

Is there any coherent measure for eternal inflation?

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An eternally inflating universe produces an infinite amount of spatial volume, so every possible event happens an infinite number of times, and it is impossible to define probabilities in terms of frequencies. This problem is usually addressed by means of a measure, which regulates the infinities and produces meaningful predictions. I argue that any measure should obey certain general axioms, but then give a simple toy model in which one can prove that no measure obeying the axioms exists. In certain cases of eternal inflation there are measures that obey the axioms, but all such measures appear to be unacceptable for other reasons. Thus the problem of defining sensible probabilities in eternal inflation seems not to be solved.

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I. INTRODUCTION

Inflation is generically eternal. While nearly every co-moving location in an inflating universe rapidly thermalizes, those that do not are rewarded with continuing exponential expansion. Except for certain carefully arranged models, the physical volume of inflating space grows forever, leading in turn to an infinite volume in thermalized regions like ours.

Such a universe makes it difficult to make predictions. There are an infinite number of regions just like ours. An infinite subset of these will have any possible future. So in what sense can we say that one future outcome is more likely than another?

In a finite universe, even a very large one, anthropic ideas such as Vilenkin's "principle of mediocrity" [1] or Bostrom's "self-sampling assumption" [2] tell us to consider ourselves a randomly chosen member of some finite *reference class* of observers. But such an approach does not work with an infinite class of indistinguishable regions. There is no probability distribution that assigns equal weight to each member of a countably infinite set, so we cannot imagine ourselves to be drawn from such a distribution. Instead, we must use a *measure*.

Many different measures have been suggested, but I will not attempt to discuss the possibilities here. Instead I will define the idea of a measure by the job it has to do. Suppose that we know the content of the entire multiverse. We also have a set of observations about conditions here and in our past light cone. Given these observations, we can enumerate the places in the multiverse that we might be. Unfortunately, there are infinitely many of them. These different places have various possible futures, say, F_1, F_2, \dots . A measure is then a function that takes the entire multiverse and our own current observations and returns a probability for each possible future observation.

In principle, any such function is a measure. But we want our measure to have some degree of coherence; that is to say, its probabilities should not be completely arbitrary. The results of applying the measure should have some

correspondence to ordinary ideas of probability. A measure should not, for example, tell me one thing based on my observations up to this moment and then without reason change to something utterly different given my observations one second from now. In Sec. II, I attempt to formalize this idea by proposing two minimal axioms that I think any reasonable candidate measure should obey.

Unfortunately, it turns out that these seemingly reasonable axioms are not easily obeyed. The difficulty is closely related to the problems discussed in Refs. [3–5]. In Sec. III, I describe a simple toy model of an infinite universe and show that in that model no measure can satisfy both the axioms. In Sec. IV, I discuss more specifically why global cutoff measures do not satisfy the axioms, unless one treats the cutoff as a true "end of time" [3] that renders the universe finite.

In Sec. V, I compare the toy model with more realistic models of eternal inflation, and find that in certain cases eternal inflation does admit measures that satisfy the axioms, but these measures are undesirable for other reasons and fail if one allows bubble collisions. In Sec. VI, I discuss the consequences of weakening the requirement that the measure give an observer the same probabilities after some time has passed without the observer acquiring new information, but find that quite anomalous situations can then arise. I conclude in Sec. VII.

II. AXIOMS

I now propose two axioms that I think any measure should satisfy in order to have the resulting probabilities conform to our understanding of how probabilities should behave.

Axiom 1a (Bayes's Rule—general case) When we make new observations, the measure should now yield the same probability distribution as we would get by Bayesian updating of the old probability distribution using the new observations.

Specifically, suppose that the measure tells you that the probability of some certain situation is $p(A)$. Now we observe some process whose outcome is determined (independently for each identical region) by chance. In case A , the chance to make some observation O is $p(O|A)$. If not A , that chance is $p(O|-A)$. If we observe O , Bayes's rule will tell us that the chance to be in situation A is now

$$p(A|O) = \frac{p(O|A)p(A)}{p(O|A)p(A) + p(O|-A)p(-A)}. \quad (1)$$

Applying the measure to our new state of knowledge should give this same value for $p(A)$.

In fact, all we will need below is the trivial case of Bayesian updating where you do not make any new observations:

Axiom 1b (Bayes's Rule—trivial case) If time passes, but you receive no other new information, your probabilities should remain unchanged.

