Discrete symmetries in covariant loop quantum gravity

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We study time-reversal and parity—on the physical manifold and in internal space—in covariant loop gravity. We consider a minor modification of the Holst action which makes it transform coherently under such transformations. The classical theory is not affected but the quantum theory is slightly different. In particular, the simplicity constraints are slightly modified and this restricts orientation flips in a spin foam to occur only across degenerate regions, thus reducing the sources of potential divergences.

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I. TIME REVERSAL IN TETRAD GRAVITY

Classically, the physics of gravity is equally well described by the Einstein-Hilbert action

$$S_{\rm EH}[g] = \frac{1}{2}\int \sqrt{-\det g}\, R d^4x, \tag{1}$$

where the gravitational field is the metric g, or by the tetrad action

$$S_T[e] = \int e^I \wedge e^J \wedge F^\star_{IJ}, \tag{2}$$

where the gravitational field is the tetrad 1-form e with components $e^I = e^I_\mu dx^\mu$ and F^{IJ} are the components of the curvature of the torsionless spin-connection $\omega = \omega[e]$ determined by the tetrad.[1] The relation between the two languages is of course $g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu$. These two actions, however, are not equivalent. This can be seen by performing an internal time-reversal operation

$$^{(\rm i)}Te^0 := -e^0, \qquad ^{(\rm i)}Te^i := e^i, \qquad i = 1, 2, 3. \tag{3}$$

Under this transformation, $S_{\rm EH}$ is clearly invariant as the metric $g = e^I e_I$ is not affected by this transformation, while S_T flips sign, $S_T[^{(\rm i)}Te] = -S_T[e]$. The difference becomes manifest by writing both actions in tensor notation and in terms of tetrads

$$S_{\rm EH}[e] = \frac{1}{2}\int |\det e| R[e] d^4x, \tag{4}$$

$$S_T[e] = \frac{1}{2}\int (\det e) R[e] d^4x. \tag{5}$$

They differ by the sign factor

$$s \equiv \mathrm{sgn}(\det e), \tag{6}$$

where for later convenience we define $\mathrm{sgn}(0) = 0$.

In loop quantum gravity, one utilizes the first-order formalism where the tetrad e and spin connection ω are treated as independent variables, and adds to the action the Holst term

$$S_H[e, \omega] = \frac{1}{\gamma}\int e^I \wedge e^J \wedge F_{IJ}, \tag{7}$$

which has no effect on the classical equations of motion. Here we take $\gamma > 0$. Thus, the action usually taken as the starting point for the definition of the quantum theory is

$$\begin{aligned} S[e, \omega] &= S_T[e, \omega] + S_H[e, \omega] \\ &= \int e^I \wedge e^J \wedge \left(F^\star_{IJ} + \frac{1}{\gamma} F_{IJ}\right) \\ &\equiv \int e^I \wedge e^J \wedge \left(\star + \frac{1}{\gamma}\right) F_{IJ}. \end{aligned} \tag{8}$$

Defining $^{(\rm i)}T\omega^{IJ}$ as (note that this is the same transformation as for $\omega[e]$ in the tetrad action)

$$^{(\rm i)}T\omega^{0i} := -\omega^{0i}, \qquad ^{(\rm i)}T\omega^{ij} := \omega^{ij}, \qquad i, j = 1, 2, 3, \tag{9}$$

we observe that the two terms in this action do not transform in the same way under an internal time reversal:

$$S[^{(\rm i)}Te, {}^{(\rm i)}T\omega] = -S_T[e, \omega] + S_H[e, \omega]. \tag{10}$$

That is, S does not transform coherently under $^{(\rm i)}T$, in spite of the fact that this transformation changes only the time orientation of the internal Minkowski space. Can we replace S with an action that transforms coherently? This can be done in two different manners: either modifying the first term, to have it behave as the Einstein-Hilbert action

$$S'[e, \omega] = \int e^I \wedge e^J \wedge \left(s\star + \frac{1}{\gamma}\right) F_{IJ}, \tag{11}$$

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[1] We use units where $8\pi G = 1$. Greek indices are space-time indices while capital latin indices denoting the four-dimensional internal space are raised and lowered with the Minkowski metric η_{IJ}. The star indicates the Hodge dual in Minkowski space: $F^\star_{IJ} \equiv {}^\star F_{IJ} := \frac{1}{2}\epsilon_{IJKL} F^{KL}$. See Ref. [1] for the rest of the notation.

