Covariant actions for models with nonlinear twisted self-duality

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We describe a systematic way of the generalization, to models with nonlinear duality, of the space-time covariant and duality-invariant formulation of duality-symmetric theories in which the covariance of the action is ensured by the presence of a single auxiliary scalar field. It is shown that the duality-symmetric action should be invariant under the two local symmetries characteristic of this approach, which impose constraints on the form of the action similar to those of Gaillard and Zumino and in the noncovariant formalism. We show that the (twisted) self-duality condition obtained from this action upon integrating its equations of motion can always be recast in a manifestly covariant form which is independent of the auxiliary scalar and thus corresponds to the conventional on-shell duality-symmetric covariant description of the same model. Supersymmetrization of this construction is briefly discussed.

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I. INTRODUCTION

Duality invariance is an important symmetry that arises in many models of physical interest. A classical example is electrodynamics without sources in $D = 4$ dimensions where the $U(1)$ duality group mixes the field strength of the electric field with its Hodge dual identified with the field strength of the (locally defined) magnetic field. Another well known example is the $SL(2, R)$ duality symmetry of $D = 10$ IIB supergravity.

Duality symmetries have been observed and the corresponding duality groups have been completely classified in $D = 1, 2, \ldots, 9$ supergravity models obtained by toroidal compactifications of $D = 11$ supergravity. An important case is the toroidal compactification of $D = 11$ supergravity on seven tori that leads to the celebrated $D = 4$, $N = 8$ supergravity with $E_{7(7)}$ duality group [1–3].

Astonishingly, explicit calculations [4–9] have proven that $N = 8$, $D = 4$ supergravity is finite at the perturbative level up to three and even four loops. These wonderful results have revived a great interest in this theory in regards to an old question of its finiteness [10–14]. Since supersymmetry alone is not sufficient to explain these results, it has been natural to assume that the $E_{7(7)}$ duality symmetry controls remarkable cancellations of divergent contributions to the supergravity amplitudes and, perhaps, ensures the possible finiteness of the theory [15,16]. At the perturbative level, this symmetry is a global continuous symmetry, though it is broken to a discrete subgroup $E_{7(7)}(Z)$ by nonperturbative stringy effects.

An explanation of the three-loop finiteness has been suggested in [17,18] by showing that the only possible supersymmetric candidate for the counterterm at three loops violates $E_{7(7)}$. The same argument holds for the candidate counterterms at five and six loops [19,20].[1]

The arguments in [19,20], as well as in [21–24], suggest that the first divergent $E_{7(7)}$ invariant counterterm can appear at seven loops.

Such a state of affairs leads one to assume that if $N = 8$, $D = 4$ supergravity is perturbatively finite, the reason should be found beyond supersymmetry and $E_{7(7)}$ duality. However, in our opinion, before accepting this conclusion once and for all, more study on the compatibility of maximal supersymmetry with $E_{7(7)}$ duality is needed. In other words, one should demonstrate whether the counterterms which may appear at higher loops are consistent from this point of view. An analogous issue has recently shown up in $N = 4$, $D = 4$ supergravity whose four-point amplitudes have been found to be free of divergences at three loops [25,26] in spite of the fact that supersymmetry admits at this order duality-invariant quantum counterterms [24].

If the classical $N = 4$ and $N = 8$ supergravities do not allow for quantum deformations consistent with supersymmetry and duality invariance, then, as argued in [15,16,27], this may be the reason for the finiteness of these theories (at least at the corresponding loops).

In practice, one should understand (i) how the possible counterterms (and their descendants) deform original linear duality relations between "electric" and "magnetic" field strengths and (ii) whether this deformation is compatible with supersymmetry. The first problem has been addressed in [28] and further developed in [29,30], where simpler examples of duality-invariant gauge theories with higher-order (Born-Infeld-like) and higher-derivative terms

[1]At four loops there seem to be no supersymmetric counterterms.

have been studied (see also [31] for a related recent analysis at the quantum level). The second (supersymmetry) problem has been recently considered in [29,32] in the case of nonlinear generalizations of $N = 1, 2$ $D = 4$ supersymmetric Abelian gauge theories with $U(1)$ as the duality group, following earlier results of [33–41] based on the superfield formalism. To these results one should add the known examples of component nonlinear duality-symmetric (Born-Infeld-type) Abelian gauge theories with 16 supersymmetries, namely, the $N = 4$, $D = 4$ supersymmetric Abelian Born-Infeld theory on the worldvolume of the $D3$-brane [42] and the $6d$ worldvolume theory of the $M5$-brane [43–47].

When studying duality invariance of a theory one faces a well known problem that this symmetry usually directly manifests itself only on the mass shell, while the conventional Lagrangians are not invariant under the duality transformations. The reason is that only electric fields enter the Lagrangian, while their magnetic duals do not. Instead, in order to guarantee the duality invariance of the field equations, the duality variation of the Lagrangian should satisfy a consistency requirement, the Gaillard-Zumino condition [48]. This approach has been used in [28–30].

To lift the duality invariance to the level of the action, the electric and magnetic fields should enter the Lagrangian on an equal footing, while the duality relation between them should arise on the mass shell as a consequence of equations of motion that follow from the Lagrangian. The latter guarantees that the number of the physical degrees of freedom remains intact. One way to do this is to renounce the manifest space-time covariance of the action in favor of duality symmetry [49–52]. Note, however, that in such a formulation space-time (diffeomorphism or Lorentz) invariance is still present but is realized in a nonconventional way. Using this noncovariant formulation, Hillmann [53] has obtained the duality invariant action of $N = 8, D = 4$ supergravity, and Bossard *et al.* [54] have proved that the $E_{7(7)}$ symmetry is anomaly free in the perturbatively quantized theory. In [28] it has been suggested how one can reconstruct a nonlinear duality-invariant action starting from a duality-invariant counterterm.

There is, however, a possibility of keeping manifest both the duality and space-time symmetries in the action. This requires the introduction of auxiliary fields into the Lagrangian (see [55] for a brief recent overview of different covariant formulations). The most economic way (dubbed the PST approach) is to introduce a single auxiliary scalar field [56]. In this formulation, in addition to the conventional gauge symmetry, the action is invariant under two extra local symmetries. One of them can be used to gauge away the auxiliary scalar and reduce the action to a nonmanifestly Lorentz invariant form of the noncovariant approach. Another symmetry implies that some of the components of the gauge fields enter the action only under a total derivative and ensures the appearance of the duality relation as the general solution of the gauge field equations of motion.

The covariant PST approach unifies different noncovariant formulations [57,58] and has proven to be extremely useful, in particular, for the construction of the action of the $M5$-brane in $D = 11$ supergravity [44], which is an example of a nonlinear self-dual (2,0) $6d$ gauge theory with 16 supersymmetries. So we hope that it may also be useful for making further progress in pursuing the issue of the $E_{7(7)}$ and supersymmetry invariance of the $N = 8$, $D = 4$ supergravity effective action and the corresponding issue in less supersymmetric supergravities.

The problem is to explicitly identify the possible divergent counterterms in $N = 8$, $D = 4$ supergravity and to show how to write a consistent nonlinear supersymmetric effective action, if any, that arises from a given counterterm and respects $E_{7(7)}$ duality symmetry.

As a preliminary study, the purpose of this paper is to solve the problem considered in [28] in the framework of the covariant approach, namely, to have a general recipe for constructing space-time covariant actions with manifest duality symmetry at the nonlinear level. Such a construction will include in the general framework the nonlinear action for the $M5$-brane [44,45] and the corresponding on-shell covariant description of the $M5$-brane in the superembedding approach [43,46], as well as the manifestly duality-symmetric Lagrangian formulation of the Born-Infeld action for the $D3$-brane [59,60]. In this setting we will also clarify how the (twisted) self-duality condition obtained from the manifestly duality-symmetric action upon integrating its equations of motion can always be recast in a manifestly covariant form which is independent of the auxiliary scalar and thus corresponds to the conventional on-shell duality-symmetric covariant description of the same model.

