Turbulence and little rip cosmology

I. Brevik,[1,*] R. Myrzakulov,[2,†] S. Nojiri,[3,4,‡] and S. D. Odintsov[5,6,2,7,§]

[1]*Department of Energy and Process Engineering, Norwegian University of Science and Technology, N-7491 Trondheim, Norway*
[2]*Eurasian International Center for Theoretical Physics and Department of General & Theoretical Physics,*
Eurasian National University, Astana 010008, Kazakhstan
[3]*Department of Physics, Nagoya University, Nagoya 464-8602, Japan*
[4]*Kobayashi-Maskawa Institute for the Origin of Particles and the Universe, Nagoya University, Nagoya 464-8602, Japan*
[5]*Consejo Superior de Investigaciones Científicas, ICE/CSIC-IEEC, Campus UAB, Facultat de Ciències,*
Torre C5-Parell-2a pl, E-08193 Bellaterra (Barcelona), Spain
[6]*Institució Catalana de Recerca i Estudis Avançats (ICREA), 08010 Barcelona, Spain*
[7]*Tomsk State Pedagogical University, Tomsk 634061, Russia*

A variety of conditions are considered under which the cosmic dark fluid may be able to develop a future Big Rip or Little Rip singularity. Both one-component and two-component models are considered. In the latter case we present a way in which the fluid can be decomposed into two components, one nonturbulent (ideal) and one turbulent part, obeying two different equations of state. For the nonturbulent part, the thermodynamical parameter, commonly called w, is assumed to be less than -1 throughout. For the turbulent part, it turns out that it is sufficient that w_{turb} lies in the quintessence region in order to lead to a singularity. Both Big Rip and Little Rip behavior for dark energy are found. In the one-component case, we examine how the universe may develop from a viscous era with constant bulk viscosity into a turbulent era, the turbulence in effect protecting the universe from encountering the singularity at all. The equivalent description of the same cosmology in terms of inhomogeneous (imperfect) fluid is also presented.

I. INTRODUCTION

It has become customary to explain the observed acceleration of the Universe [1,2] in terms of dark energy fluid (for recent reviews, see Refs. [3,4]), which is expected to have strange properties, like negative pressure and/or negative entropy. According to the latest supernovae observations the dark energy amounts to about 73% of the total mass-energy of the Universe [5]. Although astrophysical observations favor the standard ΛCDM cosmology, the equation-of-state (EoS) parameter w is still determined with uncertainty: it is not clear if w is less than -1, equal to -1, or larger than -1. Current observations suggest that $w = -1.04^{+0.09}_{-0.10}$ [6,7].

The very interesting but least theoretically understood case corresponds to $w < -1$ (phantom dark energy) where all four energy conditions are violated. Although the theory is unstable from a quantum field theoretical viewpoint, it could be stable in classical cosmology. There are observations [8] indicating that the crossing of the cosmological constant/phantom divide took place in the near past (or will occur in the near future). An essential property of (most) phantom dark energy models is the Big Rip future singularity [9] (see also Refs. [10,11]), where the scale factor becomes infinite at a finite time in the future. A softer

future singularity caused by phantom or quintessence dark energy is the sudden (Type II) singularity [12] where the scale factor is finite at Rip time (for classification of finite-time singularities, see Ref. [13]). Recently, an attempt to resolve the finite-time future singularities has been proposed in the face of mild phantom models where w asymptotically tends to -1 and where the energy density increases with the time or remains constant, but where the singularity occurs in the infinite future [14–17]. The key point here is that if w approaches -1 sufficiently fast, then it is possible to have a model in which the time required for the occurrence of a singularity is infinite, i.e., the singularity effectively does not happen. However, if the energy density grows, the disintegration of bound structures necessarily has to take place, in a way similar to the case of a Big Rip. Such Rip phenomena turn out to be common for Big Rip, Little Rip or Pseudo-Rip cosmologies, destroying all bound structures in a finite time. It is remarkable that mild phantom scenarios like Little Rip or Pseudo-Rip may easily mimic our current ΛCDM era and indicate quite a long existence (billions of years!) of our universe before the Rip occurs.

The pioneering works on the future singularity [9] considered the cosmic fluid to be nonviscous. This is an idealized model, of course: it is useful in practice in many cases but generally not able to cope with intricate micro-scale phenomena that occur, especially near solid boundaries. A next step in complexity is to allow for deviations from thermal equilibrium to the first order. That means that one

*iver.h.brevik@ntnu.no
†rmyrzakulov@gmail.com; rmyrzakulov@csufresno.edu
‡nojiri@phys.nagoya-u.ac.jp
§odintsov@ieec.uab.es

has to introduce two viscosity coefficients, namely the shear viscosity η and the bulk viscosity ζ.