But could knowing that time has passed provide new information? Indeed it could, if there were other alternatives. For a mundane example, suppose that you are about to have heart surgery. The surgeon tells you there is a 50% chance that you have a minor problem which will be easily repaired. But there is also a 50% chance that you have a very severe problem. The surgeon will try to repair it, but your chance of surviving the surgery is quite small. Now you're anesthetized for the surgery. If you wake up in the recovery room, you will consider it very likely that you had only the minor problem.

In this case, O is the observation that you are still alive. The observation $-O$, that you are dead, can never be made, but that does not matter. You should still update your probabilities according to Eq. (1), with A representing the minor problem and $-A$ the severe one.

A more esoteric idea is that in certain circumstances you could be duplicated. Suppose that at 1:00 you flip a fair coin but do not examine it. You now think that the chance of heads is $1/2$. Now suppose that at 1:10, if the coin is heads, there will be an exact duplicate made of you (in a duplicate room, with a duplicate coin, etc.) If the coin is tails, nothing special will happen.

You wait until after 1:10. Now I claim you should think the probability of heads to be $2/3$, since in the case of heads there are 2 copies of you sitting in rooms with coins showing heads, whereas in the case of tails there is only one. Again the passage of time has given you new information, in this case about the possibility of duplication.¹

The important issue here is the following. In the first example, your position is changed not simply by the passage of time but by the possibility of a different outcome, in

particular that your worldline might have come to an end before the second observation. The second example is the reverse. Your worldline might have come into existence at 1:10. In that case, you might also wish to readjust your probabilities. Thus we can update our axiom to exclude these possibilities:

Axiom 1 (Bayes's Rule—final version) If the measure gives you a certain set of probabilities at time t_1 , you receive no new information between t_1 and some later time t_2 , and every worldline that includes either time includes both, then the measure should give you the same probabilities at t_2 .

“Receiving new information” here means making an observation that could have occurred differently (or not at all). The passage of time might be accompanied by other events that you knew would take place. The fact that these events do take place as expected does not give you new information. Even though your circumstances are different after seeing them, they do not constitute new information because you knew that you would observe them, and thus they do not give a reason for you to update your probabilities.

We will need one more axiom:

Axiom 2a (Consistency—general case) If two or more observers are together and share their knowledge, they should agree on the chance of future events.

The idea here is that the measure is supposed to give probabilities of future events. If the event is to be observable by multiple observers, the measure should not tell them to expect its occurrence with different probabilities, unless some observers have relevant knowledge that others lack.

In fact, we will need this only in the following more limited form.

Axiom 2 (Consistency—final version) If two or more indistinguishable observers are together, they should agree on the chance of mutually observable events.

Since the observers are indistinguishable, the measure will automatically supply the same probability distribution, but these probabilities are relative to each observer. For example, suppose two observers know that exactly one of them has won the lottery. Then the measure will tell each observer that his own chance to be the winner is the same number p , and the chance for the other person to have won is $1 - p$. The consistency axiom then requires that $p = 1 - p$, so $p = 1/2$.

In addition to these axioms, the probabilities resulting from a measure should obey the normal axioms of probability theory, i.e., they should be non-negative real numbers, the sum of the probabilities of mutually exclusive possibilities should be the probability of their union, and the total probability should be 1.

III. TOY MODEL

Now we ask whether there is any measure that satisfies axioms 1 and 2. We begin by constructing a simple toy model of an infinite universe.

¹This problem is very similar to the “sleeping beauty problem,” which has occasioned much philosophical debate. But it makes no difference whether you agree with my claim in this paragraph, so long as you accept the converse claim that if there is no possibility of duplication you shouldn't change your probabilities.

Suppose there is a machine that produces observers. At 1:00 it produces three observers. One of the three observers has a blue card in his pocket, and the other two observers have red cards. Otherwise they are identical. Since they have not looked in their pockets, the three observers are subjectively indistinguishable.

Each observer is in his own bedroom, but the three (identical) bedrooms are part of a single apartment with a common living room. The doors between the bedrooms and living room are open, so the observers can go out to the living room and confer with each other. All the observers know all this information.

At 2:00, the machine produces another identical set of three observers, and at 3:00 yet another set, and so on forever. Thus the machine creates a countably infinite set of subjectively indistinguishable observers. To keep the observers indistinguishable, they must not know their creation time. So we will endow each bedroom with a clock, but these clocks will measure only the time elapsed for each observer. Each clock will read 12:00 at the time that the observer is created.

