PHYSICAL REVIEW D **86,** 045009 (2012)

Relativistic action-at-a-distance description of gravitational interactions?

Domingo J. Louis-Martinez*

Department of Physics and Astronomy, University of British Columbia, Vancouver, Canada, V6T 1Z1

(Received 20 May 2012; published 2 August 2012)

It is shown that certain aspects of gravitation may be described using a relativistic action-at-a-distance formulation. The equations of motion of the model presented are invariant under Lorentz transformations and agree with the equations of Einstein's theory of general relativity, at the first post-Newtonian approximation, for any number of interacting point masses.

DOI: 10.1103/PhysRevD.86.045009 PACS numbers: 03.30.+p, 04.25.−g, 04.25.Nx, 04.90.+e

I. INTRODUCTION

After the discovery of the action-at-a-distance formulation of electrodynamics [1–10], several relativistic noninstantaneous action-at-a-distance theories have been investigated [11–21]. Instantaneous action-at-a-distance formulations have been studied using a variety of approaches [22–31]. For gravity, several relativistic action-at-a-distance models have been proposed [32–46] and compared with observations [47,48]. A major difficulty with most of the models proposed is their disagreement with Einstein's theory of general relativity (GR) in the so-called "slow motion approximation" [first post-Newtonian approximation (1PN)], even for the simpler case of two point masses ($N = 2$).

The objective of this paper is to present a relativistic action-at-a-distance description of gravitational interactions for a system consisting of an arbitrary number N of point masses.

The model presented in this paper is in agreement with GR at 1PN for an arbitrary number N of interacting point masses.

Our description also agrees with GR in the so-called "fast motion approximation" for N point masses [at the first post-Minkowskian order (1PM)], and it is in agreement with GR for the one-body case ($N = 1$) at all orders (assuming the central mass is not spinning) if the Schwarzschild metric is expressed in the isotropic gauge.

II. AN ACTION FUNCTIONAL FOR GRAVITY IN THE RELATIVISTIC ACTION-AT-A-DISTANCE FORMULATION

In order to describe a relativistic system of N point masses interacting gravitationally, we consider the following action functional:

$$S = -\sum_i m_i c \int d\lambda_i \zeta_i + \sum_i \sum_{j\neq i} \frac{G m_i m_j}{c} \iint d\lambda_i d\lambda_j \delta(\rho_{ij}) F_{ij} + \sum_i \sum_{j\neq i} \sum_{k\neq i,j} \frac{G^2 m_i m_j m_k}{c^3} \times \iiint d\lambda_i d\lambda_j d\lambda_k \delta(\rho_{ij}) \delta(\rho_{jk}) F_{ijk} + \dots \quad (1)$$

In (1), m_i ($i = 1, 2, \dots, N$) is the mass of particle i, λ_i is a Poincaré invariant parameter labeling the events along the world line $z_i^\mu(\lambda_i)$ of particle i, c is the speed of light, and G is the universal gravitational constant.

The functions $F_{ij} = F_{ij}(\xi_{ij}, \gamma_{ij}, \gamma_{ji}, \zeta_i, \zeta_j)$, $F_{ijk} = F_{ijk}(\xi_{ij}, \xi_{ik}, \xi_{jk}, \gamma_{ij}, \gamma_{ji}, \gamma_{ik}, \gamma_{ki}, \gamma_{jk}, \gamma_{kj}, \zeta_i, \zeta_j, \zeta_k)$, are invariant under Poincaré transformations since they are assumed to be functions of the Poincaré invariants ξ_{ij}, γ_{ij}, and ζ_i. The Poincaré invariants ρ_{ij}, ξ_{ij}, γ_{ij}, and ζ_i are defined as follows [49]:

$$\rho_{ij} = (z_i - z_j)^2, \quad (2)$$

$$\xi_{ij} = (\dot{z}_i \dot{z}_j), \quad (3)$$

$$\gamma_{ij} = (\dot{z}_i (z_j - z_i)), \quad (4)$$

$$\zeta_i = \dot{z}_i^2. \quad (5)$$

We denote $\dot{z}_i^\mu = \frac{dz_i^\mu}{d\lambda_i}$. The metric tensor: $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$.

The action functional (1) is invariant under Lorentz transformations and does not involve any fields to mediate the interactions between the masses. The particles interact with each other directly, and we assume that the interactions propagate at the speed of light c in vacuum. The Dirac delta functions in (1) account for the interactions propagating at the speed of light forward and backward in time.

The action (1) can be written in a compact form as follows:

$$S = -\sum_i m_i c \int d\lambda_i \zeta_i + \sum_{k=2}^{N} \sum_{i_1} \sum_{i_2\neq i_1} \dots \sum_{i_k \neq i_1,\dots,i_{k-1}} \frac{G^{k-1} m_{i_1} \dots m_{i_k}}{c^{2k-3}} \times \int \dots \int d\lambda_{i_1} \dots d\lambda_{i_k} \prod_{l=1}^{k-1} \delta(\rho_{i_l i_{l+1}}) F_{i_1 \dots i_k}. \quad (6)$$

Notice that not only the two-body interactions ($k = 2$) but all possible k-body interactions ($k = 2, \dots, N$) contribute to the action.

*martinez@phas.ubc.ca

Without loss of generality we can assume that $F_{ji} = F_{ij}$, $F_{kji} = F_{ijk}$, and so on ($F_{i_k i_2 \ldots i_{k-1} i_1} = F_{i_1 i_2 \ldots i_{k-1} i_k}$). From (1), we can see that we can write the action of an individual particle i as follows:

$$S_i = -m_i c \int d\lambda_i \Big(\zeta_i - \frac{2G}{c^2} \sum_{j\neq i} m_j \int d\lambda_j \delta(\rho_{ij}) F_{ij} - \frac{G^2}{c^4} \sum_{j\neq i} \sum_{k\neq i,j} m_j m_k \iint d\lambda_j d\lambda_k (\delta(\rho_{ij})\delta(\rho_{jk}) F_{ijk} + \delta(\rho_{jk})\delta(\rho_{ki}) F_{jki} + \delta(\rho_{ki})\delta(\rho_{ij}) F_{kij}) + \ldots \Big). \tag{7}$$

The equations of motion of the relativistic particles can be derived from the action (1) [or from (7)] using the variational principle. We find

$$\ddot{z}_i^\mu + \frac{G}{c^2} \sum_{j\neq i} m_j \int d\lambda_j \Big(\frac{\partial}{\partial z_{i\mu}} (\delta(\rho_{ij}) F_{ij}) - \frac{d}{d\lambda_i} \Big(\delta(\rho_{ij}) \frac{\partial F_{ij}}{\partial \dot{z}_{i\mu}} \Big) \Big) + \frac{G^2}{2c^4} \sum_{j\neq i} \sum_{k\neq i,j} m_j m_k \iint d\lambda_j d\lambda_k \Big(\delta(\rho_{jk}) \Big(\frac{\partial}{\partial z_{i\mu}} (\delta(\rho_{ij}) F_{ijk}) - \frac{d}{d\lambda_i} \Big(\delta(\rho_{ij}) \frac{\partial F_{ijk}}{\partial \dot{z}_{i\mu}} \Big) \Big) + \delta(\rho_{jk}) \Big(\frac{\partial}{\partial z_{i\mu}} (\delta(\rho_{ki}) F_{jki}) - \frac{d}{d\lambda_i} \Big(\delta(\rho_{ki}) \frac{\partial F_{jki}}{\partial \dot{z}_{i\mu}} \Big) \Big) + \frac{\partial}{\partial z_{i\mu}} (\delta(\rho_{ki})\delta(\rho_{ij}) F_{kij}) - \frac{d}{d\lambda_i} \Big(\delta(\rho_{ki})\delta(\rho_{ij}) \frac{\partial F_{kij}}{\partial \dot{z}_{i\mu}} \Big) \Big) + \ldots = 0. \tag{8}$$

Integrating by parts and taking into account that

$$\frac{d}{d\lambda_i}(\delta(\rho_{ij})) = \frac{(\frac{d\rho_{ij}}{d\lambda_i})}{(\frac{d\rho_{ij}}{d\lambda_j})} \frac{d}{d\lambda_j}(\delta(\rho_{ij})) = \frac{\gamma_{ij}}{\gamma_{ji}} \frac{d}{d\lambda_j}(\delta(\rho_{ij})), \tag{9}$$

we can write the equations of motion (8) in the form

$$\ddot{z}_i^\mu + \frac{G}{c^2} \sum_{j\neq i} m_j \int d\lambda_j \delta(\rho_{ij}) (A_{ij}^\mu + B_{ij}^{\mu\nu} \ddot{z}_{i\nu} + C_{ij}^{\mu\nu} \ddot{z}_{j\nu}) + \frac{G^2}{c^4} \sum_{j\neq i} \sum_{k\neq i,j} m_j m_k \iint d\lambda_j d\lambda_k \delta(\rho_{ij})\delta(\rho_{jk}) (A_{ijk}^\mu + B_{ijk}^{\mu\nu} \ddot{z}_{i\nu} + C_{ijk}^{\mu\nu} \ddot{z}_{j\nu} + D_{ijk}^{\mu\nu} \ddot{z}_{k\nu}) + \frac{G^2}{2c^4} \sum_{j\neq i} \sum_{k\neq i,j} m_j m_k \iint d\lambda_j d\lambda_k \delta(\rho_{ji})\delta(\rho_{ik}) (\tilde{A}_{jik}^\mu + \tilde{B}_{jik}^{\mu\nu} \ddot{z}_{i\nu} + \tilde{C}_{jik}^{\mu\nu} \ddot{z}_{j\nu} + \tilde{D}_{jik}^{\mu\nu} \ddot{z}_{k\nu}) + \ldots = 0, \tag{10}$$