This should set a stage for further analysis of the compatibility of supersymmetry with various possible nonlinear deformations of a given duality symmetric theory, in particular, in the cases of extended supersymmetries and supergravities for which superfield methods are not applicable off the mass shell and/or have not yet been developed enough to include higher-order corrections even on the mass shell.

The paper is organized as follows. In Sec. II, to introduce our notation, we review the covariant approach to theories with a linear self-duality condition. In Sec. III we extend the approach to nonlinear systems. This is done by starting with a nonlinear action which is invariant, by construction, under a local symmetry mentioned above, i.e. in such a way that some components of the gauge fields enter the action under a total derivative only. The action is constructed as a series of local field functionals $I^{(k)}$, $k = 0, 1, \ldots$, where $I^{(0)}$ is the term in the action which is quadratic in the field strengths. Then one imposes the condition that the action is invariant also under the local symmetry which ensures the auxiliary nature of the PST scalar $a(x)$. This imposes a constraint on the local functional $I = \sum I^{(k)}$ that, given $I^{(1)}$, allows one to apply an iterative procedure to determine $I^{(k)}$. In Sec. IV we derive the relation between the twisted self-duality condition obtained from the action in terms of the functionals $I^{(k)}$, that contain the auxiliary scalar $a(x)$, and a manifestly covariant nonlinear twisted self-duality condition which only involves the gauge field strengths (and derivatives thereof) and no auxiliary scalar. Section V contains our conclusions and includes a nonexhaustive discussion of the compatibility between supersymmetry and duality.

II. PST FORMULATION OF A LINEAR DUALITY-SYMMETRIC THEORY IN $D = 4$

Consider a system of N Abelian vector fields in $D = 4$ described by the 1-forms $A^r(x)$ $(r = 1, \ldots, N)$ with the field strengths $F^r = dA^r$. Call $A^{\bar{r}}$ their magnetic duals with field strengths $F^{\bar{r}} = dA^{\bar{r}} := -\bullet \frac{2\delta S}{\delta F^r}$, where S is an action constructed of the electric field strengths F^r and $\bullet$ is the Hodge map.[2] For instance in the case of the Maxwell action $S_0 = -\int d^4x \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$ we have

$$F^{\bar{r}}_{\mu\nu} = (\bullet F^r)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{r\rho\sigma},$$
$$F^{r}_{\mu\nu} = -(\bullet F^{\bar{r}})_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\bar{r}\rho\sigma}. \quad (1)$$

Now let us define $A^i \equiv (A^r, A^{\bar{r}})$ and $F^i \equiv (F^r, F^{\bar{r}})$, $(i = 1, \ldots\ldots 2N)$. The duality group $G \subset Sp(2N, R)$ acts linearly on A^i (and F^i). The vector fields A^r can be coupled to gravity (or supergravity) and to a set of scalars, ϕ, and fermions, ψ.

In the presence of scalars and fermions the definition of the field strengths F^i can be generalized as follows

$$F^i = dA^i + C^i \quad (2)$$

where $C[\phi, \psi]^i$ are 2-forms. In supersymmetric theories such a redefinition is useful since it allows one to make the field strengths transform covariantly under supersymmetry.

[2]We have denoted the Hodge map by $\bullet$ instead of the conventional $*$, since we shall use the latter for denoting the twisted-duality conjugation. In our conventions, a p-form $\phi_{(p)}$ in D dimensions is defined, in the vielbein basis, as

$$\phi_{(p)} = \frac{1}{p!} e^{a_1} \ldots e^{a_p} \phi_{a_p \ldots a_1}$$

and its Hodge dual is

$$\bullet \phi_{(p)} = \frac{1}{(D-p)!} e^{a_1} \ldots e^{a_{D-p}} \epsilon^{b_1 \ldots b_p}_{a_{D-p} \ldots a_1} \phi_{b_p \ldots b_1},$$

so that, in $D = 4$ space-time with Minkowski signature, $\bullet\bullet = -1$. The external differential d acts on the differential forms from the left.

The scalars parametrize the coset G/H, where H is the maximal compact subgroup of G, and the fermions belong to some representation of H. The scalars ϕ are described by the "bridges" $\mathcal{V}(\phi)^q_i$ where the index q spans a representation of H whose (real) dimension is equal to that of G labeled by i. One can define $\mathcal{V}_{iq} := (\mathcal{V}^q_i)^*$ and its inverse $\mathcal{V}^{ip}$ such that $\mathcal{V}_{iq}\mathcal{V}^{ip} = \delta^p_q$. Then the scalars allow us to define an invertible metric in G given by

$$G_{ij} = \mathcal{V}^q_i \mathcal{V}_{jq} + \text{c.c.} \quad (3)$$

Since G is a subgroup of $Sp(2N, R)$ one can define a matrix $\Omega^{ij} = -\Omega^{ji}$ with the only nonvanishing elements given by $\Omega^{r\bar{r}} = -\Omega^{\bar{r}r} = \delta^{r\bar{r}}$ and in a similar way one can define Ω_{ij} so that $\Omega^{ij}\Omega_{jk} = -\delta^i_k$. Ω^{ij} and Ω_{ij} can be used to rise and lower the indices $i, j \ldots$ Then one can define the complex structure

$$J^i_j = G^{ik}\Omega_{kj} = \Omega^{ik}G_{kj} \quad (4)$$

such that $J^i{}_k J^k{}_j = -\delta^i_j$ and finally one defines the "star operation" as

$$* = J^i_j \bullet$$

so that

$$** = 1.$$

With the use of the star operation the duality relations between the electric and magnetic fields, Eq. (1), take the form of the linear *twisted self-duality condition* on the field strength of $A^i = (A^r, A^{\bar{r}})$.

$$F^i = (*F)^i. \quad (5)$$

Note that, acting on (5) with the differential d one gets the field equations of the vector fields. In addition, the constraint (5) implies that only half of the fields A^i, e.g. A^r, are independent, which ensures the correct number of the degrees of freedom of the theory.

Since A^i transform linearly under the duality symmetry G the duality constraint (5) and, hence, the equations of motion are duality invariant. However, the conventional action constructed with a half number of the fields A^i is not duality invariant. Instead, the duality symmetry manifests itself through the Gaillard-Zumino condition [48] which should be satisfied by the duality variation of the action.

For studying properties of the duality-symmetric theory it is useful to have an action that yields (5) as a (consequence of) field equations. However, since Eq. (5) is of the first order in derivatives, while usually the bosonic field equations are of the second order, constructing the duality-symmetric action turns out to be not a straightforward procedure.

One possibility is to renounce the requirement of manifest Lorentz invariance by splitting the $D = 4$ Lorentz-vector indices of the fields A^i_μ. There are several ways of splitting the components of the $D = 4$ vector, namely

$4 = (4 - n) + n$ (where $n = 1, 2, 3$). Each splitting results in a different noncovariant duality-symmetric action that produces Eq. (5) (see [55,61–63] for more details). In the original construction of [50], which is closely related to the Hamiltonian description of the theory, the time-component A^i_0 of the vectors A^i_μ $[\mu = (0, m)]$ gets separated from their spacial components and does not appear in the action. Though this breaks the manifest Lorentz invariance, the action is invariant under a modified space-time symmetry which reduces to the conventional Lorentz symmetry on the mass shell [51,52,56].