[recall that s is defined in Eq. (6)] or modifying the Holst term obtaining an action that changes sign under internal time reversal:

$$S''[e,\omega] = \int e^I \wedge e^J \wedge \left(\star + \frac{s}{\gamma}\right) F_{IJ}. \qquad (12)$$

In this paper we explore the consequences of both of these corrections upon quantization. We build on the recent work of Yasha Neiman [2] and Jon Engle [3], but also on Refs. [4,5] where these internal discrete symmetries have been studied in the context of spin foams.

Before closing this section, we add a few comments.

(1) *Other definitions of time reversal.* There also exists a time-reversal transformation that acts on the manifold (considered in the spin foam context in Ref. [6]), defined by

$$^{(m)}Te_a^I := e_a^I, \qquad ^{(m)}Te_t^I := -e_t^I, \qquad a = 1,2,3, \qquad (13)$$

and the "total" time-reversal transformation $T = {}^{(i)}T^{(m)}T$. T is the time-reversal symmetry mostly considered in the literature. $S_{\rm EH}$, S_T are both even under T, while the Holst term is odd. Note also that $Ts = s$.

(2) *Orientation.* Alternatively, ${}^{(i)}T$ and ${}^{(m)}T$ can be defined by leaving the fields untouched and flipping the orientation of the internal Minkowski space and the spacetime manifold, respectively. A change of the orientation flips the sign of the normalization of the completely antisymmetric Levi-Civita symbols. Thus, ${}^{(i)}T$ changes the sign of s and of the Hodge operator $\star$, while ${}^{(m)}T$ changes the sign of s and of the integral of a 4-form. It is easy to check that these definitions are equivalent to Eqs. (3) and (13), respectively. Then T corresponds to reversing the orientation of the Minkowski space and the manifold simultaneously.

(3) *Parity.* We have formulated the issue above in terms of time reversal, but it is similarly possible to do so in terms of parity. Define

$$^{(i)}Pe^i := -e^i, \qquad ^{(i)}Pe^0 := e^0, \qquad i = 1,2,3. \qquad (14)$$

Since all actions are invariant under ${}^{(i)}P^{(i)}Te = -e$, it is clear that we have the same structure with internal parity transformations as we had in terms of ${}^{(i)}T$. We also have the total parity transformation defined by $P = {}^{(m)}P^{(i)}P$, where ${}^{(m)}P$ is defined analogously to (13) and (14). The Holst term changes by a sign under P, and is invariant under PT.

(4) *The Ashtekar Electric Field.* In canonical loop gravity one works in the time gauge and chooses a linear combination of the connection and its Hodge dual as a canonical variable. The corresponding conjugate momentum is the Ashtekar electric field E^{ai}, but (confusingly) one finds two different expressions for this field in the literature [7,8]:

$$E^{ai} = |\det e| e^{ai}, \quad \text{or} \quad E^{ai} = \epsilon^{abc}\epsilon^i_{jk} e^j_b e^k_c. \qquad (15)$$

The two expressions differ by the sign s and can be derived from S' and S'', respectively.

II. MODIFIED SIMPLICITY CONSTRAINT

We now explore the effect of taking S' or S'' instead of S as the starting point for deriving the covariant dynamics of loop quantum gravity. We begin from the effect on the canonical structure. In this section we restrict the analysis to the regions where $s \neq 0$: we analyze the regions where $s = 0$, namely where the determinant of the metric vanishes, at the end of the section.

