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New neutrino mass sum rule from the inverse seesaw mechanism

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A class of discrete flavor-symmetry-based models predicts constrained neutrino mass matrix schemes that lead to specific neutrino mass sum rules. One of these implies a lower bound on the effective neutrinoless double beta mass parameter, even for normal hierarchy neutrinos. Here we propose a new model based on the S_4 flavor symmetry that leads to the new neutrino mass sum rule and discuss how to generate a nonzero value for the reactor angle θ_{13} indicated by recent experiments, and the resulting correlation with the solar angle θ_{12}.

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I. INTRODUCTION

The discovery of neutrino oscillations have provided a strong evidence of the nonvanishing neutrino masses, although their nature (if they are Dirac or Majorana particles) has so far remained elusive. The observation of neutrinoless double beta decay ($0\nu 2\beta$) would provide an irrefutable confirmation of the Majorana nature of neutrinos [1]. Majorana neutrinos are characterized by a symmetric mass matrix whose parameters are restricted by the experimental data: the neutrino oscillation parameters, such as mixing angles and neutrino squared mass differences [2,3], as well as the limits on the $0\nu 2\beta$ effective mass parameter [4,5]. Indeed, upcoming $0\nu\beta\beta$ experiments are expected to improve the sensitivity by up to about 1 order of magnitude [6–9].

The general neutrino mixing matrix containing the three mixing angles and the *CP* violating phases can be parametrized in different equivalent ways [10–12]. A particular *ansatz* of the mixing matrix is the tribimaximal mixing matrix (TBM) [13], which, despite the fact of the nonzero value of the θ_{13} angle indicated by recent experiments [14–17], can still be used as a good first approximation, especially so taking into account that it can receive corrections from charged lepton diagonalization and/or from renormalization effects, depending on its scale of validity.

Several flavor models based in non-Abelian discrete symmetries predict a two-parameter neutrino mass matrix, which imply a particular mixing matrix form, as is pointed out in [13]. In Ref. [18] it was noted that in these models only the following mass relations can be obtained:

$$\chi m_2^\nu + \xi m_3^\nu = m_1^\nu, \tag{1}$$

$$\frac{\chi}{m_2^\nu} + \frac{\xi}{m_3^\nu} = \frac{1}{m_1^\nu}, \tag{2}$$

$$\chi\sqrt{m_2^\nu} + \xi\sqrt{m_3^\nu} = \sqrt{m_1^\nu}, \tag{3}$$

$$\frac{\chi}{\sqrt{m_2^\nu}} + \frac{\xi}{\sqrt{m_3^\nu}} = \frac{1}{\sqrt{m_1^\nu}}, \tag{4}$$

where χ and ξ are free parameters that characterize each specific model. For previous studies on the mass sum rules (1) and (2) see [19]. A classification of all models predicting TBM mixing that generate mass relations similar to the first three are given there. The last case is completely new, and here we will present a model from first principles, implementing the inverse seesaw mechanism [20,21] as well as a non-Abelian flavor symmetry [22], along the lines of Ref. [23], but adopting S_4, instead of A_4. Non-Abelian discrete flavor symmetries may also arise from the breaking of a continuous symmetry [24–27] or in orbifold constructions [28–30].

The inverse seesaw scheme constitutes the first example of a low-scale seesaw scheme [31] with naturally light neutrinos. The particle content is the same as that of the Standard Model except for the addition of a pair of two component gauge singlet leptons, ν_i^c and S_i,[1] within each of the three generations, labeled by i. The isodoublet neutrinos ν_i and the fermion singlets S_i have the same lepton number, opposite with respect to that of the three singlets ν_i^c associated with the "right-handed" neutrinos. In the ν, ν^c, S basis the 9×9 neutral lepton mass matrix M_ν has the form

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[1]Note that here S_i is used for fermion singlets.

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$$M_\nu = \begin{pmatrix} 0 & m_D^T & 0 \\ m_D & 0 & M^T \\ 0 & M & \mu \end{pmatrix}, \tag{5}$$

where m_D and M are arbitrary 3×3 complex matrices, while μ is symmetric due to the Pauli principle. Note that, in the limit as $\mu \to 0$, the lepton number symmetry is recovered, making the three light neutrinos strictly massless. Thus the smallness of neutrino mass is natural, in the sense of 't Hooft [32], as it is protected by $U(1)_L$. Note also that the idea behind the so-called inverse seesaw model can also be realized within other extended gauge groups such as left-right symmetry [33,34]. Moreover, in specific models, the smallness of μ may arise dynamically [35].

Following the seesaw diagonalization method in [36] one sees that, in the limit as $\mu \to 0$, U the matrix that diagonalizes M_ν consists of a maximal block rotation, corresponding to the Dirac nature of the three heavy leptons made up of ν_i^c and S_i, followed by the two rotations in the light and heavy sector, respectively.

As a result of diagonalization one obtains the effective light neutrino mass matrix as

$$m_\nu \sim m_D^T M^{T-1} \mu M^{-1} m_D, \tag{6}$$

with the entry μ being very small. The diagram illustrating the mass generation through the inverse seesaw mechanism is shown in Fig. 1.

It is straightforward to show that if m_D and μ are both proportional to the identity, and

$$M \sim M_{\mathrm{TBM}} = \begin{pmatrix} x & y & y \\ y & x+z & y-z \\ y & y-z & x+z \end{pmatrix}, \tag{7}$$

in the basis where the charged lepton mass matrix is diagonal, here there is a specific (complex) relation among the parameters x, y, and z [37], leaving only two free complex parameters, and we obtain the mass sum rule in Eq. (4).

In Sec. II we give our model. In Sec. III we present the predictions regarding the lower bound on the $0\nu 2\beta$ amplitude and discuss possible departures from tribimaximality, including a finite θ_{13} value. In Appendix C we present details on the symmetry structure, Yukawa couplings as well as the scalar potential of the model.

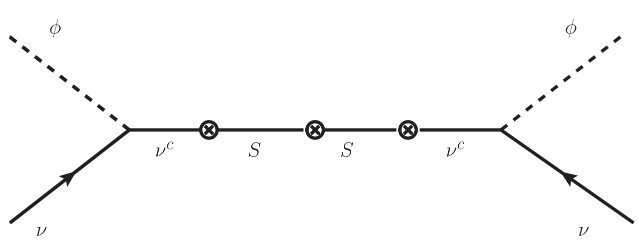

FIG. 1. Inverse seesaw mechanism.

