Neutral kaon mixing beyond the standard model with $n_f = 2 + 1$ chiral fermions

P. A. Boyle,[1] N. Garron,[1,*] and R. J. Hudspith[1]

(The RBC and UKQCD Collaborations)

[1]*SUPA, School of Physics, The University of Edinburgh, Edinburgh EH9 3JZ, United Kingdom*

We compute the hadronic matrix elements of the four-quark operators needed for the study of $K^0 - \bar{K}^0$ mixing beyond the Standard Model (SM). We use $n_f = 2 + 1$ flavors of domain-wall fermion that exhibit good chiral-flavor symmetry. The renormalization is performed nonperturbatively through the RI-MOM scheme and our results are converted perturbatively to $\overline{\text{MS}}$. The computation is performed on a single lattice spacing $a \sim 0.086$ with a lightest unitary pion mass of 290 MeV. The various systematic errors, including the discretization effects, are estimated and discussed. Our results confirm a previous quenched study, where large ratios of non-SM to SM matrix elements were obtained.

I. INTRODUCTION

Recent progress achieved by the lattice community is greatly improving our theoretical understanding of *CP* violation in kaon decays. The experimentally well-measured parameters ε_K and ε'_K, which quantify indirect and direct *CP* violation in $K \to \pi\pi$, can be confronted with a theoretical computation of the $K \to (\pi\pi)_{I=0,2}$ amplitudes and of neutral kaon mixing, providing the nonperturbative effects are correctly accounted for. The first direct and realistic computation of the $K \to (\pi\pi)_{I=2}$ amplitude has been recently performed [1,2]. A complete computation of the $\Delta I = 1/2$ amplitude is still missing, but important work is being performed in that direction [3]. The situation is much more favorable for neutral kaon oscillations: the bag parameter B_K that describes the long-distance contributions to neutral kaon mixing in the Standard Model (SM), is now computed with a precision of a few percent (see e.g. [4–6]) for recent unquenched determinations). By combining B_K with the experimental value of ε_K, one obtains important constraints on the free parameters associated with quark flavor mixing (see [7] for a recent pedagogical review). In principle, the same techniques can be applied for beyond the Standard Model (BSM) theories (see for example [8–17]), but not much is known concerning the long-distance contributions of the nonstandard operators beyond the quenched approximation.

In this paper, we present the first realistic computation of the matrix elements of neutral kaon mixing beyond the Standard Model. Previous studies were either preliminary [18,19] or suffered from the quenched approximation [20,21]. Since a noticeable disagreement was observed between Refs. [20] and [21] it is important to repeat this computation in a more realistic framework [22].

II. FORMALISM

In the SM, neutral kaon mixing is dominated by box diagrams as in Fig. 1. By performing an operator product expansion, one factorizes the long-distance effects in the weak matrix element (WME) $\langle \bar{K}^0 | O_1^{\Delta s=2} | K^0 \rangle$ of the four-quark operator $O_1^{\Delta s=2}$ given by (α and β are color indices)

$$O_1^{\Delta s=2} = (\bar{s}_\alpha \gamma_\mu (1 - \gamma_5) d_\alpha)(\bar{s}_\beta \gamma_\mu (1 - \gamma_5) d_\beta). \quad (1)$$

Only one four-quark operator appears in the SM because neutral kaon mixing occurs under *W*-boson exchange, implying a "vector-axial" Dirac structure. Since this four-quark operator is invariant under Fierz rearrangement, the two different color structures (mixed and unmixed) are equivalent.