Working on a three-dimensional Cauchy surface Σ, the momentum conjugate to the connection ω is the boundary 1-form with values in the $sl(2,\mathbb{C})$ algebra

$$\pi^{IJ} = \left(s\star + \frac{1}{\gamma}\right)(e^I \wedge e^J)|_\Sigma \qquad (16)$$

for S' and

$$\pi^{IJ} = \left(\star + \frac{s}{\gamma}\right)(e^I \wedge e^J)|_\Sigma \qquad (17)$$

for S''. The tetrad e maps the normal 1-form to the boundary surface to a (timelike) Minkowski vector n_I, which allows us to split this momentum into its electric $K^I = \pi^{IJ} n_J$ and magnetic components $L^I = -\star\pi^{IJ} n_J$. Since clearly $n_I e^I|_\Sigma = 0$, one of the two terms vanishes in each component, leaving

$$K^I = s n_J (e^I \wedge e^J)^\star|_\Sigma, \quad L^I = -\frac{1}{\gamma} n_J (e^I \wedge e^J)^\star|_\Sigma \qquad (18)$$

in the first case, and

$$K^I = n_J (e^I \wedge e^J)^\star|_\Sigma, \quad L^I = -\frac{s}{\gamma} n_J (e^I \wedge e^J)^\star|_\Sigma \qquad (19)$$

in the second. K^I and L^I are normal to n_I and live therefore in a 3d space (oriented by the n and the orientation of $\mathcal{M}$). We use the notation $\vec{K} = \{K^i, i = 1,2,3\}$ to indicate them. For S', Eq. (18) implies

$$\boxed{\vec{K} + s\gamma\vec{L} = 0} \qquad (20)$$

while for S'', Eq. (19) gives

$$\boxed{s\vec{K} + \gamma\vec{L} = 0} \qquad (21)$$

which is equivalent to (20) since $s = \pm 1$. This is the modified linear simplicity constraint for the actions S' and S''.[2]

$\vec{K}$ and $\vec{L}$ are 2-forms on the oriented 3d space Σ, that is, for instance $K^i = K^i_{ab} dx^a \wedge dx^b$. Therefore they define 3×3 matrices, like $K^{ci} := K^i_{ab}\epsilon^{abc}$, whose determinant we indicate, respectively, as $\det K$ and $\det L$. Let $s_K := \mathrm{sgn}[\det K]$; and $s_L := \mathrm{sgn}[\det L]$. Since in these coordinates we have $e^0|_\Sigma = 0$, one can check easily that

$$s = s_K, \qquad s_L = 1 \tag{22}$$

in the S' case,[3] while in S''

$$s = s_L, \qquad s_K = 1. \tag{23}$$

So far we have only considered the nondegenerate case. The degenerate case occurs when $\vec{K} = \vec{L} = 0$ and therefore $s_K = s_L = 0$ as well. For the degenerate sector, the simplicity constraints have the same form for the two actions S' and S'' and are that both $\vec{K}$ and $\vec{L}$ must vanish.

III. DISCRETIZATION

As a step towards the quantum theory, consider the discretization of the theory. Introduce an (oriented) triangulation of space-time and integrate the 2-forms π over two-cells f in the triangulation. This associates the variable

$$\pi^{IJ}_f = \int_f \pi^{IJ} \tag{24}$$

to each f. Consider one such face f sitting on the boundary of the manifold. With respect to the frame defined by n_I, determined by the normal to the boundary, this momentum splits into its electric and magnetic components $\vec{K}_f$ and $\vec{L}_f$. Consider a three-cell in Σ and let f_i, $i = 1, 2, 3$ be three of its four faces, ordered according to the orientation of the manifold. Define

$$\det L := \vec{L}_{n_1} \cdot \vec{L}_{n_2} \times \vec{L}_{n_3}, \tag{25}$$

which is the discrete analog of the determinant of $\vec{L}$ in the continuum,[4] and the sign

$$s_L := \mathrm{sgn}[\det L]. \tag{26}$$

We see that $\vec{L}$ and $\vec{K}$ live on faces in the discretized theory while s_L and s_K are associated to tetrahedra.

With these definitions, it is possible to rewrite the simplicity constraints in the discrete theory for the nondegenerate case.