II. THE MODEL

Here we follow Table I given in Ref. [23], where some possible schemes realizing the TBM pattern are summarized for the inverse seesaw case. From these we will implement case (1), since the other two cases, (2) and (3), correspond to the mass sum-rule relations (C) and (A), respectively, which already have model realizations in the existing literature,

$$M_D \propto I, \qquad \mu \propto I, \qquad M \propto \begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & C & B \end{pmatrix}. \tag{8}$$

In contrast to Ref. [23] here we adopt the S_4 flavor symmetry, instead of A_4.

To obtain the S_4-based inverse seesaw model we assign the charge matter fields as in Table I. Three right-handed neutrinos ν_R are introduced, as are three $SU(2)$ fermion singlets S_i, $i = 1, 2, 3$, the latter transforming as the $\mathbf{3}_1$ (note that ν^c and ν_R are conjugates, and hence have the opposite lepton number). All fermion fields in Table I transform as the triplet $\mathbf{3}_1$ and the Higgs doublet as the trivial singlet $\mathbf{1}_1$.

On the other hand, to generate the desired mass matrix structures we introduce five flavon fields, ϕ_ν, ϕ'_ν, ϕ_l, ϕ'_l, and ϕ''_l supplemented by the extra symmetries Z_3 and Z_2, whose assignments are given in Table II. The presence of these extra Abelian symmetries in the theory ensures the presence of adequate zeros in the neutrino mass matrix. We include in the model a scalar field σ, which breaks the lepton number through its vacuum expectation value (VEV). Such a field gives mass to the S field.[2] We keep renormalizability of the Lagrangian by adding a Frogatt–Nielsen fermion χ and its conjugate χ^c, both singlets under the weak $SU(2)$ gauge group [39–42]. In Table II we present the relevant quantum numbers of the matter fields in the theory under these extra symmetries.

The renormalizable Lagrangian relevant for neutrinos is

$$\mathcal{L}_\nu = Y_{D_{ij}} \bar{L}_i \nu_{R_j} h + Y^k_{\nu ij} \nu_{R_i} S_j \phi_{\nu_k} + Y'_{\nu ij} \nu_{R_i} S_j \phi'_\nu + \mu_{ij} S_i S_j \sigma, \tag{9}$$

while the renormalizable Yukawa terms involving the messenger fields is

$$\mathcal{L}_\chi = M_\chi \chi \chi^c + \bar{L} h \chi + \chi^c l_R \phi_l + \chi^c l_R \phi'_l + \chi^c l_R \phi''_l. \tag{10}$$

After integrating out the messenger fields χ, the effective Lagrangian for charged leptons takes the form

[2] Regarding the possible presence of a Majoron note that it can be made compatible in two ways, either by making the Majoron invisible through the addition of singlets (see, for example, extensive generic discussions of this point in the review in [38]) or by explicit soft symmetry breaking in the scalar potential that avoids it altogether. These technical details are outside the scope of this paper.

TABLE I. Fields and transformation properties under $SU(2)$, the S_4 flavor symmetry, and global lepton number $U_l(1)$.

	$\bar{L}$	ν_R	l_R	h	S	ϕ_ν	ϕ'_ν	ϕ_l	ϕ'_l	ϕ''_l	σ	χ	χ^c
$SU(2)$	2	1	1	2	1	1	1	1	1	1	1	1	1
S_4	3_1	3_1	3_1	1_1	3_1	3_1	1_1	3_1	3_2	1_1	1_1	3_1	3_1
$U(1)_L$	-1	1	1	0	-1	0	0	0	0	0	2	1	-1

TABLE II. Fields and their transformation properties under the Z_3 and Z_2 flavor symmetries.

	$\bar{L}$	ν_R	l_R	h	S	ϕ_ν	ϕ'_ν	ϕ_l	ϕ'_l	ϕ''_l	σ	χ	χ^c
Z_3	ω^2	ω	1	1	1	ω^2	ω^2	ω	ω	ω	1	ω	ω^2
Z_2	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$

$$\mathcal{L}_l = \frac{y_l}{\Lambda}(\bar{L}l_R)h\phi_l + \frac{y'_l}{\Lambda}(\bar{L}l_R)h\phi'_l + \frac{y''_l}{\Lambda}(\bar{L}l_R)h\phi''_l, \quad (11)$$

where Λ is the effective scale. This effective Lagrangian is responsible for charged lepton mass generation, as shown in Fig. 2.

To obtain the desired neutrino mixing matrix, we require the flavon fields to have the following alignments:

$$\langle\phi_\nu\rangle = v_\nu(1,0,0), \qquad \langle\phi_l\rangle = v_l(1,1,1), \qquad \langle\phi'_l\rangle = v'_l(1,1,1), \quad (12)$$

where we also define $\langle\phi'_\nu\rangle = v'_\nu$, $\langle\phi''_l\rangle = v''_l$, $\langle\sigma\rangle = v_\sigma$, and $\langle h\rangle = v$. In Appendix C we report the form of the potential (C2). We have verified that there exists a large portion of the parameter space where the required alignment is found to be a solution of the minimization of the potential.

With these alignments the three 3×3 blocks in Eq. (5) and charged lepton matrices take the form

$$(\mu) = \begin{pmatrix} \mu v_\sigma & 0 & 0 \\ 0 & \mu v_\sigma & 0 \\ 0 & 0 & \mu v_\sigma \end{pmatrix},$$

$$M_D = \begin{pmatrix} Y_D v & 0 & 0 \\ 0 & Y_D v & 0 \\ 0 & 0 & Y_D v \end{pmatrix},$$

$$M = \begin{pmatrix} Y'_\nu v'_\nu & 0 & 0 \\ 0 & Y'_\nu v'_\nu & Y_\nu v_\nu \\ 0 & Y_\nu v_\nu & Y'_\nu v'_\nu \end{pmatrix}, \quad (13)$$

and

$$M_l = \begin{pmatrix} y''_l v''_l & y_l v_l - y'_l v'_l & y_l v_l + y'_l v'_l \\ y_l v_l + y'_l v'_l & y''_l v''_l & y_l v_l - y'_l v'_l \\ y_l v_l - y'_l v'_l & y_l v_l + y'_l v'_l & y''_l v''_l \end{pmatrix} \frac{v}{\Lambda}. \quad (14)$$

