_{\min} = 11.2$ km, independently on the values of a and m. On the other hand, in this case $\kappa = 2.816$. If we take $m = 2m_n$, this corresponds to a scattering length $a = 7.93$ fm, and a coupling constant $\lambda = 1.90 \times 10^3$. In the fully-relativistic treatment, we find $(\rho_c)_{\max} = 0.091 \times 10^{16}\ \mathrm{g/cm^3}$ and $R_{\min} = 18.54$ km. For the parameter κ we obtain $\kappa = 4.578$. If we take $m = 2m_n$, this corresponds to a scattering length $a = 21.0$ fm and a coupling constant $\lambda = 5.02 \times 10^3$. Therefore, if we assume values of κ of the order of $\kappa \sim 3$ in the semirelativistic treatment, and $\kappa \sim 5$ in the fully-relativistic treatment, we obtain stellar objects with physical parameters in the range $M \sim 2\ M_\odot$, $R \sim$ 10–20 km, and $\rho_c \sim 0.3 - 0.1 \times 10^{16}\ \mathrm{g/cm^3}$. The only free parameter in the model, κ, uniquely determines the mass $M_{\max}$ of the star (or conversely).

It may be of interest to make a connection with the results of Oppenheimer and Volkoff [47]. In their model, the mass, the radius, and the central density of the critical configuration are

$$M_{\mathrm{OV}} = 0.376\left(\frac{\hbar c}{G}\right)^{3/2}\frac{1}{m_n^2} = 0.7\ M_\odot, \quad (109)$$

$$R_{\mathrm{OV}} = 9.36\frac{GM_{OV}}{c^2} = 9.6\ \mathrm{km}, \quad (110)$$

$$(\rho_c)_{\mathrm{OV}} = 3.92 \times 10^{-3}\frac{c^6}{G^3M_{OV}^2} = 5 \times 10^{15}\ \mathrm{g/cm^3}. \quad (111)$$

In our model, introducing the parameter λ, the maximum mass is given by

$$M_{\max} = 2\sigma_c v(\xi_1)\sqrt{\frac{\lambda}{8\pi}}\left(\frac{\hbar c}{G}\right)^{3/2}\frac{1}{m^2}. \quad (112)$$

If we write the boson mass as $m = km_n$, we obtain the maximum mass in the form

$$M_{\max} = C\frac{\sqrt{\lambda}}{k^2}M_{\mathrm{OV}}, \quad (113)$$

where $C = 0.265$ for a nonrelativistic equation of state and $C = 0.163$ for a relativistic equation of state. Using the previous relations, we can then easily relate $R_{\min}$ and $(\rho_c)_{\max}$ to R_{OV} and $(\rho_c)_{\mathrm{OV}}$. Equation (113) clearly shows that, with respect to the standard Oppenheimer-Volkoff model, we have an additional parameter λ (the strength of the self-interaction) which gives the possibility of obtaining higher values for the maximum mass. Using Eq. (80) which becomes $\kappa = 0.258\sqrt{\lambda}/k^2$, we can rewrite Eq. (113) as

$$M_{\max} = 3.87C\kappa M_{\mathrm{OV}}. \quad (114)$$

Presently, there is conclusive observational evidence from pulsar studies for the existence of neutron stars with masses significantly greater than 1.5 $M_\odot$ [58]. By using the Shapiro time delay to measure the inclination, the mass of PSR J1614-223048 was recently determined to be 1.97 ± 0.04 $M_\odot$ [59]. Moreover, a number of X-ray binaries seem to contain high-mass neutron stars: About 1.9 $M_\odot$ in the case of Vela X-1 and 2.4 $M_\odot$ in the case of 4U 1700-377 [58]. Even more intriguing is the case of the black widow pulsar B1957 + 20, with a best mass estimate of about 2.4 $M_\odot$ [60]. This system has both pulsar timing and optical light curve information. B1957 + 20 is located in an eclipsing binary system, consisting of the 1.6 ms pulsar in a nearly circular 9.17 h period orbit, and an extremely low mass companion, $M_c \approx 0.03\ M_\odot$. It is believed that irradiation of the companion by the pulsar strongly heats its cosmic environment to the point of ablation, leading to a cometlike tail, and a large cloud of plasma. The plasma cloud is responsible for the eclipsing. The pulsar is literally consuming its companion, hence the name black widow. The mass of the companion star has been reduced to a small fraction of its original mass [58]. On the other hand, a measured mass of 2.4 $M_\odot$ would be incompatible with hybrid star models containing *significant* proportions of exotic matter in the form of hyperons, some forms of Bose condensates, or quark matter [58].