A manifestly Lorentz-covariant formulation of the duality-symmetric action that yields (5) as a field equation can be constructed following the approach proposed in [56,64]. In this approach, in addition to the physical fields A^i_μ, the action contains an auxiliary scalar field $a(x)$. It enters the action through the 1-form $v(x)$[3]

$$v = \frac{da}{\sqrt{\partial_\mu a \partial^\mu a}} \tag{6}$$

so that $v_\mu v^\mu = 1$ and

$$v i_v + i_v v = 1, \tag{7}$$

where i_v is the contraction with the vector $v^\mu \partial_\mu$ acting from the left, i.e.

$$i_v \phi_{(p)} = i_v\left(\frac{1}{p!} e^b e^{a_1} \ldots e^{a_{p-1}} \phi_{a_{p-1}\ldots a_1 b}\right) = \frac{1}{(p-1)!} e^{a_1} \ldots e^{a_{p-1}} \phi_{a_{p-1}\ldots a_1 b} v^b. \tag{8}$$

Acting on any p-form $X^i_{(p)}$ that transforms as a vector of the group G, one has the identities[4]

$$i_v * = * v, \qquad v * = * i_v \tag{9}$$

so that

$$v i_v * = * i_v v, \qquad i_v v * = * v i_v. \tag{10}$$

Using these identities one can decompose F^i as follows

$$F^i = (v i_v + i_v v) F^i = v i_v F^i + * v i_v * F^i = v i_v (F - *F)^i + (1 + *)(v i_v * F)^i. \tag{11}$$

In what follows we shall also use the following formulas for the variations δv and δi_v (acting on a p-form) with respect to δa:

$$\delta v = \frac{1}{\sqrt{\partial_\mu a \partial^\mu a}} i_v v (d\delta a) \tag{12}$$

$$\delta i_v = \frac{1}{\sqrt{\partial_\mu a \partial^\mu a}} * i_v v (d\delta a) * . \tag{13}$$

The covariant action S_0 can be written in various equivalent ways, e.g.

$$S_0 = \frac{1}{8} \int \Omega_{ij} [F^i * F^j - (v i_v (F^i - *F^i)) * (v i_v (F^j - *F^j))], \tag{14}$$

where for simplicity we have considered, for the moment, the case with $C^i = 0$ [see Eq. (2)]. The general case will be considered later.

Using (11), as well as the identities (9) and (10), Eq. (14) can be rewritten as

$$S_0 = \frac{1}{4} \int \Omega_{ij} [v i_v F^i * (v i_v * F^j) - (v i_v * F^i) * (v i_v * F^j)] \tag{15}$$

or

$$S_0 = \frac{1}{4} \int \Omega_{ij} [v i_v (F^i - *F^i) F^j]. \tag{16}$$

Equation (15) can also be rewritten as

$$S_0 = -\frac{1}{4} \int d^4x \sqrt{g} G_{ij} [(i_v F^i)^\mu (i_v * F^j)_\mu - (i_v * F^i)^\mu (i_v * F^j)_\mu], \tag{17}$$

where $g_{\mu\nu}(x)$ is the metric of the $4D$ space-time which for generality we consider to be curved.

An important property of the action S_0 is that the $i_v A^i$ component of the gauge field enters this action only under the total derivative. Indeed, $i_v A^i$ enters only the term $\int \Omega_{ij} (v i_v F^i) F^j$ of (16) and the corresponding contribution is the total derivative, which is assumed to vanish at infinity

$$\int \Omega_{ij} v i_v (d(v i_v A^i) F^j) = -\int \Omega_{ij} da d\left(\frac{1}{\sqrt{(\partial a)^2}} i_v A^i\right) F^j = -\int d(\Omega^{ij} v i_v A^i F^j) = 0.$$

The independence of S_0 from $i_v A^i$ is analogous to the absence of the components A^i_0 in the action of the noncovariant formulation. It implies that the action is invariant under the following local transformations of the gauge fields

$$\delta_I A^i = d a \Phi^i, \qquad \delta_I a = 0, \tag{18}$$

where $\Phi^i(x)$ are scalar gauge parameters.

Another symmetry of the action ensures that the field $a(x)$ is a pure gauge. It acts only on $a(x)$ and the duality symmetric gauge fields and leaves invariant other fields of the theory (scalars, fermions, metric etc.)

$$\delta_{II} a = \varphi(x), \qquad \delta_{II} A^i = -\frac{1}{\sqrt{(\partial a)^2}} i_v (F^i - *F^i) \varphi(x), \tag{19}$$

[3]The signature of our metric is $(1, -1, -1, -1)$.
[4]For notational simplicity, sometimes in expressions like $(*X)^i$ we shall drop the parentheses and write $*X^i$.

where $\varphi(x)$ is a local gauge parameter. It is important to note that the δ_{II} variation of A^i is proportional to the self-duality constraint, which as we shall see is a consequence of the field equation (20), so that it vanishes on shell. The consequence of this fact is that on the mass shell the theory becomes manifestly Lorentz covariant without any need for the auxiliary field.

The field equations of A^i are

$$d[v i_v(F^i - *F^i)] = 0 \tag{20}$$

and the equation of motion of $a(x)$ is

$$d\bigg[\Omega_{ij}\frac{1}{\sqrt{(\partial a)^2}}v i_v(F^i - *F^i)i_v(F^j - *F^j)\bigg] = 0. \tag{21}$$

It can be obtained using Eqs. (12) and (13).

One can check that Eq. (21) is identically satisfied if Eq. (20) holds. This reflects the fact that $a(x)$ is the auxiliary field. The general solution of (20) is

$$v i_v(F^i - *F^i) = d(daX^i)) = -dadX^i \tag{22}$$

where $X^i(x)$ are arbitrary functions.[5] On the other hand, under (finite) transformations of the symmetry (18),

$$\delta_I[v i_v(F^i - *F^i)] = -dad\Phi^i \tag{23}$$

so that a transformation with the parameter $\Phi^i = -X^i$ allows us to eliminate X^i from the right-hand side of (22) and get

$$v i_v(F^i - *F^i) = 0. \tag{24}$$

Moreover, since $(1 - *)F^i$ is anti-self-dual, this equation also implies

$$i_v v(F^i - *F^i) = 0. \tag{25}$$

In view of (7), Eqs. (24) and (25) are equivalent to the twisted self-duality constraint (5).

We have thus shown that the twisted self-duality relation follows from the covariant action as the solution of its equations of motion. Using the local symmetry (19) we can gauge fix the auxiliary field $a(x)$ to be

$$a(x) = n_\mu x^\mu, \qquad v_\mu = \frac{n_\mu}{\sqrt{n_\nu n^\nu}} \tag{26}$$

where n_μ is a constant vector.[6] Depending on whether this vector is timelike or spacelike, one reduces the PST action to different noncovariant formulations. For instance, when $n_\mu = \delta^0_\mu$, one recovers the noncovariant formulation of [50–52]. The nonconventional off-shell space-time invariance of the latter is explained by the necessity to keep intact the gauge condition (26) under the Lorentz transformations, which is achieved by adding to the Lorentz variation of the gauge field $\delta_L A_\mu$ the compensating gauge transformation (19)

$$\delta A = \delta_L A + \frac{1}{\sqrt{n_\mu n^\mu}} i_n(F^i - *F^i)n_\mu L^\mu{}_\nu x^\nu \tag{27}$$

where $L^\mu{}_\nu$ are the infinitesimal parameters of the Lorentz transformation. Note that on the mass shell, i.e. when the twisted self-duality condition (24) is satisfied, the variation (27) becomes the conventional Lorentz transformation of the gauge field.