The charged lepton mass matrix, Eq. (14), is diagonalized by the "magic" matrix

$$U_\omega = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad (15)$$

where $\omega^3 = 1$ and $1+\omega+\omega^2 = 0$. On the other hand, by using Eqs. (5) and (6) it is straightforward to obtain the light neutrino mass matrix, which takes the form

$$M_\nu = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{a^2+b^2}{(b^2-a^2)^2} & -\frac{2ab}{(b^2-a^2)^2} \\ 0 & -\frac{2ab}{(b^2-a^2)^2} & \frac{a^2+b^2}{(b^2-a^2)^2} \end{pmatrix}, \quad (16)$$

where $a = Y'_\nu v'_\nu/(\sqrt{\mu v_\sigma} Y_D v)$ and $b = Y_\nu v_\nu/(\sqrt{\mu v_\sigma} Y_D v)$. In the basis where charged lepton mass matrix is diagonal, the light neutrino mass matrix is diagonalized by the TBM form, and the corresponding eigenvalues are given by

$$m_1 = \frac{1}{(a+b)^2}, \qquad m_2 = \frac{1}{(a-b)^2}, \qquad m_3 = \frac{1}{a^2}. \quad (17)$$

With these eigenvalues we obtain the neutrino mass sum rule

$$\frac{1}{\sqrt{m_1}} = \frac{2}{\sqrt{m_3}} - \frac{1}{\sqrt{m_2}}, \quad (18)$$

which is, indeed, of the type given in Eq. (4).

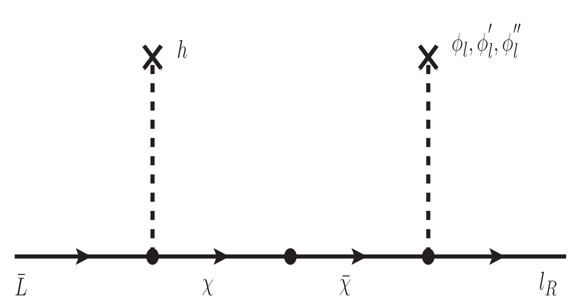

FIG. 2. Charged lepton mass generation.

III. PHENOMENOLOGY

A. Neutrinoless double beta decay

Using the symmetric parametrization of the lepton mixing matrix [10,11] we can obtain the general expression of the mass parameter $|m_{ee}|$, which determines the $0\nu 2\beta$ decay amplitude as

$$|m_{ee}| = \left|\sum_j U_{ej}^2 m_j\right| = \begin{cases} |c_{12}^2 c_{13}^2 m_1 + s_{12}^2 c_{13}^2 m_2 e^{\frac{1}{2}i\alpha_{21}} + s_{13}^2 m_3 e^{\frac{1}{2}i(\alpha_{31}-2\delta)}| & \text{(PDG[12])}, \\ |c_{12}^2 c_{13}^2 m_1 + s_{12}^2 c_{13}^2 m_2 e^{2i\phi_{12}} + s_{13}^2 m_3 e^{2i\phi_{13}}| & \text{(symmetrical)}, \end{cases} \tag{19}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$, m_i, $i = 1, 2, 3$, are the neutrino masses, and we adopt the symmetric parametrization where ϕ_{12} and ϕ_{13} are the two Majorana phases.

By varying the neutrino oscillation parameters in their allowed range, one can plot $|m_{ee}|$ in terms of the lightest neutrino mass. Depending on which is the lightest neutrino, one can have two different spectra, normal and inverse hierarchy, respectively. In the latter case one has a lower bound, on quite general grounds, as in this case there can be no destructive interference between the light neutrinos.

In the present scheme, however, as noted in Ref. [18], the neutrino mass sum rule can be interpreted geometrically as a triangle in the complex plane, its area providing a measure of the Majorana *CP* violation. Then, fixing the (ξ, χ) parameters for each model one can, in principle, determine the two Majorana *CP* violating phases; see Ref. [18] for details.

As a result there is a lower bound on $|m_{ee}|$ even in the case of normal hierarchy (for other schemes of this type see, for example, Ref. [18] and references therein), as illustrated in Fig. 3. Moreover, since the allowed ranges for normal and inverse hierarchy are much more constrained in this model than in the generic case, it becomes possible to distinguish the neutrino mass hierarchy even for lighter neutrinos lying within the nondegenerate mass regime.

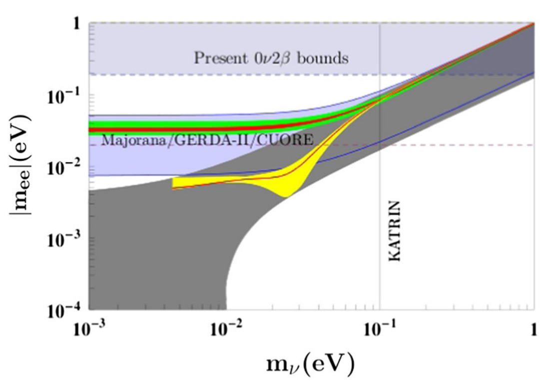

FIG. 3 (color online). $|m_{ee}|$ as a function of the lightest neutrino mass corresponding to the mass sum rule in Eq. (4). The lower band in gray and upper band in blue correspond to generic normal and inverse hierarchy regions, while the yellow (inside the lower gray band) and green (inside the upper yellow band) bands correspond to our flavor prediction varying the values of oscillation parameters in their 3σ C.L. range. The thin bands (in red) correspond to the TBM limit. The band in the top corresponds to the present bounds on $0\nu 2\beta$. For references to the experiments see [6–9,47].