However, the mass and radius of the 2–2.4 $M_\odot$ neutron stars perfectly fit the expected properties of a BEC star. For $\kappa \sim 3$, the mass of a typical general relativistic BEC star is of the order of two solar masses, with a radius of around 11 km. Therefore, we propose that the recently observed 2–2.4 $M_\odot$ mass neutron stars could be typical BEC stars.

A last comment may be in order. If we apply the same model (self-gravitating BEC with short-range interactions) to dark matter [32], and use the Newtonian approximation (which is valid in this context), the radius of a dark matter halo is given by Eq. (34), which can be rewritten

$$R = 1.746 \times 10^{-2}\left(\frac{a}{1\ \mathrm{fm}}\right)^{1/2}\left(\frac{m}{1\ \mathrm{eV}/c^2}\right)^{-3/2}\ \mathrm{kpc}. \quad (115)$$

We note that the radius R determines the ratio a/m^3 or λ/m^4. Estimating the radius of dark matter halos by $R = 10$ kpc, we obtain $m^3/a = 3.049 \times 10^{-6}$ $(\mathrm{eV}/c^2)^3/\mathrm{fm}$ and $m^4/\lambda = 23.94$ $(\mathrm{eV}/c^2)^4$ [34].

VIII. DISCUSSIONS AND FINAL REMARKS

In the present paper, we have proposed that the core of neutron stars is a superfluid in which the neutrons form equivalent of Cooper pairs, so that they act as bosons of mass $2m_n$. Therefore, once the Bose-Einstein condensation takes place, the neutron star should be modeled as a self-gravitating BEC star. In our approach, we also assume that the bosons have a self-interaction, described by a scattering length a. The basic properties of the gravitationally bounded BECs have been obtained in both Newtonian and general relativistic regimes. To obtain the physical characteristics of the system, we have used the Madelung representation in which condensates can be modeled by using the hydrodynamic Euler equations describing a gas whose density and pressure are related by a barotropic equation of state. For the study of the BEC we have adopted the Thomas-Fermi approximation, which is valid if the total number of particles N obeys the condition $N \gg R/\pi a$ [33,34,36], a condition which can be reformulated, with the use of the mass density of the condensate, as $\bar{\rho} \gg 3m/4\pi^2 R^2 a$. This restriction is obviously satisfied by condensates with densities of the same order as the nuclear density.

In the physically most interesting case, corresponding to a quartic nonlinearity term in the energy functional, the equation of state of the BEC is that of a polytrope with polytropic index $n = 1$. In this case, the radius and the mass of the Newtonian stellar condensate can be obtained in an exact form. By contrast, relativistic configurations must be constructed numerically. In a Bose-Einstein condensed neutron star, the mass m of the particles does not need to coincide with the neutron mass. For the mass of the condensed particles, we have used an effective value of the order of $m^* = 2m_n$. This value is justified by the high densities in the neutron star cores, where the process of Bose-Einstein condensation is most likely to occur via the formation of Cooper pairs. However, we have also explicitly presented the numerical values of the basic physical parameters of the stars for other values of the mass.

In the standard general relativistic theory of compact astrophysical objects (neutron stars and pulsars) one obtains first the equation of state of the dense neutron (or quark) matter, by using quantum field theoretical methods in a Newtonian framework (the effects of gravity are ignored when deriving the equation of state). Then, with the help of this equation of state, general relativistic models of stars are constructed, and the maximum mass and radius of the stars are obtained by numerically integrating the TOV equation. These procedures are described in great detail, for example, in Ref. [61]. In our paper, we have used a similar method. First, from quantum mechanical considerations we have obtained the equation of state of the condensate, which happens to be a polytrope of index $n = 1$ in the nonrelativistic limit [see Eq. (56)]. The difference between this equation of state and the other polytropic equations of state analyzed in the literature is that, in the BEC equation of state, all the constants are fixed from physical considerations (there are no free parameters in the model). Then, with the help of this equation of state $P = K\rho^2$ we have constructed independently both Newtonian and general relativistic stellar models. We have proposed that this nonrelativistic equation of state (derived from the Gross-Pitaevskii equation) could be relevant to describe neutron stars and pulsars (viewed as BEC stars) because neutrons are too massive to be relativistic. Of course, for these compact objects, gravitation must be treated with general relativity. We have also considered a relativistic equation of state (85) obtained phenomenologically from the scalar field theory of Colpi *et al.* [25]. We have found that these two equations of state give similar results (for what concerns orders of magnitudes).