For S', we must have $s_L = 1$. Also, the constraint (20) becomes

$$K_f \pm \gamma L_f = 0, \tag{27}$$

as both values of $s_K = \pm 1$ are allowed. The important point is that, since there is one s_K per tetrahedron, one of ± 1 must be chosen for all of the four faces that compose each tetrahedron.

For S'', we have $s_L = \pm 1$ in the nondegenerate sector, and the constraint (21) is

$$K_f + s_L \gamma L_f = 0, \tag{28}$$

which implies that $s_K = 1$.

Now let us consider the degenerate cases. A degenerate tetrahedron is one where $s_K = s_L = 0$, while a degenerate face is one where $K_f = L_f = 0$. Note that a tetrahedron can be degenerate without its faces being degenerate (and vice versa) and therefore a degenerate tetrahedron cannot constrain its faces.

Finally, the oriented square volume V^2 of a three-cell is determined by [1]

$$V^2 = \frac{2}{9}\gamma^3 \vec{L}_{n_1} \cdot \vec{L}_{n_2} \times \vec{L}_{n_3}, \tag{29}$$

which gives the important relation

$$s_L = \mathrm{sgn}(V^2). \tag{30}$$

IV. QUANTUM THEORY

Let us now study the effect of using the modified simplicity condition on the quantum theory. We refer the readers to Refs. [1,9,11,12] for the general construction.

In the quantum theory, π^{IJ}_f is promoted to a quantum operator which is identified as the generator of $SL(2,\mathbb{C})$ over a suitable space formed by $SL(2,\mathbb{C})$ unitary representations. $\vec{K}_f$ and $\vec{L}_f$ are then the generators of boosts and rotations respectively. The unitary representations of $SL(2,\mathbb{C})$ are labelled by the two quantum numbers ρ and k, where $\rho \in \mathbb{R}^+$ and $2k \in \mathbb{Z}$. A discrete basis in the (ρ, k) representation is obtained by diagonalizing the total angular momentum $|\vec{L}|^2$ of the rotation subgroup of $SL(2,\mathbb{C})$ and its L_3 component. The basis vectors are then denoted by $|\rho, k; j, m\rangle$, where j is a half-integer greater or equal to $|k|$ while m is a half-integer in the interval of $[-j, j]$. The Casimirs of $SL(2,\mathbb{C})$ are $C_1 = \vec{L} \cdot \vec{K}$ and $C_2 = |\vec{L}|^2 - |\vec{K}|^2$ and take the values $C_1 = \rho k$ and $C_2 = k^2 - \rho^2$ in the (ρ, k) representation. If the quantum operators K^i_f and L^i_f

[2] The conventional action S gives the linear simplicity constraint $\vec{K} + \gamma\vec{L} = 0$ instead [9,10]. But note that, consistently with what we find here, a negative sign is obtained in Ref. [2].

[3] In this case, $s_L = 1$ implies that $\vec{L}$ is a pseudovector with respect to ${}^{(i)}T$ and ${}^{(i)}P$, since it does not change sign under parity and time reversal, while $\vec{K}$ is a proper vector as its determinant can be positive or negative. In the quantum theory $\vec{L}$ generates rotations and $\vec{K}$ boosts, thus S' appears to better respect the expected transformation properties of $\vec{L}$ and $\vec{K}$.

[4] Note that it is possible to choose any three of the four edges meeting at the edge (so long as the relative orientation is taken into account) due to the closure constraint on $\vec{L}$ (which is equivalent to the Gauss constraint in the quantum theory). There is no closure constraint on $\vec{K}$ in the quantum theory without the simplicity constraint and therefore defining $\det K$ in the discrete theory is not useful for spin foam models.

are defined on the representation (ρ_f, k_f) and satisfy the modified simplicity constraint (27) or (28), the states in the quantum theory must therefore satisfy the relations (see Ref. [9] for the details of this procedure)

$$\rho_f = \gamma j_f; \qquad k_f = s j_f, \tag{31}$$

where s is a sign coming from (27) or (28). As $j > 0$, this relation determines the sign of the quantum number k, which in the literature was usually taken to be positive (although not in Ref. [2]). Therefore the key effect of the introduction of the sign s is that the quantum theory now includes both positive and negative k representations.