Suppose you are one of these observers and you are wondering about the color of your card. You know that there are infinitely many observers and that there are infinitely many with blue cards. You cannot divide these infinities to compute the chance that your card is blue. You need a measure. The measure gives you some probability p that your card is blue. Because the observers in the other bedrooms of your apartment are subjectively indistinguishable from you, it must also tell them that their cards are blue with the same probability p . But because you and they can have a discussion, and because you know the way in which you were all created, you can be certain that exactly one of you is the “blue observer” (i.e., the one with a blue card). Thus the measure must give you $p = 1/3$ to agree with the principles of probability theory.²

Now we extend the scenario. Some time after being created, each observer goes back to his own bedroom, and the machine closes the bedroom doors, separating the three observers in each apartment. Then it silently rearranges the bedrooms into two-bedroom apartments as follows. For each creation step N , one of the red observers created by the machine in that step will share an apartment with the blue observer created in step $2N - 1$, while the other red observer created in step N will share an apartment with the blue observer created in step $2N$. Thus every observer will be part of a pair, and every pairing contains one blue observer.

²A previous version of this paper reached the same conclusion based on the symmetry of the creation process. However, since the observers will later be treated differently based on their card color, it is not clear that such a principle can be used. One might add as an axiom that the measure should not depend on future events, but this would rule out some measures currently under consideration, such as the light cone time or causal patch measure [6].

When the clock in his room reads 12:30, each observer will see the bedroom door open, and he will be able to talk to the occupant of the one other bedroom in his apartment. Since red observers must generally wait to be paired with blue observers created many hours later, time must run more slowly for them, so that they reach 12:30 at the same time as their partners. This could be implemented, for example, via gravitational time delay.

All observers know all this, from the moment they are created. Thus each observer at 12:00 knows that he is now part of a triplet of observers, and that when his clock reads 12:30, he will be part of a pair of observers. He knows exactly how the pairings will be made, but he does not know with whom he himself will be paired, because he does not know which observer he is.

At 12:00, the measure told each observer that his chance to have a blue card was $1/3$. What does it tell him at 12:30? Let us check that the conditions of axiom 1 are satisfied. No observer receives any information between 12:00 and 12:30. At 12:30 the doors open, but that has no bearing on the color of the cards. It happens at 12:30 independently of the colors, and the observer knew beforehand that it would happen then. Neither is there any worldline of any observer which terminates or begins between times labeled 12:00 and 12:30. Thus observers do not receive new information at 12:30, and so by axiom 1 the measure must continue to tell each observer that his chance to have a blue card is $1/3$.

But this conflicts with axiom 2. Now that the doors are again open, the observers can discuss their situations in pairs. Each pair will conclude that exactly one of them has a blue card. Each observer thinks that he has only a $1/3$ chance to have a blue card and so the other must have a $2/3$ chance. Consistency is violated.

Thus no measure can obey axioms 1 and 2. We do not need to enumerate possible measures; we merely show what results any measure must produce in order to obey the axioms, and show that no measure can obey them all.

IV. GLOBAL TIME CUTOFF MEASURES

A common way to construct measures is to introduce a global time coordinate t . Only a finite number of observers exist at times before some particular value of t , and thus probabilities can be defined as frequencies if one considers only that subspace. Then one takes either a large value of t or the limit as t grows without bound and one has a measure that gives well-defined probabilities.

How would such measures handle our toy model? First we have to specify the choice of t . Suppose first we use “external time.” We ignore the slow flow of time in some of the rooms and just consider the time of creation, plus a fixed small amount of time after the second observer in the pair is created, to bring the rooms together. Thus an observer whose clock reads 12:00 considers herself typical among all observers with such clocks that were created in

the first N steps of the machine. One third of these observers have blue cards.

But the observer whose clock reads 12:30 reasons differently. She considers herself typical among those observers whose clocks have reached 12:30 before some external time N . Observers paired with someone created after time N are not included in this set. Thus the external time cutoff measure tells the observer whose clock reads 12:30 that her chance of having a blue card is $1/2$.

The change from $1/3$ to $1/2$ violates axiom 1. The observer has made no new observations, but she is nevertheless instructed to change her probabilities. A bizarre consequence of this sort of choice will be discussed in Sec. VI.

Now suppose instead we use a cutoff in proper time, by which we mean the time before creation plus the subjective time spent in the room. We consider all observers with proper time less than N hours. There are $3N$ of these observers with clocks reading either 12:00 or 12:30. So in either case, this measure yields probability $1/3$. The problem here is that some observers who are in the reference class are able to talk to observers who are not in the reference class. This violates axiom 2. Each observer thinks the following three possibilities are equally probable: “My card is blue and yours is red,” “My card is red and yours is blue,” and “My card is red and your experiences don’t count³ because you were created after time N .” This nonsensical situation where counted observers and noncounted observers can converse explains the failure of the consistency axiom.