B. Quark sector

Quarks are introduced as in Table III where, different from the lepton sector, we assign the first and second families to a doublet representation of S_4 and the third family to a singlet of S_4, namely, $Q_D = (Q_1, Q_2) \sim \mathbf{2}$, $q_{R_D} = (q_{R_1}, q_{R_2}) \sim \mathbf{2}$, $Q_s = Q_3 \sim \mathbf{1}_1$, and $q_{R_3} \sim \mathbf{1}_1$. We add flavons $\phi_{D,S}$ in doublet and singlet representations of the S_4,

$$\mathcal{L}_q^d = (Y_1^d \bar{Q}_S d_{R_S} \phi_S + Y_2^d \bar{Q}_D \phi_D d_{R_D} + Y_3^d \bar{Q}_D d_{R_D} \phi_S + Y_4^d \bar{Q}_D \phi_D d_{R_S} + Y_5^d \bar{Q}_S \phi_D d_{R_D}) h/\Lambda + \text{H.c.}, \tag{20}$$

$$\mathcal{L}_q^u = (Y_1^u \bar{Q}_S u_{R_S} \tilde{\phi}_S + Y_2^u \bar{Q}_D \tilde{\phi}_D u_{R_D} + Y_3^u \bar{Q}_D u_{R_D} \tilde{\phi}_S + Y_4^u \bar{Q}_D \tilde{\phi}_D u_{R_S} + Y_5^u \bar{Q}_S \tilde{\phi}_D u_{R_D}) \tilde{h}/\Lambda + \text{H.c.} \tag{21}$$

As in the charged lepton sector the dimension five operators can be given in terms of renormalizable interaction by introducing suitable messenger fields. Taking the VEV of ϕ_D in the direction (we verified that it is a possible solution of the potential)

$$\langle \phi_D \rangle \sim (-\sqrt{3}, 1), \tag{22}$$

the mass matrix for the quarks is

$$\mathbf{M}_{u(d)} = \begin{pmatrix} m_1^{u(d)} + m_2^{u(d)} & -\sqrt{3} m_2^{u(d)} & -\sqrt{3} m_5^{u(d)} \\ -\sqrt{3} m_2^{u(d)} & m_1^{u(d)} - m_2^{u(d)} & m_5^{u(d)} \\ -\sqrt{3} m_4^{u(d)} & m_4^{u(d)} & m_3^{u(d)} \end{pmatrix}, \tag{23}$$

which is very similar to the one proposed in Ref. [43] where a fit of the quark masses and mixing has been performed, and we refer to that paper for more detail.

C. Finite θ_{13} value

As we have discussed so far, the model leads to the TBM pattern. However, by coupling an extra S_4-doublet flavon field one can obtain corrections from the charged lepton sector, which induce nonzero values of θ_{13} as recently suggested by the Daya Bay experiment [14], the T2K

TABLE III. Quark sector and their transformation properties under the Z_3 and Z_2 flavor symmetries.

	$\bar{Q}_D$	$\bar{Q}_S$	u_{R_D}	u_{R_S}	d_{R_D}	d_{R_S}	ϕ_D	ϕ_S
$SU(2)$	2	2	1	1	1	1	1	1
S_4	2	1_1	2	1_1	2	1_1	2	1_1
Z_3	ω	ω	ω^2	ω^2	ω	ω	ω	ω
Z_2	+	+	−	−	−	−	−	−

[15], the Double Chooz [16], and the RENO [17] results, including also recent reactor flux calculations.

For example, consider a flavon scalar doublet under S_4, $\phi \sim \mathbf{2}$ and transforming as $(\omega, +)$ under $Z_3 \times Z_2$. In the Lagrangian we must then include the term

$$(\bar{L} l_R) h \phi. \tag{24}$$

This is a dimension five operator that can be obtained from a renormalizable Lagrangian by means of the messenger fields χ, χ^c of Table I as shown in Fig. 2. Assuming that ϕ acquires the VEV $\langle\phi\rangle = (u_1, u_2)$, a natural vacuum alignment is $u_1 = -\sqrt{3} u_2$, since this is consistent with the previous alignments in Eq. (12). Using multiplication rules in Appendix A one finds that the contribution from this term to the charged lepton mass matrix is

$$\delta M_l = \begin{pmatrix} -\sqrt{\frac{2}{3}} v u_2 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} v u_1 + \sqrt{\frac{1}{6}} v u_2 & 0 \\ 0 & 0 & -\sqrt{\frac{1}{2}} v u_1 + \sqrt{\frac{1}{6}} v u_2 \end{pmatrix}, \tag{25}$$

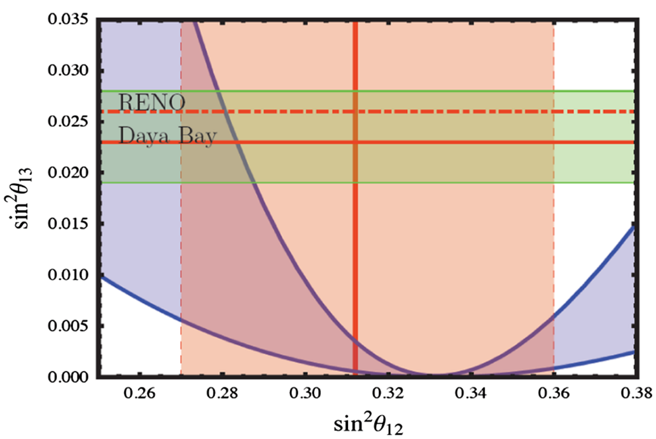

FIG. 4 (color online). The lavender region between the two curves represents the correlation between the reactor and solar neutrino mixing angles. The vertical red line corresponds to the best global determination of the solar mixing angle, while the vertical light red band corresponds to the 3σ region for the solar mixing angle. The horizontal dashed red line corresponds to the central value of the RENO measurement for the reactor mixing angle, the horizontal red line corresponds to the central value, and the horizontal green band corresponds to the 2σ of the Daya Bay Collaboration.

which modifies the diagonal entries δM_l in the charged lepton mass matrix, M_l, so that the total $M_l + \delta M_l$ is no longer diagonalized by U_ω. This way one can induce a potentially "large" value for θ_{13}, as hinted by recent experiments [15,16], and also potential departures of the solar and atmospheric angles from their TBM values. Moreover, in the presence of a nonzero θ_{13} one finds relations among these neutrino mixing angles. The most interesting of these is the correlation involving the solar and reactor angles, as illustrated in Fig. 4.[3] The horizontal green band represents the 2σ Daya Bay measurement [14], and its central value is indicated by the horizontal red line, while the horizontal dot-dashed line indicates the central value of the recent RENO measurement [17]. On the other hand, the vertical band, delimited by dotted lines, corresponds to the 3σ region for $\sin^2\theta_{12}$ found in the global analysis in Ref. [2], and the vertical line corresponds to the central value. The region in lavender shows the correlation between the reactor and solar angles. We observe that the deviation of θ_{13} from zero can be substantial provided the departure of θ_{12} from its TBM value is also large. Moreover, the model is consistent with the measurements of the two recent reactor experiments, only if the solar angle lies substantially BELOW the TBM prediction (at 2σ).