where

$$A_{ij}^\mu = \frac{\partial F_{ij}}{\partial z_{i\mu}} - \frac{\partial^2 F_{ij}}{\partial z_i^\eta \partial \dot{z}_{i\mu}} \dot{z}_i^\eta + \frac{\zeta_j}{\gamma_{ji}^2} \Big((z_i^\mu - z_j^\mu) F_{ij} + \gamma_{ij} \frac{\partial F_{ij}}{\partial \dot{z}_{i\mu}} \Big) + \frac{1}{\gamma_{ji}} \Big(-\dot{z}_j^\mu F_{ij} + (z_i^\mu - z_j^\mu) \frac{\partial F_{ij}}{\partial z_j^\eta} \dot{z}_j^\eta + \xi_{ij} \frac{\partial F_{ij}}{\partial \dot{z}_{i\mu}} + \gamma_{ij} \frac{\partial^2 F_{ij}}{\partial z_j^\eta \partial \dot{z}_{i\mu}} \dot{z}_j^\eta \Big), \tag{11}$$

$$B_{ij}^{\mu\nu} = -\frac{\partial^2 F_{ij}}{\partial \dot{z}_{i\mu} \partial \dot{z}_{i\nu}}, \tag{12}$$

$$C_{ij}^{\mu\nu} = \frac{(z_i^\mu - z_j^\mu)}{\gamma_{ji}} \Big(\frac{\partial F_{ij}}{\partial \dot{z}_{j\nu}} - \frac{(z_i^\nu - z_j^\nu)}{\gamma_{ji}} F_{ij} \Big) + \frac{\gamma_{ij}}{\gamma_{ji}} \Big(\frac{\partial^2 F_{ij}}{\partial \dot{z}_{i\mu} \partial \dot{z}_{j\nu}} - \frac{(z_i^\nu - z_j^\nu)}{\gamma_{ji}} \frac{\partial F_{ij}}{\partial \dot{z}_{i\mu}} \Big), \tag{13}$$

$$A^{\mu}_{ijk} = \frac{\partial F_{ijk}}{\partial z_{i\mu}} - \frac{\partial^2 F_{ijk}}{\partial z_i^\eta \partial \dot{z}_{i\mu}} \dot{z}_i^\eta + \frac{\zeta_j}{\gamma_{ji}^2}\left((z_i^\mu - z_j^\mu)F_{ijk} + \gamma_{ij}\frac{\partial F_{ijk}}{\partial \dot{z}_{i\mu}}\right)$$
$$+ \frac{1}{\gamma_{ji}}\left(-\dot{z}_j^\mu F_{ijk} + (z_i^\mu - z_j^\mu)\frac{\partial F_{ijk}}{\partial z_j^\eta}\dot{z}_j^\eta + \xi_{ij}\frac{\partial F_{ijk}}{\partial \dot{z}_{i\mu}} + \gamma_{ij}\frac{\partial^2 F_{ijk}}{\partial z_j^\eta \partial \dot{z}_{i\mu}}\dot{z}_j^\eta\right)$$
$$- \frac{1}{\gamma_{kj}\gamma_{ji}}\left(\xi_{jk} + \frac{\gamma_{jk}}{\gamma_{kj}}\zeta_k\right)\left((z_i^\mu - z_j^\mu)F_{ijk} + \gamma_{ij}\frac{\partial F_{ijk}}{\partial \dot{z}_{i\mu}}\right)$$
$$- \frac{\gamma_{jk}}{\gamma_{kj}\gamma_{ji}}\left((z_i^\mu - z_j^\mu)\frac{\partial F_{ijk}}{\partial z_k^\eta}\dot{z}_k^\eta + \gamma_{ij}\frac{\partial^2 F_{ijk}}{\partial z_k^\eta \partial \dot{z}_{i\mu}}\dot{z}_k^\eta\right), \tag{14}$$

$$B^{\mu\nu}_{ijk} = -\frac{\partial^2 F_{ijk}}{\partial \dot{z}_{i\mu}\partial \dot{z}_{i\nu}}, \tag{15}$$

$$C^{\mu\nu}_{ijk} = \frac{(z_i^\mu - z_j^\mu)}{\gamma_{ji}}\left(\frac{\partial F_{ijk}}{\partial \dot{z}_{j\nu}} - \frac{(z_i^\nu - z_j^\nu)}{\gamma_{ji}}F_{ijk}\right) + \frac{\gamma_{ij}}{\gamma_{ji}}\left(\frac{\partial^2 F_{ijk}}{\partial \dot{z}_{i\mu}\partial \dot{z}_{j\nu}} - \frac{(z_i^\nu - z_j^\nu)}{\gamma_{ji}}\frac{\partial F_{ijk}}{\partial \dot{z}_{i\mu}}\right), \tag{16}$$

$$D^{\mu\nu}_{ijk} = \frac{\gamma_{jk}(z_j^\nu - z_k^\nu)}{\gamma_{kj}^2}\left(\frac{(z_i^\mu - z_j^\mu)}{\gamma_{ji}}F_{ijk} + \frac{\gamma_{ij}}{\gamma_{ji}}\frac{\partial F_{ijk}}{\partial \dot{z}_{i\mu}}\right) - \frac{\gamma_{jk}}{\gamma_{kj}}\left(\frac{(z_i^\mu - z_j^\mu)}{\gamma_{ji}}\frac{\partial F_{ijk}}{\partial \dot{z}_{k\nu}} + \frac{\gamma_{ij}}{\gamma_{ji}}\frac{\partial^2 F_{ijk}}{\partial \dot{z}_{i\mu}\partial \dot{z}_{k\nu}}\right), \tag{17}$$

$$\tilde{A}^{\mu}_{jik} = \frac{\partial F_{jik}}{\partial z_{i\mu}} - \frac{\partial^2 F_{jik}}{\partial z_i^\eta \partial \dot{z}_{i\mu}}\dot{z}_i^\eta + \frac{\zeta_j}{\gamma_{ji}^2}\left((z_i^\mu - z_j^\mu)F_{jik} + \gamma_{ij}\frac{\partial F_{jik}}{\partial \dot{z}_{i\mu}}\right) + \frac{\zeta_k}{\gamma_{ki}^2}\left((z_i^\mu - z_k^\mu)F_{jik} + \gamma_{ik}\frac{\partial F_{jik}}{\partial \dot{z}_{i\mu}}\right)$$
$$+ \frac{1}{\gamma_{ji}}\left(-\dot{z}_j^\mu F_{jik} + (z_i^\mu - z_j^\mu)\frac{\partial F_{jik}}{\partial z_j^\eta}\dot{z}_j^\eta + \xi_{ij}\frac{\partial F_{jik}}{\partial \dot{z}_{i\mu}} + \gamma_{ij}\frac{\partial^2 F_{jik}}{\partial z_j^\eta \partial \dot{z}_{i\mu}}\dot{z}_j^\eta\right)$$
$$+ \frac{1}{\gamma_{ki}}\left(-\dot{z}_k^\mu F_{jik} + (z_i^\mu - z_k^\mu)\frac{\partial F_{jik}}{\partial z_k^\eta}\dot{z}_k^\eta + \xi_{ik}\frac{\partial F_{jik}}{\partial \dot{z}_{i\mu}} + \gamma_{ik}\frac{\partial^2 F_{jik}}{\partial z_k^\eta \partial \dot{z}_{i\mu}}\dot{z}_k^\eta\right), \tag{18}$$

$$\tilde{B}^{\mu\nu}_{jik} = -\frac{\partial^2 F_{jik}}{\partial \dot{z}_{i\mu}\partial \dot{z}_{i\nu}}, \tag{19}$$

$$\tilde{C}^{\mu\nu}_{jik} = \frac{(z_i^\mu - z_j^\mu)}{\gamma_{ji}}\left(\frac{\partial F_{jik}}{\partial \dot{z}_{j\nu}} - \frac{(z_i^\nu - z_j^\nu)}{\gamma_{ji}}F_{jik}\right) + \frac{\gamma_{ij}}{\gamma_{ji}}\left(\frac{\partial^2 F_{jik}}{\partial \dot{z}_{i\mu}\partial \dot{z}_{j\nu}} - \frac{(z_i^\nu - z_j^\nu)}{\gamma_{ji}}\frac{\partial F_{jik}}{\partial \dot{z}_{i\mu}}\right), \tag{20}$$

$$\tilde{D}^{\mu\nu}_{jik} = -\frac{(z_i^\nu - z_k^\nu)}{\gamma_{ki}^2}\left((z_i^\mu - z_k^\mu)F_{jik} + \gamma_{ik}\frac{\partial F_{jik}}{\partial \dot{z}_{i\mu}}\right) + \frac{1}{\gamma_{ki}}\left((z_i^\mu - z_k^\mu)\frac{\partial F_{jik}}{\partial \dot{z}_{k\nu}} + \gamma_{ik}\frac{\partial^2 F_{jik}}{\partial \dot{z}_{i\mu}\partial \dot{z}_{k\nu}}\right). \tag{21}$$