Up to now we have considered only the case where $C^i = 0$ in (2). The general case can be easily recovered by adding the Wess-Zumino term in the r.h.s. of (14) [and (15) and (16)]

$$-\frac{1}{2}\int \Omega_{ij} dA^i C^j.$$

III. PST ACTION WITH NONLINEAR DUALITY IN $D = 4$

In the previous section we considered the case in which the magnetic field strengths $F^{\bar r}$ are related to the electric ones F^r by the linear Hodge duality, or equivalently in which the field strengths $F^i = (F^r, F^{\bar r})$ satisfy the linear self-duality constraint (5). This is the case in which the conventional action $S_0[F^r]$ is quadratic in F^r. If in addition to S_0 an action $S = S_0 + \hat S$ contains terms $\hat S$ of higher order in F^r and/or derivatives of F^r, the relation between $F^{\bar r}$ and F^r, i.e.

$$\bullet\ F^{\bar r}_{\mu\nu} = \delta^{\bar r}_r \frac{\delta S}{\delta (F^r)^{\mu\nu}}$$

becomes nonlinear in F^r and/or contains derivatives of F^r. In this case the linear self-duality constraint (5) is replaced by a nonlinear (deformed) twisted self-duality condition that in general can be expressed as follows:

$$F^i - \lambda\Big(\frac{\delta W[F]}{\delta F}\Big)^i = *\Big(F - \lambda\frac{\delta W[F]}{\delta F}\Big)^i,$$
$$\Big(\frac{\delta W[F]}{\delta F}\Big)^i \equiv G^{ij}\frac{1}{2}dx^\mu dx^\nu \frac{\delta W[F]}{\delta (F^j)^{\nu\mu}}, \tag{28}$$

where $W[F]$ is a local functional of F^i and their derivatives (as well as of other fields) which is invariant under the transformations of the duality group G and λ is a parameter of dimension l^2 which plays the role of a coupling constant characterizing a nonlinear deformation of the Maxwell-like theory for which $\lambda = 0$. The functional $W[F]$ is, in general, a series in λ and F [29]

[5]Strictly speaking this is true only locally. In topologically nontrivial backgrounds $v i_v(F^i - *F^i)$ may be closed but not an exact 1-form.

[6]Note that, though the gauge $a(x) = 0$ is not directly admissible, since the action (14) contains $\sqrt{(\partial a)^2}$ in the denominator, on can nevertheless reach this gauge by handling a singularity in the action in such a way that the ratio $\partial_\mu a \partial^\nu a/\partial_\rho a \partial^\rho a$ remains finite. This can be achieved by first imposing the gauge fixing condition $a(x) = \epsilon x^\mu n_\mu$ and then sending the parameter ϵ to 0.

$$W[F] = \sum_0^\infty \lambda^k W^{(k)}[F]. \quad (29)$$

The order k of λ is associated with the dimension of terms in $W^{(k)}$ in such a way that λW has the dimension l^{-4}. Duality-invariant counterterms of a quantum theory are examples of sources of the nonlinearly deformed self-duality condition. Simple counterterm deformations considered in [28] are

$$W[F] \sim C^2(\partial F)^2, \qquad W[F] \sim (F)^4,$$

where C is the 4d Weyl tensor.

In this section we would like to extend the PST approach to the generic nonlinear case. As we have seen in the previous section, the self-duality condition which is derived from the PST action contains the auxiliary field $a(x)$ [see Eq. (24)]. We have then shown that this relation is equivalent to the conventional covariant twisted self-duality condition (5) which does not contain $a(x)$. In the nonlinear case we shall encounter and solve a similar problem; namely in the next section we will demonstrate how the covariant nonlinear twisted self-duality condition (28) is related to the one which we will now derive from the nonlinear PST action.[7]

In the linear case one of the possible forms of the PST action was given in Eq. (15) [or (17)]. Let us rewrite it as follows:

$$S_0 = -\frac{1}{2}\int d^4x\sqrt{g}\Big[G_{ij}\frac{1}{2}(i_v F^i)^\mu (i_v * F^j)_\mu - \mathcal{L}^{(0)}\Big], \quad (30)$$

where

$$\mathcal{L}^{(0)} = \frac{1}{2}G_{ij}(i_v * F^i)^\mu (i_v * F^j)_\mu. \quad (31)$$

As was shown in the previous section, the action (30) is invariant under the two local symmetries (18) and (19). This suggests that we consider in the nonlinear case the action

$$S = -\frac{1}{2}\int d^4x\sqrt{g}\Big[G_{ij}\frac{1}{2}(i_v F^i)^\mu (i_v * F^j)_\mu - \mathcal{L}\Big] \quad (32)$$

where now

$$\mathcal{L} = \sum_0^\infty \lambda^k \mathcal{L}^{(k)}. \quad (33)$$

$\mathcal{L}^{(k)}$ are local functions of $i_v(*F)^i$ (and, possibly, of their derivatives and of the other fields of the theory), and $\mathcal{L}^{(0)}$ is defined in (31). We shall also denote

$$I^{(k)} = \int d^4x \mathcal{L}^{(k)}$$

and

$$I = \sum_0^\infty \lambda^k I^{(k)}.$$

Since $\mathcal{L}$ depends on $*F^i$ only through their contraction with v, i.e. $i_v * F^i$, by construction the action (32) is invariant under the symmetry (18). We should also find the conditions under which this action is invariant under a nonlinear generalization of the symmetry (19). To find the form of this symmetry let us look at the equations of motion of the vector fields $A^i(x)$ and the auxiliary field $a(x)$. The vector field equations are

$$d\Big[v\Big((i_v F)^i - \Big(\frac{\delta I}{\delta(i_v * F)}\Big)^i\Big)\Big] = d\Big[v\Big((i_v(1-*)F)^i - \lambda\Big(\frac{\delta \hat{I}}{\delta(i_v * F)}\Big)^i\Big)\Big] = 0, \quad (34)$$

where

$$\lambda\hat{I} = I - I^{(0)} = \lambda\sum_{k=1}^\infty \lambda^{k-1} I^{(k)} \quad (35)$$

and $\frac{\delta I^{(k)}}{\delta(i_v * F))}$ are the 1-forms

$$\Big(\frac{\delta I^{(k)}}{\delta(i_v * F)}\Big)^i = dx^\mu G^{ij}\frac{\delta I^{(k)}}{\delta(i_v * F)^{j\mu}}. \quad (36)$$

Since $I^{(k)}$ (actually) depend on $v i_v * F^i$, one can write $\frac{\delta I^{(k)}}{\delta(i_v * F))} = i_v\frac{\delta I^{(k)}}{\delta(v i_v * F))}$ and present Eq. (34) in the form

$$d\Big[v i_v(1-*)F^i - \lambda v i_v\Big(\frac{\delta \hat{I}}{\delta(v i_v * F)}\Big)^i\Big] = 0, \quad (37)$$

where $\frac{\delta I^{(k)}}{\delta(v i_v * F)}$ denote the 2-forms defined as in (28).

As in the linear case, Eqs. (34) or (37) can be integrated and with the use of the local symmetry (18) result in the duality-like relations

$$v\Big(i_v F^i - \Big(\frac{\delta I}{\delta(i_v * F)}\Big)^i\Big) = v i_v\Big((1-*)F^i - \lambda\Big(\frac{\delta \hat{I}}{\delta(v i_v * F)}\Big)^i\Big) = 0. \quad (38)$$

The $a(x)$-field equation of motion is obtained from the action (32) using Eqs. (12) and (13) and has the form

[7] An example of the 6d counterpart of the condition (28) is the nonlinearly self-dual field strength on the worldvolume of the M5-brane in the superembedding formulation [43,46]. In [65] it was shown that the covariant nonlinear self-duality condition, which is a consequence of a superembedding constraint, is related to a self-duality condition which follows from the M5-brane action [44,45,47,66]. The latter either contains the (derivatives of) the auxiliary field $a(x)$, or (upon its gauge fixing) is not manifestly invariant under diffeomorphism (or Lorentz) transformations.

$$d\left\{\frac{1}{\sqrt{(\partial a)^2}}\Omega_{ij}v\left[((i_v * F^i)(i_v * F^j) + (i_v F^i)(i_v F^j)) - 2(i_v F^i)\left(\frac{\delta I}{\delta(i_v * F)}\right)^j\right]\right\} = 0. \tag{39}$$

Notice that when $\mathcal{L}$ reduces to $\mathcal{L}^{(0)}$, at $\lambda = 0$, Eqs. (34) and (39) reduce, respectively, to (20) and (21).