Thus, given j_f and s, it is possible to determine ρ_f and k_f. As one can easily check from (27) and (28), it is necessary to know s_L in order to implement the simplicity constraints and therefore, one must calculate s_L for each edge. In order to do this, we first diagonalize the state with respect to the operator corresponding to V^2 for each edge, which is equivalent to diagonalizing the states with respect to the s_L operator given by (30). This determines s_L for every edge and in the next section, we show how to use this in order to implement the simplicity constraints in the vertex amplitude.

V. AMPLITUDE

Let us now see the effect of the above on the amplitude that defines the quantum theory [1]. We start by recalling the usual form of the covariant loop quantum gravity amplitude [9,10,13–17]. Among the numerous equivalent manners of writing this amplitude, we choose the "Polish" one [18]: Let Δ be a 2-complex with faces f, edges e and vertices v. For simplicity we assume here that Δ is the dual of a four-dimensional triangulation and without boundaries. The amplitude associated to this triangulation is

$$A_\Delta = \sum_{j_f} \mu(j_f)\, \mathrm{Tr}_\Delta \prod_{\mathrm{e}} P_{\mathrm{e}}. \tag{32}$$

Here the half-integer j_f is the assignment of a spin to each face, $\mu(j_f) = \prod_f (2j_f + 1)$ is a measure factor and the operators P_e are defined on the space

$$H_e = \otimes_{f\in e} H_f, \tag{33}$$

where H_f is the Hilbert space carrying the $SL(2,\mathbb{C})$ representation $(\rho, k) = (\gamma j_f, j_f)$. The trace is obtained by tracing over all Hilbert spaces H_f at couples of edges sharing the same face at the vertices. The model is then defined by

$$P_e = P_g P_h P_g, \tag{34}$$

where P_g is the projection on the $SL(2,\mathbb{C})$ invariant subspace of H_e (the intertwiner space), and $P_h = (\otimes_{f\in e} P_h^f)$ is the projection on the $SU(2)$ invariant substance of H_f with $SU(2)$ spin j_f. This defines covariant loop quantum gravity. This is the amplitude that has been shown to be related to the general relativity action in the large distance limit [4,19–21].

Let us now define the variant of the theory that takes the orientation into account. The first step is to introduce the projectors $P(s_L = 0, \pm 1)$. The projector $P(s_L = 0)$ annihilates all states with a nonzero volume, while $P(s_L = 1)$ and $P(s_L = -1)$ project onto the subspaces where the oriented square volume is positive and negative, respectively.

The next step is to determine the relation between s_K and s_L in the $SU(2)$ invariant subspace of an $SL(2,\mathbb{C})$ representation. From the definitions (26) and by looking at the action of the operators K_i and L_i in $SL(2,\mathbb{C})$ given for instance in Ref. [12], it is easy to derive that

$$s_K = \mathrm{sgn}(k_{f_1} k_{f_2} k_{f_3}) s_L, \tag{35}$$

where the signs of k_{f_1}, k_{f_2}, k_{f_3} and k_{f_4} are all the same, as can be seen from the discretized simplicity constraints (27) and (28). An important consequence of this relation is that for a state where $s_L = 0$, the relation $s_K = 0$ also necessarily holds in the $SU(2)$ invariant subspace.

A. The amplitude for S'

To define the quantum theory for the action S' we have to change the above definition in order to implement two modifications: (i) k_f should be allowed positive as well as negative, and (ii) s_L, which is equal to the sign of V^2, must be positive. These are easily implanted by defining the amplitude[5]

$$A'_\Delta = \sum_{k_f} \mu(j_f)\, \mathrm{Tr}_\Delta \prod_{\mathrm{e}} P'_{\mathrm{e}}, \tag{36}$$

where $j_f = |k_f|$, the operators P'_e are defined on the Hilbert space

$$H'_e = \otimes_{f\in e} H'_f, \tag{37}$$

where H'_f is the Hilbert space carrying the $SL(2,\mathbb{C})$ representation $(\rho, k) = (\gamma |k_f|, k_f)$. The P'_e operator is defined by

$$P'_e = P_g P_h P'_s P_h P_g, \tag{38}$$

where P'_s is the additional projector defined as follows:

$$P'_s = P(s_L = 1) \times \prod_{f_1, f_2 \in e} \delta(\mathrm{sgn}(k_{f_1}), \mathrm{sgn}(k_{f_2})) + P(s_L = 0), \tag{39}$$