In the present paper we consider the dark energy era with a future Big Rip or Little Rip singularity under a variety of conditions, assuming a more realistic form for the cosmic fluid than what has commonly been the case. We start in the next section by reviewing essentials of the theory of a dark fluid satisfying the condition $p < -\rho$, i.e., a phantom fluid. While in the general case one has to account for both coefficients η and ζ, as mentioned, we shall assume—in conformity with usual practice—that spatial anisotropies are smoothed out. Thus only ζ has to be included. Viscous Little Rip cosmology in an isotropic fluid has recently been worked out; c.f., Eq. (7) below. In Sec. III we focus attention on the description of a *turbulent* state of the fluid in the late universe. Such a possibility would seem physically most natural, in view of the violent motions expected in the vicinity of the singularity. We propose a two-component model in which, for physical reasons, the turbulent energy component is set proportional to the scalar expansion. We consider in this context two different proposals for the thermodynamic parameter w_{turb} occurring in the equation of state. First, we put w_{turb} equal to the usual parameter w for nonturbulent matter, meaning that the turbulent matter component behaves as a passive ingredient as far as the equation of state is concerned. Second, we allow for $w_{\text{turb}} > -1$, meaning that we cover the region $-1 < w_{\text{turb}} < 0$ as well. Both Big Rip and Little Rip evolutions for dark energy are found. In Sec. VI we consider an approach which is quite different, namely, we model the universe as a one-component dark fluid that becomes suddenly transformed into a turbulent state. A noteworthy property of this model is that the turbulence in effect protects the universe from encountering the future singularity at all. In Sec. VII we present an equivalent description in terms of an inhomogeneous imperfect fluid, including the effect of viscosity but not that of turbulence. Some summary and outlook is given in the Discussion VIII section.

II. DARK FLUID WITH A BULK VISCOSITY

Consider the following model for the cosmic fluid in the later stages of the development of the universe, assuming that the fluid is dark and satisfies the inequality

$$p < -\rho, \tag{1}$$

so that the equation

$$\dot{H} = -\frac{1}{2}\kappa^2(\rho + p), \tag{2}$$

with $H = \dot{a}/a$ and $\kappa^2 = 8\pi G$ implies the property

$$\dot{H} > 0. \tag{3}$$

This is the phantom region, corresponding to the thermodynamic parameter $w = p/\rho$ being less than -1. In the

usual case of this type of universe the future singularity is a true mathematical singularity, reached in a finite time, and is called the Big Rip. It is notable, however, that in the limiting case when $w \rightarrow -1$ from below, the future singularity is only asymptotically reached. This scenario is called the Little Rip.

It is noteworthy that in a cosmological context the value of η appears to be very much larger than that of ζ. At least that is so in the earlier stages of the development of the universe where the physical conditions are better known than in the later stages and conventional kinetic theory can be used to calculate the viscosities. A calculation of the viscosity coefficients was made by Caderni and Fabbri [18]. For instance, considering the instant $t = 1000$ s after the Big Bang, it turns out that

$$\eta = 2.8 \times 10^{14} \text{ gcm}^{-1}\text{s}^{-1},$$
$$\zeta = 7.0 \times 10^{-3} \text{ gcm}^{-1}\text{s}^{-1}, \tag{4}$$

(c.f., also Ref. [19]) showing the large difference in magnitude between η and ζ. Yet it appears that spatial anisotropies in the universe are effectively smoothed out, at least on a large scale, so that in most current models the universe is assumed to be spatially isotropic. It means that ζ is retained, but η omitted in the Friedmann equations.

A theory of viscous Little Rip cosmology was recently given in Ref. [20]. Let us recapitulate one of the characteristic results from that investigation: if the effective pressure p_{eff} is assumed to have the explicit form

$$p_{\text{eff}} = -\rho - A\sqrt{\rho} - 3\zeta H, \tag{5}$$

with A a positive constant, and if moreover the bulk viscosity ζ is assumed to satisfy the condition

$$3\zeta H \equiv \xi_0 = \text{constant}, \tag{6}$$

then the following expression is found for the time-dependent energy density:

$$\rho(t) = \left[\left(\frac{\xi_0}{A} + \sqrt{\rho_0}\right)\exp(\sqrt{6\pi G}At) - \frac{\xi_0}{A}\right]^2. \tag{7}$$

Here subscript zero refers to the present time. Thus an infinite time is needed to reach the infinite energy density case. This is precisely the characteristic property of the Little Rip phenomenon.

III. THE TURBULENT APPROACH

Let us now apply a physical point of view to the dark energy universe in its later stages when it approaches the future singularity. The simple description above, in terms of macroscopic bulk viscosity in the fluid, cannot be considered to be satisfactory due to the following reason: in the assumed states of violent local fluid element motion a transition into *turbulent* motion seems to be inevitable. The local Reynolds number must be expected to be very high. That brings, in fact, the *shear* back into the analysis,

not in a macroscopic sense as before, but in a local sense, causing the distribution of local eddies over the wave number spectrum. What kind of turbulence should we expect? The natural choice is that of isotropic turbulence, which is a topic reasonably well understood. Thus, we should expect a Loitziankii region for low wave numbers where the energy density varies proportionally to k^4; for higher k we should expect an inertial subrange characterized by the formula

$$E(k) = \alpha \epsilon^{2/3} k^{-5/3}, \qquad (8)$$

with α the Kolmogorov constant and ϵ the mean energy dissipation per unit time and unit mass; and finally, when the values of k become as high as the inverse Kolmogorov length η_L,

$$k \to k_L = \frac{1}{\eta_L} = \left(\frac{\epsilon}{\nu^3}\right)^{1/4}, \qquad (9)$$

with ν the kinematic viscosity, we enter the dissipative region where the local Reynolds number is of order unity and heat dissipation occurs. In accordance with common usage we shall consider the fluid system as quasistationary, and omit the production of heat energy. In practical cases it may be useful to combine these elements into the useful von Kármán interpolation formula which covers the whole wave number spectrum (c.f., for instance, Refs. [21–23]).