Multiplying (8) by $\dot{z}_{i\mu}$ (and performing the summation over μ), we find that the solutions of the equations of motion (8) must satisfy the following N conditions ($i = 1, 2, \ldots, N$):

$$\frac{d}{d\lambda_i}\left(\zeta_i + \frac{2G}{c^2}\sum_{j\neq i} m_j \int d\lambda_j \delta(\rho_{ij})\left(F_{ij} - \dot{z}_i^\mu \frac{\partial F_{ij}}{\partial \dot{z}_i^\mu}\right) + \frac{G^2}{c^4}\sum_{j\neq i}\sum_{k\neq i,j} m_j m_k \iint d\lambda_j d\lambda_k \left(\delta(\rho_{ij})\delta(\rho_{jk})\left(F_{ijk} - \dot{z}_i^\mu \frac{\partial F_{ijk}}{\partial \dot{z}_i^\mu}\right)\right.\right.$$
$$\left.\left. + \delta(\rho_{jk})\delta(\rho_{ki})\left(F_{jki} - \dot{z}_i^\mu \frac{\partial F_{jki}}{\partial \dot{z}_i^\mu}\right) + \delta(\rho_{ki})\delta(\rho_{ij})\left(F_{kij} - \dot{z}_i^\mu \frac{\partial F_{kij}}{\partial \dot{z}_i^\mu}\right)\right) + \ldots\right) = 0. \tag{22}$$

Let us assume that $F_{i_1\ldots i_k}$ ($k = 2, \ldots, N$) are homogeneous functions of degree two in $\dot{z}_{i_1}, \ldots, \dot{z}_{i_k}$; i.e., we assume that they satisfy the following conditions:

$$\dot{z}_{i_1}^\mu \frac{\partial F_{i_1\ldots i_k}}{\partial \dot{z}_{i_1}^\mu} = \ldots = \dot{z}_{i_k}^\mu \frac{\partial F_{i_1\ldots i_k}}{\partial \dot{z}_{i_k}^\mu} = 2F_{i_1\ldots i_k}. \tag{23}$$

The conditions (22) combined with (23) guarantee that, for the solutions of the equations of motion, the expressions

$$\zeta_i - \frac{2G}{c^2}\sum_{j\neq i} m_j \int d\lambda_j \delta(\rho_{ij}) F_{ij} - \frac{G^2}{c^4}\sum_{j\neq i}\sum_{k\neq i,j} m_j m_k \iint d\lambda_j d\lambda_k (\delta(\rho_{ij})\delta(\rho_{jk})F_{ijk} + \delta(\rho_{jk})\delta(\rho_{ki})F_{jki} + \delta(\rho_{ki})\delta(\rho_{ij})F_{kij}) + \ldots = c_i \tag{24}$$

are constants (which by simple scaling can be made equal to 1):

$$\zeta_i - \frac{2G}{c^2}\sum_{j\neq i} m_j \int d\lambda_j \delta(\rho_{ij}) F_{ij} - \frac{G^2}{c^4}\sum_{j\neq i}\sum_{k\neq i,j} m_j m_k \iint d\lambda_j d\lambda_k (\delta(\rho_{ij})\delta(\rho_{jk})F_{ijk} + \delta(\rho_{jk})\delta(\rho_{ki})F_{jki} + \delta(\rho_{ki})\delta(\rho_{ij})F_{kij}) + \ldots = 1. \tag{25}$$

From (23) it immediately follows that

$$F_{i_1\ldots i_k} = \frac{1}{2}\frac{\partial^2 F_{i_1\ldots i_k}}{\partial \dot z^\mu_{i_1}\partial \dot z^\nu_{i_1}} \dot z^\mu_{i_1}\dot z^\nu_{i_1} = \ldots = \frac{1}{2}\frac{\partial^2 F_{i_1\ldots i_k}}{\partial \dot z^\mu_{i_k}\partial \dot z^\nu_{i_k}} \dot z^\mu_{i_k}\dot z^\nu_{i_k}. \tag{26}$$

Using (26) we immediately see that the action for particle i (7) can be rewritten in a compact form as

$$S_i = -m_i c \int d\lambda_i g^{(i)}_{\mu\nu} \dot z^\mu_i \dot z^\nu_i, \tag{27}$$

where

$$g^{(i)}_{\mu\nu} = \eta_{\mu\nu} - \frac{G}{c^2}\sum_{j\neq i} m_j \int d\lambda_j \delta(\rho_{ij}) \frac{\partial^2 F_{ij}}{\partial \dot z^\mu_i \partial \dot z^\nu_i} - \frac{G^2}{2c^4}\sum_{j\neq i}\sum_{k\neq i,j} m_j m_k \iint d\lambda_j d\lambda_k \Big(\delta(\rho_{ij})\delta(\rho_{jk})\frac{\partial^2 F_{ijk}}{\partial \dot z^\mu_i \partial \dot z^\nu_i} + \delta(\rho_{jk})\delta(\rho_{ki})\frac{\partial^2 F_{jki}}{\partial \dot z^\mu_i \partial \dot z^\nu_i} + \delta(\rho_{ki})\delta(\rho_{ij})\frac{\partial^2 F_{kij}}{\partial \dot z^\mu_i \partial \dot z^\nu_i}\Big) + \ldots. \tag{28}$$

From (28), (23), and (26) it follows that

$$\frac{\partial g^{(i)}_{\alpha\beta}}{\partial \dot z^\mu_i} \dot z^\alpha_i = 0, \tag{29}$$

$$\frac{\partial^2 g^{(i)}_{\alpha\beta}}{\partial \dot z^\mu_i \partial \dot z^\nu_i} \dot z^\alpha_i \dot z^\beta_i = 0, \tag{30}$$

$$\frac{\partial^2 g^{(i)}_{\alpha\beta}}{\partial \dot z^\mu_i \partial z^\nu_i} \dot z^\alpha_i = 0. \tag{31}$$

Using (28)–(31) the equations of motion (8) can also be written in a more compact form:

$$g^{(i)}_{\mu\nu}\ddot z^\nu_i + \frac{1}{2}\left(\frac{\partial g^{(i)}_{\mu\alpha}}{\partial z^\beta_i} + \frac{\partial g^{(i)}_{\mu\beta}}{\partial z^\alpha_i} - \frac{\partial g^{(i)}_{\alpha\beta}}{\partial z^\mu_i}\right)\dot z^\alpha_i \dot z^\beta_i = 0. \tag{32}$$

From (10), (22), and (28), we find that the conditions (22) can be simply expressed as

$$\frac{d}{d\lambda_i}(g^{(i)}_{\mu\nu}\dot z^\mu_i \dot z^\nu_i) = 0. \tag{33}$$

From (25), (26), and (28), it follows that for the solutions of the equations of motion

$$d\lambda_i^2 = g^{(i)}_{\mu\nu} dz^\mu_i dz^\nu_i. \tag{34}$$

Notice that $g^{(i)}_{\mu\nu}$ is not a field. It depends not only on z_i, z_j, and $\dot z_j$ ($j \neq i$) but also on $\dot z_i$.

The main task in our formulation is to determine the functions F_{ij}, F_{ijk}, etc., in (6) and to verify that the predictions of the theory are in agreement with observations.

III. TEST PARTICLES AND THE FORMULATION OF THE ACTION-AT-A-DISTANCE MODEL AS A FIELD THEORY

Let us assume that in the limit $m_i \to 0$ the tensor $g^{(i)}_{\mu\nu}$ does not depend on $\dot z_i$. Only in this limit, in which m_i is a test particle, we may have a field interpretation for the metric tensor $g^{(i)}_{\mu\nu}$.

Let us consider a system of $N+1$ point particles, one of them being a test particle of mass m and the other N particles having masses m_i ($i = 1, \ldots, N$). Let $z(\lambda)$ be the worldline of the test particle. From (27) we see that we can write the action for the test particle as follows:

$$S = -mc \int d\lambda g_{\mu\nu}\dot z^\mu \dot z^\nu. \tag{35}$$

In (35), the metric tensor $g_{\mu\nu}$ depends on z [and on z_i and $\dot z_i$ ($i = 1, \ldots, N$)] but does not depend on $\dot z$. It can be given a field interpretation, if one desires to do so [50].

From (32) we see that for a test particle the equations of motion are

$$g_{\mu\nu}\ddot z^\nu + \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial z^\beta} + \frac{\partial g_{\mu\beta}}{\partial z^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial z^\mu}\right)\dot z^\alpha \dot z^\beta = 0. \tag{36}$$

Assuming that the matrix $g_{\mu\nu}$ is invertible, these are, of course, the well-known equations for geodesics.

It may be possible to impose conditions on the functions F_{ij}, F_{ijk}, etc., in (6) if, for example, one demands that the metric tensor $g_{\mu\nu}$ (which is associated with a test particle) obeys Einstein's field equations. Of course, there is no guarantee that this can be done at all orders in the post-Minkowskian expansion, either due to the mathematical complexity of the equations or due to the possibility that Einstein's theory of general relativity may not exactly admit a dual (action-at-a-distance) formulation, or at least not one described by an action of the form (6).