Needless to stress, a nonzero θ_{13} would also open the way also for the phenomenon of CP violation in neutrino oscillations, one of the central goals of the upcoming generation of long baseline oscillation studies [44,45].

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[3]Note also that the correlation is very sharp close to the TBM limit.

APPENDIX A: S_4 GROUP

The S_4 group is the discrete group given by the four objects' permutations. It contains 24 elements and can be obtained from two generators, S and T, satisfying

$$S^4 = T^3 = 1, \qquad ST^2S = T. \tag{A1}$$

The S_4 irreducible representations are two singlets, $\mathbf{1}_1$, $\mathbf{1}_2$, one doublet, $\mathbf{2}$, and two triplets, $\mathbf{3}_1$, $\mathbf{3}_2$. The product rules are given by (for more details see [46])

$$\begin{aligned}
\mathbf{1}_i \times \mathbf{1}_j &= \mathbf{1}_{(i+j)\mathrm{mod}2+1} \quad \forall\ i, j,\\
\mathbf{2} \times \mathbf{1}_i &= 2 \quad \forall\ i,\\
\mathbf{3}_i \times \mathbf{1}_j &= \mathbf{1}_{(i+j)\mathrm{mod}2+1} \quad \forall\ i, j,\\
\mathbf{3}_i \times \mathbf{2} &= \mathbf{3}_1 + \mathbf{3}_2 \quad \forall\ i,\\
\mathbf{3}_1 \times \mathbf{3}_2 &= \mathbf{1}_2 + \mathbf{2} + \mathbf{3}_1 + \mathbf{3}_2,\\
\mathbf{2} \times \mathbf{2} &= \mathbf{1}_1 + \mathbf{2} + \mathbf{1}_2,\\
\mathbf{3}_i \times \mathbf{3}_i &= \mathbf{1}_1 + \mathbf{2} + \mathbf{3}_1 + \mathbf{3}_2 \quad \forall\ i,
\end{aligned} \tag{A2}$$

where we can introduce the notation $[\mu \times \mu]$ and $\{\mu \times \mu\}$ for the symmetric and antisymmetric parts of $\mu \times \mu$, respectively:

$$\begin{aligned}
&[\mathbf{2}\times\mathbf{2}] = \mathbf{1}_1 + \mathbf{2}, \qquad \{\mathbf{2}\times\mathbf{2}\} = \mathbf{1}_2,\\
&[\mathbf{3}_i\times\mathbf{3}_i] = \mathbf{1}_1 + \mathbf{2} + \mathbf{3}_1, \qquad \{\mathbf{3}_i\times\mathbf{3}_i\} = \mathbf{3}_2 \quad \forall\ i.
\end{aligned} \tag{A3}$$

Given the following representations:

$$\begin{aligned}
&A, \qquad A' \sim \mathbf{1}_1, \qquad B, \qquad B' \sim \mathbf{1}_2,\\
&\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} \sim \mathbf{2}, \qquad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},\\
&\begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix} \sim \mathbf{3}_1, \qquad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \qquad \begin{pmatrix} c'_1 \\ c'_2 \\ c'_3 \end{pmatrix} \sim \mathbf{3}_2,
\end{aligned} \tag{A4}$$

the conjugate representations transform in the same way, as the representation matrices can be chosen all real.

For the product of one-dimensional representations the Clebsch–Gordan coefficients are the trivial products of representations, and also for the product of the 1_1 singlet with any nontrivial representation

$$\begin{pmatrix} Aa_1 \\ Aa_2 \end{pmatrix} \sim \mathbf{2}, \qquad \begin{pmatrix} Ab_1 \\ Ab_2 \\ Ab_3 \end{pmatrix} \sim \mathbf{3}_1, \qquad \begin{pmatrix} Ac_1 \\ Ac_2 \\ Ac_3 \end{pmatrix} \sim \mathbf{3}_2. \tag{A5}$$

For the product with the 1_2 singlet

$$\begin{pmatrix} -Ba_2 \\ Ba_1 \end{pmatrix} \sim \mathbf{2}, \qquad \begin{pmatrix} Ab_1 \\ Ab_2 \\ Ab_3 \end{pmatrix} \sim \mathbf{3}_2, \qquad \begin{pmatrix} Ac_1 \\ Ac_2 \\ Ac_3 \end{pmatrix} \sim \mathbf{3}_1. \tag{A6}$$

The Clebsch–Gordan coefficients for the product $\mathbf{2} \times \mathbf{2}$ are

$$\begin{aligned}
&a_1a'_1 + a_2a'_2 \sim \mathbf{1}_1, \qquad -a_1a'_2 + a_2a'_1 \sim \mathbf{1}_2,\\
&\begin{pmatrix} a_1a'_2 + a_2a'_1 \\ a_1a'_1 - a_2a'_2 \end{pmatrix} \sim \mathbf{2},
\end{aligned} \tag{A7}$$

for $\mathbf{3}_1 \times \mathbf{3}_1$

$$\begin{aligned}
&\sum_{j=1}^{3} b_jb'_j \sim \mathbf{1}_1, \qquad \begin{pmatrix} \frac{1}{\sqrt{2}}(b_2b'_2 - b_3b'_3) \\ \frac{1}{\sqrt{6}}(-2b_1b'_1 + b_2b'_2 + b_3b'_3) \end{pmatrix} \sim \mathbf{2},\\
&\begin{pmatrix} b_2b'_3 + b_3b'_2 \\ b_1b'_3 + b_3b'_1 \\ b_1b'_2 + b_2b'_1 \end{pmatrix} \sim \mathbf{3}_1, \qquad \begin{pmatrix} b_3b'_2 - b_2b'_3 \\ b_1b'_3 - b_3b'_1 \\ b_2b'_1 - b_1b'_2 \end{pmatrix} \sim \mathbf{3}_2,
\end{aligned} \tag{A8}$$

and for $\mathbf{3}_2 \times \mathbf{3}_2$

$$\begin{aligned}
&\sum_{j=1}^{3} c_jc'_j \sim \mathbf{1}_1, \qquad \begin{pmatrix} \frac{1}{\sqrt{2}}(c_2c'_2 - c_3c'_3) \\ \frac{1}{\sqrt{6}}(-2c_1c'_1 + c_2c'_2 + c_3c'_3) \end{pmatrix} \sim \mathbf{2},\\
&\begin{pmatrix} c_2c'_3 + c_3c'_2 \\ c_1c'_3 + c_3c'_1 \\ c_1c'_2 + c_2c'_1 \end{pmatrix} \sim \mathbf{3}_1, \qquad \begin{pmatrix} c_3c'_2 - c_2c'_3 \\ c_1c'_3 - c_3c'_1 \\ c_2c'_1 - c_1c'_2 \end{pmatrix} \sim \mathbf{3}_2.
\end{aligned} \tag{A9}$$