However, the full spectral theory of isotropic turbulence will not be needed in our first approach to the problem. Rather, we shall in the following focus attention on how the turbulent part of the energy density, called ρ_{turb}, can be estimated to vary from present time t_0 onwards. First, we write the effective energy density as a sum of two terms,

$$\rho_{\text{eff}} = \rho + \rho_{\text{turb}}, \qquad (10)$$

where ρ denotes the conventional macroscopic energy density in the local rest inertial system of the fluid. It is natural to assume that ρ_{turb} is proportional to ρ itself. Further, we shall assume that ρ_{turb} is proportional to the scalar expansion $\theta = U^\mu_{;\mu} = 3H$. This is because physically speaking the transition to turbulence is expected to be more pronounced in the violent later stages, and a proportionality to the scalar expansion is mathematically the most simple way in which to represent the effect. Calling the proportionality factor τ, we can thus write the effective energy density as

$$\rho_{\text{eff}} = \rho(1 + 3\tau H). \qquad (11)$$

Consider next the effective pressure p_{eff}. We split it into two terms,

$$p_{\text{eff}} = p + p_{\text{turb}}, \qquad (12)$$

analogously as above. For the conventional nonturbulent quantities p and ρ we assume the standard relationship

$$p = w\rho, \qquad (13)$$

where $-1 < w < -1/3$ in the quintessence region and $w < -1$ in the phantom region. The question now is how does p_{turb} depend on ρ_{turb}? There seems to be no definite physical guidance to that problem, so we shall make the simplest possible choice in the following; namely, we write

$$p_{\text{turb}} = w_{\text{turb}}\rho_{\text{turb}}, \qquad (14)$$

with w_{turb} a constant.

We shall consider two different possibilities for the value of w_{turb}. The first is to put w_{turb} equal to w in Eq. (13), meaning that the turbulent matter behaves in the same way as the nonturbulent matter as far as the equation of state is concerned. This option is straightforward and natural, and is not quite trivial since ρ_{turb} and ρ behave differently, in view of Eq. (11). Our second option will be to assume that w_{turb} takes another, prescribed value. In view of the expected violent conditions near the future singularity, it might even be natural here to choose the value $w_{\text{turb}} = +1$, i.e., the Zel'dovich fluid option.

The first and the second Friedmann equations can now be written:

$$H^2 = \frac{1}{3}\kappa^2\rho(1 + 3\tau H), \qquad (15)$$

$$\frac{2\ddot{a}}{a} + H^2 = -\kappa^2\rho(w + 3\tau H w_{\text{turb}}), \qquad (16)$$

(recall that $\kappa^2 = 8\pi G$). This may be compared with an earlier attempt to introduce the turbulence in dark energy [24].

Equations (15) and (16) determine our physical model. Recall that its input parameters are $\{w, w_{\text{turb}}, \tau\}$, all assumed constant. From these equations we can now describe the development of the Hubble parameter. For convenience we introduce the quantities γ and γ_{turb}, defined as

$$\gamma = 1 + w, \qquad \gamma_{\text{turb}} = 1 + w_{\text{turb}}. \qquad (17)$$

We can then write the governing equation for H as

$$(1 + 3\tau H)\dot{H} + \frac{3}{2}\gamma H^2 + \frac{9}{2}\tau\gamma_{\text{turb}}H^3 = 0. \qquad (18)$$

This equation is in principle to be integrated from present time $t = t_0 = 0$ onwards, with initial value $H = H_0 = \dot{a}_0/a_0$.

A. On the energy balance equation

If $T^{\mu\nu}_{\text{tot}}$ denotes the total energy-momentum tensor for the cosmic fluid, we must have

$$T^{\mu\nu}_{\text{tot};\nu} = 0, \qquad (19)$$

as a consequence of Einstein's equation.

In most cases studied, the expression for $T^{\mu\nu}_{\text{tot}}$ can be written down explicitly: this is so for nonviscous fluids as well as with macroscopic viscous fluids. In the present case this no longer true, however, since the turbulent energy is

produced by shear stresses on a *small scale*, much less than the scale of the macroscopic fluid equations. That is, we are dealing with a nonclosed physical system of essentially the same kind as encountered in phenomenological electrodynamics in a continuous medium in special relativity. It implies that the source term in the energy balance equation has to be put in by hand.

Henceforth, let $T^{\mu\nu}$ refer to the nonviscous part of the fluid. We may express the energy balance as

$$\dot{\rho} + 3H(\rho + p) = -Q, \qquad (20)$$

where the source term Q is positive, corresponding to an energy sink for the nonviscous fluid. We shall put Q equal to $\epsilon\rho$, where the specific energy dissipation ϵ however shall be taken to involve the large Hubble parameter H in the later stage of the development. Let us assume the form

$$\epsilon = \epsilon_0(1 + 3\tau H), \qquad (21)$$

ϵ_0 being the specific energy dissipation at present time. This equation is seen to contain the same kind of development as assumed before; cf. the analogous Eq. (11). Thus, our ansatz for the energy balance reads

$$\dot{\rho} + 3H(\rho + p) = -\rho\epsilon_0(1 + 3\tau H). \qquad (22)$$

We shall in the following consider some examples. First, however, it is of interest to compare Eq. (22) with the generic equation commonly accepted in order to deal with the extra *pressure* in a fluid,

$$T^{\mu\nu} = \rho U^\mu U^\nu + (p + \Pi)h^{\mu\nu}, \qquad (23)$$

with $h^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu$ the projection tensor. Here Π is the extra pressure brought about by a variety of effects, among them typically viscosity, matter creation, or eventually a combination of these. (For a more detailed discussion along the lines cf., for instance, Refs. [25–27].) In the case of bulk viscosity, it is known that $\Pi = -3\zeta H$. Thus, Eq. (22) may be regarded as an energy equation analogous to the pressure equation (23).