The form of the field equations (34), (38), and (39) prompts us to consider that the nonlinear generalization of the field variations under the second local symmetry (19) should take the following form

$$\delta_{II}A^i = -\frac{1}{\sqrt{(\partial a)^2}}\left[i_v F^i - \left(\frac{\delta I}{\delta(i_v * F)}\right)^i\right]\varphi(x);$$
$$\delta_{II}a = \varphi(x). \tag{40}$$

The variation of the action under (40) is

$$4\delta_{II}S = \int \delta_{II}ad\left\{\frac{1}{\sqrt{(\partial a)^2}}\Omega_{ij}v\left[((i_v * F^i)(i_v * F^j) + (i_v F^i)(i_v F^j)) - 2(i_v F^i)\left(\frac{\delta I}{\delta(i_v * F)}\right)^j\right]\right\} + 2\int \Omega_{ij}\delta_{II}A^i d\left[v\left(i_v F^j - \left(\frac{\delta I}{\delta(i_v * F)}\right)^j\right)\right]. \tag{41}$$

For this variation to vanish, the following condition should hold:

$$d\left\{\frac{1}{\sqrt{(\partial a)^2}}\Omega_{ij}\left[v((i_v * F^i)(i_v * F^j) + (i_v F^i)(i_v F^j)) - 2v(i_v F^i)\left(\frac{\delta I}{\delta(i_v * F)}\right)^j\right] - v\left(i_v F^i - \left(\frac{\delta I}{\delta(i_v * F)}\right)^i\right)\left(i_v F^j - \left(\frac{\delta I}{\delta(i_v * F)}\right)^j\right)\right\} = 0, \tag{42}$$

which can be simplified to

$$d\left[\frac{1}{\sqrt{(\partial a)^2}}\Omega_{ij}v\left((i_v * F^i)(i_v * F^j) - \left(\frac{\delta I}{\delta(i_v * F)}\right)^i\left(\frac{\delta I}{\delta(i_v * F)}\right)^j\right)\right] = 0. \tag{43}$$

This equation is the fundamental consistency condition which is necessary for the action (32) to be invariant under the local variations (40). It ensures that $a(x)$ is a pure gauge degree of freedom. A similar condition has been found by Bossard and Nicolai [28] in the noncovariant approach. The latter is obtained from (43) upon gauge fixing $a(x) = x^0$. This condition is clearly related to the space-time invariance of the duality-symmetric construction and to the Gaillard-Zumino condition [48].

Equation (43) is automatically satisfied at zero's order in λ. At first order in λ one has

$$d\left[\frac{1}{\sqrt{(\partial a)^2}}\Omega_{ij}v(i_v * F^i)\left(\frac{\delta I^{(1)}}{\delta(i_v * F)}\right)^j\right] = 0. \tag{44}$$

If this condition is satisfied by a certain choice of $I^{(1)}$, the consistency condition (43) imposes the constraint on the possible form of $I^{(2)}$ at order λ^2

$$d\left[\frac{1}{\sqrt{(\partial a)^2}}\Omega_{ij}v\left(2(i_v * F^i)\left(\frac{\delta I^{(2)}}{\delta(i_v * F)}\right)^j + \left(\frac{\delta I^{(1)}}{\delta(i_v * F)}\right)^i\left(\frac{\delta I^{(1)}}{\delta(i_v * F)}\right)^j\right)\right] = 0, \tag{45}$$

on $I^{(3)}$ at order λ^3 and so on. Solving these constraints one can reconstruct $I = \int \mathcal{L}$ order by order.

This iteration procedure, however, does not determine $I = \int \mathcal{L}$ unambiguously. Indeed, if at some order k, there exists an action $\bar{I}^{(k)} = \int \bar{\mathcal{L}}^{(k)}$ that satisfies Eq. (44) (with $I^{(1)}$ replaced by $\bar{I}^{(k)}$), writing $I^{(k)} + c_k\bar{I}^{(k)}$ one can carry out the same procedure for $k' > k$ which will result in a consistent action that now depends on the arbitrary constant c_k. This arbitrariness repeats over and over for any $\bar{I}^{k'}$ that satisfies the condition (44).

Note that the invariance of the action under the gauge transformations (40) implies conditions on the form of the higher-order terms. Using the relations (10) one can rewrite Eq. (38) as follows:

$$v i_v(1 - *)F^i = \lambda v\left(\frac{\delta\hat{I}}{\delta(i_v * F)}\right)^i \Rightarrow -i_v v(1 - *)F^i = \lambda * v\left(\frac{\delta\hat{I}}{\delta(i_v * F)}\right)^i \Rightarrow (1 - *)F^i = \lambda(1 - *)v\left(\frac{\delta\hat{I}}{\delta(i_v * F)}\right)^i = \lambda(1 - *)v i_v\left(\frac{\delta\hat{I}}{\delta(v i_v * F)}\right)^i. \tag{46}$$

Since the left-hand side of (46) does not depend on v, its right-hand side should also be v independent, which imposes restrictions on the possible forms of $\hat{I}$. These restrictions are controlled by the local symmetry (40) and, hence, are a consequence of Eq. (43). Namely, the symmetry (40) can be used to gauge fix v_μ to be a constant vector as in (26). Then Eq. (46) implies that its right-hand side must be Lorentz invariant on the mass shell, i.e. when the duality condition (38) is satisfied. This should be automatically so, since, as we have explained in the case of the linear self-duality, the on-shell Lorentz transformation (27) of the gauge fields is the conventional one. If such, the right-hand side of (46) must transform covariantly under the Lorentz symmetry and, therefore, can only be constructed of the Lorentz-covariant combinations of F^i (and their derivatives).

This observation allows us to relate the higher-order terms in the action (32) to those of the nonlinear twisted self-duality condition (28). Indeed, comparing Eq. (46) with (28) we see that

$$(1-*)v\frac{\delta\hat{I}}{\delta(i_v * F)} = (1-*)\frac{\delta W[F]}{\delta F} \quad \text{or}$$
$$v\frac{\delta\hat{I}}{\delta(i_v * F)} = v i_v(1-*)\frac{\delta W[F]}{\delta F}. \tag{47}$$

Thus, knowing a higher-order deformation $W[F]$ of the original duality-symmetric theory, e.g. by quantum counterterms, one can obtain the form of the corresponding nonlinear contributions to the duality-symmetric action and vice versa.

IV. RELATION BETWEEN THE TWO FORMS OF THE NONLINEAR SELF-DUALITY CONDITION

In this section we shall demonstrate how to relate the self-duality constraint (28) and the Eq. (38) obtained from the action (32), i.e. between $W[F]$ and $\hat{I}[F]$.