General relativistic effects impose strong constraints on the maximum mass. In the framework of the general relativistic approach one must numerically integrate the structure equations of the star. In this way, we obtain a large class of stable astrophysical objects, whose basic parameters (mass and radius) depend on the particle mass m and scattering length a. Since the values of a and m are not well-known, this offers the possibility to obtain a maximum mass for neutron stars that is larger than the Oppenheimer-Volkoff limit of 0.7 $M_{\odot}$, and may be compatible with recent observational determinations of the masses of some neutron stars. This is possible because we have two new parameters in our model, the boson mass m and the scattering length a, which give additional freedom (although these parameters should be ultimately determined by fundamental physics). We have found that the maximum mass $M_{\max}(\kappa)$ of the condensate star, given by Eqs. (74) or (93), depends in fact on a *single* parameter $\kappa(a, m)$ which is proportional to the ratio a/m^3 (or, equivalently, to the ratio λ/m^4). Basically, this ratio comes from the constant K appearing in the equations of state (56) and (85). Since the radius $R_{\min}$ and the density $(\rho_c)_{\max}$ only depend on $M_{\max}$, all the physical properties of the BEC stars are determined by the parameter κ. Condensates with particle masses of the order of two neutron masses and scattering length of the order of 10–20 fm (corresponding to $\kappa \sim 3$–5) have maximum masses of the order of 2 $M_{\odot}$, minimum radii in the range of 10–20 km, and maximum central density $\rho_c \sim 0.3 - 0.1 \times 10^{16}$ g/cm^3 in the semi-relativistic and fully-relativistic treatments, respectively. On the other hand, for $a = 1$ fm, the maximum mass of the condensate varies between 0.4–0.7 $M_{\odot}$ for $m = 2m_n$ (corresponding to $\kappa = 1$), between 1–2 $M_{\odot}$ for $m = m_n$ (corresponding to $\kappa = 2.8$), and between 10–16 $M_{\odot}$ for $m = m_n/4$ (corresponding to $\kappa = 22$), a value which may correspond, for example, to the effective (density dependent) kaon mass m_K^* in the interior of neutron stars. Kaon condensation may provide an important example of Bose-Einstein type stellar condensate [13]. Attraction from nuclear matter could bring down the mass of the kaon to an effective value of $m_K^* \approx 200\ \mathrm{MeV}/c^2 \approx m_n/9.38$. Relativistic kaon condensates with kaon effective mass of the order of $m_K^* = m_n/10$ and scattering length $a = 1$ fm, leading to $\kappa = 89.44$ could have masses as high as 39 $M_{\odot}$, and radii of the order of 362 km [see Eqs. (92) and (93)]. On the other hand, smaller mass condensed particles can have significantly higher maximum relativistic masses. In addition, if we consider kaons with mass $m_K^* = m_n/10$ and scattering length $a = 10^8$ fm (corresponding to the order of magnitude of the values of a observed in terrestrial laboratory experiments [6–8]) we have $\kappa = 8.944 \times 10^5$. With the use of Eqs. (92) and (93), we obtain a maximum mass of $M = 4 \times 10^5\ M_{\odot}$ for the kaon condensate star with a radius $R = 4 \times 10^6$ km ($\sim R_{\odot}$), corresponding to a super-massive black hole. Thus, BEC stars, formed from small mass particles, may represent viable candidates for the super-massive "black holes" that reside at the galactic centers. Hence the Bose-Einstein condensation process in the early universe may have provided the seeds from which super-massive black holes were eventually formed through accretion of interstellar matter.

An alternative model of a condensed compact astrophysical object is represented by the boson stars [23–28], self-gravitating compact solitonic objects made up of bosonic fields. Noninteracting complex scalar fields were originally considered for the constituents composing boson stars. However, the resultant configurations are typically mini-boson stars, which have small size and mass. This result originates from the following specific feature of boson stars: The boson star is protected from gravitational collapse by the Heisenberg uncertainty principle, instead of the Pauli exclusion principle that applies to fermionic stars, and the characteristic length scale of the former is much smaller than that of the latter.