IV. THE CASE $w_{\text{turb}} = w < -1$

This case means that the turbulent component of the fluid is regarded as a passive ingredient as far as the equation-of-state (EoS) parameter is concerned. The time development of ρ and ρ_{turb} will however be different. Equation (18) reduces to

$$\dot{H} + \frac{3}{2}\gamma H^2 = 0, \qquad (24)$$

leading to

$$H = \frac{H_0}{Z}, \qquad (25)$$

where we have defined

$$Z = 1 + \frac{3}{2}\gamma H_0 t, \qquad (26)$$

(note that $w < -1$ implies $\gamma < 0$). Thus we have a Big Rip cosmology, where the future singularity time t_s is given by

$$t_s = \frac{2}{3|\gamma|H_0}. \qquad (27)$$

The scale factor becomes correspondingly

$$a = a_0 Z^{2/3\gamma}, \qquad (28)$$

and from the first Friedmann equation (15) we get the nonturbulent energy density as

$$\rho = \frac{3H_0^2}{\kappa^2} \frac{1}{Z} \frac{1}{Z + 3\tau H_0}. \qquad (29)$$

The ratio between turbulent and nonturbulent energy becomes

$$\frac{\rho_{\text{turb}}}{\rho} = 3\tau H = \frac{3\tau H_0}{Z}. \qquad (30)$$

It is of main interest to consider the behavior near t_s. As $Z = 1 - t/t_s$ we see that

$$H \sim \frac{1}{t_s - t}, \qquad a \sim \frac{1}{(t_s - t)^{2/3|\gamma|}}, \qquad (31)$$

$$\rho \sim \frac{1}{t_s - t}, \qquad \frac{\rho_{\text{turb}}}{\rho} \sim \frac{1}{t_s - t}. \qquad (32)$$

Notice the difference from conventional cosmology: the behavior of H and a near the singularity is as usual, while the singularity of ρ has become *weakened*. The reason for this is, of course, the nonvanishing value of the parameter τ. Moreover, as $t \to t_s$ all the nonturbulent energy has been converted into turbulent energy. From a physical point of view, this is just as we would expect.

Let us examine how this formalism compares with our ansatz (22) for the energy balance of the nonturbulent fluid. The left-hand side of that equation is explicitly calculable by means of the formulas just derived. It is convenient here to make use of the mathematical relationship

$$\dot{\rho} + 3H(\rho + p) = a^{-3\gamma} \frac{d}{dt}(\rho a^{3\gamma}), \qquad (33)$$

by means of which we obtain

$$\dot{\rho} + 3H(\rho + p) = -\frac{27}{2} \frac{H_0^4 |\gamma| \tau}{\kappa^2} \frac{1}{Z^2(Z + 3\tau H_0)^2}. \qquad (34)$$

In the limit $t \to t_s$, this reduces to

$$\dot{\rho} + 3H(\rho + p) \to -\frac{3}{2} \frac{H_0^2 |\gamma|}{\kappa^2 \tau} \frac{1}{Z^2}. \qquad (35)$$

In Eq. (22) we calculate the right-hand side,

$$-\rho\epsilon_0(1 + 3\tau H) = -\frac{3H_0^2}{\kappa^2}\frac{\epsilon_0}{Z(Z + 3\tau H_0)}\left(1 + \frac{3\tau H_0}{Z}\right),$$

$$(36)$$

which for $t \to t_s$ reduces to

$$-\rho\epsilon_0(1 + 3\tau H) \to -\frac{3H_0^2}{\kappa^2}\frac{\epsilon_0}{Z^2}.$$

$$(37)$$

This has actually the same form as Eq. (35), thus supporting the physical consistency of the ansatz. We obtain the relationship

$$\epsilon_0 = \frac{1}{2}\frac{|\gamma|}{\tau}.$$

$$(38)$$

This could hardly have been seen in advance. The specific energy dissipation ϵ_0 at the present time is related to the EoS parameter γ and the parameter τ introduced in Eqs. (11) and (12) in a simple way. In geometric units, the dimension of ϵ_0 is cm^{-1}.

A. A comment on Little Rip cosmology

The theory given above concerns Big Rip, a singularity obtained in the future at a finite time. A milder variant of this is the Little Rip scenario, corresponding to an infinite span of time needed to reach the singularity (c.f., for instance, Refs. [20,28]). To achieve a Little Rip, we have to modify to some extent the above basic assumptions. One such modification is the following:

(1) Take the equation of state to be

$$p = -\rho - A\sqrt{\rho},$$

$$(39)$$

with A a positive constant.

(2) Put $\tau = 0$ in the first Friedmann equation (15).