At the first post-Minkowskian order we can write [51–54]

$$F_{ij} = \alpha_0 \xi_{ij}^2 + \beta_0 \zeta_{ij}^2, \tag{37}$$

where α_0 and β_0 are constants, and

$$\zeta_{ij} = \sqrt{\zeta_i \zeta_j}. \tag{38}$$

At the first post-Minkowskian approximation there is no need to consider the functions F_{ijk} since the terms associated with these functions in the action (1) give contributions only at the second post-Minkowskian order.

From (28) and (37) it follows that, at the first post-Minkowskian order, in the presence of N massive particles the metric tensor associated with a test particle is given by the formula

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1)}, \tag{39}$$

where

$$h_{\mu\nu}^{(1)} = -\frac{2G}{c^2}\sum_i m_i \int d\lambda_i \delta((z-z_i)^2)[\alpha_0 \dot{z}_{i\mu}\dot{z}_{i\nu} + \beta_0 \dot{z}_i^2 \eta_{\mu\nu}]. \tag{40}$$

From (40) it immediately follows that, at the first post-Minkowskian order, the quantities $h_{\mu\nu}^{(1)}$ obey the equations

$$\frac{\partial h^{(1)\mu\nu}}{\partial z^\nu} - \frac{1}{2}\eta^{\mu\nu}\frac{\partial h_\lambda^{(1)\lambda}}{\partial z^\nu} = \frac{G}{c^2}(\alpha_0 + 2\beta_0)\eta^{\mu\nu}\sum_i m_i \int d\lambda_i \frac{\partial}{\partial z^\nu}(\delta((z-z_i)^2))\dot{z}_i^2. \tag{41}$$

In this approximation, the Ricci tensor is given by the expression

$$R_{\mu\nu} \approx R_{\mu\nu}^{(1)} = \frac{1}{2}\left(\frac{\partial^2 h_{\nu\lambda}^{(1)}}{\partial z^\mu \partial z_\lambda} + \frac{\partial^2 h_{\mu\lambda}^{(1)}}{\partial z^\nu \partial z_\lambda} - \frac{\partial^2 h_\lambda^{(1)\lambda}}{\partial z^\mu \partial z^\nu} - \frac{\partial^2 h_{\mu\nu}^{(1)}}{\partial z^\lambda \partial z_\lambda}\right), \tag{42}$$

and the scalar curvature

$$R \approx R^{(1)} = \eta^{\alpha\beta}R_{\alpha\beta}^{(1)} = \frac{\partial^2 h_\lambda^{(1)\rho}}{\partial z^\rho \partial z_\lambda} - \frac{\partial^2 h_\rho^{(1)\rho}}{\partial z^\lambda \partial z_\lambda}. \tag{43}$$

At the first post-Minkowskian approximation Einstein's equations take the form [51–54]:

$$R_{\mu\nu}^{(1)} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} = \frac{8\pi G}{c^4}T_{\mu\nu}^{(0)}, \tag{44}$$

where $T_{\mu\nu}^{(0)}$ is the energy-momentum tensor in this approximation:

$$T_{\mu\nu}^{(0)}(z) = \sum_i m_i c^2 \int d\lambda_i \dot{z}_{i\mu}\dot{z}_{i\nu}\delta^4(z-z_i). \tag{45}$$

Substituting (40) into (42)–(44) we find that, at the first post-Minkowskian approximation, Einstein's equations reduce to the following:

$$\sum_i m_i \int d\lambda_i \frac{\partial^2(\delta((z-z_i)^2))}{\partial z^\alpha \partial z^\beta}[\alpha_0 \dot{z}_{i\mu}\dot{z}_{i\nu}\eta^{\alpha\beta} + (\alpha_0 + 2\beta_0)\dot{z}_i^2(\delta_\mu^\alpha \delta_\nu^\beta - \eta_{\mu\nu}\eta^{\alpha\beta})] = 8\pi \sum_i m_i \int d\lambda_i \dot{z}_{i\mu}\dot{z}_{i\nu}\delta^4(z-z_i). \tag{46}$$

Recalling that

$$\Box\delta((z-z_i)^2) = \eta^{\alpha\beta}\frac{\partial^2(\delta((z-z_i)^2))}{\partial z^\alpha \partial z^\beta} = 4\pi\delta^4(z-z_i), \tag{47}$$

from (47) and (46), we find agreement with the equations of general relativity, in the first post-Minkowskian approximation, if [51–54]

$$\alpha_0 = 2, \tag{48}$$

$$\beta_0 = -1. \tag{49}$$

From (48), (49), and (41), it follows that

$$\frac{\partial h^{(1)\mu\nu}}{\partial z^\nu} - \frac{1}{2}\eta^{\mu\nu}\frac{\partial h_\lambda^{(1)\lambda}}{\partial z^\nu} = 0. \tag{50}$$

In field theory, the linearized Einstein's equations (44), together with the conditions (50), are the equations obeyed by spin 2 fields [55].

IV. A POSSIBLE EXPRESSION FOR F_{ij}

More generally, beyond the first post-Minkowskian order, let us assume that the functions F_{ij} can be expressed as follows:

$$F_{ij} = \alpha(\epsilon_{ij}, \epsilon_{ji})\xi_{ij}^2 + \beta(\epsilon_{ij}, \epsilon_{ji})\zeta_{ij}^2, \tag{51}$$

where

$$\epsilon_{ij} = \frac{Gm_i}{2c^2|\eta_{ij}|} \tag{52}$$

and [49]

$$\eta_{ij} = \frac{\gamma_{ij}}{\sqrt{\zeta_i}}. \tag{53}$$

We assume that the functions α and β are symmetric:

$$\alpha(\epsilon_{ij}, \epsilon_{ji}) = \alpha(\epsilon_{ji}, \epsilon_{ij}), \tag{54}$$

$$\beta(\epsilon_{ij}, \epsilon_{ji}) = \beta(\epsilon_{ji}, \epsilon_{ij}). \tag{55}$$

V. THE ONE-BODY PROBLEM

Let us consider the case of a test particle interacting with a particle of mass M. This is the case $N = 1$ (the one-body problem). The motion of the mass M is not affected by the presence of the test particle. The mass M moves with constant velocity in any inertial reference frame.

Let us, for simplicity, consider the inertial frame in which the mass M is at rest and positioned at the origin of the coordinate system. In this frame of reference the world line of the test particle is described by the four-vector $z^\mu = (ct, \vec{r})$. From (28) and (51) we find the components of the metric tensor $g_{\mu\nu}$ in this reference frame to be as follows:

$$g_{00} = 1 - \frac{2GM(\alpha(0,\epsilon) + \beta(0,\epsilon))}{c^2 r}, \tag{56}$$

$$g_{0i} = 0, \tag{57}$$

$$g_{ij} = -\delta_{ij}\left(1 - \frac{2GM\beta(0,\epsilon)}{c^2 r}\right). \tag{58}$$

In (56)–(58)

$$\epsilon = \frac{GM}{2c^2 r}. \tag{59}$$

If we choose the functions α and β as follows:

$$\alpha(0,\epsilon) = \frac{(1+\epsilon)^4 - \frac{(1-\epsilon)^2}{(1+\epsilon)^2}}{4\epsilon}, \tag{60}$$

$$\beta(0,\epsilon) = \frac{1-(1+\epsilon)^4}{4\epsilon}, \tag{61}$$

one can easily check that the metric (56)–(58), with α and β given by (60) and (61), coincides with the well-known Schwarzschild metric of GR in isotropic form [55].

Since α, β, and ϵ_{ji} are Poincaré invariants, we can write the functional relations

$$\alpha(0,\epsilon_{ji}) = \frac{(1+\epsilon_{ji})^4 - \frac{(1-\epsilon_{ji})^2}{(1+\epsilon_{ji})^2}}{4\epsilon_{ji}}, \tag{62}$$

$$\beta(0,\epsilon_{ji}) = \frac{1-(1+\epsilon_{ji})^4}{4\epsilon_{ji}}. \tag{63}$$

At the second post-Minkowskian order (up to terms proportional to G^2 in the metric), we can write

$$\alpha(0,\epsilon_{ji}) \approx \alpha_0 + \alpha_1\epsilon_{ji}, \tag{64}$$

$$\beta(0,\epsilon_{ji}) \approx \beta_0 + \beta_1\epsilon_{ji}, \tag{65}$$

where α_0, β_0, α_1, and β_1 are constants.