Bousso *et al.* [3] argue that when we use a geometric cutoff, we must take it seriously as an end of time. In our case, this could be done by saying that when the external time reaches, say, 10:59, the universe comes to an end. All observers’ worldlines terminate there and the machine ceases operation. There are 30 observers created with clocks reading 12:00, and 10 of those have blue cards. Of these observers, 20, including the 10 blue ones, are paired in apartments, while 10 of them terminate at external time 10:59 without ever being paired. Thus observers use probability $1/3$ at 12:00 and $1/2$ at 12:30. There is no contradiction, because in this case the observers do learn new information at 12:30: they learn that they have not reached the end of time.

This measure satisfies all the axioms. That is possible because in this case the universe is finite. But for the same reason, no measure is necessary. Once the universe is finite, we can just consider ourselves typical among all observers, without need for regularization.

Any finite universe will be free of measure problems, but I do not think that constitutes a demonstration that our

³There is some resemblance here to what philosophers call a “zombie”, i.e., an entity who acts in every way as a person would, but somehow has no consciousness, so it is impossible to imagine that one is oneself that entity.

Universe is finite. I argue here that we don’t know what to do to construct sensible probabilities in the case of an infinite universe, but the fact that one does not know how to proceed in any given situation does not mean that situation cannot exist.

Furthermore, even if the right answer is that the Universe is finite, I see no reason to introduce a cutoff intended to regulate infinite universes and then argue that it is a physical boundary that renders the Universe finite. One could equally well conclude that inflation is not eternal or even that the Universe will come to an end because it is a giant simulation performed on a very large but finite computer.

V. MORE REALISTIC MODELS

Could it be that eternal inflation is substantially different from this toy model, so that an eternally inflating universe could be handled by a measure, even though the toy model cannot? One difference is that the toy model generates universes linearly, whereas an inflating universe grows exponentially. But we can make a toy model with exponential growth, as follows.

Instead of creating one blue and two red observers at each time, let the machine create 2^{N-1} blue observers and 2^N red observers at time N . Each red observer created at N can be paired with one of the blue observers created at $N + 1$. There is still need for a time delay in pairing the observers, but now it is merely that red observers need to be delayed for 1 hour before being paired. Otherwise the issues are exactly as before and the measure cannot satisfy the axioms.

Another difference is that in eternal inflation observers are grouped into different thermalized regions (“pocket universes”), which in simple cases are each infinitely large. The infinite number of observers in each region can be handled by a measure which considers only those in a very large spherical subregion. As long as each region is homogeneous, this part of the measure will pose no problem.

This technique handles the probabilities of the different observers in the same region. We still must handle the probability to be in one region or another. Techniques such as this, which treat the chance to be a particular observer as the product of the chance to be in a particular thermalized region and the chance to be one particular observer within that region are called “pocket-based measures.” Several such ideas have been proposed [7–9], but they have mostly been abandoned because of several problems.

One such problem is that the pocket-based measures are prone to a “ Q catastrophe” in which the density perturbation amplitude Q is very strongly driven to either extremely large [10] or extremely small [11] values, giving a universe very much unlike ours.

Another problem for these measures is that of Boltzmann brains [12]. Each thermalized region begins

with a big bang that leads to the development of ordinary observers. But eventually each region with positive cosmological constant contains an infinite, empty universe asymptotically approaching de Sitter space, and an infinite number of Boltzmann brains can nucleate in any comoving volume of that universe. The reason pocket-based measures avoid the problems discussed here (in particular violation of axiom 1) is that they never cut through the worldline of any observer, so they do not require observers to change their probabilities at different times in their lives. But it is precisely the property of not separating past from future in the same region that leads to Boltzmann brain domination in the infinite future.

Finally, it is not really the case that within a thermalized region conditions are the same everywhere. Instead, the bubbles that nucleate in an eternally inflating space-time generally collide with each other and so break the equivalence between one part of the thermalized bubble interior and another.

VI. CAN WE ACCEPT TIME-DEPENDENT PROBABILITIES?

Another possibility is that the axioms are wrong. In particular, Guth and Vanchurin [4] argue that one should simply deny axiom 1. Instead, they explicitly allow situations in which the probability one assigns to some state of affairs can change merely because of the passage of time, even without any possibility of death or duplication.