It has been shown that this situation can be dramatically changed by introducing self-interacting complex scalar fields. The self-interaction effectively generates a repulsive force and the maximum mass of stable relativistic boson stars can be enhanced up to a size of the order of ordinary fermionic stars [25,26]. However, the mass of the boson star made of noninteracting scalar fields depends on the mass $m_{\rm scal}$ of the scalar field, so that with an appropriate choice of $m_{\rm scal}$ (very small) even the mass of a mini-boson star can be as large as the mass of a boson star consisting of self-interacting particles [23]. This is the case, in particular, of axion stars that could account for the mass of MACHOs (between 0.3 and 0.8 $M_{\odot}$) if the axion mass is of order $10^{-10}\ \mathrm{eV}/c^2$ (see Fig. 1 of Ref. [27]). In the case of the repulsive scalar interaction, larger boson star masses can be achieved and the scalar field mass $m_{\rm scal}$ can be chosen larger. Based on these arguments, it has been proposed since the 1990s that boson stars (self-interacting or not) could produce the mass of neutron stars by using the Bose-Einstein condensation of the relativistic Klein-Gordon model (see the review [28]).

In the present paper we are considering "self-interacting" Bose-Einstein condensed systems in the Thomas-Fermi limit, which consists in neglecting the kinetic energy of the particles (or equivalently the quantum pressure). The star is stabilized by the pressure term generated by the presence of the self-interaction potential. There is a fundamental difference between the physical properties of the complex scalar field boson stars and the BEC stars considered in the present model. This difference is related to the rotational properties of these two classes of objects. It has been shown in Ref. [26] that a slow rotation of a boson star is impossible (the angular momentum is quantized). However, rapidly rotating boson stars exist in general relativity and they rotate differentially (not uniformly). The energy density of the star ρ vanishes at the axis of rotation due to the presence of the centrifugal forces. The nearby maximum of the energy density extends to a mass torus, thereby modeling, to some extent, also a relativistic accretion disk, which is thickened along the equator. These properties of the complex scalar field stars are a result of the assumption of the asymptotic flatness of the spacetime, which implies the existence of conserved quantities for solutions with a Killing vector field ξ^μ [26]. On the other hand, in the present model, the BEC star satisfies a polytropic equation of state, with polytropic index $n = 1$ (in the nonrelativistic limit). The slow rotation of Newtonian polytropic stars was investigated in detail in Ref. [40], and, as shown in Sec. IV B, the Lane-Emden equation can be solved exactly for a continuous range of angular velocities, starting with the static case. General relativistic barotropic stars can also have a slow rotation. Therefore, as opposed to the scalar field description, BEC stars can evolve continuously to a given rotational state. Moreover, for BEC stars, the ratio of the conserved angular momentum and of the particle number is not an integer, as it is in the case of the complex scalar field boson stars [26]. Finally, the general relativistic configurations of BEC stars do not have asymptotic flatness, but they are continuously matched to an exterior Schwarzschild geometry. It is expected, however, that in the strong coupling limit $\Lambda \gg 1$, the self-interacting boson stars behave as a fluid with an isotropic barotropic equation of state [25] so they should have a continuous range of rotational states like BEC stars. There may remain differences between BEC stars and complex scalar field boson stars due to the differences in the physical nature of the two types of stars, and they can show up in surface structure, radiation emission, interior vortices, etc.

There is another important physical difference between the present model (BEC star) and the scalar field model. In our model, the star consists of a fluid obtained by averaging microscopic quantities associated with its constituent particles, and the fluid laws can be obtained via a kinetic theory using microscopic models of the fluid particles, and of their interactions. The statistical averaging procedure depends on the particles' energy: in the nonrelativistic case ($E = p^2/2m$) it is based on the Schrödinger quantum mechanical description, while in the relativistic case ($E^2 = p^2c^2 + m^2c^4$) it is based on the Klein-Gordon equation. By contrast, a boson star, made of a scalar field, does not derive from an average. The star is not made of pointlike particles but is described by a field (self-interacting or not) of unspecified physical nature, assumed to obey the Klein-Gordon equation.[7] Its physical properties are not obtained by statistical averaging, and its thermodynamic properties (equation of state, for example) do not have a clear physical meaning, and cannot be interpreted unambiguously. Thus, from a conceptual point of view, a scalar field and a BEC fluid are very different physical systems.