For the couplings $\mathbf{2} \times \mathbf{3}_1$ and $\mathbf{2} \times \mathbf{3}_2$, we have, respectively,

$$\begin{aligned}
&\begin{pmatrix} a_2b_1 \\ -\frac{1}{2}(\sqrt{3}a_1b_2 + a_2b_2) \\ \frac{1}{2}(\sqrt{3}a_1b_3 - a_2b_3) \end{pmatrix} \sim \mathbf{3}_1,\\
&\begin{pmatrix} a_1c_1 \\ \frac{1}{2}(\sqrt{3}a_2c_2 - a_1c_2) \\ -\frac{1}{2}(\sqrt{3}a_2c_3 + a_1c_3) \end{pmatrix} \sim \mathbf{3}_1,\\
&\begin{pmatrix} a_1b_1 \\ \frac{1}{2}(\sqrt{3}a_2b_2 - a_1b_2) \\ -\frac{1}{2}(\sqrt{3}a_2b_3 + a_1b_3) \end{pmatrix} \sim \mathbf{3}_2,\\
&\begin{pmatrix} a_2c_1 \\ -\frac{1}{2}(\sqrt{3}a_1c_2 + a_2c_2) \\ \frac{1}{2}(\sqrt{3}a_1c_3 - a_2c_3) \end{pmatrix} \sim \mathbf{3}_2.
\end{aligned} \tag{A10}$$

And finally, for the $\mathbf{3}_1 \times \mathbf{3}_2$ product

$$\begin{aligned}
&\sum_{j=1}^{3} b_jc_j \sim \mathbf{1}_2, \qquad \begin{pmatrix} \frac{1}{\sqrt{6}}(2b_1c_1 - b_2c_2 - b_3c_3) \\ \frac{1}{\sqrt{2}}(b_2c_2 - b_3c_3) \end{pmatrix} \sim \mathbf{2},\\
&\begin{pmatrix} b_3c_2 - b_2c_3 \\ b_1c_3 - b_3c_1 \\ b_2c_1 - b_1c_2 \end{pmatrix} \sim \mathbf{3}_1, \qquad \begin{pmatrix} b_2c_3 + b_3c_2 \\ b_1c_3 + b_3c_1 \\ b_1c_2 + b_2c_1 \end{pmatrix} \sim \mathbf{3}_2.
\end{aligned} \tag{A11}$$

APPENDIX B: YUKAWA COUPLINGS

Each term in the Lagrangian for neutrinos and charged leptons can be decomposed by components as

$$\mu S \cdot S\sigma = \mu(S_1S_1 + S_2S_2 + S_3S_3)\sigma, \tag{B1}$$

$$Y_D\bar{L} \cdot \nu_R h = Y_D(\bar{L}_1\nu_{1R} + \bar{L}_2\nu_{2R} + \bar{L}_3\nu_{3R})h, \tag{B2}$$

$$Y_\nu(\nu_R \cdot S)\phi_\nu = Y_\nu[(\nu_{2R}S_{3R} + \nu_{3R}S_2)\phi_{\nu 1} + (\nu_{1R}S_{3R} + \nu_{3R}S_1)\phi_{\nu 2} + (\nu_{1R}S_{2R} + \nu_{2R}S_1)\phi_{\nu 3}], \tag{B3}$$

$$Y'_\nu(\nu_R \cdot S)\phi'_\nu = Y'_\nu(\nu_{1R}S_1 + \nu_{2R}S_2 + \nu_{3R}S_3)\phi'_\nu, \tag{B4}$$

$$\frac{y_l}{\Lambda}(\bar{L}l_R)h\phi_l = \frac{y_l}{\Lambda}[(\bar{L}_2l_{3R} + \bar{L}_3l_{2R})h\phi_{l_1} + (\bar{L}_1l_{3R} + \bar{L}_3l_{1R})h\phi_{l_2} + (\bar{L}_1l_{2R} + \bar{L}_2l_{1R})h\phi_{l_3}], \tag{B5}$$

$$\frac{y'_l}{\Lambda}(\bar{L}l_R)h\phi'_l = \frac{y'_l}{\Lambda}[(\bar{L}_3l_{2R} - \bar{L}_2l_{3R})h\phi'_{l_1} + (\bar{L}_1l_{3R} - \bar{L}_3l_{1R})h\phi'_{l_2} + (\bar{L}_2l_{1R} - \bar{L}_1l_{2R})h\phi'_{l_3}], \tag{B6}$$

$$\frac{y''_l}{\Lambda}(\bar{L}l_R)h\phi''_l = \frac{y''_l}{\Lambda}(\bar{L}_1l_{1R} + \bar{L}_2l_{2R} + \bar{L}_3l_{3R})h\phi''_l]. \tag{B7}$$