(3) Assume a milder form for the sink term in the energy balance equation than the form given in (22). A simple option is to take Q to be proportional to H itself. Calling the proportionality constant $\xi_0(>0)$, we get the following modified ansatz:

$$\dot{\rho} + 3H(\rho + p) = -\xi_0 H.$$

$$(40)$$

From Eqs. (39) and (40), we now have

$$\dot{\rho} = (3A\sqrt{\rho} - \xi_0)H.$$

$$(41)$$

Comparing this with the modified first Friedmann equation we get

$$t = \frac{\sqrt{3}}{\kappa}\int_{\rho_0}^{\rho}\frac{d\rho}{\sqrt{\rho}(3A\sqrt{\rho} - \xi_0)}$$

$$= \frac{2}{\sqrt{3}}\frac{1}{\kappa A}\ln\frac{3A\sqrt{\rho} - \xi_0}{3A\sqrt{\rho_0} - \xi_0}.$$

$$(42)$$

Inverting this equation we have

$$\rho = \frac{\xi_0^2}{9A^2}\left[1 + \left(\frac{3A\sqrt{\rho_0}}{\xi_0} - 1\right)\exp\left(\frac{1}{2}\sqrt{3}\kappa At\right)\right]^2.$$

$$(43)$$

This expression shows just the characteristic property of the Little Rip phenomenon: the universe reaches the state $\rho \to \infty$, but needs an infinite time to do so.

One may ask: are the physical assumptions underlying the Big Rip scenario stronger than those underlying the Little Rip one? In our opinion this is most likely so, although there are of course considerable uncertainties connected with the future time development of the universe.

V. THE CASE $w < -1$, $w_{\text{turb}} > -1$

This case is thermodynamically quite different from the preceding one as the turbulent component of the fluid is no longer a passive ingredient. We have now $\gamma_{\text{turb}} = 1 + w_{\text{turb}} > 0$, which means that we cover the region $-1 < w_{\text{turb}} < 0$ also. In the latter region, the turbulent contribution to the pressure is still negative as above, while if $w_{\text{turb}} > 0$ the turbulent pressure becomes *positive*, just as in ordinary hydrodynamical turbulence.

The governing equation (18), written as

$$(1 + 3\tau H)\dot{H} = \frac{3}{2}H^2(|\gamma| - 3\tau\gamma_{\text{turb}}H),$$

$$(44)$$

tells us that at the present time $t = 0$ the condition

$$|\gamma| > 3\tau\gamma_{\text{turb}}H_0,$$

$$(45)$$

must hold. This is so because at the present time, the turbulent part is regarded as unimportant. This corresponds to the inequality $\dot{H} > 0$ [c.f., Eq. (3)].

Equation (44) can be integrated to give t as a function of H,

$$t = \frac{2}{3|\gamma|}\left(\frac{1}{H_0} - \frac{1}{H}\right) - \frac{2\tau}{|\gamma|}\left(1 + \frac{\gamma_{\text{turb}}}{|\gamma|}\right)$$

$$\times \ln\left[\frac{|\gamma| - 3\tau\gamma_{\text{turb}}H}{|\gamma| - 3\tau\gamma_{\text{turb}}H_0}\frac{H_0}{H}\right].$$

$$(46)$$

A striking property of this expression is that it describes a *Little Rip* scenario. As $t \to \infty$, the Hubble parameter reaches a finite critical value,

$$H_{\text{crit}} = \frac{1}{3\tau}\frac{|\gamma|}{\gamma_{\text{turb}}}.$$

$$(47)$$

The physical role of γ_{turb} is thus to postpone and weaken the development towards the future singularity.

A natural choice for the EoS parameter w_{turb} in the vicinity of the singularity, in view of the violent motions expected, would be

$$w_{\text{turb}} = +1,$$

$$(48)$$

that is, a Zel'dovich fluid. This is an extreme case, where the velocity of sound equals the velocity of light.

A. Vacuum nonturbulent fluid component ($w = -1$), $w_{\mathrm{turb}} > -1$

This case, corresponding to a vacuum nonturbulent fluid ($p = -\rho$), has to be considered separately. From Eq. (44) we can solve directly for H as a function of t,

$$\frac{H}{H_0} = \frac{3\tau H_0 + \sqrt{1 + 6\tau H_0 + 9\tau H_0^2(\tau + \gamma_{\mathrm{turb}}t)}}{1 + 6\tau H_0 + 9\tau H_0^2 \gamma_{\mathrm{turb}}t}. \quad (49)$$

Thus as $t \to \infty$, H decreases smoothly to zero as $t^{-1/2}$. We conclude that at least quintessence conditions are necessary to produce a future singularity.

VI. A ONE-COMPONENT DARK FLUID

We now turn to an approach that is quite different from the one above, namely to consider the cosmic fluid as a *one-component* fluid. Thus the distinction between a nonturbulent and a turbulent fluid component is avoided altogether. This new approach is actually closer to the usual picture in hydrodynamics, where a fluid is known to shift suddenly from a laminar to a turbulent state.

Consider the following picture: the universe starts from present time $t = 0$ as an ordinary viscous fluid with a bulk viscosity called ζ, and develops according to the Friedmann equations. We assume as before that the EoS parameter $w < -1$, meaning that the universe develops in the viscous era towards a future singularity. Before this happens, however, at some instant $t = t_*$, we assume that there is a sudden transition of the whole fluid into a turbulent state, after which the EoS parameter is w_{turb} and the pressure accordingly $p_{\mathrm{turb}} = w_{\mathrm{turb}}\rho_{\mathrm{turb}}$. As before we assume that $w_{\mathrm{turb}} > -1$, and for simplicity we take ζ, as well as w and w_{turb}, to be constants. One may ask: what is the resulting behavior of the fluid, especially at later stages?