The values of these constants can easily be determined by expanding (62) and (63). We find

$$\alpha_0 = 2, \tag{66}$$

$$\beta_0 = -1, \tag{67}$$

$$\alpha_1 = -\frac{1}{2}, \tag{68}$$

$$\beta_1 = -\frac{3}{2}. \tag{69}$$

VI. THE SECOND POST-MINKOWSKIAN APPROXIMATION

Let us now consider the gravitational N-body problem described by the action (6). Assume that F_{ij} are given by (51), (54), and (55). At the second post-Minkowskian (2PM) order the functions α and β will be given by the expressions

$$\alpha(\epsilon_{ij},\epsilon_{ji}) \approx \alpha_0 + \alpha_1(\epsilon_{ij} + \epsilon_{ji}), \tag{70}$$

$$\beta(\epsilon_{ij},\epsilon_{ji}) \approx \beta_0 + \beta_1(\epsilon_{ij} + \epsilon_{ji}). \tag{71}$$

Let us consider the case where the functions F_{ijk} can be written as

$$\begin{aligned} F_{ijk} &= a(\epsilon_{ij},\epsilon_{ji},\epsilon_{ki},\epsilon_{ik},\epsilon_{jk},\epsilon_{kj})\xi_{ij}\xi_{jk}\xi_{ki} \\ &+ b(\epsilon_{ij},\epsilon_{ji},\epsilon_{ki},\epsilon_{ik},\epsilon_{jk},\epsilon_{kj})\zeta_i\zeta_j\zeta_k \\ &+ c(\epsilon_{ij},\epsilon_{ji},\epsilon_{ki},\epsilon_{ik},\epsilon_{jk},\epsilon_{kj})\xi_{ki}^2\zeta_j. \end{aligned} \tag{72}$$

We assume that the functions a, b, and c are symmetric in the indexes (ik):

$$a(\epsilon_{kj},\epsilon_{jk},\epsilon_{ik},\epsilon_{ki},\epsilon_{ji},\epsilon_{ij}) = a(\epsilon_{ij},\epsilon_{ji},\epsilon_{ki},\epsilon_{ik},\epsilon_{jk},\epsilon_{kj}), \tag{73}$$

$$b(\epsilon_{kj},\epsilon_{jk},\epsilon_{ik},\epsilon_{ki},\epsilon_{ji},\epsilon_{ij}) = b(\epsilon_{ij},\epsilon_{ji},\epsilon_{ki},\epsilon_{ik},\epsilon_{jk},\epsilon_{kj}), \tag{74}$$

$$c(\epsilon_{kj},\epsilon_{jk},\epsilon_{ik},\epsilon_{ki},\epsilon_{ji},\epsilon_{ij}) = c(\epsilon_{ij},\epsilon_{ji},\epsilon_{ki},\epsilon_{ik},\epsilon_{jk},\epsilon_{kj}). \tag{75}$$

At the second post-Minkowskian order we have

$$a(\epsilon_{ij},\epsilon_{ji},\epsilon_{ki},\epsilon_{ik},\epsilon_{jk},\epsilon_{kj}) \approx a_0, \tag{76}$$

$$b(\epsilon_{ij},\epsilon_{ji},\epsilon_{ki},\epsilon_{ik},\epsilon_{jk},\epsilon_{kj}) \approx b_0, \tag{77}$$

$$c(\epsilon_{ij},\epsilon_{ji},\epsilon_{ki},\epsilon_{ik},\epsilon_{jk},\epsilon_{kj}) \approx c_0, \tag{78}$$

where a_0, b_0, and c_0 are constants.

Therefore, at the second post-Minkowskian order we can write the action, for a system of N particles interacting gravitationally, as follows:

$$S = -\sum_i m_i c \int d\lambda_i \zeta_i + \sum_i \sum_{j\neq i} \frac{G m_i m_j}{c} \iint d\lambda_i d\lambda_j \delta(\rho_{ij}) F_{ij} + \sum_i \sum_{j\neq i} \sum_{k\neq i,j} \frac{G^2 m_i m_j m_k}{c^3} \times \iiint d\lambda_i d\lambda_j d\lambda_k \delta(\rho_{ij})\delta(\rho_{jk}) F_{ijk}, \quad (79)$$

where

$$F_{ij} = (\alpha_0 + \alpha_1(\epsilon_{ij} + \epsilon_{ji}))\xi_{ij}^2 + (\beta_0 + \beta_1(\epsilon_{ij} + \epsilon_{ji}))\zeta_{ij}^2, \quad (80)$$

$$F_{ijk} = a_0 \xi_{ij}\xi_{jk}\xi_{ki} + b_0 \zeta_i \zeta_j \zeta_k + c_0 \xi_{ki}^2 \zeta_j. \quad (81)$$

At the second post-Minkowskian approximation there is no need to consider the functions $F_{i_1\ldots i_k}$ for $k > 3$ since the terms associated with these functions in the action (6) give contributions only at the $(k-1)$-post-Minkowskian order.

At the second Post-Minkowskian order the equations of motion are

$$\ddot{z}_i^\mu + \frac{G}{c^2}\sum_{j\neq i} m_j \int d\lambda_j \delta(\rho_{ij})(A_{ij}^{(0)\mu} + A_{ij}^{(1)\mu} + B_{ij}^{(0)\mu\nu}\ddot{z}_{i\nu} + C_{ij}^{(0)\mu\nu}\ddot{z}_{j\nu}) + \frac{G^2}{c^4}\sum_{j\neq i}\sum_{k\neq i,j} m_j m_k \iint d\lambda_j d\lambda_k \delta(\rho_{ij})\delta(\rho_{jk}) A_{ijk}^{(0)\mu} + \frac{G^2}{2c^4}\sum_{j\neq i}\sum_{k\neq i,j} m_j m_k \iint d\lambda_j d\lambda_k \delta(\rho_{ji})\delta(\rho_{ik}) \tilde{A}_{jik}^{(0)\mu} = 0. \quad (82)$$

Substituting (80) and (81) into (11)–(14) and (18) we find (in this approximation)

$$A_{ij}^{(0)\mu} = \frac{(z_i^\mu - z_j^\mu)}{\eta_{ji}^2}(\alpha_0 \xi_{ij}^2 + \beta_0 \zeta_{ij}^2) + \frac{2\dot{z}_i^\mu}{\zeta_i^{1/2}\eta_{ji}}\zeta_{ij}^2\left(\frac{\xi_{ij}}{\zeta_{ij}} + \frac{\eta_{ij}}{\eta_{ji}}\right)\beta_0 + \frac{\dot{z}_j^\mu}{\zeta_j^{1/2}\eta_{ji}}\left[\alpha_0\left(\xi_{ij}^2 + 2\xi_{ij}\zeta_{ij}\frac{\eta_{ij}}{\eta_{ji}}\right) - \beta_0\zeta_{ij}^2\right], \quad (83)$$

$$A_{ij}^{(1)\mu} = \frac{2(z_i^\mu - z_j^\mu)}{\eta_{ji}^2}\left(\epsilon_{ji} - \epsilon_{ij}\left(1 + \frac{\xi_{ij}\eta_{ij}}{\zeta_{ij}\eta_{ji}} - \frac{\eta_{ij}^2}{\eta_{ji}^2}\right)\right)(\alpha_1\xi_{ij}^2 + \beta_1\zeta_{ij}^2) + \frac{\dot{z}_i^\mu}{\zeta_i^{1/2}\eta_{ji}}\left[4\epsilon_{ji}\zeta_{ij}^2\left(\frac{\xi_{ij}}{\zeta_{ij}} + \frac{\eta_{ij}}{\eta_{ji}}\right)\beta_1 - \epsilon_{ij}\left(1 - \frac{\eta_{ij}^2}{\eta_{ji}^2}\right)(\alpha_1\xi_{ij}^2 + 3\beta_1\zeta_{ij}^2)\right] + \frac{2\dot{z}_j^\mu}{\zeta_j^{1/2}\eta_{ji}}\left[\epsilon_{ji}\left(\alpha_1\left(2\xi_{ij}\zeta_{ij}\frac{\eta_{ij}}{\eta_{ji}} + \xi_{ij}^2\right) - \beta_1\zeta_{ij}^2\right) - \epsilon_{ij}\left(\alpha_1\left(\xi_{ij}^2 + \xi_{ij}\zeta_{ij}\left(\frac{\eta_{ji}}{\eta_{ij}} - \frac{\eta_{ij}}{\eta_{ji}}\right)\right) + \beta_1\zeta_{ij}^2\right)\right], \quad (84)$$

$$B_{ij}^{(0)\mu\nu} = -2\alpha_0\dot{z}_j^\mu\dot{z}_j^\nu - 2\beta_0\zeta_j\eta^{\mu\nu}, \quad (85)$$

$$C_{ij}^{(0)\mu\nu} = -\frac{(z_i^\mu - z_j^\mu)(z_i^\nu - z_j^\nu)}{\zeta_j\eta_{ji}^2}(\alpha_0\xi_{ij}^2 + \beta_0\zeta_{ij}^2) + \frac{2(z_i^\mu - z_j^\mu)}{\zeta_j^{1/2}\eta_{ji}}(\alpha_0\xi_{ij}\dot{z}_i^\nu + \beta_0\zeta_i\dot{z}_j^\nu) - \frac{2(z_i^\nu - z_j^\nu)\zeta_i^{1/2}\eta_{ij}}{\zeta_j\eta_{ji}^2}(\alpha_0\xi_{ij}\dot{z}_j^\mu + \beta_0\zeta_j\dot{z}_i^\mu) + 2\frac{\eta_{ij}\zeta_i^{1/2}}{\eta_{ji}\zeta_j^{1/2}}(\alpha_0(\eta^{\mu\nu}\xi_{ij} + \dot{z}_j^\mu\dot{z}_i^\nu) + 2\beta_0\dot{z}_i^\mu\dot{z}_j^\nu), \quad (86)$$