APPENDIX C: SCALAR POTENTIAL

The most general renormalizable scalar potential is (without writing the S_4 products explicitly)

$$V = V(h) + V(\sigma) + V(\phi_\nu) + V(\phi'_\nu) + V(\phi_l) + V(\phi'_l) + V(\phi''_l) \tag{C1}$$

$$+ V(\phi_\nu, \phi'_\nu, \phi_l, \phi'_l, \phi''_l) + V(\sigma, h, \phi_\nu, \phi'_\nu, \phi_l, \phi'_l, \phi''_l), \tag{C2}$$

with

$$\begin{aligned}
V(h) &= \mu_h h^\dagger h + \lambda_h(h^\dagger h)(h^\dagger h), \qquad V(\sigma) = \mu_\sigma \sigma^\dagger\sigma + \lambda_\sigma(\sigma^\dagger\sigma)(\sigma^\dagger\sigma),\\
V(\phi_\nu) &= \mu_1(\phi_\nu^\dagger\phi_\nu) + \sum_i \lambda_j^\nu\{\phi_\nu^\dagger\phi_\nu\phi_\nu^\dagger\phi_\nu\}_i, \qquad V(\phi'_\nu) = \mu_2(\phi'^\dagger_\nu\phi'_\nu) + \sum_i \lambda_j^{\nu'}\{\phi'^\dagger_\nu\phi'_\nu\phi'^\dagger_\nu\phi'_\nu\}_i,\\
V(\phi_l) &= \mu_3(\phi_l^\dagger\phi_l) + \sum_i \lambda_i^l\{\phi_l^\dagger\phi_l\phi_l^\dagger\phi_l\}_i + \sum_i \kappa_i\{(\phi_l\phi_l)\phi_l + \text{H.c.}\}_i,\\
V(\phi'_l) &= \mu_4(\phi'^\dagger_l\phi'_l) + \sum_i \lambda_i^{l'}\{\phi'^\dagger_l\phi'_l\phi'^\dagger_l\phi'_l\}_i + \sum_i \kappa_i\{(\phi'_l\phi'_l)\phi'_l + \text{H.c.}\}_i,\\
V(\phi''_l) &= \mu_5(\phi''^\dagger_l\phi''_l) + \sum_i \lambda_i^{l''}\{\phi''^\dagger_l\phi''_l\phi''^\dagger_l\phi''_l\}_i + \text{H.c.},
\end{aligned}$$

$$\begin{aligned}
V(\sigma, h, \phi_\nu, \phi'_\nu, \phi_l, \phi'_l, \phi''_l) &= \lambda^{h\sigma}(h^\dagger h)(\sigma^\dagger\sigma) + \lambda^{\nu\sigma}(\phi_\nu^\dagger\phi_\nu)(\sigma^\dagger\sigma) + \lambda^{\nu'\sigma}(\phi'^\dagger_\nu\phi'_\nu)(\sigma^\dagger\sigma) + \lambda^{l\sigma}(\phi_l^\dagger\phi_l)(\sigma^\dagger\sigma)\\
&\quad + \lambda^{l'\sigma}(\phi'^\dagger_l\phi'_l)(\sigma^\dagger\sigma) + \lambda^{l''\sigma}(\phi''^\dagger_l\phi''_l)(\sigma^\dagger\sigma),\\
V(\phi_\nu, \phi'_\nu, \phi_l, \phi'_l, \phi''_l) &= \sum_i \kappa_i\{\phi_l\phi_l\phi'_l\}_i + \sum_i \kappa_i\{\phi_l\phi'_l\phi'_l\}_i + \sum_i \kappa_i\{\phi_l\phi_l\phi''_l\}_i + \sum_i \kappa_i\{\phi'_l\phi'_l\phi''_l\}_i + \sum_i \kappa_i\{\phi_l^\dagger\phi_\nu\phi_\nu\}_i\\
&\quad + \sum_i \kappa_i\{\phi'^\dagger_l\phi_\nu\phi_\nu\}_i + \sum_i \kappa_i\{\phi''^\dagger_l\phi_\nu\phi_\nu\}_i + \sum_i \kappa_i\{\phi_l^\dagger\phi_\nu\phi'_\nu\}_i + \sum_i \kappa_i\{\phi''^\dagger_l\phi'_\nu\phi'_\nu\}_i\\
&\quad + \sum_i \lambda_i^{lll'l'}\{\phi_l^\dagger\phi_l^\dagger\phi'_l\phi'_l\}_i + \sum_i \lambda_i^{ll'l'l'}\{\phi_l\phi'^\dagger_l\phi'_l\phi'_l\}_i + \sum_i \lambda_i^{llll'}\{\phi_l^\dagger\phi_l^\dagger\phi_l\phi'_l\}_i\\
&\quad + \sum_i \lambda_i^{llll'}\{\phi'^\dagger_l\phi_l^\dagger\phi_l\phi'_l\}_i + \sum_i \lambda_i^{ll'l'l'}\{\phi_l^\dagger\phi'^\dagger_l\phi'_l\phi'_l\}_i + \sum_i \lambda_i^{llll'}\{\phi_l^\dagger\phi_l^\dagger\phi_l\phi''_l\}_i\\
&\quad + \sum_i \lambda_i^{llll''}\{\phi_l^\dagger\phi_l^\dagger\phi'_l\phi''_l\}_i + \sum_i \lambda_i^{ll'll''}\{\phi_l^\dagger\phi'^\dagger_l\phi_l\phi''_l\}_i + \sum_i \lambda_i^{ll'l'l''}\{\phi_l^\dagger\phi'^\dagger_l\phi'_l\phi''_l\}_i\\
&\quad + \sum_i \lambda_i^{l'l'll''}\{\phi'^\dagger_l\phi'^\dagger_l\phi_l\phi''_l\}_i + \sum_i \lambda_i^{l'l'l'l''}\{\phi'^\dagger_l\phi'^\dagger_l\phi'_l\phi''_l\}_i + \sum_i \lambda_i^{lll''l''}\{\phi_l^\dagger\phi_l^\dagger\phi''_l\phi''_l\}_i\\
&\quad + \sum_i \lambda_i^{l'l'l''l''}\{\phi'^\dagger_l\phi'^\dagger_l\phi''_l\phi''_l\}_i + \sum_i \lambda_i^{ll''ll''}\{\phi_l^\dagger\phi''^\dagger_l\phi_l\phi''_l\}_i + \sum_i \lambda_i^{l'l''l'l''}\{\phi'^\dagger_l\phi''^\dagger_l\phi'_l\phi''_l\}_i\\
&\quad + \sum_i \lambda_i^{ll\nu\nu}\{\phi_l\phi_l\phi_\nu\phi_\nu\}_i + \sum_i \lambda_i^{ll'\nu\nu}\{\phi_l\phi'_l\phi_\nu\phi_\nu\}_i + \sum_i \lambda_i^{l'l'\nu\nu}\{\phi'_l\phi'_l\phi_\nu\phi_\nu\}_i\\
&\quad + \sum_i \lambda_i^{ll''\nu\nu}\{\phi_l\phi''_l\phi_\nu\phi_\nu\}_i + \sum_i \lambda_i^{l'l''\nu\nu}\{\phi'_l\phi''_l\phi_\nu\phi_\nu\}_i + \sum_i \lambda_i^{l''l''\nu\nu}\{\phi''_l\phi''_l\phi_\nu\phi_\nu\}_i
\end{aligned}$$