The problem can easily be solved, making use of the condition that the density of the fluid has to be continuous at $t = t_*$. In the viscous era $0 < t < t_*$ the energy-momentum tensor of the fluid is

$$T_{\mu\nu} = \rho U_\mu U_\nu + (p - 3H\zeta)h_{\mu\nu}, \quad (50)$$

where

$$h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu, \quad (51)$$

is the projection tensor (the shear viscosity is omitted because of spatial isotropy). Solving the Friedmann equations one gets [29,30]

$$H = \frac{H_0 e^{t/t_c}}{1 - \frac{3}{2}|\gamma|H_0 t_c(e^{t/t_c} - 1)}, \quad (52)$$

$$a = \frac{a_0}{[1 - \frac{3}{2}|\gamma|H_0 t_c(e^{t/t_c} - 1)]^{2/3|\gamma|}}, \quad (53)$$

$$\rho = \frac{\rho_0 e^{2t/t_c}}{[1 - \frac{3}{2}|\gamma|H_0 t_c(e^{t/t_c} - 1)]^2}, \quad (54)$$

where t_c is the "viscosity time",

$$t_c = \left(\frac{3}{2}\kappa^2\zeta\right)^{-1}. \quad (55)$$

The values H_*, a_*, ρ_* at $t = t_*$ are thereby known.

In the turbulent era $t > t_*$ we can make use of the same expressions (52)–(54) as above, only with substitutions $t_c \to \infty(\zeta \to 0)$, $t \to t - t_*$, $w \to w_{\mathrm{turb}}$, $H_0 \to H_*$, $a_0 \to a_*$, $\rho_0 \to \rho_*$. Thus,

$$H = \frac{H_*}{1 + \frac{3}{2}\gamma_{\mathrm{turb}}H_*(t - t_*)}, \quad (56)$$

$$a = \frac{a_*}{[1 + \frac{3}{2}\gamma_{\mathrm{turb}}H_*(t - t_*)]^{2/3\gamma_{\mathrm{turb}}}}, \quad (57)$$

$$\rho = \frac{\rho_*}{[1 + \frac{3}{2}\gamma_{\mathrm{turb}}H_*(t - t_*)]^2}, \quad (58)$$

(recall that $\gamma_{\mathrm{turb}} > 0$). Thus the density ρ, at first increasing with increasing t according to Eq. (54), decreases again once the turbulent era has been entered, and goes smoothly to zero as t^{-2} when $t \to \infty$. In this way the transition to turbulence protects the universe from entering the future singularity.

It should be noted that whereas the density is continuous at $t = t_*$ the pressure is not: in the laminar era $p_* = w\rho_* < 0$, while in the turbulent era $p_* = w_{\mathrm{turb}}\rho_*$ will even be positive, if $w_{\mathrm{turb}} > 0$. Thus, we demonstrated the possible role of turbulence to protect the universe from the future singularity. In the same fashion, one can consider its role in protecting the universe from Rip, i.e., the disintegration of bound structures.

VII. INHOMOGENEOUS (IMPERFECT) DARK FLUID DESCRIPTION

Let us present the equivalent formulation in terms of an inhomogeneous (imperfect) fluid. By including the effect of the viscosity but not including that of the turbulence, we may consider the following EoS:

$$p = -\rho + f(\rho) - 3\zeta H. \quad (59)$$

Here $f(\rho)$ is an appropriate function of the energy density ρ. The bulk viscosity ζ can be a function of ρ and H, $\zeta = \zeta(\rho, H)$. Let us assume that the Hubble rate H is given by a function $H = h(t)$. Then, by using the first FRW equation, we find

$$t = h^{-1}(H) = h^{-1}\left(\kappa\sqrt{\frac{\rho}{3}}\right). \quad (60)$$

Then, by using the second FRW equation, we obtain

$$f(\rho) = -\frac{2}{\kappa^2}\dot{H} + 3\zeta(\rho, H)H$$

$$= -\frac{2}{\kappa^2}h'\left(h^{-1}\left(\kappa\sqrt{\frac{\rho}{3}}\right)\right) + \kappa\sqrt{3\rho}\zeta\left(\rho, \kappa\sqrt{\frac{\rho}{3}}\right). \quad (61)$$

Conversely, if $f(\rho)$ is given by (61), we obtain a solution of the FRW equation as $H = h(t)$.

Just for the simplicity, we may assume $3\zeta H$ is a constant, as in (6):

$$3\zeta(\rho, H)H = \kappa\sqrt{3\rho}\zeta\left(\rho, \kappa\sqrt{\frac{\rho}{3}}\right) = \xi_0. \quad (62)$$

As a simple example, we may consider the model (25) with (26), which is realized by including the turbulence. Then we obtain

$$f(\rho) = \gamma\rho + \xi_0. \quad (63)$$

As another example, we consider the model (44), which gives (45). Since

$$h'(t) = \dot{H} = \frac{3H^2(|\gamma| - 3\tau\gamma_{\text{turb}}H)}{2(1 + 3\tau H)}$$

$$= \frac{\kappa^2\rho(|\gamma| - \tau\gamma_{\text{turb}}\kappa\sqrt{3\rho})}{2(1 + \tau\kappa\sqrt{3\rho})}, \quad (64)$$

we find that

$$f(\rho) = -\frac{\rho(|\gamma| - \tau\gamma_{\text{turb}}\kappa\sqrt{3\rho})}{(1 + \tau\kappa\sqrt{3\rho})} + \xi_0. \quad (65)$$

Then the models with turbulence can be equivalent to the inhomogeneous (imperfect) fluid models without turbulence with respect to the expansion history of the universe. In other words, the effect of turbulence and/or of viscosity may always be traded via the change of the effective equation of state.