$$A_{ijk}^{(0)\mu} = \frac{(z_i^\mu - z_j^\mu)}{\gamma_{ji}}\left(\frac{\zeta_j}{\gamma_{ji}} - \frac{1}{\gamma_{kj}}\left(\xi_{jk} + \frac{\gamma_{jk}}{\gamma_{kj}}\zeta_k\right)\right)(a_0\xi_{ij}\xi_{jk}\xi_{ki} + b_0\zeta_i\zeta_j\zeta_k + c_0\xi_{ki}^2\zeta_j) + \frac{2\dot{z}_i^\mu}{\gamma_{ji}}\left(\zeta_j\frac{\gamma_{ij}}{\gamma_{ji}} + \xi_{ij} - \frac{\gamma_{ij}}{\gamma_{kj}}\left(\xi_{jk} + \frac{\gamma_{jk}}{\gamma_{kj}}\zeta_k\right)\right)\zeta_j\zeta_k b_0 + \frac{\dot{z}_j^\mu}{\gamma_{ji}}\left(\left(\zeta_j\frac{\gamma_{ij}}{\gamma_{ji}} - \frac{\gamma_{ij}}{\gamma_{kj}}\left(\xi_{jk} + \frac{\gamma_{jk}}{\gamma_{kj}}\zeta_k\right)\right)\xi_{jk}\xi_{ki}a_0 - \zeta_i\zeta_j\zeta_k b_0 - \xi_{ki}^2\zeta_j c_0\right) + \frac{\dot{z}_k^\mu}{\gamma_{ji}}\left(\zeta_j\frac{\gamma_{ij}}{\gamma_{ji}} + \xi_{ij} - \frac{\gamma_{ij}}{\gamma_{kj}}\left(\xi_{jk} + \frac{\gamma_{jk}}{\gamma_{kj}}\zeta_k\right)\right)(a_0\xi_{jk}\xi_{ij} + 2c_0\xi_{ki}\zeta_j), \quad (87)$$

$$\tilde{A}^{(0)\mu}_{jik} = \frac{(z^\mu_i - z^\mu_j)\zeta_j}{\gamma^2_{ji}}(a_0\xi_{ji}\xi_{ik}\xi_{kj} + b_0\zeta_j\zeta_i\zeta_k + c_0\xi^2_{kj}\zeta_i) + \frac{(z^\mu_i - z^\mu_k)\zeta_k}{\gamma^2_{ki}}(a_0\xi_{ji}\xi_{ik}\xi_{kj} + b_0\zeta_j\zeta_i\zeta_k + c_0\xi^2_{kj}\zeta_i)$$
$$+ 2\dot{z}^\mu_i\left(\zeta_j\frac{\gamma_{ij}}{\gamma^2_{ji}} + \zeta_k\frac{\gamma_{ik}}{\gamma^2_{ki}} + \frac{\xi_{ij}}{\gamma_{ji}} + \frac{\xi_{ik}}{\gamma_{ki}}\right)(\zeta_j\zeta_k b_0 + \xi^2_{kj}c_0) + \dot{z}^\mu_j\left(\left(\zeta_j\frac{\gamma_{ij}}{\gamma^2_{ji}} + \zeta_k\frac{\gamma_{ik}}{\gamma^2_{ki}} + \frac{\xi_{ik}}{\gamma_{ki}}\right)\xi_{kj}\xi_{ik}a_0 - \frac{1}{\gamma_{ji}}(b_0\zeta_j\zeta_i\zeta_k + c_0\xi^2_{kj}\zeta_i)\right)$$
$$+ \dot{z}^\mu_k\left(\left(\zeta_j\frac{\gamma_{ij}}{\gamma^2_{ji}} + \zeta_k\frac{\gamma_{ik}}{\gamma^2_{ki}} + \frac{\xi_{ij}}{\gamma_{ji}}\right)\xi_{kj}\xi_{ji}a_0 - \frac{1}{\gamma_{ki}}(b_0\zeta_j\zeta_i\zeta_k + c_0\xi^2_{kj}\zeta_i)\right). \tag{88}$$

From (80), (81), and (28) it follows that, at the second post-Minkowskian approximation, the metric tensor associated with the particle i (with non-negligible mass m_i) is given by the formula

$$g^{(i)}_{\mu\nu} = \eta_{\mu\nu} - \frac{2G}{c^2}\sum_{j\neq i} m_j \int d\lambda_j \delta(\rho_{ij})[\alpha_0\dot{z}_{j\mu}\dot{z}_{j\nu} + \beta_0\zeta_j\eta_{\mu\nu}] - \frac{G^2}{c^4}\sum_{j\neq i} m^2_j \int d\lambda_j \frac{\delta(\rho_{ij})}{|\eta_{ji}|}[\alpha_1\dot{z}_{j\mu}\dot{z}_{j\nu} + \beta_1\zeta_j\eta_{\mu\nu}]$$
$$- \frac{G^2 m_i}{c^4}\sum_{j\neq i} m_j \int d\lambda_j \frac{\delta(\rho_{ij})}{|\eta_{ij}|}\Bigg[\alpha_1\dot{z}_{j\mu}\dot{z}_{j\nu} + \beta_1\zeta_j\eta_{\mu\nu} + \left(\frac{\dot{z}_{i\mu}}{\zeta_i} + \frac{(z_{i\mu} - z_{j\mu})}{\gamma_{ij}}\right)(\alpha_1\xi_{ij}\dot{z}_{j\nu} + \beta_1\zeta_j\dot{z}_{i\nu})$$
$$+ \left(\frac{\dot{z}_{i\nu}}{\zeta_i} + \frac{(z_{i\nu} - z_{j\nu})}{\gamma_{ij}}\right)(\alpha_1\xi_{ij}\dot{z}_{j\mu} + \beta_1\zeta_j\dot{z}_{i\mu}) + \frac{1}{2}\Bigg[\left(\frac{\dot{z}_{i\mu}}{\zeta_i} + \frac{(z_{i\mu} - z_{j\mu})}{\gamma_{ij}}\right)\left(\frac{\dot{z}_{i\nu}}{\zeta_i} + \frac{(z_{i\nu} - z_{j\nu})}{\gamma_{ij}}\right)$$
$$+ \frac{1}{\zeta_i}\left(\eta_{\mu\nu} - \frac{2\dot{z}_{i\mu}\dot{z}_{i\nu}}{\zeta_i} + \frac{(z_{i\mu} - z_{j\mu})(z_{i\nu} - z_{j\nu})}{\eta^2_{ij}}\right)\Bigg](\alpha_1\xi^2_{ij} + \beta_1\zeta^2_{ij})\Bigg] - \frac{G^2}{c^4}\sum_{j\neq i}\sum_{k\neq i,j} m_j m_k \iint d\lambda_j d\lambda_k$$
$$\times\Bigg[\delta(\rho_{ij})\delta(\rho_{jk})\left(\frac{1}{2}a_0\xi_{jk}(\dot{z}_{j\mu}\dot{z}_{k\nu} + \dot{z}_{k\mu}\dot{z}_{j\nu}) + b_0\zeta_j\zeta_k\eta_{\mu\nu} + c_0\zeta_j\dot{z}_{k\mu}\dot{z}_{k\nu}\right)$$
$$+ \delta(\rho_{jk})\delta(\rho_{ki})\left(\frac{1}{2}a_0\xi_{jk}(\dot{z}_{k\mu}\dot{z}_{j\nu} + \dot{z}_{j\mu}\dot{z}_{k\nu}) + b_0\zeta_j\zeta_k\eta_{\mu\nu} + c_0\zeta_k\dot{z}_{j\mu}\dot{z}_{j\nu}\right)$$
$$+ \delta(\rho_{ki})\delta(\rho_{ij})\left(\frac{1}{2}a_0\xi_{jk}(\dot{z}_{k\mu}\dot{z}_{j\nu} + \dot{z}_{j\mu}\dot{z}_{k\nu}) + (b_0\zeta_k\zeta_j + c_0\xi^2_{jk})\eta_{\mu\nu}\right)\Bigg]. \tag{89}$$

For the case of a test particle (the mass of which can be neglected) in the presence of N particles with non-negligible masses m_i ($i = 1, \ldots, N$) the above expression simplifies to the following:

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{2G}{c^2}\sum_i m_i \int d\lambda_i \delta((z - z_i)^2)[\alpha_0\dot{z}_{i\mu}\dot{z}_{i\nu} + \beta_0\dot{z}^2_i\eta_{\mu\nu}]$$
$$- \frac{G^2}{c^4}\sum_i m^2_i \int d\lambda_i \frac{\delta((z - z_i)^2)(\dot{z}^2_i)^{1/2}}{|(\dot{z}_i(z - z_i))|}[\alpha_1\dot{z}_{i\mu}\dot{z}_{i\nu} + \beta_1\dot{z}^2_i\eta_{\mu\nu}]$$
$$- \frac{G^2}{c^4}\sum_i\sum_{j\neq i} m_i m_j \iint d\lambda_i d\lambda_j$$
$$\times\Bigg[\delta((z - z_i)^2)\delta((z_i - z_j)^2)\left(\frac{1}{2}a_0(\dot{z}_i\dot{z}_j)(\dot{z}_{i\mu}\dot{z}_{j\nu} + \dot{z}_{j\mu}\dot{z}_{i\nu}) + b_0\dot{z}^2_i\dot{z}^2_j\eta_{\mu\nu} + c_0\dot{z}^2_i\dot{z}_{j\mu}\dot{z}_{j\nu}\right)$$
$$+ \delta((z_i - z_j)^2)\delta((z - z_j)^2)\left(\frac{1}{2}a_0(\dot{z}_i\dot{z}_j)(\dot{z}_{i\mu}\dot{z}_{j\nu} + \dot{z}_{j\mu}\dot{z}_{i\nu}) + b_0\dot{z}^2_i\dot{z}^2_j\eta_{\mu\nu} + c_0\dot{z}^2_j\dot{z}_{i\mu}\dot{z}_{i\nu}\right)$$
$$+ \delta((z - z_j)^2)\delta((z - z_i)^2)\left(\frac{1}{2}a_0(\dot{z}_i\dot{z}_j)(\dot{z}_{i\mu}\dot{z}_{j\nu} + \dot{z}_{j\mu}\dot{z}_{i\nu}) + (b_0\dot{z}^2_i\dot{z}^2_j + c_0(\dot{z}_i\dot{z}_j)^2)\eta_{\mu\nu}\right)\Bigg]. \tag{90}$$

VII. THE FIRST POST-NEWTONIAN APPROXIMATION

The equations of motion (82) involve multiple times. The force acting on mass i depends on the state of motion of particle i at time t and, to account for the time needed for the transmission of the interactions, on the states of motion of the remaining $N - 1$ particles at the past and future times $t^{(i,s)}_j$ ($j \neq i$, $s = -, +$) and also on $t^{(j,s)}_k$ ($k \neq i, j$, $s = -, +$).