[5]We can restrict the sum to be over nonzero k_f. Even though degenerate faces $k_f = j_f = 0$ are allowed by the simplicity constraints, we know from canonical loop quantum gravity that links with $j = 0$ can be erased from the spin network. The same will be done for the amplitude of the action S'' as well.

where the Kronecker delta imposes the signs of all of the k_f meeting at a nondegenerate edge to agree (there is no such constraint for faces meeting at a degenerate edge).

Since P_h projects on j_f, we have immediately that $k_j = s j_f$, where $s \equiv \mathrm{sgn}(k_f)$ which is the same no matter which face is chosen due to the Kronecker delta. It is easy to see from Eq. (35) that s_K can be positive or negative, as wanted.

Notice that on the one hand the states in the sum have doubled because k_f can take both signs, but on the other hand they are halved as all states with $s_L = -1$ are killed.

Therefore, for S' it will be necessary to work with Hilbert spaces carrying the representations ($\rho_f = \gamma j_f$, $k_f = j_f$) and ($\rho_f = \gamma j_f$, $k_f = -j_f$). Note that only the first is considered in the usual Engle-Pereira-Rovelli-Livine (EPRL) model [9], though in that case there is no projector P'_s. (We shall see that for S'' these two representations will again be needed although the extra projector P''_s is different.)

We close with a brief discussion of the gluing conditions. By looking at the vertex amplitude and in particular the form of the projector P'_s, it is easy to see that it is impossible to connect two nondegenerate edges with opposite values of s_K. However, it is possible to connect degenerate edges with any other type of edge. Therefore, regions with an opposite sign of s_K can only be connected by passing through a "boundary region" composed of degenerate edges.

B. The amplitude for S''

In this case, we have to change the vertex amplitude in order to implement the following two modifications: (i) k_f should be allowed positive as well as negative, and (ii) s_K must be positive. This can be obtained by defining

$$A''_\Delta = \sum_{k_f} \mu(j_f)\, \mathrm{Tr}_\Delta \prod_{\mathrm{e}} P''_{\mathrm{e}}, \tag{40}$$

where the operators P''_e are defined on the same Hilbert space as P'_e and the P''_e operator is

$$P''_e = P_g P_h P''_s P_h P_g, \tag{41}$$

where the new projector P''_s is defined by

$$P''_s = P(s_L = 0) + P(s_L = 1) \times \prod_{f\in e} \delta(k_f, j_f) + P(s_L = -1) \times \prod_{f\in e} \delta(k_f, -j_f). \tag{42}$$

Now there is no restriction regarding the sign of the oriented volume squared operator, but the simplicity constraint (28) must be imposed. Due to the relation (35), it follows that $s_K \neq -1$ follows automatically.

Once again, it is easy to see that two regions where the s_L have opposite signs cannot be glued together directly. Instead, it is necessary to pass through a degenerate edge in order to travel from a region with $s_L = 1$ to another where $s_L = -1$.

Thus, just as for S', there must be a "bridge" of degenerate edges between regions with opposite signs of s. In particular, in a connected, nondegenerate region, we must have constant s everywhere. This is not particularly surprising as it is very similar to what we find in the classical, continuous theory: the relative orientation between the physical manifold and the internal space can only change at singularities.