This idea leads to bizarre situations. Once one accepts it, one can construct a toy multiverse model which will lead to any desired probability shift. For example, let us consider the following thought experiment.

It's exactly noon. You are standing outside your house, which has two doors. You know that exactly one of these doors is locked. Because you are part of a multiverse of observers cooked up to produce probability shifts, you believe at this moment that there is a 99% chance that the front door is the unlocked one. But in 10 seconds you will believe that the back door is the unlocked one with 99% probability. Furthermore, you know that you will change your beliefs at 12:00:10. (Locked doors will remain locked and unlocked doors will remain unlocked. Your change of beliefs is entirely due to reevaluating your likely position in the multiverse as a measure instructs you to do.)

Just at this moment a hungry tiger runs into your yard. Your only hope of survival is to get inside your house rapidly. Naturally you run for the front door, which you believe has a 99% chance to be unlocked. On the other hand, if the tiger does not appear until 12:00:11, you run for the back door, which you now believe is the one very likely to be unlocked.

But suppose instead that you are a 10-second run from either door when the tiger appears at 12:00:01. Because you have a head start, you'll be able to reach either door a few seconds before the tiger. Where should you run?

Since when you see the tiger you think that the front door is unlocked, perhaps you should run there. But when you reach the front door you'll be very unhappy because now you'll be nearly sure it's locked. Another possibility, recommended by Guth and Vanchurin [4],⁴ is that you should run for the back door, because you know that by the time you arrive you will think it likely to be unlocked. But why, then, are you at 12:00:01 trying to escape a tiger by running toward a door you're nearly sure is locked? I think both possibilities are nonsense, and therefore that one should not permit measures that change probabilities only by the passage of time.

If the probability change were due to certain parts of the multiverse ending, there would be no paradox. The change of probability could occur if there is a finite multiverse with a 99:1 preponderance of regions with the front door unlocked, but nearly all those regions terminate at 12:00:10, leaving a 1:99 distribution in favor of the back door being unlocked. In that case, you should think as follows. "Most likely my worldline will end in 9 seconds, and this tiger does not matter. But in the small fraction of cases where my worldline continues, I'll be better off with the back door."

There would also be no paradox if there were a timer scheduled to lock and unlock doors at 12:00:10. In that case your reasoning should be, "The back door is probably now locked, but the timer will operate before I reach the door, so the door will be unlocked when I reach it."

The situation with the multiverse measure is very different. In that case, you think the back door is likely locked, and you know that if you run there you will arrive (not run into the end of time on the way) and that the state of the door when you arrive will be the same as it is now. Thus when you set out for the back door, you feel nearly certain of dying upon arrival. Nevertheless, 1 second before arriving, you will suddenly feel nearly sure of survival. It is this change, with no reason for it, that leads me to find such measures unacceptable.

VII. CONCLUSION

To make sense of probabilities in an eternally inflating universe, we need to introduce a measure, considered here as any procedure by which probabilities can be assigned to an observer's circumstances. I have exhibited two simple axioms that I claim should be obeyed by any such measure, and showed no measure can satisfy them in a toy model. There are some differences between eternal inflation and the toy model, but it appears that these still do not allow us to construct any acceptable eternal inflation measure.

⁴See their discussion of a subject who, before going to sleep for a variable period of time, makes a bet that will be paid when he wakes up. In fact the present example is strongly analogous to theirs, except in this case you must bet with your life.

What can we do to make sense of the situation? One might try arguing that anthropic ideas are wrong, and we should not consider ourselves typical among all observers in a multiverse. But then it does not seem possible to recover ordinary ideas of probability. In an infinite universe, everything which can happen will happen an infinite number of times, so what does it mean to say that one thing is more likely than another?

Another idea is that of complementarity [13–15]. Perhaps it is not meaningful to describe widely separated observers as simultaneously having classical existence. In the usual description, one could say that there is an observer A at some point in the multiverse and an indistinguishable observer B in some other place outside A's horizon, and one needs a measure to decide one's chances of being A or B. Complementarity claims that one can give a description of the universe in which observer A exists, or a complementary description where B

exists, but one cannot describe them as existing at the same time, and thus there is no need to decide which of these observers one is. Complementarity does evade the difficulties I discuss here, but one still needs some specific procedure to determine probabilities.

Finally, perhaps we just do not know how to proceed when the Universe is infinite (and not homogeneous). I do not think one should conclude from this that the Universe must be finite, but rather that new ideas are needed to make sense of probabilities in an infinite universe.

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