We also seek to understand whether new physics beyond the Standard Model could play a detectable role in kaon *CP* violation. Assuming that this new physics is perturbative, we might also describe this by contributions to the effective Hamiltonian, and other four-quark operators are induced by such extensions of the SM. It is conventional to introduce the so-called SUSY basis $O_{i=1\ldots5}^{\Delta s=2}$: in addition to the SM operator $O_1^{\Delta s=2}$, we define [24,25]

$$
\begin{aligned}
O_2^{\Delta s=2} &= (\bar{s}_\alpha (1 - \gamma_5) d_\alpha)(\bar{s}_\beta (1 - \gamma_5) d_\beta), \\
O_3^{\Delta s=2} &= (\bar{s}_\alpha (1 - \gamma_5) d_\beta)(\bar{s}_\beta (1 - \gamma_5) d_\alpha), \\
O_4^{\Delta s=2} &= (\bar{s}_\alpha (1 - \gamma_5) d_\alpha)(\bar{s}_\beta (1 + \gamma_5) d_\beta), \\
O_5^{\Delta s=2} &= (\bar{s}_\alpha (1 - \gamma_5) d_\beta)(\bar{s}_\beta (1 + \gamma_5) d_\alpha).
\end{aligned}
\quad (2)
$$

These four-quark operators appear in the generic $\Delta s = 2$ Hamiltonian

$$H^{\Delta s=2} = \sum_{i=1}^{5} C_i(\mu) O_i^{\Delta s=2}(\mu), \quad (3)$$

where μ is a renormalization scale. The Wilson coefficients C_i, which encode the short-distance effects, depend on the new physics model under consideration. The long-distance effects are factorized into the matrix elements of

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*Now at School of Mathematics, Trinity College, Dublin 2, Ireland.
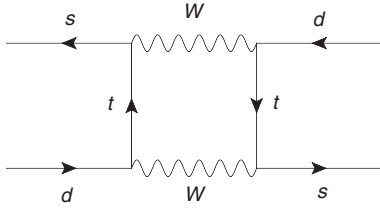
FIG. 1.   Diagram contributing to $K^0 - \bar{K}^0$ mixing in the SM.

the four-quark operators $O_i^{\Delta s=2}$. Lattice QCD offers a unique opportunity to quantify these effects in a model-independent way.

In phenomenological applications, by combining our results for the BSM matrix elements with experimental observables (typically the mass difference $\Delta M_K = m_{K_L} - m_{K_S}$ and ϵ_K), one obtains important constraints on the model under consideration. The Wilson coefficients and the bare matrix elements have to be converted into a common scheme, at a given scale μ (typically $\overline{\text{MS}}$ at a scale of 2 or 3 GeV). In the SM case ($i = 1$) it is conventional to introduce the kaon bag parameter B_K, which measures the deviation of the SM matrix element from its vacuum saturation approximation

$$B_K = \frac{\langle \bar{K}^0 | O_1 | K^0 \rangle}{\frac{8}{3} m_K^2 f_K^2}, \tag{4}$$

where the normalization for the decay constant is such that $f_{K^-} = 156.1$ MeV. Several normalizations for the BSM operators have been proposed in the literature (see for example [20]); in this work we follow [21] and define the ratios

$$R_i^{\text{BSM}} = \left[\frac{f_K^2}{m_K^2} \right]_{\text{expt}} \left[\frac{m_P^2}{f_P^2} \frac{\langle \bar{P} | O_i | P \rangle}{\langle \bar{P} | O_1 | P \rangle} \right]_{\text{latt}}, \tag{5}$$

where P is a pseudoscalar particle of mass m_P and decay constant f_P. The term $[\ ]_{\text{latt}}$ is obtained from our lattice simulations for different values of m_P.

As advocated in [21], there are various reasons to choose this normalization rather than, for example, the standard bag parameters. In particular, we expect some systematic errors to cancel in the ratio of the bare WMEs; there is no need to introduce the light quark masses and the partial conservation of the axial current. Furthermore, the mass factors in Eq. (5) compensate the leading-order chiral behavior of the WMEs, making all of the chiral extrapolations smoother. Finally, at the physical point $P = K^0$, the R_i^{BSM}'s give directly the ratios of the non-SM to SM contributions.