C. Comparison to similar results

In Ref. [2], the simplicity constraint was modified from $\vec{K} + \gamma\vec{L} = 0$ to $\vec{K} - \gamma\vec{L} = 0$. This corresponds to the case here when $s = -1$. In the model proposed in Ref. [2], the value of s can still flip from one cell to the next without restriction and also the simplicity constraint is not affected by the value of s. On the other hand, in the models presented here, the value of s can only change across a degenerate region and then this change plays a role in the simplicity constraints (21).

Reference [3] suggests a modification of the quantum theory very similar to (36). However, the two differences are: (i) the additional projector introduced in Ref. [3] (in the Euclidean setting) is in fact $P(s_L = 1)$ as it only allows states where $V_e^2 > 0$ (note that $V_e^2 = 0$ is not allowed, another difference with the prescription we give here) and (ii) there is no sum over both signs for k_f. Thus the prescription given in Ref. [3] is different as it imposes $s_K = 1$ (in addition to $s_L = 1$), while $s_K = \pm 1$ are both allowed configurations for the amplitude coming from S' presented in this paper.[6]

VI. ANALYSIS

The first key consequence of the alternate definitions of the vertex amplitude given above is that in every connected nondegenerate region, i.e., where $V_e^2 \neq 0$, the sign of s remains constant. As is clear from Sec. I, s indicates the relative orientation between the physical manifold $\mathcal{M}$ and the auxiliary Minkowski space M and thus this result

[6]The problem raised by the sign of s is related to several sign issues that have been discussed in the quantum gravity literature. See for instance the analysis of causality in spin foams in Ref. [22]; the restriction to positive physical-time energy in the reconstruction of the spin foam formalism from loop cosmology [23]; the need to select a phase picking up one component of the amplitude in reconstructing semiclassical transition amplitudes [24–26]; the interpretation of the early versions of the bounce loop cosmology [27]; the analysis of parity in the Bianchi models [28]; and in certain subtle and controversial points of the canonical quantization [29]; and the effect of orientation flip in gluing, for spin foam amplitudes [21]. In fact, uncertainties about the physical interpretation of this sign factor have been present since the very early calculations in loop gravity [30].

indicates that the relative orientation cannot flip without going through a degenerate region in the triangulation.

In the asymptotic analysis in Ref. [19], one finds a sum of two terms in the semiclassical limit of the amplitude (32),

$$\lim_{\text{semi-class}} A_\Delta \sim e^{iS_R} + e^{-iS_R}, \tag{43}$$

where S_R is the Regge action. These two terms correspond to the two possible relative orientations between $\mathcal{M}$ and M. In the EPRL model, one of these two terms appears for each edge, depending upon the relative orientation chosen at that particular edge. Neighbouring edges do not need to be glued consistently and therefore a mixing occurs between the two terms in the semiclassical limit. This mixing is directly responsible for some of the divergences in the semiclassical limit [31].

The vertex amplitude studied here behaves differently. In the case S', one might hope that the restriction to the positive eigenspaces of V^2 selects only one of the two sectors in the saddle point approximation of the vertex [4,19–21,32], leading to just one critical point instead of two, as the theory S' is essentially the Einstein-Hilbert one where the sign of the action of a time-reversed configuration does not flip. However, a new critical point might be picked up corresponding to $s_K = -1$, a configuration that was not included in previous studies. This critical point would give the same contribution as the surviving critical point corresponding to $s_L = s_K = 1$ and then the asymptotics would have the form

$$\lim_{\text{semi-class}} A'_\Delta \sim 2e^{-iS_R}. \tag{44}$$

If so, the action S' would realize the objective sought for in Ref. [3] (in fact, in a very similar manner), but in a context in which both orientations exist in the theory.

Considering the properties of S'' under the discrete transformations, we expect both terms to appear (as in the EPRL model)

$$\lim_{\text{semi-class}} A''_\Delta \sim e^{iS_R} + e^{-iS_R}, \tag{45}$$

but there is an important difference between the vertex amplitudes as now each of the terms in the asymptotic analysis corresponds to a connected, nondegenerate region, rather than the orientation of each cell. In other words, the weight associated to connected regions with nondegenerate configurations should be the cosine of the total Regge action, and not the product of the individual cosines of the Regge action of each cell. This is because connected edges must have the same relative orientation between $\mathcal{M}$ and M, unless they are separated by a degenerate region.