In general, duality-invariant $W[F]$ depends on F^i and $*F^i$ or, equivalently, on

$$F^i_{\pm} = \frac{1}{2}(F \pm *F)^i = \frac{1}{2}(1 \pm *)F^i,$$

so that $W[F] = W[F^+, F^-]$ and

$$\frac{\delta W[F]}{\delta F} = \frac{1}{2}(1-*)\frac{\delta W[F_+, F_-]}{\delta F_+} + \frac{1}{2}(1+*)\frac{\delta W[F_+, F_-]}{\delta F_-}. \tag{48}$$

Substituting this equation into (28) we see that $\frac{\delta W[F_+,F_-]}{\delta F_-}$ does not contribute, and the self-duality constraint (28) becomes

$$(1-*)\left(F^i - \lambda\left(\frac{\delta W[F_+, F_-]}{\delta F_+}\right)^i\right) = 0. \tag{49}$$

Modulo different notation and approach, Eq. (49) corresponds to Eq. (4.2) of [29].

Comparing (49) with (47) we have

$$(1-*)v\left(\frac{\delta\hat{I}}{\delta(i_v * F)}\right)^i = (1-*)\frac{\delta W[F]}{\delta F}$$
$$= (1-*)\left(\frac{\delta W[F_+, F_-]}{\delta F_+}\right)^i.$$

To analyze the relation (47), let us introduce the identity [see Eq. (11)]

$$F^i = v i_v(F^i - *F^i) - \lambda v\left(\frac{\delta\hat{I}}{\delta(i_v * F)}\right)^i + (1+*)(v i_v * F^i)$$
$$+ \lambda v\left(\frac{\delta\hat{I}}{\delta(i_v * F)}\right)^i. \tag{50}$$

Then on the mass shell (46) we have

$$F^i = (1+*)(v i_v * F^i) + \lambda v\left(\frac{\delta\hat{I}}{\delta(i_v * F)}\right)^i, \tag{51}$$

$$F^i_+ = (1+*)v i_v * F^i + \frac{\lambda}{2}(1+*)v\left(\frac{\delta\hat{I}}{\delta(i_v * F)}\right)^i, \tag{52}$$

and

$$F^i_- = \frac{\lambda}{2}(1-*)v\left(\frac{\delta\hat{I}}{\delta(i_v * F)}\right)^i,$$

which naturally coincides with (46). Equation (51) tells us that, when the twisted self-duality relation holds, F^i is a series in $v i_v * F^i$ and λ. Using this fact, one can carry out the following iteration procedure to reconstruct $\hat{I} = \sum_{k=1}^{\infty}\lambda^{k-1}I^{(k)}$ from a given counterterm $W[F]$ (29). Possible nonvanishing terms $W^{(k)}$, $k \geq 1$, are responsible for the arbitrariness in I, pointed out at the end of Sec. II. Of course for consistency also these $W^{(k)}$, on shell and at $\lambda = 0$, must satisfy the condition (44).

At the zero order in λ

$$(1-*)\frac{\delta W[F]}{\delta F}\bigg|_{\lambda=0} = (1-*)f^{(0)}[v i_v * F^i],$$

where $f^{(0)}$ is a known 2-form functional of $(1+*)v i_v * F^i$. This allows us, using (47), to reconstruct the first term $I^{(1)}$ of $\hat{I}$. Knowing $I^{(1)}$ we expand $\frac{\delta W[F]}{\delta F}$ to the first order in λ

$$(1-*)\frac{\delta W[F]}{\delta F} = (1-*)\bigg(f^{(0)}[v i_v * F] + \lambda f^{(1)}[v i_v * F]$$
$$+ \lambda\frac{\delta W^{(1)}[F]}{\delta F}\bigg|_{\lambda=0}\bigg), \tag{53}$$

where

$$f^{(1)}[v i_v * F] = \left[v\frac{\delta I^{(1)}}{\delta(i_v * F)}\right]^{\mu\nu i}$$
$$\times \frac{\delta^2 W^{(0)}}{\delta[(1+*)v i_v * F]^{\mu\nu i}\delta F}\bigg|_{\lambda=0}$$

is a known 2-form functional of $v i_v * F$. Substituting Eq. (53) into (47) one reconstructs the second term $I^{(2)}$ of $\hat{I}$.

At the quadratic order in λ the procedure for reconstructing $I^{(3)}$ becomes much more complicated since the expansion of $\frac{\delta W[F]}{\delta F}$ will have terms containing

$$\left[\frac{\delta I^{(1)}}{\delta(i_v * F)}\right]^2, \qquad \frac{\delta I^{(1)}}{\delta(i_v * F)}\frac{\delta^2 W^{(1)}[F]}{\delta[(1+*)v i_v * F]\delta F}\bigg|_{\lambda=0},$$
$$\frac{\delta I^{(2)}}{\delta(i_v * F)} \quad \text{and} \quad \frac{\delta W^{(2)}[F]}{\delta F}\bigg|_{\lambda=0}.$$

At the third and higher orders in λ the complexity increases even more.

As a consistency check of the relations between $I^{(k)}$ and W, one should verify that the action functional $\hat{I}$ obtained in this way satisfies the consistency condition (43) and whether this may impose additional restrictions on a possible form of W. Let us recall that, in the action (32) this condition ensures that $a(x)$ is the completely auxiliary (pure gauge) field and that on the mass shell the self-duality condition can be brought to a space-time covariant form in terms of a duality-invariant functional $W[F]$ which does not depend on $a(x)$. To derive the constraint on $W[F]$ imposed by the consistency condition (43) note that on the mass shell (38) the latter takes the following form:

$$d\left[\frac{1}{\sqrt{(\partial a)^2}}\Omega_{ij}v((i_v * F^i)(i_v * F^j) - ((i_v F^i)(i_v F^j))\right] = d\left[\frac{1}{\sqrt{(\partial a)^2}}\Omega_{ij}v i_v(1+*)F^i)(i_v(1-*)F^j)\right] = 0, \tag{54}$$

which in turn, in view of (28) and (48), reduces to

$$\lambda d\left[\frac{1}{\sqrt{(\partial a)^2}}\Omega_{ij}v(i_v F_+)^i\left(i_v\frac{\delta W}{\delta F_+}\right)^j\right] = 0. \tag{55}$$

Though the statement that given any duality invariant $W[F]$ one can always reconstruct a corresponding duality-symmetric action looks plausible we have not found the generic proof that the constraint (55) is satisfied by any choice of the duality invariant $W[F]$. We have checked the validity of (55) for known examples of $W[F]$ which do not contain terms with derivatives of F. When $W[F]$ contains derivatives of F, the analysis becomes technically much more involved and we leave it for further study.

V. CONCLUSION

In this paper we have described, in a systematic way, how to extend the covariant and duality- invariant PST approach to models with nonlinear duality. It has been shown that the duality-symmetric action should be invariant under the two local symmetries (18) and (40) characteristic of this approach, which require that the action is given by Eq. (32) where the local functional $I = \int \mathcal{L}$ depends on the field strengths F^i only through $v i_v * F^i$ and satisfies the quadratic constraint (43). This constraint is related to the Gaillard-Zumino constraint and, after a suitable gauge fixing, coincides with the constraint found in [28], in the framework of the noncovariant but duality-invariant approach.

In the models with nonlinear duality, gauge fields are constrained by the *deformed twisted self-duality condition*, Eq. (28). It means that there exists a self-dual 2-form $h^i = F^i - (\frac{\delta W}{\delta F})^i$ such that $h^i = *h^i$, where $W[F, \ldots]$ is a covariant and duality-invariant local functional of F^i (and the other fields). As a further result, in this paper we have exploited the relation between the functional $W[F, \ldots]$ and the functional I that constitutes the PST action.