The R_i^{BSM}'s will be the main result of this work, but for completeness we will also give the BSM bag parametrization, defined as (where $N_{2,...5} = \frac{5}{3}, -\frac{1}{3}, -2, -\frac{2}{3}$) [26]

$$B_i = -\frac{\langle \bar{K}^0 | O_i | K^0 \rangle}{N_i \langle \bar{K}^0 | \bar{s} \gamma_5 d | 0 \rangle \langle 0 | \bar{s} \gamma_5 d | K^0 \rangle}, \quad i = 2, \ldots, 5. \tag{6}$$

III. COMPUTATION DETAILS

This computation is performed on the finer of the two ensembles described in detail in [4,27]. We use $32^3 \times 64 \times 16$ (the last number corresponds to the length of the fifth dimension of the domain-wall action in lattice units) Iwasaki gauge configurations with an inverse lattice spacing $a^{-1} = 2.28(3)$ GeV, corresponding to $a \sim 0.086$ fm. The quarks are described by the domain-wall action [28–30], both in the valence and in the sea sectors. We have three different values of the light sea quark mass $am_{\text{light}}^{\text{sea}} = 0.004, 0.006, 0.008$ corresponding to unitary pion masses of approximately 290, 340, and 390 MeV, respectively. The simulated strange sea quark mass is $am_{\text{strange}}^{\text{sea}} = 0.03$, which is close to its physical value 0.0273(7) [27]. For the main results of this work, we consider only unitary light quarks, $am_{\text{light}}^{\text{valence}} = am_{\text{light}}^{\text{sea}}$, whereas for the physical strange we interpolate between the unitary ($am_{\text{strange}}^{\text{valence}} = am_{\text{strange}}^{\text{sea}} = 0.03$) and the partially quenched ($am_{\text{strange}}^{\text{valence}} = 0.025$) data.

To extract the bare matrix elements we follow [4,27], where the procedure for the evaluation of the two-point functions and of the three-point function of the SM operator has been explained in detail. In particular, we have used Coulomb gauge fixed wall sources to obtain very good statistical precision. From the measurement of B_K, the generalization to the BSM operators is straightforward. We define the three-point function

$$c_i = \langle \bar{P}(t_f) O_{i=2,...5}^{\Delta S=2}(t) \bar{P}(t_i) \rangle. \tag{7}$$
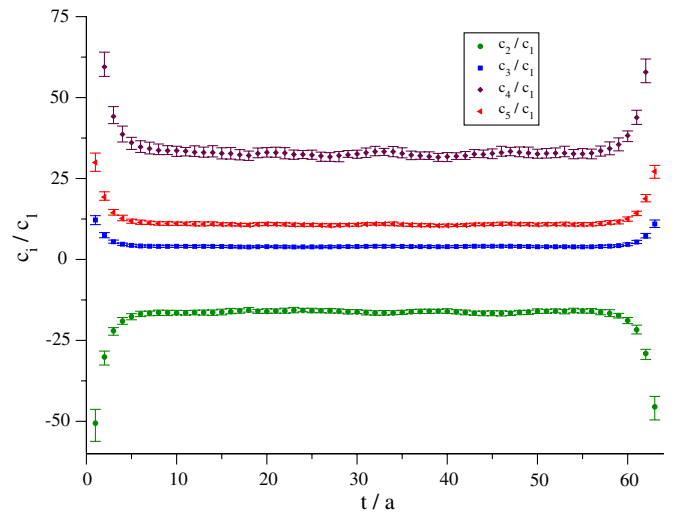
FIG. 2 (color online).   Ratios of the bare three-point functions from which we extract R_i^{BSM}. Results are shown for our lightest simulated unitary kaon.

From the asymptotic Euclidean time behavior of the ratios of three-point functions c_i/c_1 (we fit these ratios to a constant in the interval $t/a = [12, 52]$ [31]), we obtain the bare matrix elements of the four-quark operators normalized by the SM one: $[\langle \bar{P}|O_i^{\Delta S=2}|P\rangle/\langle \bar{P}|O_1^{\Delta S=2}|P\rangle]^{\text{bare}}$. In Fig. 2, we show the corresponding plateaus obtained for our lightest unitary kaon $am_{\text{light}}^{\text{sea}} = am_{\text{light}}^{\text{valence}} = 0.004$, $am_{\text{strange}}^{\text{sea}} = am_{\text{strange}}^{\text{valence}} = 0.03$.