VIII. DISCUSSION

In summary, we have studied a dark fluid universe where a finite-time future Big Rip or a Little Rip cosmology occurs. Special attention is paid to the role of viscosity via its elementary physical properties. The possibility of a viscous Little Rip cosmology where a singularity occurs in the infinite future is confirmed. We propose to take into account a turbulent state for the dark fluid in the late universe. An explicit two-component model with a turbulent component proportional to the Hubble rate is developed. A conventional as well as a quintessence value for the turbulent component equation of state parameter is considered, and the occurrence of a Big Rip and a Little Rip cosmology is demonstrated. However, for a one-component fluid which suddenly transforms into a turbulent state, the late-time acceleration universe is found to be qualitatively different: the turbulence may protect the universe from encountering a future singularity at all. It is interesting that an equivalent description in terms of an inhomogeneous fluid may also be developed.

Turbulence is a fundamental property of classical fluids. It is known to be important not only theoretically but also for a number of practical applications in everyday life (airplane flight turbulence is a well-known example). The mysterious dark energy is often considered as some kind of classical fluid with unusual properties. It is then natural to expect (if such a picture is correct) that turbulence phenomena may be important also for dark energy, especially in the very late violent universe. In this work we have made a first step towards the construction of a bridge between turbulence and dark energy. It turns out that turbulent dark energy may have quite rich properties, in particular in predicting the possible absence of future singularities. However, only forthcoming precise observational data about dark energy may confirm the role of turbulence in the dark energy paradigm.

[1] A. G. Riess *et al.* (Supernova Search Team Collaboration), Astron. J. **116,** 1009 (1998).

[2] S. Perlmutter *et al.* (Supernova Cosmology Project Collaboration), Astrophys. J. **517,** 565 (1999).

[3] K. Bamba, S. Capozziello, S. Nojiri, and S. D. Odintsov, arXiv:1205.3421.

[4] M. Li, X.-D. Li, S. Wang, and Y. Wang, Commun. Theor. Phys. **56,** 525 (2011).

[5] M. Kowalski *et al.* (Supernova Cosmology Project Collaboration), Astrophys. J. **686,** 749 (2008).

[6] K. Nakamura *et al.* (Particle Data Group Collaboration), J. Phys. G **37,** 075021 (2010).

[7] R. Amanullah *et al.* (Supernova Cosmology Project Collaboration), Astrophys. J. **716,** 712 (2010).

[8] S. Nesseris and L. Perivolaropoulos, J. Cosmol. Astropart. Phys. 01 (2007) 018; H. K. Jassal, J. S. Bagla, and T. Padmanabhan, Phys. Rev. D **72,** 103503 (2005).

[9] R. R. Caldwell, Phys. Lett. B **545,** 23 (2002); R. R. Caldwell, M. Kamionkowski, and N. N. Weinberg, Phys. Rev. Lett. **91,** 071301 (2003).

[10] B. McInnes, J. High Energy Phys. 08 (2002) 029; S. Nojiri and S. D. Odintsov, Phys. Lett. B **562,** 147 (2003); V. Faraoni, Int. J. Mod. Phys. D **11,** 471 (2002); P. F. Gonzalez-Diaz, Phys. Lett. B **586,** 1 (2004); E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **70,** 043539 (2004); P. Singh, M. Sami, and N. Dadhich, Phys. Rev. D **68,** 023522 (2003); C. Csaki, N. Kaloper, and J. Terning, Ann. Phys. (N.Y.) **317,** 410 (2005); P. X. Wu and H. W. Yu, Nucl. Phys. **B727,** 355 (2005); S. Nesseris and L. Perivolaropoulos, Phys. Rev. D **70,** 123529 (2004); H. Stefancic, Phys. Lett. B **586,** 5 (2004); L. P. Chimento and R. Lazkoz, Phys. Rev. Lett. **91,** 211301 (2003); J. G. Hao and X. Z. Li, Phys. Lett. B **606,** 7 (2005); M. P. Dabrowski and T. Stachowiak, Ann. Phys. (N.Y.) **321,** 771 (2006); I. Y. Aref'eva, A. S. Koshelev, and S. Y. Vernov, Phys. Rev. D **72,** 064017 (2005); W. Godlowski and M. Szydlowski, Phys. Lett. B **623,** 10 (2005); J. Sola and H. Stefancic, Phys. Lett. B **624,** 147 (2005).

[11] S. Nojiri and S. D. Odintsov, Phys. Rep. **505,** 59 (2011).