A possible application of the approach developed in this paper is the study of the consistent counterterms in supersymmetric duality-invariant models and in particular in $N = 8$, $D = 4$ supergravity. This is relevant to the issue of the finiteness of this theory. The question is whether $N = 8$ supersymmetry is preserved upon a certain nonlinear deformation of the classical theory. The authors of [28] argued, on general grounds similar to those ensuring the diffeomorphism invariance and the absence of corresponding anomalies, that there might be no obstructions to find a deformed theory which is supersymmetric. To give more direct evidence for this argument, one should show that the Gaillard-Zumino or similar conditions, like Eq. (43), restricting the form of the action of the duality-symmetric theory are compatible with (deformed) supersymmetry transformations.

So far the compatibility of supersymmetry with nonlinear self-duality has been explicitly demonstrated only for $N = 1, 2$ [29,32–41] and $N = 4$ [42] ($D3$-brane) Born-Infeld-like deformations of Abelian gauge theories with the duality group $U(1)$, and in the case of the $M5$-brane [44,45,67] which is the nonlinear (2,0) self-dual $6d$ gauge theory with 16 supersymmetries (i.e. $N = 4$, from the $D = 4$ perspective). However, supersymmetric examples of nonlinear theories (including supergravities) with non-Abelian duality groups of the E_7 type have not been given yet. It should be mentioned that consistent couplings of *external* supersymmetric Born-Infeld-like models to $N = 1$ and 2 supergravities are known [68,69]; however, an important issue which remains is whether nonlinear deformations are possible for vector fields *inside* supergravity multiplets, in particular, in $N = 4, 8$ supergravities.

At this point we would like to make a comment that nonlinearities in field theories and, in particular, in supersymmetric ones are often associated with spontaneous symmetry and supersymmetry breaking. For instance, the Born-Infeld structure is a manifestation of partial supersymmetry breaking of a rigid extended supersymmetry [34]. In this respect, Born-Infeld-like nonlinearities in duality-symmetric effective action of $N = 8$, $D = 4$ supergravity, if they appear, should have a different nature (e.g. stringy corrections), since there are no conventional field theories with more than 32 supersymmetries whose spontaneous breaking would result in a nonlinear generalization of $N = 8$, $D = 4$ supergravity. Restrictions on possible sources of the nonlinear deformation of the twisted self-duality condition of $N = 8$, $D = 4$ supergravity imposed by supersymmetry and $E_{7(7)}$ are discussed in [70].

There are two complementary approaches to deal with supersymmetric extensions of the duality-symmetric actions. The first is the standard approach, in which the action depends only on the electric fields. In this approach supersymmetry is manifest both at the linear level and in the form of possible candidate counterterms, but the duality symmetry of the deformed action is not manifest and should be verified. Given a supersymmetric counterterm constructed of the electric and magnetic fields in a duality-invariant way, Ref. [28] has described an iterative procedure further developed in [29] to construct a nonlinear action for the electric fields only, that satisfies the nonlinear Gaillard-Zumino condition and, hence, retains the duality invariance. Since for the consistency with duality symmetry the nonlinear deformation brings about an (infinite) series of new higher-order terms, the supersymmetry of the whole construction should be rechecked.

Other approaches deal with covariant or noncovariant formulations in which duality symmetry is manifest. It is clear that in these formulations supersymmetry is not manifest since the number of vector fields is doubled and only half of them should appear in the supersymmetry transformations of the fermions. Since the noncovariant formulation comes from a gauge fixing of the covariant one, let us discuss the supersymmetry issue in the framework of the covariant formulation. In models with linear duality there is a simple recipe [52,56] for how to modify the supersymmetric transformations of the fermions so that the PST action is invariant under this modified supersymmetry. In the supersymmetry variations of the fermions, the recipe prescribes replacing the field strengths F^i with the following 2-form:

$$K_0^q = [F^i - v i_v (F^i - *F^i)] V_i^q(\phi) = (1 + *) v i_v * F^i V_i^q(\phi), \quad (56)$$

where $V_i^q(\phi)$ is the G/H bridge scalar field matrix determined in (3). Notice that K_0^i is self-dual, $K_0^q = (*K_0)^q$, and that on the shell of the linear duality constraint $F^i = *F^i$ the 2-form K_0^q coincides with $F^i V_i^q(\phi)$. The property of K_0^q to be self-dual ensures that the supersymmetry transformations involve the right number of independent gauge fields.

For instance, in the simplest case of a $U(1)$-duality symmetric $N = 1$ theory with no scalars ($V_i^q = \delta_i^q$) and one vector supermultiplet, duality-covariant $N = 1$ supersymmetry variations look as follows:

$$\delta A_\mu^q = i\bar{\psi}\gamma_\mu \epsilon^q, \qquad \delta\psi = \frac{1}{8} K_0^{\mu\nu q}\gamma_{\mu\nu}\epsilon^q, \qquad q = 1, 2 \quad (57)$$

where $\psi(x)$ is the Majorana spinor and

$$\epsilon^q = i\varepsilon^{qs}\gamma_5 \epsilon^s \qquad (\varepsilon^{12} = -\varepsilon^{21} = 1) \quad (58)$$

is the self-dual parameter of the rigid $N = 1$, $D = 4$ supersymmetry. It is easy to see that when the duality relation $F^2_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}F^{1\rho\lambda}$ holds, the supersymmetry transformations (57) reduce to the conventional ones relating A^1 and ψ with the Majorana spinor parameter ϵ^1

$$\delta A^1_\mu = i\bar{\psi}\gamma_\mu\epsilon^1,$$
$$\delta\psi = \frac{1}{8}(F^{1\mu\nu}\gamma_{\mu\nu}\epsilon^1 + F^{2\mu\nu}\gamma_{\mu\nu}\epsilon^2) = \frac{1}{4}F^{\mu\nu 1}\gamma_{\mu\nu}\epsilon^1.$$

Let us also note that since the auxiliary field $a(x)$ does not have a superpartner, it should be invariant under the action of supersymmetry $\delta a(x) = 0$. This, however, does not contradict the supersymmetry algebra, if one assumes that the translation of $a(x)$ produced by the commutator of two supersymmetry transformations acting on $a(x)$ is compensated by the local symmetry (19) [71–74]

$$(\delta_1\delta_2 - \delta_2\delta_1)a(x) = \xi^\mu\partial_\mu a(x) - \varphi(x) = 0.$$

Now the problem is how to extend the above prescription to a nonlinear case. An obvious ansatz would be to replace K_0^q in (56) with

$$K^q = F^q - v i_v\left[(1 - *)F^q - \lambda(1 - *)\left(\frac{\delta W[F]}{\delta F}\right)^q\right] = (1 + *)v i_v\left[*F^q + \frac{\lambda}{2}(1 - *)\left(\frac{\delta W[F]}{\delta F}\right)^q\right] + \frac{1}{2}\lambda(1 - *)\left(\frac{\delta W[F]}{\delta F}\right)^q. \quad (59)$$

Again, on the duality shell (28), $F^q = K^q$ but now K^q is not self-dual. However, the anti-self-dual part of K^q does not enter the supersymmetry transformation (57) of the fermions, since its gamma contraction with the self-dual supersymmetry parameter (58) vanishes.

One may expect that this ansatz is incomplete and, in general, should also include terms of higher orders in fermionic fields. This is implicitly indicated by the analysis of rigid (2,0) supersymmetry transformations of the worldvolume fields of the kappa-symmetry gauge-fixed $M5$-brane carried out in [67]. We hope to address the problem of supersymmetry in theories with nonlinear duality in a future work.