IV. RENORMALIZATION

The four-quark operators given in Eqs. (1) and (2) mix under renormalization. In a scheme that preserves chiral symmetry, the mixing pattern is given by the transformation properties of these operators under chiral rotations $SU(3)_L \times SU(3)_R$. The SM operator $O_1^{\Delta S=2}$ belongs to a $(27, 1)$ irreducible representation of $SU(3)_L \times SU(3)_R$ and renormalizes multiplicatively. The BSM operators fall into two categories: $O_2^{\Delta s=2}$ and $O_3^{\Delta s=2}$ transform like $(6, \bar{6})$ and mix together. Likewise $O_4^{\Delta s=2}$ and $O_5^{\Delta s=2}$ belong to $(8, 8)$ and also mix. If chiral symmetry is realized, the five-by-five renormalization matrix Z_{ij} is block diagonal: a single factor for the $(27, 1)$ operator and two 2×2 matrices for the BSM operators. Because we work with the domain-wall fermions formulation, in which the explicit breaking of chiral symmetry can be made arbitrarily small (and in practice numerically irrelevant), we expect to find this continuum pattern, up to small discretization effects.

We perform the renormalization of the four-quark operators $O_i^{\Delta s=2}$ nonperturbatively in the RI-MOM scheme [32]. We compute the forward, amputated, vertex functions of the relevant operators between external quark states in the lattice Landau gauge for a given set of momenta. As a renormalization condition, we impose these to be equal to their tree-level values once projected onto their color-Dirac structures and extrapolated to the chiral limit. By using both momentum sources [33] and partially twisted boundary conditions, we obtain smooth functions of the external momentum with very good statistical accuracy [34]. Although we have performed this computation in various intermediate schemes (including the nonexceptional (γ_μ, γ_μ)-scheme introduced in [4]), we quote here the results obtained via the RI-MOM scheme because only in this case are the conversion factors to $\overline{\text{MS}}$ (computed in continuum perturbation theory) available for the whole set of operators. In such a scheme, the presence of exceptional channels enhances the Goldstone pole contaminations [18,36], which have then been subtracted explicitly [37]. We choose to impose the renormalization conditions at $\mu = 3$ GeV. At this scale, perturbation theory (PT) converges rather quickly ($\alpha_s \sim 0.25$), the chiral symmetry breaking effects are small, and we still have good control on the discretization effects [4,38]. Matching at this scale is important for our error budget since the perturbative conversion factors between the lattice schemes and $\overline{\text{MS}}$ of the
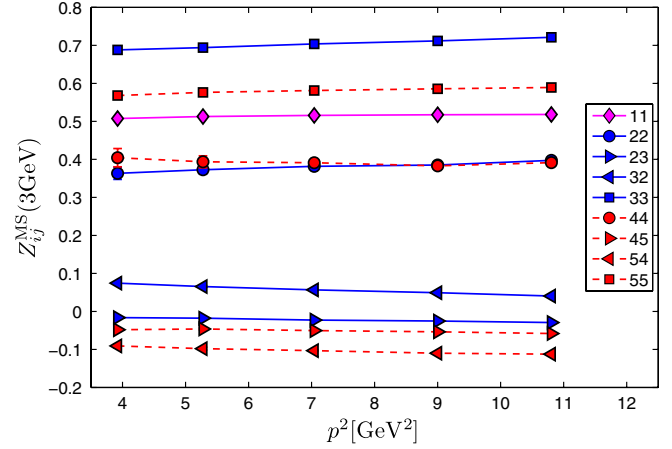

FIG. 3 (color online). Renormalization factors of the four-quark operators. At each value of the simulated momentum p, we run to the scale of 3 GeV and convert to $\overline{\text{MS}}$. The remaining scale dependence can be imputed to the truncation of the perturbative expansion. We show only the Z factors allowed by chiral symmetry.