Therefore we still have both orientations playing a role (both in S' and S''), but not cell by cell. Instead, this would occur patch by patch, on the basis of connected nondegenerate regions (see Ref. [4]).

VII. DISCUSSION

Based on the behavior of the Holst action under internal time reversal and parity transformations, we have considered two distinct modifications to the Holst action which lead to the modified actions S' and S''. In both cases, the simplicity constraints are slightly changed and the spin foam quantization of the actions is a little different from the conventional one defined for instance in Ref. [1]. These modifications might reduce one of the sources of divergences in the semiclassical limit, as the relative orientation can flip only across degenerate regions, thus removing some of the problematic mixing terms [31].

The alternative between the actions S' and S'' reflects the alternative between the Einstein-Hilbert action S_{EH} and the tetrad action S_T. In the classical theory the choice does not matter, but the two actions appear to lead to inequivalent quantum theories. In a Feynman sum-over-histories approach, summing over tetrads with both signs of the determinant in S_T is like considering each metric spacetime in S_{EH} twice: once future-oriented and once past-oriented, and weight the two with two opposite signs of the action. The cosine rather than the exponential that appears in the Ponzano-Regge asymptotics can be interpreted as having this origin. In the theory defined by S' both the future- and past-oriented configurations are summed over, just as in S'', but they are weighted with the same sign. Is there a reason to prefer one action rather than the other?

One argument in favour of S_{EH} and S' is the consideration that under internal time-reversal and parity transformations, the generators of boosts transform as proper internal vectors ($\vec{K} \to -\vec{K}$) while the generators of rotations transform as pseudo internal vectors ($\vec{L} \to \vec{L}$) in S', as one would expect on geometric grounds. The situation is reversed in S'' where $\vec{K}$ transforms as a pseudo internal vector and $\vec{L}$ as a proper internal vector.

One argument by analogy in favour of S_T and S'', on the other hand, is the fact that in nonrelativistic physics the action of a trajectory moving backward in time and that of the same trajectory going forward have opposite signs. The action for a process is $S = E\Delta T$, and if ΔT changes sign, so does S. This property is lost in S_{EH} because of general covariance, which implies that there is no way of distinguishing a forward moving spacetime from backward moving one. But it is present in S_T and S'' as they depend on the sign of s.

We close with a comment on the interpretation of regions with opposite s. In Feynman's picture one obtains quantum amplitudes summing over the particle's paths in space. The idea that in this context particles running backward in time represent antiparticles forms the intuitive basis of the Stückelberg-Feynman form of positron theory [33,34]. According to a beautiful argument given by Feynman in Ref. [35], special relativity requires such particles running back in time to exist, if the energy must be positive. This is because positive energy propagation spills necessarily outside the light cone. But a propagation of this kind is spacelike and therefore can be reinterpreted as backward in time in a different Lorentz frame. Therefore there must exist propagation backward in time in the theory and this represents a (forward propagating) antiparticle. Thus, according to Feynman, the existence of antiparticles follows directly from quantum mechanics and special relativity. Can an analogous argument be formulated in quantum gravity?

Consider a gas of particles in space-time used to define a physical comoving coordinate system. These define a time function with respect to which the gravitational field can be seen as evolving. In the quantum theory, however, the gravitational field can fluctuate off shell so that the trajectories are somewhere spacelike. But then there is a coordinatization of space-time with respect to which the particles run backward in time. In turn, the metric in this coordinatization runs backwards in time with respect to the time defined by the physical reference field. In other words, we are again in the situation where a solution running backward in time must be included in the path integral. These are only speculative remarks, but they suggest that the contribution of the tetrad fields with negative determinant—negative internal time—should perhaps not be dismissed lightly *a priori*.

Can this intuition be relevant for the dynamics of space-time itself and shed some light on the physical interpretation of a region with a flipped internal time direction? Can a region with the opposite internal time direction be thought of as a space-time running backward in time, or an "anti-space-time"?

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