Using Taylor series expansions involving the particles' present motions at time t, one can rewrite Eqs. (82) using just the one time variable t [56]. We use series expansions up to terms of second order ($\frac{v^2}{c^2}$) (1PN).

From the definition (5) it follows that

$$d\lambda_i = \frac{cdt(1-\frac{v_i^2}{c^2})^{1/2}}{\zeta_i^{1/2}}, \quad (91)$$

$$d\lambda_j = \frac{cdt_j(1-\frac{v_j^2(t_j)}{c^2})^{1/2}}{\zeta_j^{1/2}}. \quad (92)$$

The Dirac delta function can be expressed as follows [10]:

$$\delta(c^2(t-t_j)^2 - (\vec{r}_i - \vec{r}_j)^2) = \frac{1}{2c}\left(\frac{\delta(t_j - t_j^{(i,-)})}{(R_{ij}^{\rm ret} - \frac{(\vec{R}_{ij}^{\rm ret}\vec{v}_j^{(i,-)})}{c})} + \frac{\delta(t_j - t_j^{(i,+)})}{(R_{ij}^{\rm adv} + \frac{(\vec{R}_{ij}^{\rm adv}\vec{v}_j^{(i,+)})}{c})}\right). \quad (93)$$

In (93), $t_j^{(i,s)}$ $(s=-,+)$ are the two roots of the equation:

$$c^2(t-t_j)^2 - (\vec{r}_i(t) - \vec{r}_j(t_j))^2 = 0 \quad (94)$$

and

$$R_{ij}^{\rm ret} = c(t - t_j^{(i,-)}), \quad (95)$$

$$R_{ij}^{\rm adv} = c(t_j^{(i,+)} - t), \quad (96)$$

$$\vec{R}_{ij}^{\rm ret} = \vec{r}_i - \vec{r}_j^{(i,-)}, \quad (97)$$

$$\vec{R}_{ij}^{\rm adv} = \vec{r}_i - \vec{r}_j^{(i,+)}. \quad (98)$$

$t - t_j^{(i,-)}$ is the time it takes for a signal to travel forward in time at the speed of light from particle j to particle i.

$t_j^{(i,+)} - t$ is the time it takes for a signal to travel backward in time at the speed of light from particle j to particle i.

To terms of second order we can write

$$\vec{r}_i - \vec{r}_j^{(i,-)} \approx \vec{r}_{ij} + \vec{v}_j\frac{r_{ij}}{c} + \vec{v}_j\frac{(\vec{r}_{ij}\vec{v}_j)}{c^2} - \vec{a}_j\frac{r_{ij}^2}{2c^2}, \quad (99)$$

$$\vec{r}_i - \vec{r}_j^{(i,+)} \approx \vec{r}_{ij} - \vec{v}_j\frac{r_{ij}}{c} + \vec{v}_j\frac{(\vec{r}_{ij}\vec{v}_j)}{c^2} - \vec{a}_j\frac{r_{ij}^2}{2c^2}, \quad (100)$$

$$\vec{v}_j^{(i,-)} \approx \vec{v}_j - \vec{a}_j\frac{r_{ij}}{c}, \quad (101)$$

$$\vec{v}_j^{(i,+)} \approx \vec{v}_j + \vec{a}_j\frac{r_{ij}}{c}. \quad (102)$$

From (94)–(102), we find (to terms of second order)

$$\frac{(1-\frac{v_j^{(i,-)2}}{c^2})^{1/2}}{(R_{ij}^{\rm ret} - \frac{(\vec{R}_{ij}^{\rm ret}\vec{v}_j^{(i,-)})}{c})} \approx \frac{1}{r_{ij}}\left(1 - \frac{(\vec{n}_{ij}\vec{v}_j)^2}{2c^2} - \frac{(\vec{r}_{ij}\vec{a}_j)}{2c^2}\right), \quad (103)$$

$$\frac{(1-\frac{v_j^{(i,+)2}}{c^2})^{1/2}}{(R_{ij}^{\rm adv} + \frac{(\vec{R}_{ij}^{\rm adv}\vec{v}_j^{(i,+)})}{c})} \approx \frac{1}{r_{ij}}\left(1 - \frac{(\vec{n}_{ij}\vec{v}_j)^2}{2c^2} - \frac{(\vec{r}_{ij}\vec{a}_j)}{2c^2}\right). \quad (104)$$

From the definitions (3)–(5), (38), and (53), it is not difficult to see that, to terms of second order, we can write

$$\frac{\xi_{ij}}{\zeta_{ij}} \approx 1 + \frac{v_i^2}{2c^2} + \frac{v_j^2}{2c^2} - \frac{(\vec{v}_i\vec{v}_j)}{c^2}, \quad (105)$$

$$\eta_{ji}^{(i,-)} \approx r_{ij}\left(1 + \frac{(\vec{n}_{ij}\vec{v}_j)^2}{2c^2} + \frac{(\vec{r}_{ij}\vec{a}_j)}{2c^2}\right), \quad (106)$$

$$\eta_{ji}^{(i,+)} \approx -r_{ij}\left(1 + \frac{(\vec{n}_{ij}\vec{v}_j)^2}{2c^2} + \frac{(\vec{r}_{ij}\vec{a}_j)}{2c^2}\right), \quad (107)$$

$$\eta_{ij}^{(i,-)} \approx -r_{ij}\left(1 - \frac{(\vec{n}_{ij}\vec{v}_i)}{c} + \frac{(\vec{n}_{ij}\vec{v}_j)}{c} + \frac{(\vec{v}_i - \vec{v}_j)^2}{2c^2} + \frac{(\vec{n}_{ij}\vec{v}_j)^2}{2c^2} - \frac{(\vec{r}_{ij}\vec{a}_j)}{2c^2}\right), \quad (108)$$

$$\eta_{ij}^{(i,+)} \approx r_{ij}\left(1 + \frac{(\vec{n}_{ij}\vec{v}_i)}{c} - \frac{(\vec{n}_{ij}\vec{v}_j)}{c} + \frac{(\vec{v}_i - \vec{v}_j)^2}{2c^2} + \frac{(\vec{n}_{ij}\vec{v}_j)^2}{2c^2} - \frac{(\vec{r}_{ij}\vec{a}_j)}{2c^2}\right). \quad (109)$$

In (99)–(109), $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ is the relative position of particle i with respect to particle j, $\vec{n}_{ij} \equiv \frac{\vec{r}_{ij}}{r_{ij}}$, $\vec{v}_i$ is the velocity of particle i, and $\vec{v}_j$, $\vec{a}_j$ the velocity and the acceleration of particle j. All these quantities are given at time t.