four-quark operators are only known at one loop (as discussed later, matching to PT is actually one of our dominant sources of systematic error). We observe that the effects of chiral symmetry breaking are not completely negligible, even at 3 GeV [35]. Therefore we must assess a systematic error to the mixing of operators of different chirality (see next section). We have checked that in a nonexceptional scheme this small chirally forbidden mixing is strongly reduced and becomes numerically irrelevant at 3 GeV [18,35]. Thus we conclude that this effect is a physical manifestation of the infrared behavior of the exceptional intermediate scheme. The results for the chirally allowed renormalization factors $Z_{ij}^{\overline{\text{MS}}}(3$ GeV$)$ are shown in Fig. 3. They relate the bare four-quark operators to the renormalized ones through the usual relation (Z_q is the renormalization factor of the quark wave function and cancels in the ratios)

$$O_i^{\Delta s=2}(3 \text{ GeV}) = \frac{Z_{ij}}{Z_q^2}(3 \text{ GeV})O_j^{\Delta s=2}. \qquad (8)$$

V. PHYSICAL RESULTS AND ERROR ESTIMATION

Once the bare ratios have been renormalized, we extrapolate them to the physical kaon mass. We have explored different strategies, such as a simple polynomial form or next-to-leading-order chiral perturbation theory (ChPT) predictions [39]. The value of the simulated unitary strange mass differs somewhat from the physical one; therefore, we perform an interpolation using a partially quenched strange quark. We find that the R^{BSM}'s exhibit a very mild quark mass dependence (see Fig. 4 [43]); therefore, we take the results obtained with the polynomial Ansatz as our central values [44].
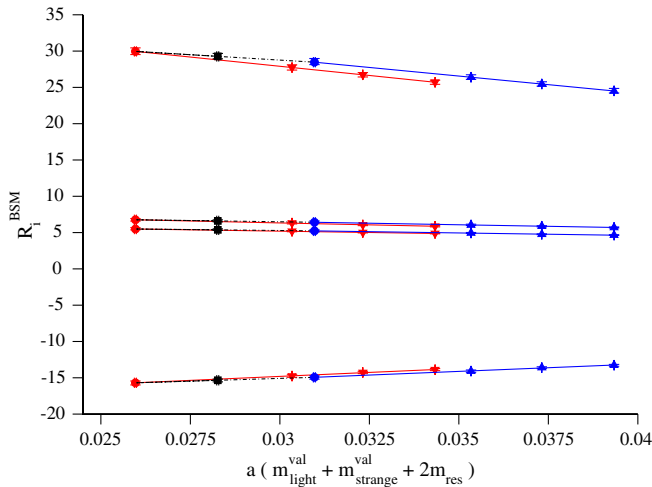
FIG. 4 (color online). Renormalized BSM ratios $R_i^{\rm BSM}$ in function of the bare valence quark mass. We show the three unitary light quarks for both the unitary strange (upward blue triangles) and the partially quenched strange (downward blue triangles), together with their extrapolations in the light sector (blue and red circles) and their interpolation to the physical kaon mass (black squares).

Our final results are the $R_i^{\rm BSM}$ quoted in $\overline{\rm MS}$ at 3 GeV given in Table I. For completeness, we also convert these to the BSM bag parameters, using Eq. (6) [45]. We also note that, using the same framework, the SM contribution is found to be $B_1 = B_K = 0.517(4)_{\rm stat}$ in the $\overline{\rm MS}$ scheme at 3 GeV, whereas a continuum value of $0.529(5)_{\rm stat}(19)_{\rm syst}$ was quoted in [4]. The difference comes from the fact that a different intermediate scheme was used in [4] (such a difference is accounted for in our estimation of the systematic errors). From the same reference, the discretization effects for B_K on this lattice are seen to be of the order of 1.5%. Since we have only one lattice spacing for the BSM ratios, we make the assumption that the discretization errors are of the same size as those affecting B_K, and estimate a 1.5% error to all the different operators. This might appear like a crude estimate, but this effect is expected to be subdominant compared to other sources of systematic errors. The next systematic error (called "extr.") represents the spread of the results obtained from different extrapolation strategies to the physical point. The systematic error associated with the nonperturbative renormalization (NPR) has been estimated from the breaking of chiral symmetry. The mixing of the $(6, \bar{6})$ with the $(8, 8)$ operators is forbidden by chiral symmetry but likely to be enhanced by the exchange of light pseudoscalar particles. As the matrix element of $O_4^{\Delta s=2}$ is numerically large, this nonphysical mixing has an effect on $O_2^{\Delta s=2}$ and $O_3^{\Delta s=2}$ of the order of 8%–9%. As discussed in the previous section, this unwanted infrared effect is absent if a nonexceptional scheme is used. The last error we quote ("PT") arises from the matching between the intermediate