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[1] B. de Wit and D. Z. Freedman, Nucl. Phys. **B130**, 105 (1977).
[2] E. Cremmer and B. Julia, Nucl. Phys. **B159**, 141 (1979).
[3] B. de Wit and H. Nicolai, Nucl. Phys. **B208**, 323 (1982).
[4] Z. Bern, L. J. Dixon, and R. Roiban, Phys. Lett. B **644**, 265 (2007).
[5] Z. Bern, J. Carrasco, L. J. Dixon, H. Johansson, D. Kosower, and R. Roiban, Phys. Rev. Lett. **98**, 161303 (2007).
[6] Z. Bern, J. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban, Phys. Rev. D **78**, 105019 (2008).
[7] Z. Bern, J. Carrasco, and H. Johansson, arXiv:0902.3765.
[8] Z. Bern, J. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban, Phys. Rev. Lett. **103**, 081301 (2009).
[9] L. J. Dixon, arXiv:1005.2703.
[10] S. Deser, J. Kay, and K. Stelle, Phys. Rev. Lett. **38**, 527 (1977).
[11] S. Ferrara and B. Zumino, Nucl. Phys. **B134**, 301 (1978).
[12] S. Deser and J. Kay, Phys. Lett. **76B**, 400 (1978).
[13] R. Kallosh, Phys. Lett. **99B**, 122 (1981).
[14] P. S. Howe and U. Lindstrom, Nucl. Phys. **B181**, 487 (1981).
[15] R. Kallosh, J. High Energy Phys. 03 (2012) 083.
[16] R. Kallosh, J. High Energy Phys. 06 (2011) 073.
[17] J. Broedel and L. J. Dixon, J. High Energy Phys. 05 (2010) 003.
[18] H. Elvang and M. Kiermaier, J. High Energy Phys. 10 (2010) 108.
[19] G. Bossard, P. Howe, and K. Stelle, J. High Energy Phys. 01 (2011) 020.
[20] N. Beisert, H. Elvang, D. Z. Freedman, M. Kiermaier, A. Morales, and S. Stieberger, Phys. Lett. B **694**, 265 (2010).
[21] N. Berkovits, Phys. Rev. Lett. **98**, 211601 (2007).
[22] M. B. Green, J. G. Russo, and P. Vanhove, Phys. Rev. Lett. **98**, 131602 (2007).
[23] M. B. Green, J. G. Russo, and P. Vanhove, J. High Energy Phys. 06 (2010) 075.
[24] G. Bossard, P. Howe, K. Stelle, and P. Vanhove, Classical Quantum Gravity **28**, 215005 (2011).
[25] Z. Bern, S. Davies, T. Dennen, and Y.-t. Huang, Phys. Rev. Lett. **108**, 201301 (2012).
[26] P. Tourkine and P. Vanhove, Classical Quantum Gravity **29**, 115006 (2012).
[27] R. Kallosh, Phys. Rev. D **85**, 081702 (2012).
[28] G. Bossard and H. Nicolai, J. High Energy Phys. 08 (2011) 074.
[29] J. J. M. Carrasco, R. Kallosh, and R. Roiban, Phys. Rev. D **85**, 025007 (2012).
[30] W. Chemissany, R. Kallosh, and T. Ortin, Phys. Rev. D **85**, 046002 (2012).
[31] R. Roiban and A. Tseytlin, arXiv:1205.0176.
[32] J. Broedel, J. J. M. Carrasco, S. Ferrara, R. Kallosh, and R. Roiban, arXiv:1202.0014.
[33] S. Cecotti and S. Ferrara, Phys. Lett. B **187**, 335 (1987).
[34] J. Bagger and A. Galperin, Phys. Rev. D **55**, 1091 (1997).
[35] M. Rocek and A. A. Tseytlin, Phys. Rev. D **59**, 106001 (1999).
[36] S. V. Ketov, Mod. Phys. Lett. A **14**, 501 (1999).
[37] S. V. Ketov, Nucl. Phys. **B553**, 250 (1999).
[38] S. M. Kuzenko and S. Theisen, J. High Energy Phys. 03 (2000) 034.
[39] S. M. Kuzenko and S. Theisen, Fortschr. Phys. **49**, 273 (2001).
[40] S. Bellucci, E. Ivanov, and S. Krivonos, Phys. Rev. D **64**, 025014 (2001).
[41] S. V. Ketov, arXiv:hep-th/0108189.
[42] M. Cederwall, A. von Gussich, B. E. Nilsson, and A. Westerberg, Nucl. Phys. **B490**, 163 (1997).
[43] P. S. Howe and E. Sezgin, Phys. Lett. B **394**, 62 (1997).
[44] I. A. Bandos, K. Lechner, A. Y. Nurmagambetov, P. Pasti, D. P. Sorokin, and M. Tonin, Phys. Rev. Lett. **78**, 4332 (1997).
[45] M. Aganagic, J. Park, C. Popescu, and J. H. Schwarz, Nucl. Phys. **B496**, 191 (1997).
[46] P. S. Howe, E. Sezgin, and P. C. West, Phys. Lett. B **399**, 49 (1997).
[47] I. A. Bandos, K. Lechner, A. Y. Nurmagambetov, P. Pasti, D. P. Sorokin, and M. Tonin, Phys. Lett. B **408**, 135 (1997).
[48] M. K. Gaillard and B. Zumino, Nucl. Phys. **B193**, 221 (1981).
[49] D. Zwanziger, Phys. Rev. D **3**, 880 (1971).
[50] S. Deser and C. Teitelboim, Phys. Rev. D **13**, 1592 (1976).
[51] M. Henneaux and C. Teitelboim, Phys. Lett. B **206**, 650 (1988).
[52] J. H. Schwarz and A. Sen, Nucl. Phys. **B411**, 35 (1994).
[53] C. Hillmann, J. High Energy Phys. 04 (2010) 010.
[54] G. Bossard, C. Hillmann, and H. Nicolai, J. High Energy Phys. 12 (2010) 052.
[55] P. Pasti, I. Samsonov, D. Sorokin, and M. Tonin, Phys. Rev. D **80**, 086008 (2009).
[56] P. Pasti, D. P. Sorokin, and M. Tonin, Phys. Rev. D **52**, R4277 (1995).
[57] P. Pasti, D. P. Sorokin, and M. Tonin, Phys. Rev. D **55**, 6292 (1997).
[58] A. Maznytsia, C. R. Preitschopf, and D. P. Sorokin, Nucl. Phys. **B539**, 438 (1999).
[59] D. Berman, Phys. Lett. B **409**, 153 (1997).
[60] A. Nurmagambetov, Phys. Lett. B **436**, 289 (1998).
[61] P.-M. Ho and Y. Matsuo, J. High Energy Phys. 06 (2008) 105.
[62] W.-M. Chen and P.-M. Ho, Nucl. Phys. **B837**, 1 (2010).
[63] W.-H. Huang, Nucl. Phys. **B861**, 403 (2012).
[64] P. Pasti, D. P. Sorokin, and M. Tonin, Phys. Lett. B **352**, 59 (1995).
[65] P. S. Howe, E. Sezgin, and P. C. West, Phys. Lett. B **400**, 255 (1997).
[66] M. Perry and J. H. Schwarz, Nucl. Phys. **B489**, 47 (1997).
[67] P. Claus, R. Kallosh, and A. Van Proeyen, Nucl. Phys. **B518**, 117 (1998).
[68] S. M. Kuzenko and S. A. McCarthy, J. High Energy Phys. 02 (2003) 038.
[69] S. M. Kuzenko, J. High Energy Phys. 06 (2012) 012.
[70] R. Kallosh and T. Ortin, arXiv:1205.4437.
[71] I. A. Bandos, N. Berkovits, and D. P. Sorokin, Nucl. Phys. **B522**, 214 (1998).
[72] G. Dall'Agata, K. Lechner, and M. Tonin, Nucl. Phys. **B512**, 179 (1998).
[73] G. Dall'Agata, K. Lechner, and M. Tonin, J. High Energy Phys. 07 (1998) 017.
[74] G. De Pol, H. Singh, and M. Tonin, Int. J. Mod. Phys. A **15**, 4447 (2000).