[12] J. D. Barrow, Classical Quantum Gravity **21,** L79 (2004); Y. Shtanov and V. Sahni, Classical Quantum Gravity **19,** L101 (2002); S. Nojiri and S. D. Odintsov, Phys. Lett. B **595,** 1 (2004); Phys. Rev. D **70,** 103522 (2004); S. Cotsakis and I. Klaoudatou, J. Geom. Phys. **55,** 306 (2005); M. P. Dabrowski, Phys. Rev. D **71,** 103505 (2005); L. Fernandez-Jambrina and R. Lazkoz, Phys. Rev. D **70,** 121503 (2004); Phys. Lett. B **670,** 254 (2009); J. D. Barrow and C. G. Tsagas, Classical Quantum Gravity **22,** 1563 (2005); H. Stefancic, Phys. Rev. D **71,** 084024 (2005); C. Cattoen and M. Visser, Classical Quantum Gravity **22,** 4913 (2005); P. Tretyakov, A. Toporensky, Y. Shtanov, and V. Sahni, Classical Quantum Gravity **23,** 3259 (2006); A. Balcerzak and M. P. Dabrowski, Phys. Rev. D **73,** 101301 (2006); M. Sami, P. Singh, and S. Tsujikawa, Phys. Rev. D **74,** 043514 (2006); A. V. Yurov, A. V. Astashenok, and P. F. Gonzalez-Diaz, Gravitation Cosmol. **14,** 205 (2008); T. Koivisto, Phys. Rev. D **77,** 123513 (2008); J. D. Barrow and S. Z. W. Lip, Phys. Rev. D **80,** 043518 (2009); M. Bouhmadi-Lopez, Y. Tavakoli, and P. V. Moniz, J. Cosmol. Astropart. Phys. 04 (2010) 016; S. J. M. Houndjo, Europhys. Lett. **92,** 10004 (2010).

[13] S. Nojiri, S. D. Odintsov, and S. Tsujikawa, Phys. Rev. D **71,** 063004 (2005); S. Nojiri and S. D. Odintsov, Phys. Rev. D **72,** 023003 (2005).

[14] P. H. Frampton, K. J. Ludwick, and R. J. Scherrer, Phys. Rev. D **84,** 063003 (2011); **85,** 083001 (2012).

[15] P. H. Frampton, K. J. Ludwick, S. Nojiri, S. D. Odintsov, and R. J. Scherrer, Phys. Lett. B **708,** 204 (2012).

[16] A. V. Astashenok, S. Nojiri, S. D. Odintsov, and A. V. Yurov, Phys. Lett. B **709,** 396 (2012); A. V. Astashenok, S. Nojiri, S. D. Odintsov, and R. J. Scherrer, Phys. Lett. B **713,** 145 (2012).

[17] P. H. Frampton and K. J. Ludwick, Eur. Phys. J. C **71,** 1735 (2011); S. Nojiri, S. D. Odintsov, and D. Saez-Gomez, AIP Conf. Proc. 1458, 207 (2011).Y. Ito, S. Nojiri, and S. D. Odintsov, arXiv:1111.5389; L. N. Granda and E. Loaiza, Int. J. Mod. Phys. D **21,** 1250002 (2012); P. Xi, X.-H. Zhai, and X.-Z. Li, Phys. Lett. B **706,** 482 (2012); M.-H. Belkacemi, M. Bouhmadi-Lopez, A. Errahmani, and T. Ouali, Phys. Rev. D **85,** 083503 (2012); A. N. Makarenko, V. V. Obukhov, and I. V. Kirnos, arXiv:1201.4742; K. Bamba, R. Myrzakulov, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **85,** 104036 (2012); R. Saitou and S. Nojiri, Eur. Phys. J. C **72,** 1946 (2012); Z.-G. Liu and Y.-S. Piao, Phys. Lett. B **713,** 53 (2012); E. Elizalde, A. N. Makarenko, S. Nojiri, V. V. Obukhov, and S. D. Odintsov, arXiv:1206.2702; P. C. Stavrinos and S. I. Vacaru, arXiv:1206.3998; A. V. Astashenok, E. Elizalde, S. D. Odintsov, and A. V. Yurov, arXiv:1206.2192.

[18] N. Caderni and R. Fabbri, Phys. Lett. B **69,** 508 (1977).

[19] I. Brevik and L. T. Heen, Astrophys. Space Sci. **219,** 99 (1994).

[20] I. Brevik, E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **84,** 103508 (2011).

[21] S. Panchev, *Random Functions and Turbulence* (Pergamon Press, Oxford, 1971).

[22] I. Brevik, Z. Angew. Math. Mech. **72,** 145 (1992).

[23] R. A. Carhart and A. B. Kostinski, Phys. Lett. A **133,** 149 (1988).

[24] I. Brevik, O. Gorbunova, S. Nojiri, and S. D. Odintsov, Eur. Phys. J. C **71,** 1629 (2011).

[25] M. O. Calvao, J. A. S. Lima, and I. Waga, Phys. Lett. A **162,** 223 (1992).

[26] J. A. S. Lima and A. S. M. Germano, Phys. Lett. A **170,** 373 (1992).

[27] I. Brevik and G. Stokkan, Astrophys. Space Sci. **239,** 89 (1996).

[28] P. H. Frampton, K. J. Ludwick, and R. J. Scherrer, Phys. Rev. D **84,** 063003 (2011).

[29] I. Brevik and O. Gorbunova, Gen. Relativ. Gravit. **37,** 2039 (2005).

[30] I. Brevik, O. Gorbunova, and D. Saez-Gomez, Gen. Relativ. Gravit. **42,** 1513 (2010).