TABLE I. Final results of this work: the first two columns show the ratios $R_i^{\rm BSM}$ and the corresponding bag parameters B_i in $\overline{\rm MS}$ at 3 GeV, together with their total error, combining systematics and statistics. In the remaining columns, we give our error budget for the $R^{\rm BSM}$, detailing the contributions in percentage of the different sources of systematics (see text for more details).

	$R_i^{\rm BSM}$	B_i	Stat.	Discr.	Extr.	NPR	PT	Total
2	$-15.3(1.7)$	0.43(5)	1.3	1.5	4.0	9.4	4.7	11.3
3	5.4(0.6)	0.75(9)	2.0	1.5	3.9	7.8	7.6	12.0
4	29.3(2.9)	0.69(7)	1.3	1.5	4.1	3.0	8.2	9.8
5	6.6(0.9)	0.47(6)	2.1	1.5	3.8	3.2	12.6	13.8

RI-MOM scheme and $\overline{\rm MS}$, which is performed at one-loop order in perturbation theory [46,47] in the three-flavor theory. The associated error is obtained by taking half the difference between the leading-order and the next-to-leading-order result [48]. We note that this error is one of the dominant ones in our budget, and we expect this error to be reduced by an important factor if a nonexceptional scheme were used, since the latter are known to converge faster in perturbation theory. We neglect the finite volume effects that have been found to be small in [4], as one can expect from the value of $m_\pi L \sim 4$ for our lightest pion mass $m_\pi \sim 290$ MeV.

VI. CONCLUSIONS

We have computed the electroweak matrix elements of the $\Delta s = 2$ four-quark operators that contribute to neutral kaon mixing beyond the SM. Our work improves on the previous studies by using both dynamical and chiral fermions. We confirm the effect seen in a previous quenched computation [21], where a large enhancement of the nonstandard matrix elements were observed. The errors quoted in this work are of the order of 10%. We note that the main limitation of this study comes from the lack of matching factors between nonexceptional renormalization schemes (such as SMOM [4]) and $\overline{\rm MS}$. Once these factors are available, we expect to reach a precision better than 5%. We also plan to utilize another lattice spacing in order to have a better handle on the discretization effects.

ACKNOWLEDGMENTS

APPENDIX

We have computed the nonperturbative scale evolution of the $R^{\rm BSM}$'s between 3 and 2 GeV and then converted the results to $\overline{\rm MS}$ using one-loop perturbation theory [46,47]:

$$U^{\mathrm{MS}}(2 \text{ GeV}, 3 \text{ GeV}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.87 & 0.02 & 0 & 0 \\ 0 & 0.09 & 1.09 & 0 & 0 \\ 0 & 0 & 0 & 0.86 & -0.01 \\ 0 & 0 & 0 & -0.03 & 0.98 \end{pmatrix}. \tag{A1}$$

Our conventions are such that

$$R^{\mathrm{BSM}}(2 \text{ GeV}) = U^{\overline{\mathrm{MS}}}(2 \text{ GeV}, 3 \text{ GeV}) R^{\mathrm{BSM}}(3 \text{ GeV}). \tag{A2}$$

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