$$
\begin{aligned}
&+\sum_i \lambda_i^{ll\nu\nu}\{\phi_l^\dagger\phi_l\phi_\nu^\dagger\phi_\nu\}_i+\sum_i \lambda_i^{ll'\nu\nu}\{\phi_l^\dagger\phi_l'\phi_\nu^\dagger\phi_\nu\}_i+\sum_i \lambda_i^{l'l'\nu\nu}\{\phi_l'^\dagger\phi_l'\phi_\nu^\dagger\phi_\nu\}_i\\
&+\sum_i \lambda_i^{ll''\nu\nu}\{\phi_l^\dagger\phi_l''\phi_\nu^\dagger\phi_\nu\}_i+\sum_i \lambda_i^{l'l''\nu\nu}\{\phi_l'^\dagger\phi_l''\phi_\nu^\dagger\phi_\nu\}_i+\sum_i \lambda_i^{ll\nu\nu'}\{\phi_l\phi_l\phi_\nu\phi_\nu'\}_i\\
&+\sum_i \lambda_i^{ll'\nu\nu'}\{\phi_l\phi_l'\phi_\nu\phi_\nu'\}_i+\sum_i \lambda_i^{l'l'\nu\nu'}\{\phi_l'\phi_l'\phi_\nu\phi_\nu'\}_i+\sum_i \lambda_i^{l'l''\nu\nu'}\{\phi_l'\phi_l''\phi_\nu\phi_\nu'\}_i\\
&+\sum_i \lambda_i^{ll\nu\nu'}\{\phi_l^\dagger\phi_l\phi_\nu^\dagger\phi_\nu'\}_i+\sum_i \lambda_i^{ll'\nu\nu'}\{\phi_l^\dagger\phi_l'\phi_\nu^\dagger\phi_\nu'\}_i+\sum_i \lambda_i^{l'l\nu\nu'}\{\phi_l'^\dagger\phi_l\phi_\nu^\dagger\phi_\nu'\}_i\\
&+\sum_i \lambda_i^{l'l'\nu\nu'}\{\phi_l'^\dagger\phi_l'\phi_\nu^\dagger\phi_\nu'\}_i+\sum_i \lambda_i^{ll''\nu\nu'}\{\phi_l^\dagger\phi_l''\phi_\nu^\dagger\phi_\nu'\}_i+\sum_i \lambda_i^{\nu\nu\nu\nu'}\{\phi_\nu^\dagger\phi_\nu\phi_\nu^\dagger\phi_\nu'\}_i\\
&+\sum_i \lambda_i^{\nu\nu'\nu\nu'}\{\phi_\nu^\dagger\phi_\nu'\phi_\nu^\dagger\phi_\nu'\}_i+\sum_i \lambda_i^{\nu\nu\nu'\nu'}\{\phi_\nu^\dagger\phi_\nu\phi_\nu'^\dagger\phi_\nu'\}_i+\sum_i \lambda_i^{ll\nu'\nu'}\{\phi_l\phi_l\phi_\nu'\phi_\nu'\}_i\\
&+\sum_i \lambda_i^{l'l'\nu'\nu'}\{\phi_l'\phi_l'\phi_\nu'\phi_\nu'\}_i+\sum_i \lambda_i^{l''l''\nu'\nu'}\{\phi_l''\phi_l''\phi_\nu'\phi_\nu'\}_i+\sum_i \lambda_i^{ll''\nu\nu'}\{\phi_l^\dagger\phi_l''\phi_\nu'^\dagger\phi_\nu\}_i\\
&+\sum_i \lambda_i^{ll\nu'\nu'}\{\phi_l^\dagger\phi_l\phi_\nu'^\dagger\phi_\nu'\}_i+\sum_i \lambda_i^{l'l'\nu'\nu'}\{\phi_l'^\dagger\phi_l'\phi_\nu'^\dagger\phi_\nu'\}_i+\sum_i \lambda_i^{l''l''\nu\nu}\{\phi_l''^\dagger\phi_l''\phi_\nu^\dagger\phi_\nu\}_i\\
&+\sum_i \lambda_i^{l''l''\nu'\nu'}\{\phi_l''^\dagger\phi_l''\phi_\nu'^\dagger\phi_\nu'\}_i+\mathrm{H.c.},
\end{aligned}
$$

where $\sum_i\lambda_i\{\}_i$, $\sum_i\kappa_i\{\}_i$ sums over all possible ways to group the fields inside the brackets and make the product of representations in order to obtain a singlet.

APPENDIX D: SCALE OF THE FLAVON FIELD VEVS

In this appendix, we give an example for the possible scale of the flavon fields. From the Lagrangian of the charged leptons in Eqs. (B5)–(B7) we take, for instance, the Yukawa couplings to be of order one and all the field VEVs of the same order of magnitude, which we set at the scale of TeV. For definiteness let us take the term in Eq. (B5)

$$
y_l v\frac{\langle\phi_l\rangle}{\Lambda}\sim\frac{1}{3}m_\tau, \tag{D1}
$$

where the scale Λ corresponds to the that of the mass of the fields χ. Taking $y_l\sim 1$ in Eq. (D1) gives

$$
\frac{\langle\phi_l\rangle}{\Lambda}\sim\frac{1}{3}\frac{m_\tau}{v}\approx 3.4\times 10^{-3}. \tag{D2}
$$

Of course, we can always tune the Yukawa couplings in order to change this ratio. The same is valid for the flavon VEVs in Eqs. (B6) and (B7).

Let us continue with this example: from the neutrino sector, we have that the neutrino mass is of the order[4]

$$
m_\nu\sim\mu v_\sigma\frac{v^2}{v_\nu^2}\sim\mu v_\sigma\frac{(100\ \mathrm{GeV})^2}{(3000\ \mathrm{GeV})^2}\sim 10^{-3}\mu v_\sigma, \tag{D3}
$$

where we choose a mass for the right-handed neutrinos of the order of 3 TeV. Then a neutrino mass matrix of 1 eV would correspond to $\mu v_\sigma\sim 1$ KeV.

[4] We again set the Yukawa couplings to be order one.

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