From (25) and (80) at the first post-Minkowskian order, for the solutions of the equations of motion we obtain

$$\zeta_i = 1 + \frac{2G}{c^2}\sum_{j\neq i} m_j \int d\lambda_j \delta(\rho_{ij})\zeta_j\left(\alpha_0\frac{\xi_{ij}^2}{\zeta_{ij}^2} + \beta_0\right). \quad (110)$$

Now, substituting (92), (93), and (103)–(105) into (110), to terms of second order, for the solutions of the equations of motion we can write

$$\zeta_i \approx 1 + \frac{2G}{c^2}(\alpha_0 + \beta_0)\sum_{j\neq i}\frac{m_j}{r_{ij}}. \quad (111)$$

Substituting (91)–(109) and (111) into (82)–(88) we find the equations of motion to terms of second order (in $\frac{v^2}{c^2}$) (first post-Newtonian approximation):

$$\vec{a}_i + G(\alpha_0+\beta_0)\sum_{j\neq i}\frac{m_j}{r_{ij}^2}\vec{n}_{ij} + \frac{v_i^2}{c^2}\vec{a}_i + \frac{(\vec{v}_i\vec{a}_i)}{c^2}\vec{v}_i + \frac{2G\alpha_0}{c^2}\vec{a}_i\sum_{j\neq i}\frac{m_j}{r_{ij}} - \frac{G}{2c^2}(3\alpha_0-\beta_0)\sum_{j\neq i}\frac{m_j}{r_{ij}}\vec{a}_j$$
$$-\frac{G}{2c^2}(\alpha_0+\beta_0)\sum_{j\neq i}\frac{m_j}{r_{ij}}\vec{n}_{ij}(\vec{n}_{ij}\vec{a}_j) + \frac{G}{c^2}\sum_{j\neq i}\frac{m_j}{r_{ij}^2}\vec{n}_{ij}\left(\alpha_0(v_i^2+v_j^2-2(\vec{v}_i\vec{v}_j)) - \frac{3}{2}(\alpha_0+\beta_0)(\vec{n}_{ij}\vec{v}_j)^2\right)$$
$$+\frac{G}{c^2}(\beta_0-\alpha_0)\vec{v}_i\sum_{j\neq i}\frac{m_j}{r_{ij}^2}((\vec{n}_{ij}\vec{v}_i)-(\vec{n}_{ij}\vec{v}_j)) + \frac{G}{c^2}\sum_{j\neq i}\frac{m_j}{r_{ij}^2}\vec{v}_j(2\alpha_0(\vec{n}_{ij}\vec{v}_i)-(\alpha_0-\beta_0)(\vec{n}_{ij}\vec{v}_j))$$
$$+\frac{G^2m_i}{c^2}((\alpha_0+\beta_0)^2+\alpha_1+\beta_1)\sum_{j\neq i}\frac{m_j}{r_{ij}^3}\vec{n}_{ij} + \frac{G^2}{c^2}(2(\alpha_0+\beta_0)^2+\alpha_1+\beta_1)\sum_{j\neq i}\frac{m_j^2}{r_{ij}^3}\vec{n}_{ij}$$
$$+\frac{G^2}{c^2}\sum_{j\neq i}\sum_{k\neq i,j}\frac{m_jm_k}{r_{ij}^2}\vec{n}_{ij}\left(\frac{1}{r_{ik}}(2(\alpha_0+\beta_0)^2+a_0+b_0+c_0)+\frac{1}{r_{jk}}((\alpha_0+\beta_0)^2+a_0+b_0+c_0)\right)=0. \tag{112}$$

Complete agreement with the equations of motion of general relativity [57,58], at the first post-Newtonian order, is achieved if

$$\alpha_0 = 2, \tag{113}$$

$$\beta_0 = -1, \tag{114}$$

$$\alpha_1 + \beta_1 = -2, \tag{115}$$

$$a_0 + b_0 + c_0 = -2. \tag{116}$$

VIII. CONCLUSIONS

We have obtained Lorentz invariant equations of motion describing the gravitational interactions of a system consisting of N point masses. The equations are derived explicitly from a Lorentz invariant action. Contrary to general relativity, which is a field theory, the model presented here is a relativistic action-at-a-distance description (the interactions are not mediated by a field). We have shown that the equations of motion for N point masses agree with those of general relativity at the first post-Newtonian approximation. Agreement with general relativity for the N body problem at orders beyond 1.5PN has not been established. The model presented is in agreement with general relativity for the one-body case, at all orders. At the first post-Minkowskian approximation our model reduces to the model of Havas and Goldberg [51,52], which is known to be in agreement with general relativity in this approximation. Because of this agreement, gravitational radiation effects in our model begin to appear at the 2.5PN order ($\frac{v^5}{c^5}$) [59,60].

[1] K. Schwarzschild, Gottingen Nachrichten **128**, 132 (1903).
[2] H. Tetrode, Z. Phys. **10**, 317 (1922).
[3] A. D. Fokker, Physica **12**, 145 (1932).
[4] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945).
[5] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **21**, 425 (1949).
[6] R. A. Moore, T. C. Scott, and M. B. Monagan, Phys. Rev. Lett. **59**, 525 (1987).
[7] F. Hoyle and J. V. Narlikar, *Lectures on Cosmology and Action at a Distance Electrodynamics* (World Scientific, Singapore, 1996).
[8] A. Schild, Phys. Rev. **131**, 2762 (1963).
[9] D. J. Louis-Martinez, Phys. Lett. A **320**, 103 (2003).
[10] A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Dover Publications Inc., New York, 1980).
[11] J. W. Dettman and A. Schild, Phys. Rev. **95**, 1057 (1954).
[12] C. M. Andersen and H. C. von Baeyer, Ann. Phys. (N.Y.) **60**, 67 (1970).
[13] A. Schild, Ann. Phys. (N.Y.) **93**, 88 (1975).
[14] P. Ramond, Phys. Rev. D **7**, 449 (1973).
[15] A. Katz, J. Math. Phys. (N.Y.) **10**, 1929 (1969); **10**, 2215 (1969).
[16] A. Degasperis, Phys. Rev. D **3**, 273 (1971).
[17] J. M. Cornwall, Nucl. Phys. **B128**, 75 (1977).
[18] A. Rivacoba, Nuovo Cimento Soc. Ital. Fis. B **84**, 35 (1984).
[19] J. Weiss, J. Math. Phys. (N.Y.) **27**, 1015 (1986).
[20] M. Kalb and P. Ramond, Phys. Rev. D **9**, 2273 (1974).
[21] B. Jensen and U. Lindstrom, Phys. Lett. B **398**, 83 (1997).
[22] R. N. Hill, J. Math. Phys. (N.Y.) **8**, 201 (1967).
[23] E. H. Kerner, J. Math. Phys. (N.Y.) **9**, 222 (1968).
[24] Ph. Droz-Vincent, Lett. Nuovo Cimento **1**, 839 (1969).
[25] L. Bel and X. Fustero, Ann. Inst. Henri Poincaré A **25**, 411 (1976).

[26] I. T. Todorov, in *Relativistic Action at a Distance: Classical and Quantum Aspects*, Lectures Notes in Physics 162, edited by J. Llosa (Springer-Verlag, Berlin, 1982).
[27] L. P. Horwitz and F. Rohrlich, Phys. Rev. D **24**, 1528 (1981).
[28] H. Sazdjian, Ann. Phys. (N.Y.) **136**, 136 (1981).
[29] G. Longhi, D. Dominici, J. Gomis, and J. A. Lobo, in *Relativistic Action at a Distance: Classical and Quantum Aspects*, Lectures Notes in Physics 162, edited by J. Llosa (Springer-Verlag, Berlin, 1982).
[30] J. Bijtebier, Nucl. Phys. **A696**, 581 (2001).
[31] Ph. Droz-Vincent, Int. J. Theor. Phys. **42**, 1809 (2003).
[32] H. Poincare, Rend. Circ. Mat. Palermo **21**, 129 (1906).
[33] W. De Sitter, Mon. Not. R. Astron. Soc. Pac. Conf. Ser. **71**, 388 (1911).
[34] A. N. Whitehead, *The Principle of Relativity* (Cambridge Univ. Press, Cambridge, 1922).
[35] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge Univ. Press, Cambridge, 1924).
[36] J. L. Synge, Proc. R. Soc. A **211**, 303 (1952).
[37] J. Dyer and A. Schild, J. Math. Anal. Appl. **4**, 328 (1962).
[38] F. Hoyle and J. V. Narlikar, Proc. R. Soc. A **282**, 191 (1964).
[39] S. W. Hawking, Proc. R. Soc. A **286**, 313 (1965).
[40] P. C. W. Davis, Nature (London) **228**, 270 (1970).
[41] E. G. Harris, Am. J. Phys. **49**, 1051 (1981).
[42] Yu. S. Vladimirov and A. Yu. Turygin, *Theory of Direct Interparticle Interaction* (Energoatomizdat, Moscow, 1986, in Russian).
[43] A. Yu. Turygin, Gen. Relativ. Gravit. **18**, 333 (1986).
[44] R. P. Gaida, Yu. B. Klyuchkovskii, and V. I. Tretyak, Russ. Phys. J. **33**, 40 (1990).
[45] G. N. Afanasiev and R. A. Asanov, Phys. Part. Nucl. **27**, 296 (1996).
[46] Yu. S. Vladimirov, Gravitation and Cosmology **14**, 41 (2008).
[47] G. J. Whitrow and G. E. Morduch, in *Vistas in Astronomy*, edited by A. Beer (Pergamon, New York, 1965), Vol. 6, p. 1.
[48] G. Gibbons and C. M. Will, Stud. Hist. Phil. Mod. Phys. **39**, 41 (2008).
[49] D. J. Louis-Martinez, Phys. Lett. B **632**, 733 (2006).
[50] D. J. Louis-Martinez, Found. Phys. **42**, 215 (2012).
[51] J. N. Goldberg, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
[52] P. Havas and J. N. Goldberg, Phys. Rev. **128**, 398 (1962).
[53] J. L. Anderson, *Principles of Relativity Physics* (Academic Press, New York, 1967).
[54] J. Friedman and K. Uryu, Phys. Rev. D **73**, 104039 (2006).
[55] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[56] D. J. Louis-Martinez, Phys. Lett. A **364**, 93 (2007).
[57] A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 65 (1938).
[58] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Butterworth- Heinenann, Oxford, 1996), 4th revised English ed.
[59] S. F. Smith and P. Havas, Phys. Rev. **138**, B495 (1965).
[60] M. Walker and C. M. Will, Astrophys. J. **242**, L129 (1980).