We may argue that BEC star models are more physical and transparent than boson star models. Indeed, BEC stars consist of physical particles (Cooper paired neutrons, pions, kaons, quarks etc.) which may have well-known physical characteristics, and are described by the standard laws of nature, as we know them from Earth. The thermodynamic properties of the system are obtained by averaging over a quantum ensemble of self-interacting particles, obeying a well-known statistical distribution, and they are obtained in a transparent, clear, and physical way, starting from a second quantization procedure. There are no similar methods for determining the physical properties of the boson stars, where the nature of the scalar particle is largely unknown (Higgs bosons are not particularly good candidates as elementary constituents of the boson stars, and the form of the self-interaction potential is mostly obtained phenomenologically). By contrast, BECs are currently obtained in the terrestrial laboratories, and this gives a much better insight in their physical properties, as compared to the scalar field. However, since at relativistic energies, BECs satisfy the Klein-Gordon equation with a self-interaction term, the possibility to describe the relativistic BEC via a scalar field representation cannot be excluded *a priori*. Scalar fields are useful tools in many physical applications, and they are sometimes used as effective descriptions of usual standard thermodynamic fluids. Indeed, an effective description of fluid systems in terms of a scalar field was considered by several authors. For a minimally coupled real scalar field, it was shown that the energy-momentum tensor always has the structure of the energy-momentum tensor of a perfect fluid [63,64]. For spherically symmetric complex scalar fields (boson stars), the radial and tangential pressure are in general different ($P_r \neq P_\perp$), but the anisotropy drops to zero in the strong coupling limit [65,66]. This justifies the adoption of a fluid description, based on an isotropic barotropic equation of state, for relativistic scalar field systems with $\Lambda \gg 1$ [25].

Presently, the mass of the neutron stars can be determined very accurately, and many of them have masses in the range of 2–2.4 solar masses, which are very difficult to explain by the standard neutron matter models, including those with exotic matter like quarks. However, these mass values could be very easily explained by our model if neutron stars can be considered as BEC stars. In addition, the Bose-Einstein condensation model as formulated in this paper has a firm physical basis since its theoretical features have been extensively tested and observed in terrestrial laboratory experiments. Therefore, the possibility that the 2–2.4 solar mass neutron stars are BEC stars cannot be eliminated *a priori* in the favor of a more exotic model. Moreover, since our model is described by a polytropic equation of state, it admits both slow and rapid (general relativistic) rotation modes, which make BEC stars reasonable candidates for describing pulsar properties, which are known to have a superfluid core.

Bose-Einstein condensate stars could have a normal matter crust, since we expect that the condensation cannot take place at densities smaller than the nuclear density or quark deconfinement density. The presence of the thin crust increases the mass and the radius of the condensate star by a factor of 10% or 17%, respectively. Therefore, the presence of a neutron crust does not modify significantly the basic physical properties of the star. Distinguishing between BEC stars and "standard" neutron stars or other types of condensate or quark stars could be an extremely difficult observational task. Similarly to the case of quark stars [67,68], we suggest that high energy radiation processes from the surface of the condensate may provide some distinctive features allowing a clear differentiation of these different types of stellar objects.

In a very general approach, one may assume that the masses m of the particles forming the stellar type condensate are anisotropic, and they should be described by a mass tensor m_{ij}. Such anisotropic masses are known from condensed matter physics where they are encountered in effective mass calculations for electrons immersed in a band structure, in the case of excitons (electron-hole couples held together by the Coulomb attraction) and in BECs

[7]The physical interpretation of the field ϕ in the scalar field model is different from that of the wave function ψ in the BEC model. As a result, the energy density and the pressure of the scalar field are obtained directly from the energy-momentum tensor of the field, while in the BEC model they can be obtained in the standard thermodynamic way from the hydrodynamic representation in both the nonrelativistic [32–36] and relativistic cases [62].

for semiconductors. The doping structure of the semiconductor and its anisotropies would give place to an effective mass matrix for the paraexcitons (singlet excitons) at least in the low momentum approximation [5]. A different value for the effective mass m may considerably increase (or decrease) the total mass of the condensate.

A rotating BEC may exhibit a very complex internal structure and dynamics, mainly due to the presence of vortex lattices. The vortex lattices may evolve kinetically, with each vortex following the streamline of a quadrupolar flow. The quadrupolar distortions can lead to a disordering of the vortex lattice, and to an instability due to interparticle collisions, finite temperature effects or to the quadrupolar distortions induced by the external potential. On the other hand, due to the high neutrino emissivity, which is significantly enhanced due to the condensation, kaon condensate stars are very dark objects. Hence their observational detection may prove to be an extremely difficult task. The possible astrophysical/observational relevance of these processes will be considered in a future publication.

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